MAST10018
Linear Algebra
Extension Studies

Edward Wang

Semester 1, 2024

Contents

1 Matrices

A **matrix** is a rectangular array of numbers. If a matrix has m rows and n columns, then we call it a $(m \times n)$-matrix. The set of all square $(n \times n)$-matrices with real entries is denoted $M_n(\mathbb{R})$. The entry in the i^{th} row and j^{th} column of a matrix A is denoted A_{ij}.

1.1 Matrix multiplication

Matrix multiplication can be thought of as taking the dot product of the rows of the first matrix with the columns of the second matrix.

Definition (Matrix multiplication)

Given a $(m \times n)$-matrix A and a $(n \times p)$-matrix B, their product is a $(m \times p)$-matrix AB, where the entries are

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Example 1.1

We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

Remark — The matrix product AB is only defined if the number of columns of A is equal to the number of rows of B.

1.2 Matrix transpose

The matrix transpose is a useful operation on a single matrix. There are many properties of the matrix transpose.

Definition

Given a matrix A, the **transpose** of A is denoted A^T, such that

$$(A^T)_{ij} = A_{ji}.$$

Definition

A matrix A is **symmetric** if $A = A^T$.

Theorem 1.1

For matrices A and B where AB is defined, we have

$$(AB)^T = B^T A^T.$$

Proof. We know by definition that $((AB)^T)_{ij} = (AB)_{ji}$. Using the definition of matrix multiplication, we have

$$\begin{aligned}(AB)_{ji} &= \sum_{k=1}^{n} A_{jk}B_{ki} \\ &= \sum_{k=1}^{n} B_{ki}A_{jk} \\ &= \sum_{k=1}^{n} (B^T)_{ik}(A^T)_{kj} \\ &= (B^T A^T)_{ij}.\end{aligned}$$

As this holds for all i, j, we must have that $(AB)^T = B^T A^T$. □

1.3 Trace

The **trace** of a square matrix A is the sum of the diagonal entries of A, and is denoted $\mathrm{tr}(A)$.

Theorem 1.2

Given $(n \times n)$-matrices A and B, we have

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

1.4 Matrix inverses

A square matrix A has an **inverse** if there exists a matrix B such that $AB = BA = I$, where I is the identity matrix.

Remark — Matrix multiplication is **not** commutative, that is $AB \neq BA$ in general.

If there exists such a matrix B, then we call it the **inverse** of A, and denote it $B = A^{-1}$. A matrix is **invertible** if it has an inverse, and **singular** if it does not have an inverse.