Quantum many-body spin Hamiltonians

Learning outcomes

- Understand the difference between classical and quantum magnetism
- Rationalize the meaning of expectation values in a quantum magnet
- Identify phase transitions in quantum magnets

From classical to quantum magnetism

The Ising dimer

What is the ground state of this Hamiltonian

$$\mathcal{H} = S_0^z S_1^z$$

The Hamiltonian has two ground states (related by time-reversal symmetry)

$$|GS_1\rangle = |\uparrow\downarrow\rangle \qquad |GS_2\rangle = |\downarrow\uparrow\rangle$$

Each ground state breaks time-reversal symmetry

A symmetry broken antiferromagnet is a macroscopic version of this

The quantum Heisenberg dimer

What is the ground state of this quantum Hamiltonian?

$$\mathcal{H} = \vec{S}_0 \cdot \vec{S}_1$$

The ground state is unique, and does not break time-reversal

$$|GS\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle] \qquad \langle \vec{S}_i \rangle = 0$$

The state is maximally entangled

How does this generalize to a macroscopic system?

The quantum many-body problem

Let us go back to a simple many-body problem

$$\mathcal{H} = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

And let us imagine that we have L different sites on our system and S=1/2

For example, for L=2 sites the elements of the basis are

$$|\uparrow\uparrow\rangle \quad |\uparrow\downarrow\rangle \quad |\downarrow\uparrow\rangle \quad |\downarrow\downarrow\rangle$$

For L=3 sites the elements of the basis are

$$|\uparrow\uparrow\uparrow\rangle \quad |\uparrow\uparrow\downarrow\rangle \quad |\uparrow\downarrow\uparrow\rangle \quad |\uparrow\downarrow\downarrow\rangle$$

$$|\downarrow\uparrow\uparrow\rangle \quad |\downarrow\uparrow\downarrow\rangle \quad |\downarrow\downarrow\uparrow\rangle \quad |\downarrow\downarrow\downarrow\rangle$$

The quantum many-body problem

Let us go back to a simple many-body problem

$$\mathcal{H} = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

And let us imagine that we have L different sites on our system and S=1/2

For L=4 sites, the elements of the basis are

$$
\begin{array}{cccc}
|\uparrow\uparrow\uparrow\uparrow\rangle & |\uparrow\uparrow\uparrow\downarrow\rangle & |\uparrow\uparrow\downarrow\uparrow\rangle & |\uparrow\uparrow\downarrow\downarrow\rangle \\
|\uparrow\downarrow\uparrow\uparrow\rangle & |\uparrow\downarrow\uparrow\downarrow\rangle & |\uparrow\downarrow\downarrow\uparrow\rangle & |\uparrow\downarrow\downarrow\downarrow\rangle \\
|\downarrow\uparrow\uparrow\uparrow\rangle & |\downarrow\uparrow\uparrow\downarrow\rangle & |\downarrow\uparrow\downarrow\uparrow\rangle & |\downarrow\uparrow\downarrow\downarrow\rangle \\
|\downarrow\downarrow\uparrow\uparrow\rangle & |\downarrow\downarrow\uparrow\downarrow\rangle & |\downarrow\downarrow\downarrow\uparrow\rangle & |\downarrow\downarrow\downarrow\downarrow\rangle
\end{array}
$$

The quantum many-body problem

Let us go back to a simple many-body problem

$$\mathcal{H} = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

And let us imagine that we have L different sites on our system and S=1/2

The dimension of the Hilbert space grows as

$$d = 2^L$$

The quantum many-body problem

Let us go back to a simple many-body problem

$$\mathcal{H} = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

A typical wavefunction is written as

$$|\Psi\rangle = \sum c_{s_1, s_2, \ldots, s_L} |s_1, s_2, \ldots s_L\rangle$$

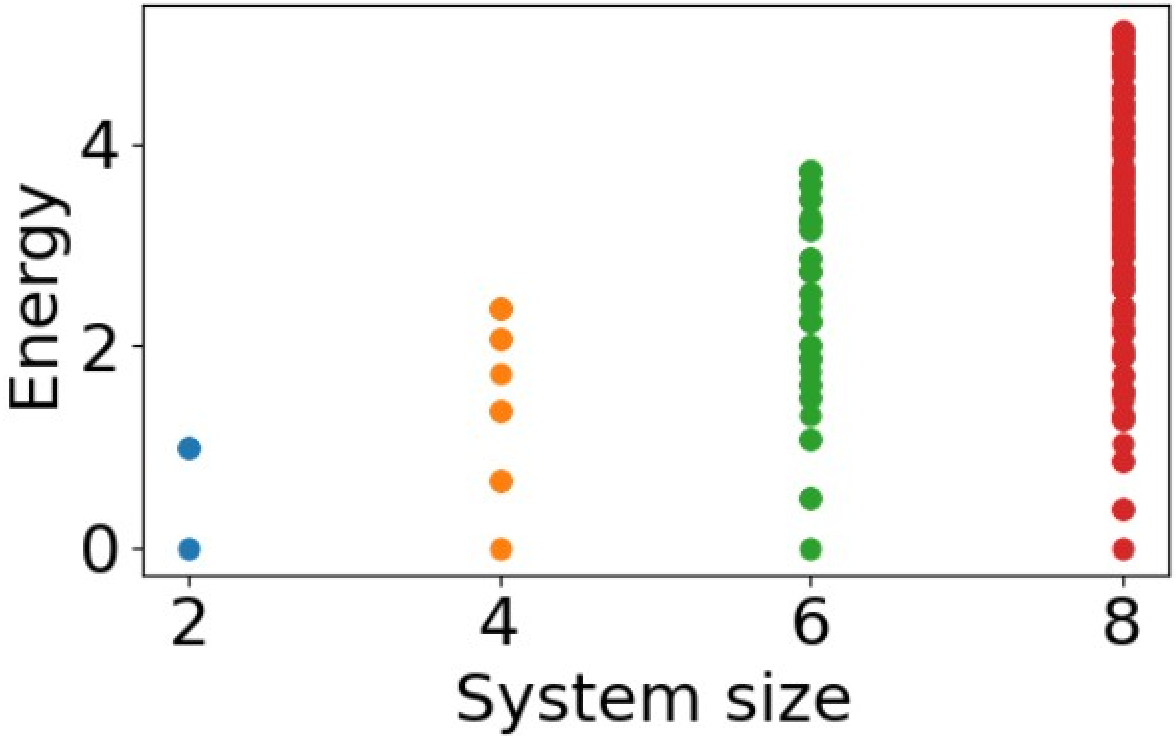
We need to determine in total 2^L coefficients

The Hamiltonian is a $2^L \times 2^L$ matrix

Understanding the many-body spectra

Energies of a many-body model

Let us take a Heisenberg model $H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1}$



From classical to quantum magnetism

Let us take a modified Heisenberg model

$$H = \sum_n J_{xy} S_n^x S_{n+1}^x + J_{xy} S_n^y S_{n+1}^y + S_n^z S_{n+1}^z$$

For $J_{xy} = 0$ we have an Ising model

$$H = \sum_n S_n^z S_{n+1}^z$$

For $J_{xy} = 1$ we have a Heisenberg model

$$H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1}$$

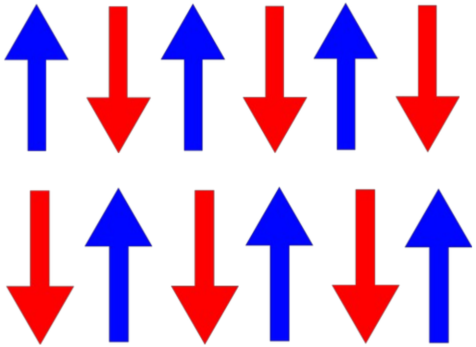
From classical to quantum magnetism

Let us take a modified Heisenberg model

$$H = \sum_n J_{xy} S_n^x S_{n+1}^x + J_{xy} S_n^y S_{n+1}^y + S_n^z S_{n+1}^z$$

Ising model

Two ground states



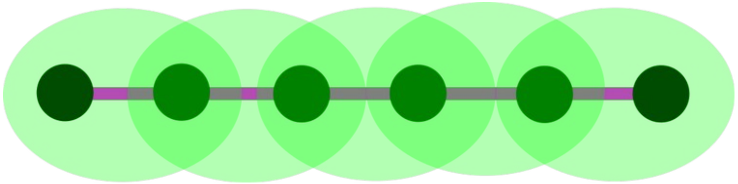
Transition at

$$J_{xy} = 0.5$$

in the thermodynamic limit

Heisenberg model

One ground state



From classical to quantum magnetism

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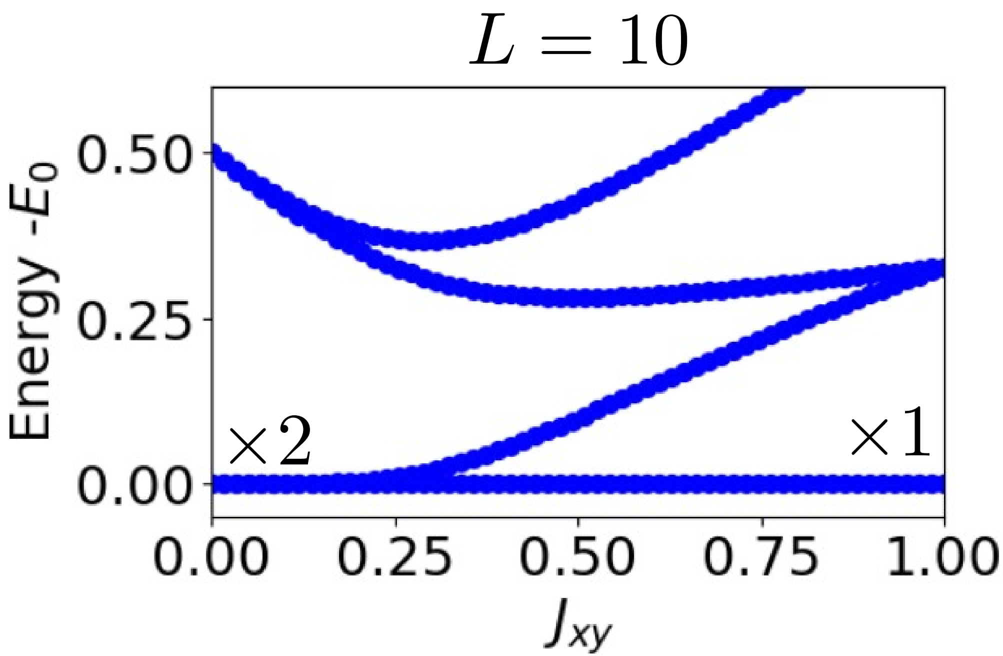
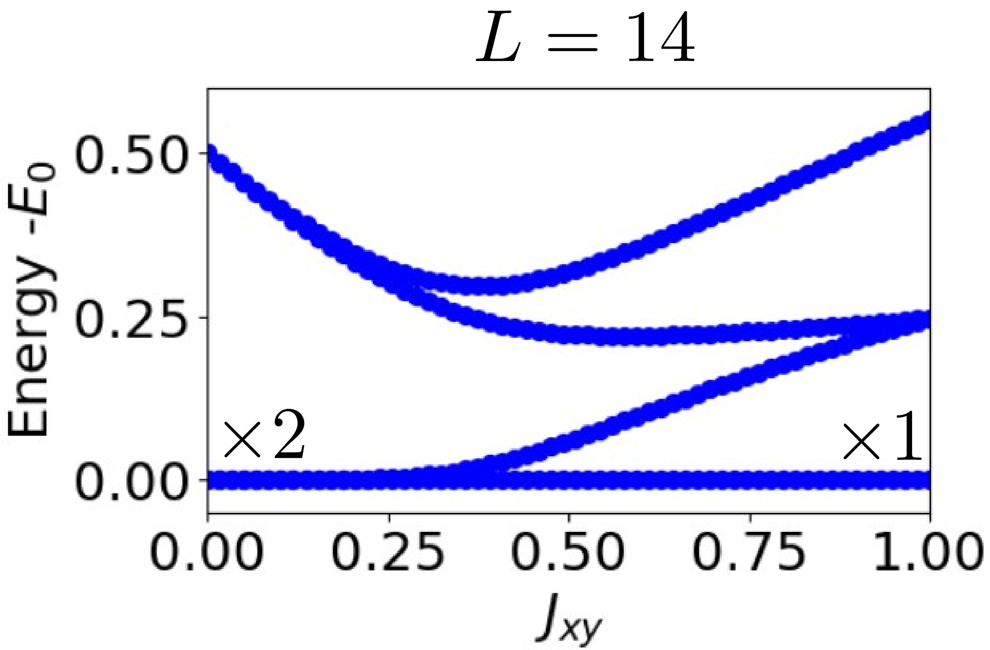




A minimal quantum magnet

Let us start by taking the Heisenberg dimer

$$H = \vec{S}_0 \cdot \vec{S}_1$$

$$S = 1/2$$

We can define a collective spin as

$$\vec{L} = \vec{S}_0 + \vec{S}_1$$

So that the Hamiltonian takes the form

$$H \sim \vec{L} \cdot \vec{L} \sim l(l+1)$$

$$L = 0, 1$$

$l = 0$ singlet

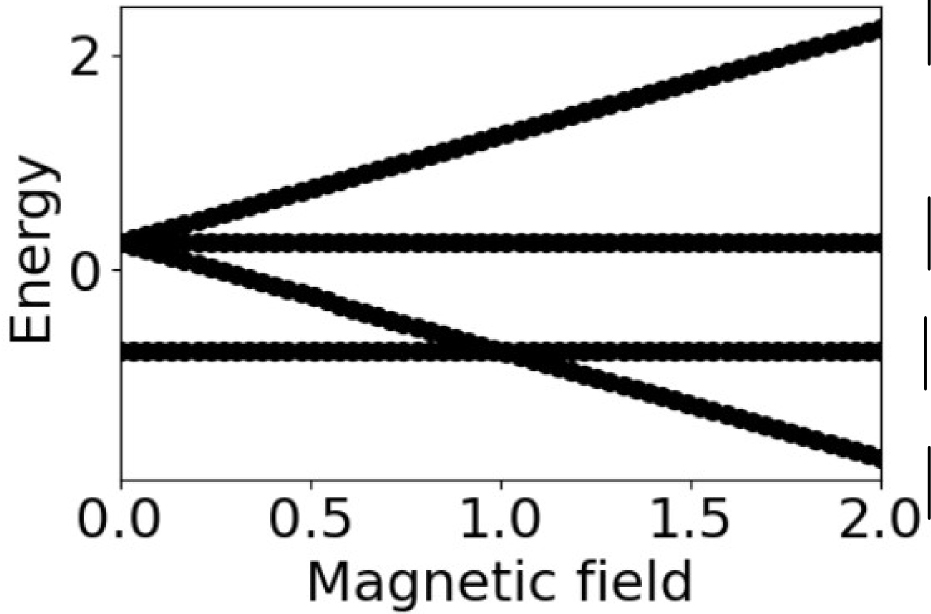
$l = 1$ triplet

Energies of a many-body model

Let us look how the energies depend on the external magnetic field

$$H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1} + B_z \sum_n S_n^z$$

$$L = 2$$



$$|T,+1\rangle = |\uparrow\uparrow\rangle$$

$$|T,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|S\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

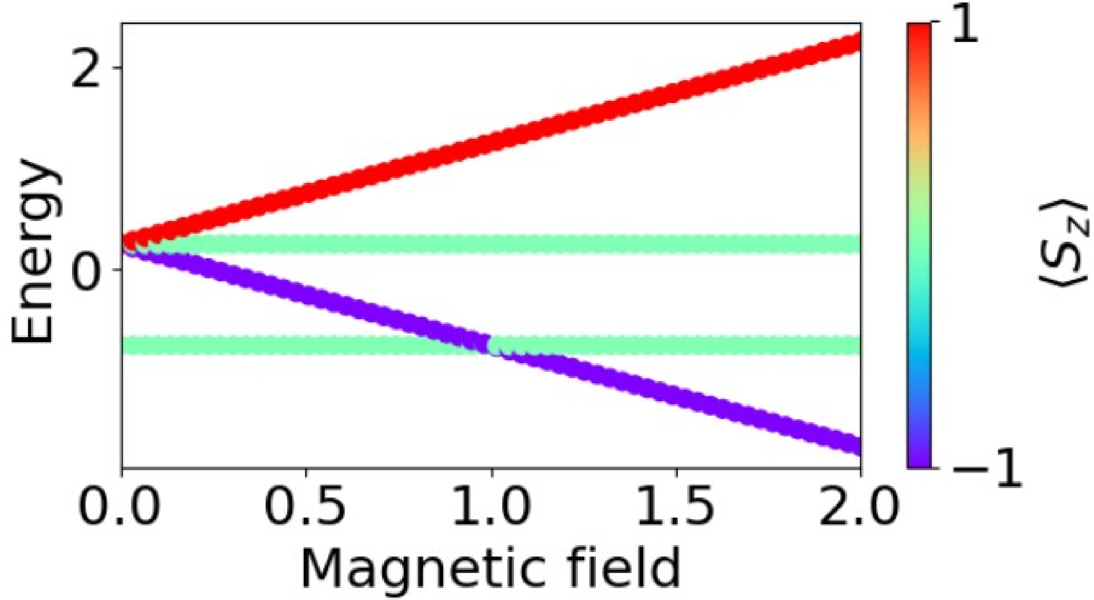
$$|T,-1\rangle = |\downarrow\downarrow\rangle$$

Energies of a many-body model

Let us look how the energies depend on the external magnetic field

$$\langle S_z \rangle = \langle \Omega_n | \sum_n S_n^z | \Omega_n \rangle \qquad H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1} + B_z \sum_n S_n^z$$

$$L = 2$$



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$$|S\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

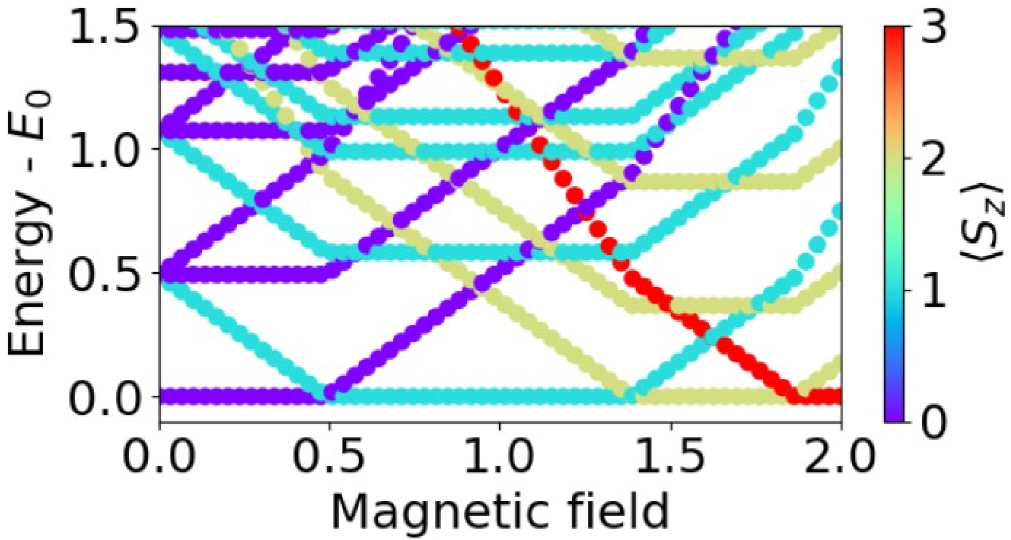
$$|T,-1\rangle = |\downarrow\downarrow\rangle$$

Energies of a many-body model

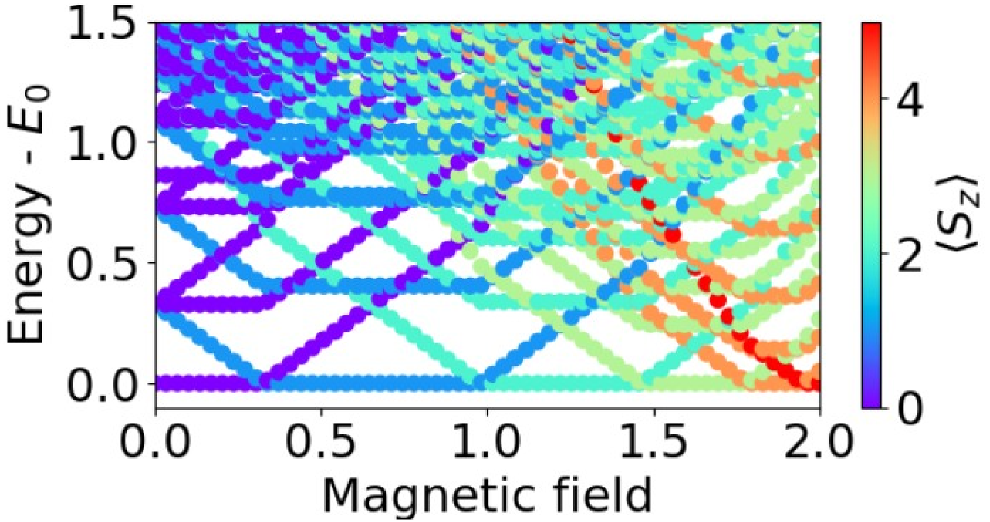
Let us now look at bigger systems

$$H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1} + B_z \sum_n S_n^z$$

$$L = 6$$



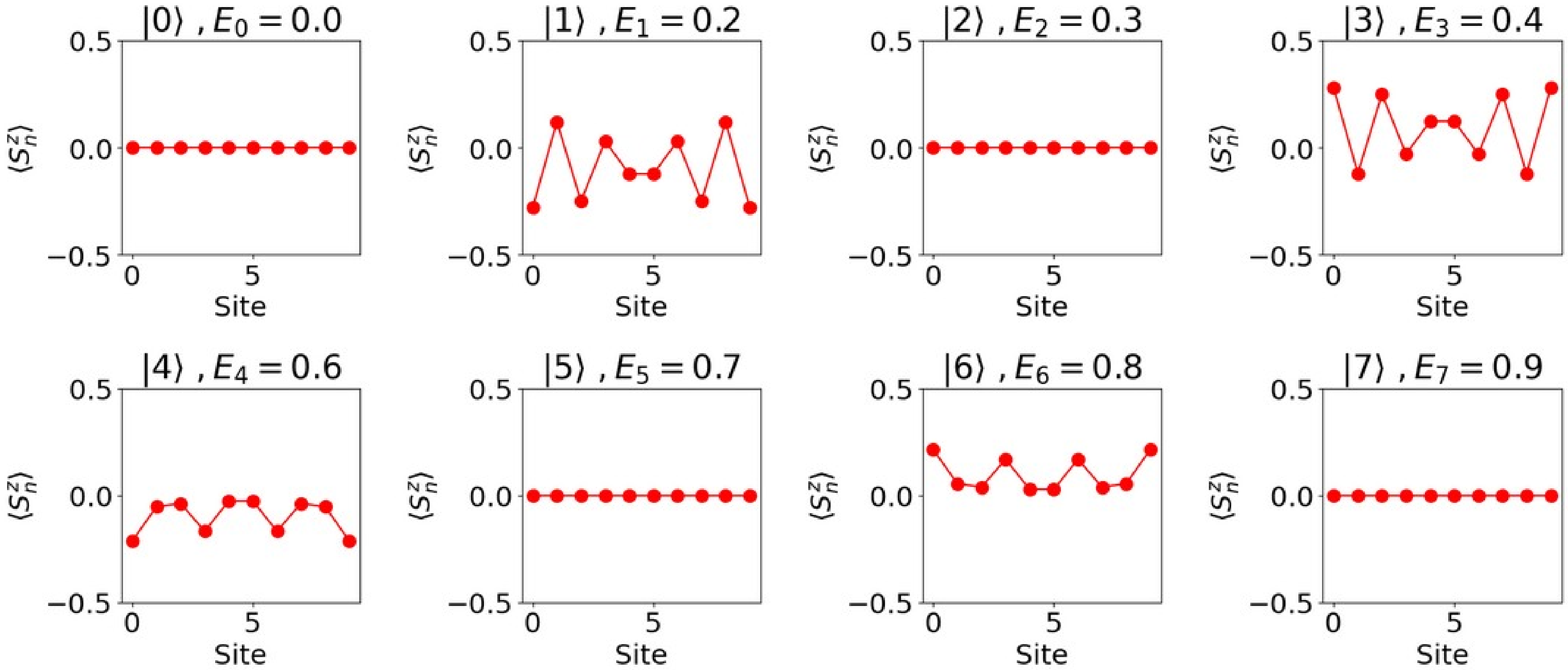
$$L = 10$$



Excitation energies and magnetization

Let us now consider a system with L=10

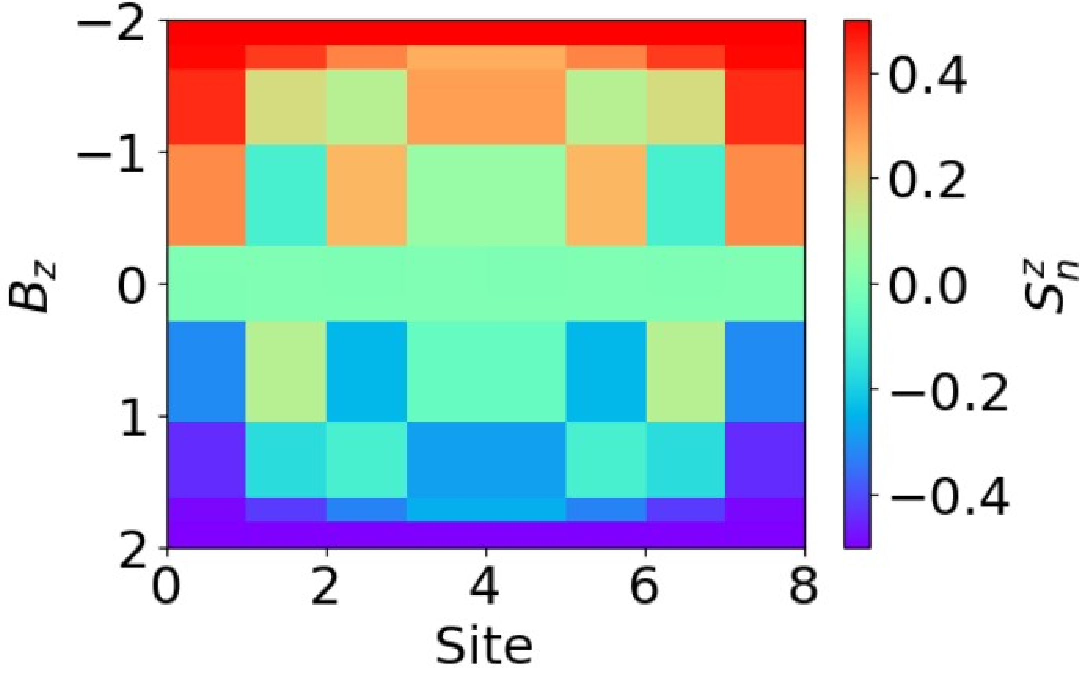
$$H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1} + B_z \sum_n S_n^z$$



Ground state of a many-body model

Let us look how the ground state depends on the external magnetic field

$$H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1} + B_z \sum_n S_n^z$$



$$H|\Omega_n\rangle = E_0|\Omega_n\rangle$$

$$\langle S_n^z\rangle = \langle \Omega_n|S_n^z|\Omega_n\rangle$$

Dynamical spin correlators

The many-body excitations of spin Hamiltonian can be characterized by the dynamical spin correlator

$$A(\omega) = \langle GS|S_n^z \delta(\omega - H + E_{GS}) S_n^z|GS\rangle$$

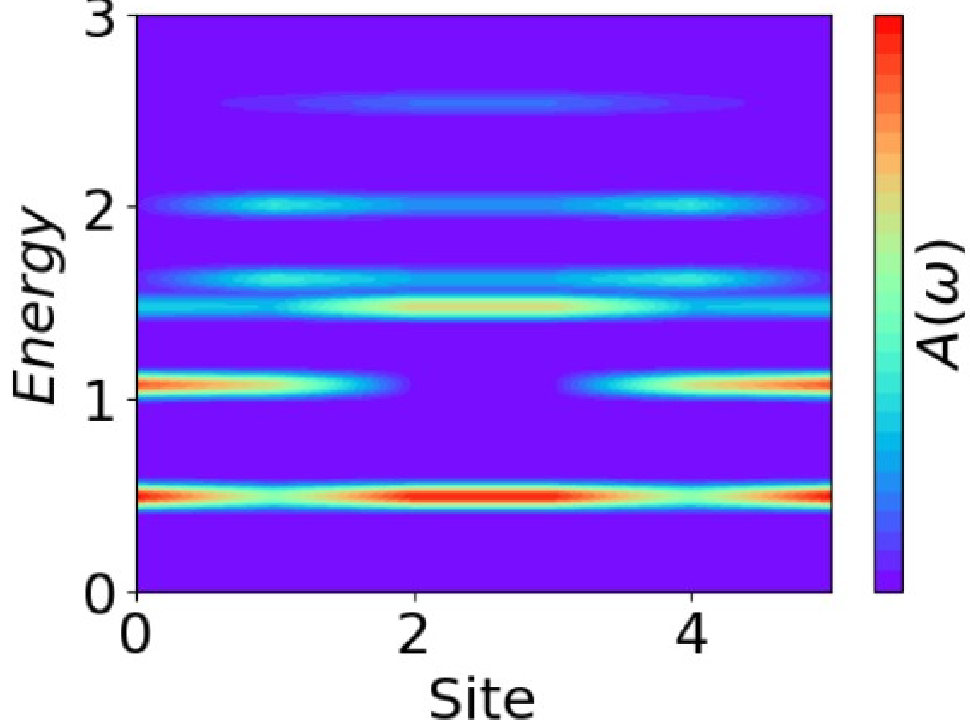
The spectral function above signal excited states that have one more spin excitation than the ground state

$$\delta(\omega - H + E_{GS}) = |\alpha\rangle\langle\alpha|\delta(\omega - E_\alpha + E_{GS})$$

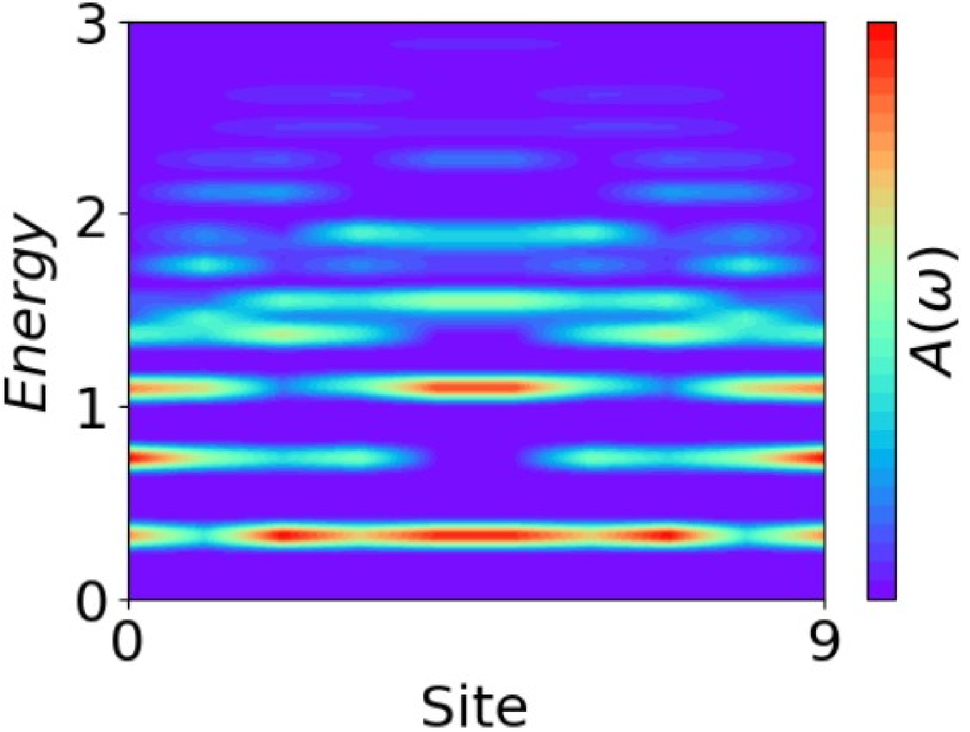
Dynamical correlator of the Heisenberg model

Lets look at the Heisenberg model with $S = 1/2$ $H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1}$

$L = 6$



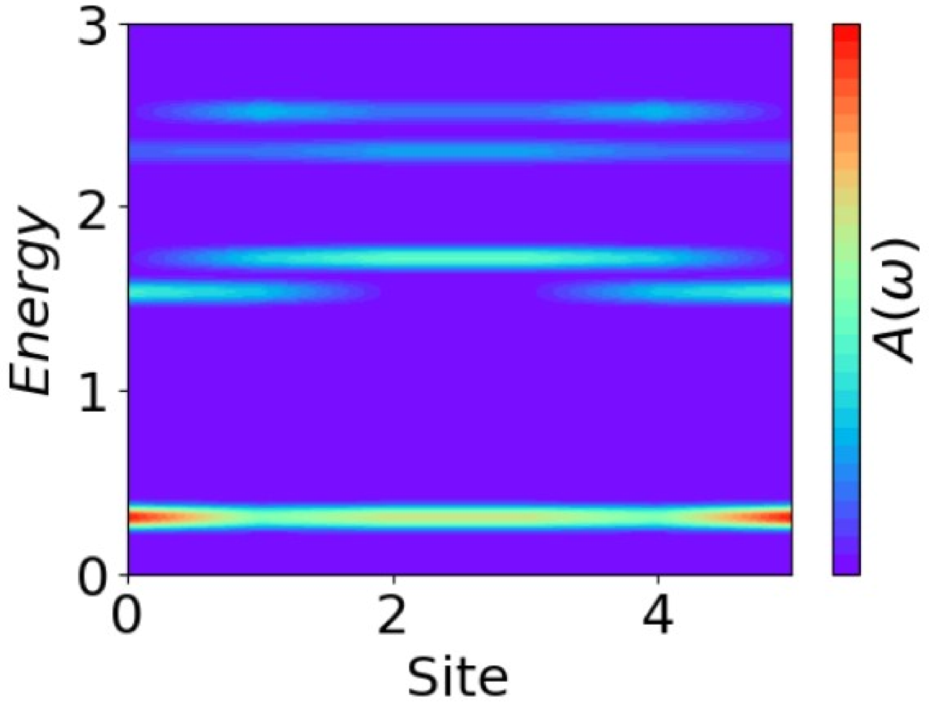
$L = 10$



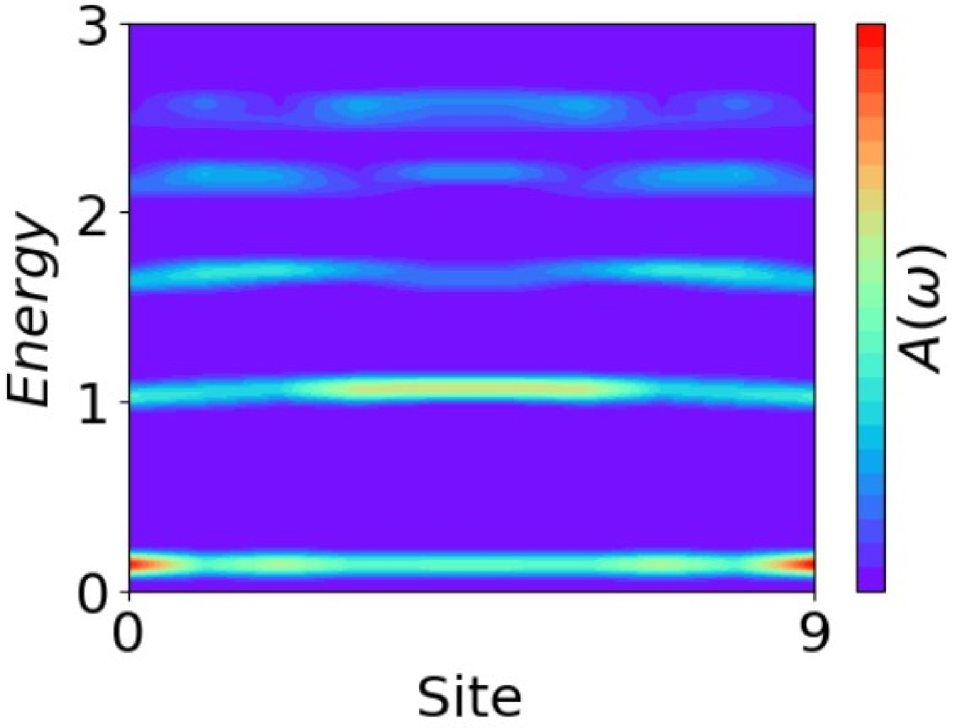
Dynamical correlator of the Heisenberg model

Lets look at the Heisenberg model with $S = 1$ $H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1}$

$L = 6$



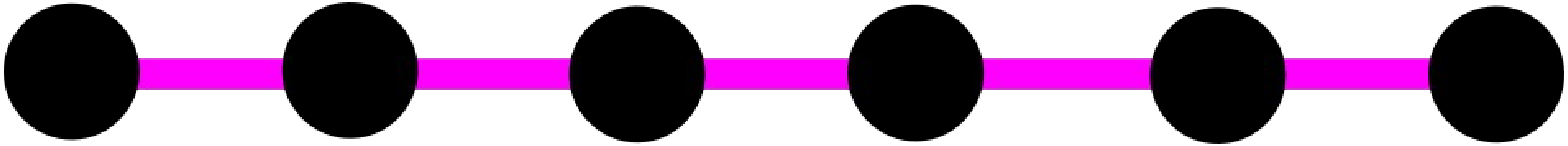
$L = 10$



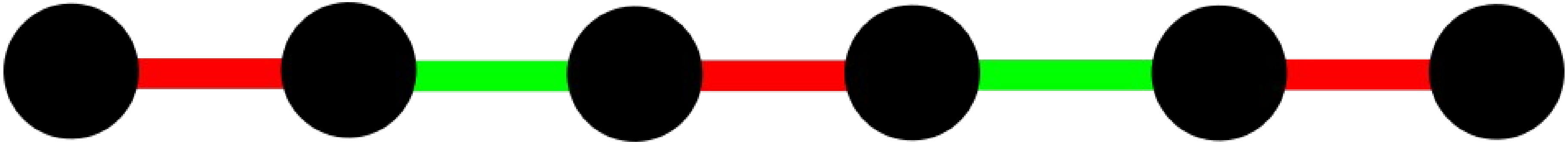
Two types of spin chains

Let us consider two different types of many-body spin chains

Uniform exchange coupling $H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1}$

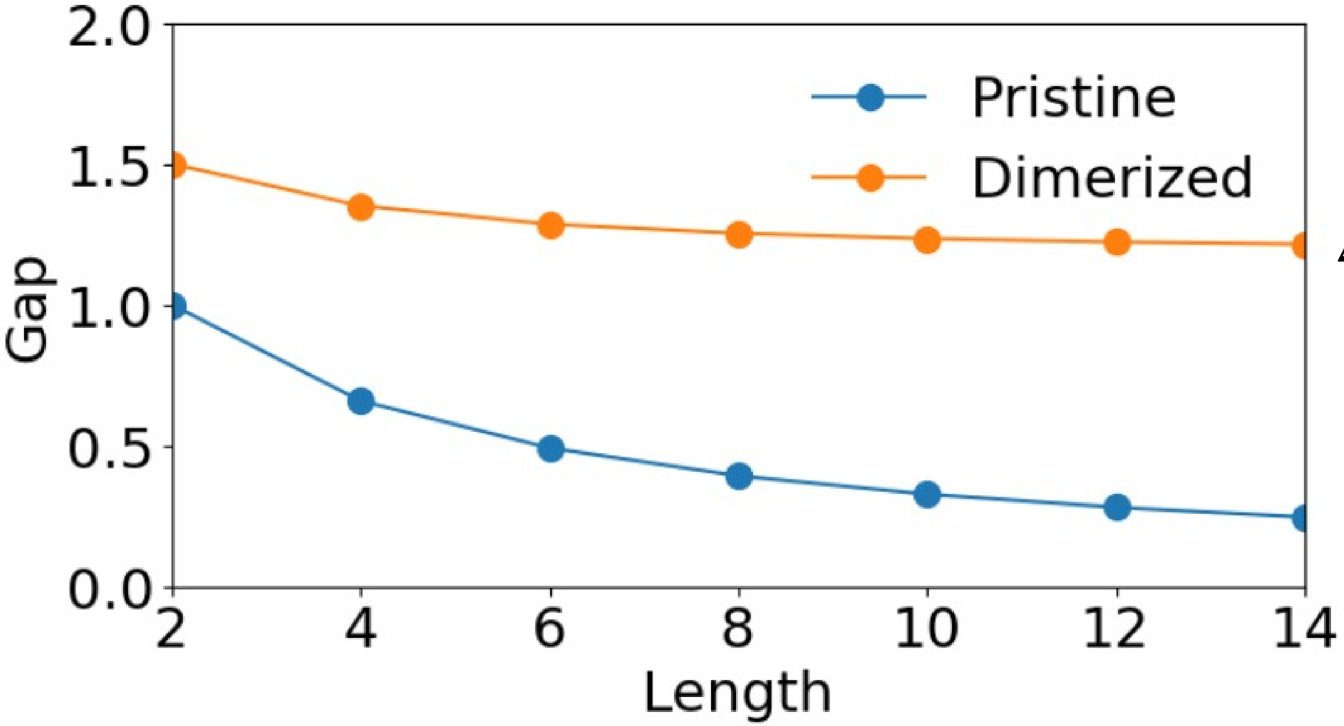


Dimerized coupling $H = \sum_n (1 + \delta(-1)^n) \vec{S}_n \cdot \vec{S}_{n+1}$



Pristine and dimerized chains

$$H = \sum_n (1 + \delta(-1)^n) \vec{S}_n \cdot \vec{S}_{n+1}$$



$$\Delta(L) \sim \Delta_0 + \Delta_1 e^{-\lambda L}$$

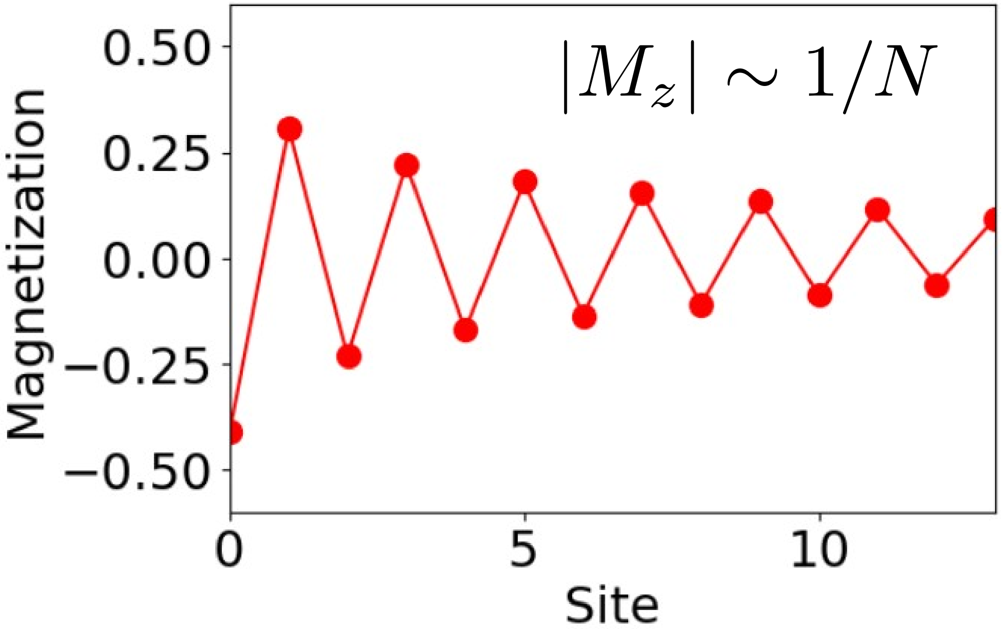
$$\Delta(L) \sim 1/L$$

Response of a quantum magnet to a magnetic impurity

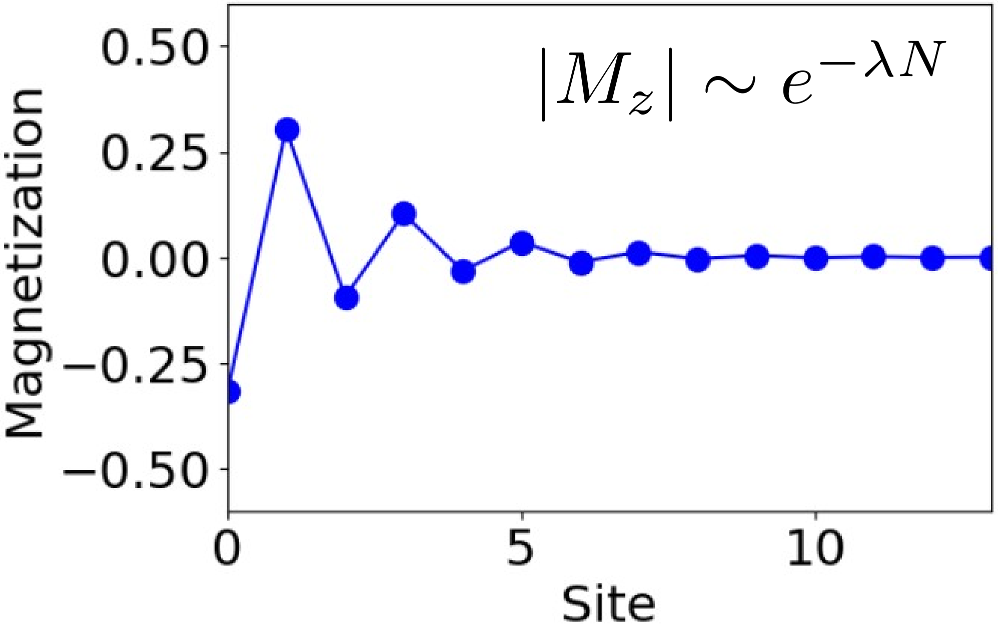
Quantum chain with a local field

$$H = J_{ij}\sum_{\langle ij\rangle}\vec{S}_i\cdot\vec{S}_j + B_z S_0^z$$

Uniform chain



Dimerized chain



Non-local static correlators

The non-local static correlator allows probing how the many-body wavefunction is entangled between different regions of the system

$$\chi_{ij} \equiv \langle \vec{S}_i \cdot \vec{S}_j \rangle - \langle \vec{S}_i \rangle \cdot \langle \vec{S}_j \rangle$$

For a product state, the correlator above is zero

Two different types of decays are possible in the correlator

$$\chi_{ij} \sim 1/|r_i - r_j|$$

Gapless spectrum

$$\chi_{ij} \sim e^{-\lambda |r_i - r_j|}$$

Gapped spectrum

Non-local static correlators

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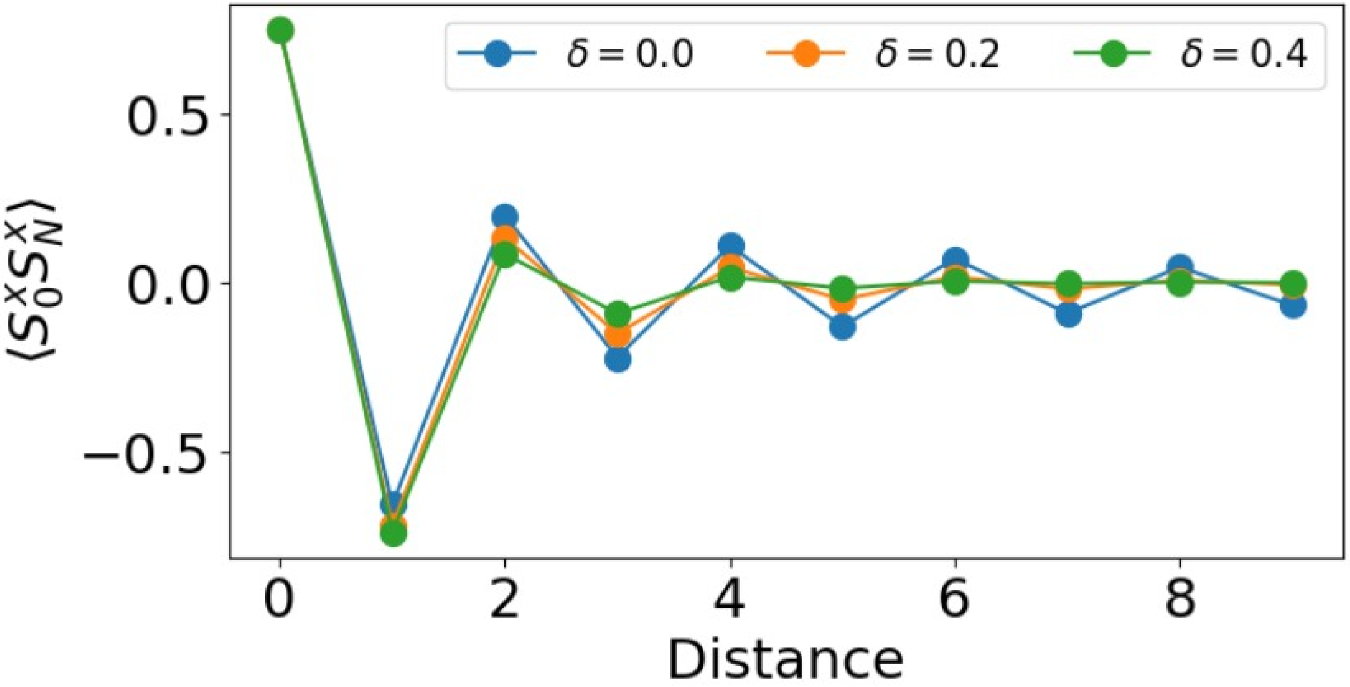


Quasiparticles in quantum magnets

Quasiparticles in a quantum magnet

Let us assume that a certain Hamiltonian realizes a quantum magnet

$$\mathcal{H} = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

Quantum magnets have a ground state $\langle \vec{S}_i \rangle = 0$

We will make an approximate algebraic replacement to transform a quantum many body Hamiltonian into an effective single particle one

The parton transformation

Transform spin operators to auxiliary fermions (Abrikosov fermions)

$$S_i^{\alpha} = \frac{1}{2}\sigma^{\alpha}_{s,s'} f^{\dagger}_{i,s} f_{i,s'}$$

The fermions f (spinons) have S=1/2 but no charge

This transformation artificially enlarges the Hilbert space, thus we have to put the constraint

$$\sum_s f^{\dagger}_{i,s} f_{i,s} = 1$$

This transformation allow to turn a spin Hamiltonian into a fermionic Hamiltonian

The spinon Hamiltonian

We can insert the auxiliary fermions $S_i^\alpha \sim \sigma^\alpha_{s,s'} f^\dagger_{i,s} f_{i,s'}$

And perform a mean-field in the auxiliary fermions (spinons)

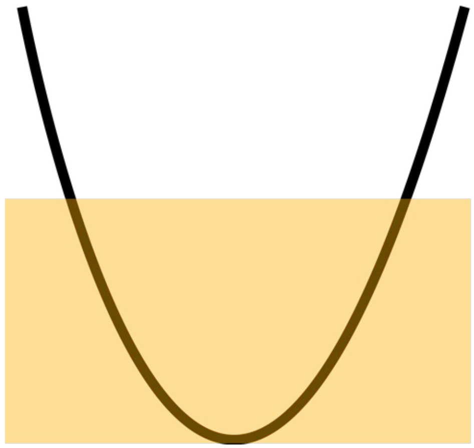
$$\mathcal{H} = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j \longrightarrow \mathcal{H} = \sum_{ij,s} \chi_{ij} f^\dagger_{i,s} f_{j,s}$$

Enforcing time-reversal symmetry $\langle \vec{S}_i \rangle = 0$

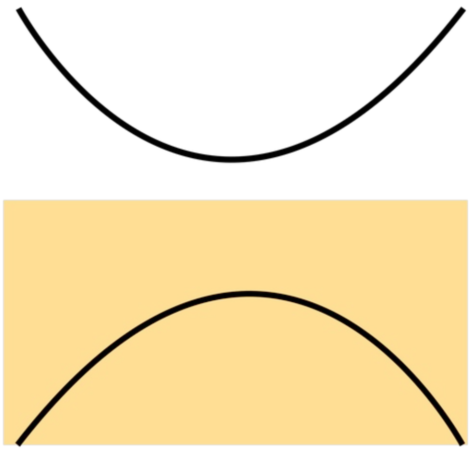
The excitations of the quantum magnet are described by a single particle spinon Hamiltonian

Spinon dispersions

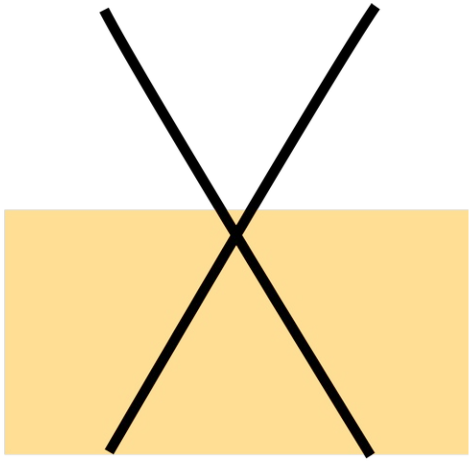
Gapless spinons



Gapped spinons



Dirac spinons



$$\mathcal{H} = \sum_{n,s} f^{\dagger}_{n,s} f_{n+1,s} + h.c.$$

Spinon dispersions in a chain

Uniform exchange coupling $H = \sum_n \vec{S}_n \cdot \vec{S}_{n+1}$

Gapless spinons $\mathcal{H} = \sum_{n,s} f^\dagger_{n,s} f_{n+1,s} + h.c.$

Dimerized coupling $H = \sum_n (1 + \delta(-1)^n) \vec{S}_n \cdot \vec{S}_{n+1}$

Gapped spinons $\mathcal{H} = \sum_{n,s} (1 + \gamma(-1)^n) f^\dagger_{n,s} f_{n+1,s} + h.c.$