Relational Dynamics From Entangled Eigenstates

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Certificate of Examination

This is to certify that the dissertation titled **Relational Dynamics From Entangled Eigenstates** submitted by **Aditya Dev** (Reg. No. MS19022) for the partial fulfillment of BS-MS Dual Degree programme of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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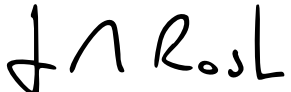

Dr. Abhishek Chaudhuri
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Prof. Dr. Jan Micheal Rost
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Dated:

Declaration

The work presented in this dissertation has been carried out by me under the joint supervision of **Dr. Abhishek Chaudhuri** at the Indian Institute of Science Education and Research, Mohali, **Prof. Dr. Jan Micheal Rost** at Max Planck Institute for the Physics of Complex Systems, Dresden Germany.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgment of collaborative research and discussions. This thesis is a bonafide record of my original work, and all sources listed within have been detailed in the bibliography.

Aditya Dev
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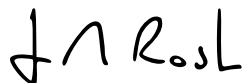
Dated:

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.



Dr. Abhishek Chaudhuri
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I dedicate this thesis to my beloved Nani, whose unwavering love and care made me the person I am today. Her humble nature and motherly affection guided me through life's challenges and her kindhearted spirit still inspires me. I will forever be grateful for the role she played in my life.

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– Aditya Dev

Abstract

One of the major challenges in unifying quantum mechanics with general relativity lies in reconciling the principle of general covariance, or background independence, with the quantum framework. In such a unified theory, the Hamiltonian is constrained to vanish identically ($\hat{H} \equiv 0$). This constraint presents the "problem of time" - implying that the quantum states should not evolve with respect to an external, classical time coordinate. On the other hand, just as in local Quantum Field theories where a superselection rule applies to charges, there has been speculation about the existence of a similar super-selection rule for energy. This is because energy couples to long-range gravitational fields in the same way that charge couples to the Coulomb field. Consequently, one would expect to observe no explicit time dependence in any quantities that can be experimentally measured.

Two main approaches address these problems. The first relies on semiclassical approximations which inherently carry over the classical notion of time. The second approach, known as Relational Quantum Mechanics, takes a purely quantum-mechanical perspective. This framework, pioneered by Page and Wootters [Page 83], focuses on the relationship between a system of interest and a reference system, often called a "clock" or "environment." By conditioning the global quantum state on the state of the environment, one can extract the dynamics of the system relative to its reference "clock". Recent work by [Gemsheim 23] has shown that this relational approach can be extended to handle complex interacting quantum systems.

This thesis investigates the classical limit of the quantum relational approach. Specifically, we focus on analyzing the JC-Hamiltonian within the relational paradigm and identify the key conditions under which the environment can be considered classical. Finally, we compare the results obtained through this method with those predicted by established semiclassical theories. Our analysis demonstrates that, under appropriate limits, the classical limit of relational dynamics recovers the well-known results expected from semi-classical approaches. This outcome strengthens the connection between these two frameworks, opening exciting new avenues for future research.

List of Figures

List of Tables

Contents

Chapter 1

Introduction

> “ Time is an illusion ”
>
> – Albert Einstein

The unification of Quantum Mechanics and General Relativity has long been the Holy Grail of physics since their development. Despite extensive research, no satisfactory theory has yet been proposed to solve this challenge. General relativity states that a physical theory should not depend on background structures. However, conventional quantization methods often introduce background structures, such as imposing the canonical commutation relations on a constant-time hypersurface. The Hamiltonian of a generally covariant theory, such as general relativity, is constrained to vanish in the absence of boundaries [Gielen 23]. Attempts to quantize gravity using canonical quantization lead to a Hamiltonian constraint, known as the Wheeler-DeWitt equation (i.e., $\hat{H}\,|\Psi\rangle\rangle = 0$). This gives rise to a well-known issue referred to as the “Problem of Time” within the canonical approach to quantum gravity. Further details regarding the Problem of Time are elaborated in Appendix A. The issue is that quantum states of space-time (and matter in it) do not seem to undergo any time evolution, as dictated by the constraints of the theory.

In quantum cosmology, the universe is described by a wave function whose dynamics are governed by the Wheeler-DeWitt equation, the quantized Hamiltonian constraint of the system. Due to inherent mathematical ambiguities in the Wheeler-DeWitt equation, it is often unreliable to extend beyond the semi-classical approximation [Cooke 10]. The wave function in this approximation is provided by the WKB wave function, $\Psi_{\mathrm{WKB}} \approx \exp\left[iS_0/\hbar\right]\psi(\{x_n\})$; where S_0 is a function which obeys the classical Hamilton-Jacobi equation for the gravitational

field [Gielen 23]. Then, we substitute this WKB ansatz into the Wheeler-DeWitt equation, and applying the relevant approximations leads to the time-dependent Schrödinger equation. The work of Briggs et al. [Briggs 01] has proposed a similar solution to the problem, but from an atomic and molecular physics perspective, and their approach is discussed in Chapter 2 and Chapter 4. Their approach derives the time dependence by partitioning the entire system into a "quantum" system and a classical environment, with the latter governed by the Hamilton-Jacobi equation. The time dependence is introduced into the description as a classical parameter.

On the other hand, for locally relativistic quantum field theories, it has been demonstrated that the long-range Coulomb field causes the total "charge" operator to commute with all quasi-local observables, implying a superselection rule for charges [Strocchi 74] [refer to Appendix B for further details]. Analogously, due to the coupling of energy with the long-range gravitational field, there is speculation about the existence of a superselection rule for energy [Page 83]. This implies that only operators that commute with the Hamiltonian can be observable. However, such observables are stationary (i.e., constants of motion). How can we observe time evolution if the only observables we can observe are stationary, i.e., they do not evolve at all?

One can argue and ask why one should consider time a fundamental property; rather, what if it's an emergent property of a more fundamental theory of the universe? The above arguments and problems discussed hint toward a more fundamental nature of time. One approach is to argue that the problem of time is a manifestation of background independence rather than the "timelessness." of quantum mechanics, and means that physical states do not evolve relative to an external background time. Instead, time evolution must be extracted relationally, by selecting certain quantized degrees of freedom to serve as an internal timekeeper, or in other words, pick some quantized degrees of freedom to serve as an internal time [Höhn 21].

Even in our everyday "classical" world, we never directly measure time; instead, we measure the position or angular displacement of a pendulum or a dial and use that measurement to *define* a unit of time. The use of relational time in Quantum Mechanics is a framework in which one promotes all variables to quantum operators and later chooses one of the variables to operate like a "clock" degree of freedom [Gambini 06].

Apart from the semiclassical approaches employed in quantum cosmology,

various other avenues exist to address the time problem from a purely quantum mechanical standpoint [Höhn 21]. One such approach is the "Page-Wootters Formalism" (abbreviated P&W formalism), which defines relational dynamics in terms of conditional probabilities for the clock and evolving degrees of freedom [Page 83]. To date, no one has successfully tackled the problem involving general interaction potentials [with the exception of [Smith 19] to a certain extent]. The original formulation due to P&W is no exception, as it also neglects the interaction between the environment and the system. However, recent work by Gemsheim and Rost has developed a relational formalism based on the P&W formalism, which now includes the environment-system interaction [Gemsheim 23]. This advancement provides the necessary tools to study relational dynamics for a general system-environment setting. Their approach is discussed in detail in Chapter 3. The primary objective of this work is to investigate the transition from a quantum mechanical relational description to the classical limit. The concept of a "large" or semiclassical environment will be further elucidated in subsequent chapters, specifically in Chapter 4 and Chapter 2. Moreover, this chapter will also delve into the emergence of time dependence within the context of our model system, employing coherent states.

Coherent states hold a significant position in quantum mechanics due to their close resemblance to classical states. Thus, it is unsurprising that coherent states emerged as the most suitable candidates for representing clock states within our relational description. We will explore their properties and relevance to our model in incoming chapters.

Finally, Chapter 5 will delve into the quantum relational description of our model system, building upon the methods outlined in Chapter 3, and will address the transition to the semiclassical limits as discussed in Chapter 4 and identify the conditions one requires for treating the environment classically.

Chapter 2

Emergence of Time dependence through interaction with a Semi-classical Environment

In modern textbooks, the Time-Dependent Schrödinger equation (TDSE) is either introduced as a *fundamental* equation governing the time evolution of a quantum wave function, with the Time-independent Schrödinger equation (TISE) as a special case, or it is derived from the correspondence with classical Hamilton-Jacobi theory. This approach borrows the idea of "time" as a classical parameter with no fundamental meaning, except that it parametrizes the dynamics of the quantum wavefunction.

This chapter presents an alternative derivation of the TDSE, based on [Briggs 01], that partitions the "global" Hilbert space into a system and an environment. We then employed the WKB ansatz for the environment's wave function, effectively treating the environment semi-classically. This analysis offers a clear physical interpretation of the time parameter, which arises from a directional gradient of the classical action along the environment trajectory.

To begin with , we consider a global system with Hamiltonian H, which comprises a system $\mathcal{S}$ and an environment $\mathcal{E}$. The TISE for this system is given by

$$\mathrm{H}\,\Psi = E\Psi, \quad \mathrm{H} = \mathrm{H}_{\mathcal{S}} + \mathrm{H}_{\mathcal{E}} + \mathrm{H}_{\mathcal{SE}}\,. \tag{2.1}$$

where $\mathrm{H}_{\mathcal{S}}$ and $\mathrm{H}_{\mathcal{E}}$ are the Hamiltonians of the system and the environment respectively and $\mathrm{H}_{\mathcal{SE}}$ is the interaction Hamiltonian between the system and the

environment[1]. The total wave function Ψ can be written as

$$\Psi(x,R) = \sum \psi_m(x,R)\chi_m(R). \tag{2.2}$$

Where $\{\psi' s\}$ represent the energy eigenstates of $\mathrm{H}_{\mathcal{S}} + \mathrm{H}_{\mathcal{SE}}$ at a fixed R, with $\epsilon_n(R) = \langle\psi_n|\,\mathrm{H}_{\mathcal{S}} + \mathrm{H}_{\mathcal{SE}}\,|\psi_n\rangle$, and x and R represent the coordinates of the system and the environment, respectively. [2] The environment Hamiltonian is assumed to be of the form

$$\mathrm{H}_{\mathcal{E}} = \mathrm{K} + \mathrm{V}_{\mathcal{E}}(R). \tag{2.3}$$

with:

$$\mathrm{K} = \frac{-\hbar^2}{2M}\sum_i \frac{\partial^2}{\partial R_i^2} = \frac{-\hbar^2}{2M}\nabla_R^2. \tag{2.4}$$

Substituting Eqn. 2.2 in Eqn. 2.1 and projecting onto $\psi_n(x,R)$ gives a coupled TISE for the environment wavefunction[3] [Briggs 01]:

$$\begin{aligned}\sum_m \langle\psi_n|\left(\frac{-\hbar^2}{2M}\nabla_R^2\right)|\psi_m\rangle\,\chi_m(R) + \mathrm{V}_{\mathcal{E}}(R)\chi_n(R)\\ + \sum_m \langle\psi_n|\,\mathrm{H}_{\mathcal{S}} + \mathrm{H}_{\mathcal{SE}}\,|\psi_m\rangle\,\chi_m(R) = E\chi_n(R).\end{aligned} \tag{2.5}$$

The "potentials":

$$\mathcal{V}_{mn}(R) = \langle\psi_m|\,\mathrm{H}_{\mathcal{S}} + \mathrm{H}_{\mathcal{SE}}\,|\psi_n\rangle\,. \tag{2.6}$$

depending on the state of the quantum system, provide the energy surface which determines the dynamics of the environment. The coupling from the kinetic term is

$$\begin{aligned}&\langle\psi_m|\left(\frac{-\hbar^2}{2M}\nabla_R^2\right)|\psi_n\rangle\,\chi_n(R)\\ &= -\frac{\hbar^2}{2M}\sum_k \left(\delta_{mk}\nabla_R + \langle\psi_m|\,\nabla_R\,|\psi_k\rangle\right)\left(\delta_{kn}\nabla_R + \langle\psi_k|\,\nabla_R^2\,|\psi_n\rangle\right)\chi_n(R).\end{aligned} \tag{2.7}$$

We define

$$\Lambda_{mn}(R) = i\hbar\,\langle\psi_m|\,\nabla_R\,|\psi_n\rangle\,. \tag{2.8}$$

Now, using above definitions, Eqn. 2.5 can be written as

$$\sum_m \left[\frac{1}{2M}(\hat{P}^2)_{nm} + \mathcal{V}_{nm}(R)\right]\chi_m(R) + \mathrm{V}_{\mathcal{E}}(R)\chi_n(R) = E\chi_n(R). \tag{2.9}$$

[1] $\mathrm{H}_{\mathcal{SE}} \equiv \mathrm{H}_{\mathcal{SE}}(x,R)$

[2] We have assumed that the environment is large enough so the system has a no effect on the environment's states. Furthermore, $\langle\chi_m|\chi_n\rangle \neq \delta_{mn}$.

[3] Notice that, we integrate only over x.

Where

$$\hat{P}_{nm} = -i\hbar\left(\mathbb{I}\nabla_R - \frac{i}{\hbar}\hat{\Lambda}\right) = \left(\mathbb{I}P_R - \hat{\Lambda}\right) \quad \text{or} \quad P_{ij} = \left(\delta_{ij}P_R - \Lambda_{ij}\right). \tag{2.10}$$

Notice that the kinematic coupling $\hat{\Lambda}$ appears as a vector potential. The set of equations for system wavefunction is given by

$$\sum_m \chi_m(R)\left[\mathrm{H}_{\mathcal{S}} + \mathrm{H}_{\mathcal{S}\mathcal{E}}(x,R) - \left(E - \mathrm{V}_{\mathcal{E}}(R) + \frac{1}{\chi_m}\frac{\hbar^2}{2M}\nabla_R^2\chi_m\right)\right.$$
$$\left. -\frac{\hbar^2}{2M}\nabla_R^2 - \frac{\hbar^2}{M\chi_m}\nabla_R\chi_m\cdot\nabla_R\right]\psi_m(x,R) = 0. \tag{2.11}$$

The main approximation in this derivation would be to disentangle Eqn. 2.9 and Eqn. 2.11. To do so, we assume that the environment is large enough so that the changes in the system, i.e. the variation in matrix elements $\mathcal{V}_{mn}$ and Λ_{mn}, have no effect on the environment dynamics, i.e., we neglect the off-diagonal terms.

The fist step would be to neglect in Eqn. 2.9 all the off-diagonal matrix elements, which gives:

$$\left[\frac{1}{2M}\left(P_R - \Lambda_{nn}(R)\right)^2 + \mathrm{V}_{\mathcal{E}} + E_n(R)\right]\chi_n(R) = E\chi_n(R). \tag{2.12}$$

with $E_n(R) = \mathcal{V}_{nn}(R)$. The vector potential Λ_{nn}, since now diagonal in the above case, can be included in the definition of an effective environment momentum operator.

The second step would be to use a semi-classical approximation for each $\chi_m(R)$, i.e., we write

$$\chi_n(R) = a_n(R)\exp\left(\frac{i}{\hbar}W_n(R)\right) \equiv \exp\left(\frac{i}{\hbar}W(R, E-\epsilon_n)\right). \tag{2.13}$$

with

$$\nabla_R W_n = \boldsymbol{P}_n. \tag{2.14}$$

where the classical momentum $\boldsymbol{P}_n$ and position $\boldsymbol{R}_n$ are decided by Hamilton's equations:

$$\frac{d\boldsymbol{P}_n}{dt} = -\nabla_{R_n}H = \nabla_{R_n}\left(\mathrm{V}_{\mathcal{E}}(R) + E_n(R)\right) \tag{2.15}$$
$$\frac{d\boldsymbol{R}_n}{dt} = \nabla_{P_n}H. \tag{2.16}$$

For the standard kinetic energy $\frac{\boldsymbol{P}^2}{2M}$, one obtains from Eqn. 2.16 that $\boldsymbol{P}_n = M\frac{d\boldsymbol{R}_n}{dt}$. It is at this point that the time parameter enters the picture. To the leading order in $\hbar$, one can write:

$$\frac{\hbar}{iM}\nabla_R\chi_n = \frac{\chi_n}{a_n}\frac{\hbar}{iM}\nabla_R a_n + \chi_n\frac{1}{M}\nabla_R W_n \approx \chi_n\frac{d\boldsymbol{R}_n}{dt}. \tag{2.17}$$

For the system $\mathcal{S}$ the equation coupled to Eqn. 2.12 now reads

$$\sum_m \chi_m(R) \left[\mathrm{H}_{\mathcal{S}} + \mathrm{H}_{\mathcal{SE}}(x,R) - E_m(R) - \frac{\hbar^2}{2M}\nabla_R^2 - i\hbar \frac{d\boldsymbol{R}_m}{dt} \cdot \nabla_R \right] \psi_m(x,R) = 0. \tag{2.18}$$

It has been shown [Briggs 01] that in the above equation, the term $(\hbar^2/2M)\nabla_R^2$ with higher-order gradient couplings can be neglected in comparison with the term

$$i\hbar \frac{d\boldsymbol{R}_n}{dt} \cdot \nabla_R := i\hbar \frac{d}{d\tau_n}.$$

Hence, Eqn. 2.18 can be written as

$$\sum_m \chi_m(R) \left[\mathrm{H}_{\mathcal{S}} + \mathrm{H}_{\mathcal{SE}}(x,\{\tau_m\}) - E_m(\{\tau_m\}) - i\hbar \frac{\partial}{\partial \tau_m} \right] \psi_m(x,R) = 0. \tag{2.19}$$

We have replaced the quantum R dependence by a "classical time" like dependence on $\{\tau_m\}$.

In the approximation represented by Eqn. 2.14 and Eqn. 2.19, the environment shows classical dynamics, but the state of the quantum system determines its motion and, hence, the interaction time with the system. This represents the final impact of the quantum influence on the environment. This influence diminishes as the environment becomes fully disentangled from the system, allowing it to function as an external clock that reads a unique time [Briggs 01].

This simplification is achieved in the approximation when Eqn. 2.13 becomes

$$\chi_n(R) = a_n(R) \exp\left(\frac{i}{\hbar} W(R)\right). \tag{2.20}$$

which is valid in the limit $\epsilon_n \ll E$. At this point, the environment gets fully disentangled from the system. Then we get from Eqn. 2.20, a unique time derivative

$$\nabla_R W = M \frac{d\boldsymbol{R}}{dt} \tag{2.21}$$

Also, one can eliminate $E_m(t)$ using a purely time-dependent phase transformation and writing $\psi_{\mathcal{S}}(x) = \sum_n a_n \psi_n(x)$, Eqn. 2.19 becomes:

$$\left[\mathrm{H}_{\mathcal{S}} + \mathrm{H}_{\mathcal{SE}}(x,t) - i\hbar \frac{\partial}{\partial t} \right] \psi_m(x,t) = 0 \tag{2.22}$$

This gives the TDSE for the quantum system alone. Now, the dynamics of the environment is given by the classical equation of motion, with the system having no effect on it. However, the quantum system is affected by the environment through the term $\mathrm{H}_{\mathcal{SE}}$. We would use this analysis to describe the emergence of time in the context of light-matter interaction in Chapter 4, along with an alternative (but similar) derivation based on coherent states.

Chapter 3

Relational Quantum Mechanics: With System and Environment Interaction

We start our realtional description, by *reformulating* the time independent Schrödinger equation as an *invariance principle* [Gemsheim 23]

$$\exp\left[i\lambda(\hat{H} - E\hat{I})\right] |\Psi\rangle\rangle = |\Psi\rangle\rangle \,. \tag{3.1}$$

Where λ is any complex-valued parameter, $\hat{H}$ is the Hamiltonian of the global system, and $|\Psi\rangle\rangle$ is the global state vector. Differentiating (Eqn. 3.1) with respect to λ one gets the usual Time Independent Schrödinger Equation (TISE)

$$\hat{H} |\Psi\rangle\rangle = E |\Psi\rangle\rangle \,.$$

To facilitate our analysis, we assume that the value of λ is a real number. There is no fundamental reason why λ It should be real. However, the assumption of the realness of λ is a phenomenological observation based on the fact that the time evolution operator is the unitary operator for a closed quantum system. The unitarity of the time evolution operator and the hermiticity of the Hamiltonian implies that the parameter λ has to be real.

To extract the dynamics, we partition the total Hilbert space $\mathcal{H}$ into two components. A system $\mathcal{H}_s$ and a clock $\mathcal{H}_c$. We re-write

$$\hat{H} = \hat{H}_s \otimes \hat{I}_c + \hat{I}_s \otimes \hat{H}_c + \hat{V}. \tag{3.2}$$

Since the system and environment[1] are embedded in the global Hilbert space, one can single out the system state by projecting the global state partially onto a static state of the environment.

[1]Please note that the use of the words 'clock' and 'environment' will be synonymous. The term "system" denotes a subsystem (excluding the environment) of a global Hilbert space unless otherwise specified.

For non-interacting system and environment: Let $|\chi\rangle$ be a state vector in clock Hilbert space.

We choose some state $|\chi_0\rangle \in \mathcal{H}_c$. If one projects the state vector $|\chi_0\rangle$ onto the invariance equation (Eqn. 3.1) (assuming the absence of interaction, represented by $\hat{V} = \hat{0}$), one gets

$$\begin{aligned} \langle\chi_0|\, e^{i\lambda(\hat{H}-E)}\, |\Psi\rangle\rangle &= \langle\chi_0|\Psi\rangle\rangle \\ \langle\chi_0|\, e^{i\lambda(\hat{H}_c-E)}\, |\Psi\rangle\rangle &= e^{-i\lambda\hat{H}_s}\, \langle\chi_0|\Psi\rangle\rangle\,. \end{aligned} \tag{3.3}$$

We define

$$|\chi_\lambda\rangle = e^{-i\lambda(\hat{H}_c-E)}\, |\chi_0\rangle\,. \tag{3.4}$$

By conditioning the global state $|\Psi\rangle\rangle$ onto a clock state $|\chi_\lambda\rangle$, we utilize the state

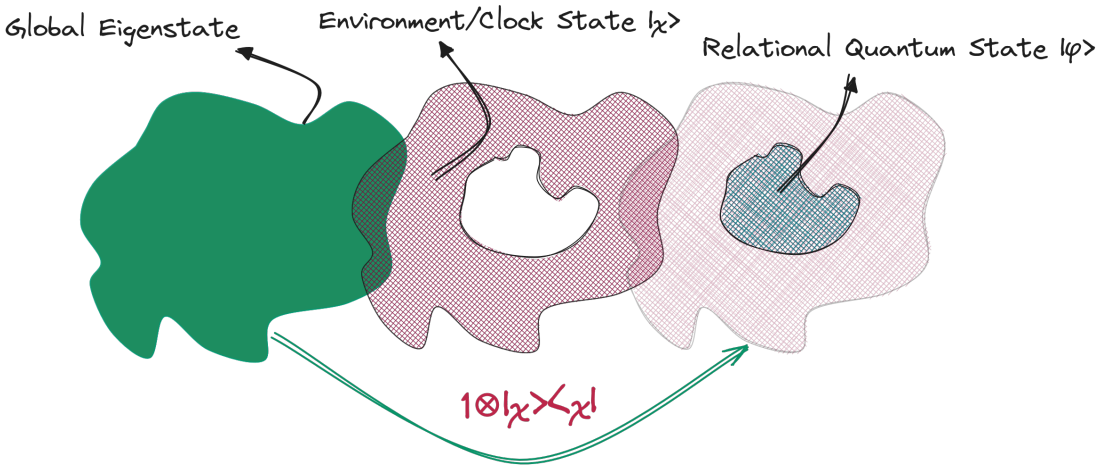

Figure 3.1: Illustration of the relational state formalism. The relational quantum state ($|\varphi\rangle$) of the system of interest is obtained once we project the Eigenstate of the Global Hilbert State onto a conditional clock state $|\chi\rangle$

$|\chi_\lambda\rangle$ as a label to associate the system state with the particular value of λ, i.e.,

$$|\varphi(\lambda)\rangle_s \equiv \langle\chi_\lambda|\Psi\rangle\rangle\,. \tag{3.5}$$

So,

$$|\varphi(\lambda)\rangle_s = e^{-i\lambda\hat{H}_s}\, \langle\chi_0|\Psi\rangle\rangle \equiv e^{-i\lambda\hat{H}_s}\, |\varphi(0)\rangle_s\,. \tag{3.6}$$

Notice that the choice of $|\chi_0\rangle$ fixes the system's initial state. Since λ in (Eqn. 3.5) is assumed to be a continuous parameter, the above equation can be interpreted as a solution to the differential equation

$$i\frac{d}{d\lambda}\, |\varphi(\lambda)\rangle_s = \hat{H}_s\, |\varphi(\lambda)\rangle_s\,. \tag{3.7}$$

This is equivalent to (Time-dependent Schrödinger Equation) TDSE in units $\hbar = 1$. It is crucial for the system and the environment within the global Hilbert space

to exhibit entanglement. Otherwise, the system and the environment would uphold separate "global" invariance principles, each with its own parameter, λ_s and λ_c, respectively. This would leave the relationship between λ_s and λ_c unresolved [Gemsheim 23].

Role of parameter λ: The variable λ introduced in the above derivation has no physical significance. It only serves as a parameter to track the evolution of our system. Any reparametrization of $\lambda \to t(\lambda)$ has no change one the equation governing the the system's state evolution, highlighting the time reparametrization invariance of the global system. One can, in principle [2] parameterize the evolution using an observable of our environment $\mathrm{A}_c(\lambda) \equiv \langle\chi_\lambda|\, \hat{A}_c\, |\chi_\lambda\rangle$. An ideal choice would be to use such a $\hat{A}_c$ for which the relation between λ and A_c is simple, for example, linear.

High-resolution clock states $|\chi_\lambda\rangle \propto \sum_k a_k \exp\left(-i\lambda E_c^k\right) |E_c\rangle_k$ require a broad distribution over the clock energy eigenstates, ideally with uniform coefficients (a_k) [Gemsheim 23, Smith 19]. This condition is readily met when the clock's physical dimensions exceed those of the system.

For interacting system and environment: So far in our analysis, we have assumed no interaction, i.e. $\hat{V} = \hat{0}$. However, in real scenarios, there is always some interaction within components of a global closed system. To extend the derivation to non-zero $\hat{V}$, we modify our clock state as

$$\begin{aligned} |\chi_\lambda\rangle = e^{-i\lambda(\hat{H}_c - E)}\, |\chi_0\rangle \to \\ |\chi_\lambda\rangle = e^{-iS(\lambda)} e^{-i\lambda(\hat{H}_c - E)}\, |\chi_0\rangle \end{aligned} \tag{3.8}$$

where $S(\lambda) = \int^\lambda \xi(\tilde{\lambda}) d\tilde{\lambda}$ is an extra factor introduced for simplifying upcoming derivations. When projected onto this, the global state can be written as

$$\left(-\hat{H}_s + \xi(\lambda) + i\frac{d}{d\lambda}\right) \langle\chi_\lambda|\Psi\rangle\rangle = \langle\chi_\lambda|\, \hat{V}\, |\Psi\rangle\rangle\,. \tag{3.9}$$

We can rearrange the above equation to write it as

$$i\frac{d}{d\lambda} \langle\chi_\lambda|\Psi\rangle\rangle = \hat{H}_s \langle\chi_\lambda|\Psi\rangle\rangle - \xi(\lambda) \langle\chi_\lambda|\Psi\rangle\rangle + \langle\chi_\lambda|\, \hat{V}\, |\Psi\rangle\rangle\,. \tag{3.10}$$

We now decompose $\langle\chi_\lambda|\, \hat{V}\, |\Psi\rangle\rangle$ into a Hermitian potential and a c-number. To facilitate the decomposition, we define the following:

$$\begin{aligned} \hat{P}_\Psi = |\Psi\rangle\rangle \langle\langle\Psi|\,, \quad \hat{P}_\chi = \hat{I}_s \otimes |\chi_\lambda\rangle \langle\chi_\lambda| \\ \hat{P}_{\Psi\chi} = \hat{P}_\Psi \hat{P}_\chi / \mathcal{N}_\lambda \end{aligned} \tag{3.11}$$

[2] As one does it always. We never measure time directly; rather, we measure the angular position of a clock dial or the number of transition electrons made in a cesium atom.

where $\mathcal{N}_\lambda = \langle\langle\Psi|\ \hat{P}_\chi\ |\Psi\rangle\rangle$. One observes that

$$\hat{P}_{\Psi\chi}\ |\Psi\rangle\rangle = \frac{\hat{P}_\Psi \hat{P}_\chi}{\mathcal{N}_\lambda}\ |\Psi\rangle\rangle = |\Psi\rangle\rangle \tag{3.12}$$

Using above-defined operators, we decompose the

$$\begin{aligned} \langle\chi_\lambda|\ \hat{V}\ |\Psi\rangle\rangle &= \langle\chi_\lambda|\ \hat{V}\hat{P}_{\Psi\chi}\ |\Psi\rangle\rangle \\ &= \left[\hat{V}_s(\lambda) - \langle\langle\Psi|\ \hat{V}\hat{P}_\chi\ |\Psi\rangle\rangle / \mathcal{N}_\lambda\right] \langle\chi_\lambda|\Psi\rangle\rangle \end{aligned} \tag{3.13}$$

where

$$\hat{V}_s(\lambda) = \frac{\langle\chi_\lambda|\left(\hat{V}\hat{P}_\Psi + \hat{P}_\Psi\hat{V}\right)|\chi_\lambda\rangle}{\mathcal{N}_\lambda} \tag{3.14}$$

Inserting (Eqn. 3.13) into (Eqn. 3.10) and setting $\xi(\lambda) = -\langle\langle\Psi|\ \hat{V}\hat{P}_\chi\ |\Psi\rangle\rangle / \mathcal{N}_\lambda$ we obtain

$$i\frac{d}{d\lambda}\ |\varphi(\lambda)\rangle_s = \left[\hat{H}_s + \hat{V}_s(\lambda)\right]\ |\varphi(\lambda)\rangle_s, \tag{3.15}$$

where we defined $\langle\chi_\lambda|\Psi\rangle\rangle = |\varphi(\lambda)\rangle_s$. The phase factor $S(\lambda) = \int^\lambda \xi(\tau)d\tau$ introduced helps eliminate the c-number after the decomposition of the "effective" potential.

We find that the final form of the Equation is equivalent to the TDSE for the remaining subsystem ($\mathcal{H}_s$) of the global Hilbert space $\mathcal{H}$.

§ 3.1 Example: Dynamics of two interacting two level system

To provide an illustrative example of the approach, let us consider an interaction two spin-1/2 particles [Gemsheim 23]. The Hamiltonians for system and clock are taken as, $\hat{H}_s = 0$, $\hat{H}_C = E_C\hat{\sigma}_{C,x}$ and the interaction $\hat{V} = V_0\,(\hat{\sigma}_x + \hat{\sigma}_z)_S \otimes (\hat{\sigma}_x)_C$. Taking $E_C = V_0 \equiv 1$, we get

$$\hat{H} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \tag{3.16}$$

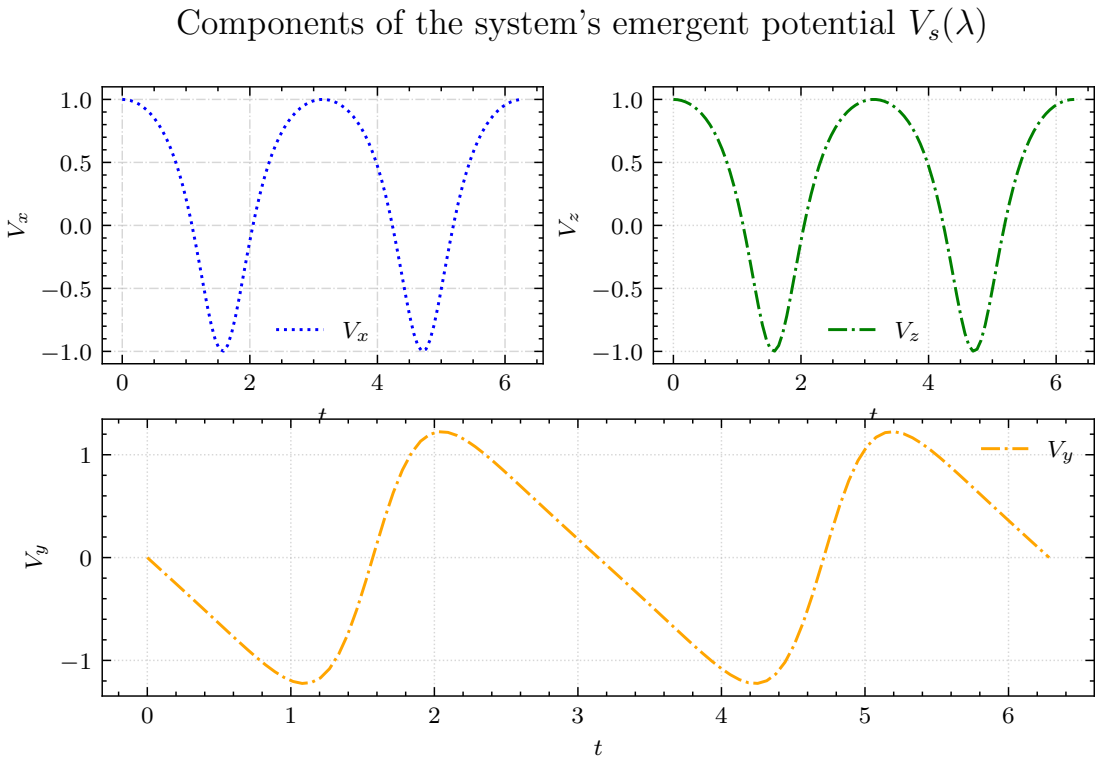

Figure 3.2: Components of the effective potential $\boldsymbol{V}_S(\lambda)$ for the two interacting spin-1/2 particles.

with eigenvalues $E = \{-\sqrt{3}, \sqrt{3}\}$, where both of them are doubly degenerate. One of the eigenvector corresponding to $E_- = -\sqrt{3}$, in the standard basis, is $\boldsymbol{\Psi} = \left(1, 0, -1, -(1+\sqrt{3})\right)^T$ [refer to Appendix C for detailed calculations]. Taking

$$|\chi(\lambda)\rangle = \frac{\exp\left(iE_-\lambda\right)}{2\sqrt{1+a\cos^2(\lambda)}}\left(e^{-i\lambda}\,|\uparrow\rangle + e^{i\lambda}\,|\downarrow\rangle\right) \tag{3.17}$$

where $a = \sqrt{3}+1$, we obtain from Eqn. 3.13, the effective potential

$$\hat{V}_s = \boldsymbol{V}_S(\lambda)\cdot\hat{\boldsymbol{\sigma}}_S. \tag{3.18}$$

The components of $\boldsymbol{V}_S(\lambda)$ are shown in Figure 3.2 and are explicitly given by

$$\begin{aligned} V_x(\lambda) = V_z(\lambda) &= \frac{\cos(2\lambda)+a\cos^2(\lambda)}{1+a\cos(\lambda)} \\ V_y(\lambda) &= -\frac{(a/2)\sin(2\lambda)}{1+a\cos^2(\lambda)} \end{aligned} \tag{3.19}$$

One can calculate the evolution of the system state by projecting the global state onto the clock state, which gives

$$|\varphi(\lambda)\rangle_s \equiv \langle\chi(\lambda)|\Psi\rangle\rangle = \frac{\exp\left(ia\lambda\right)}{2\sqrt{1+a\cos^2(\lambda)}}\left[\,|\uparrow\rangle - \left(ae^{-2i\lambda}+1\right)\,|\downarrow\rangle\right]. \tag{3.20}$$

The dynamics is shown in Figure 3.3. One of the features of utilizing the relational approach is that, it enables us to obtain analytical solutions for the evolution of

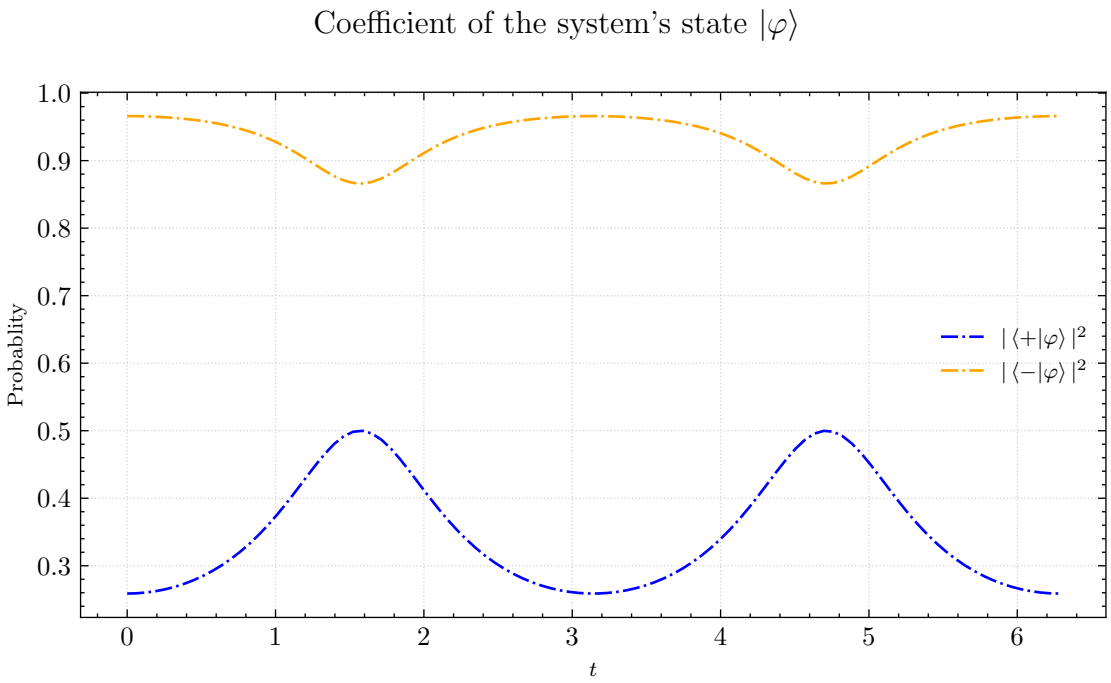

Figure 3.3: Evolution of the system state $|\varphi(\lambda)\rangle_s$ for the two interacting spin-1/2 particles.

system states, even in the presence of complex time-dependent potentials that pose challenges for conventional methods. Consequently, the relational approach not only lends significance to the time parameter but also offers a viable means to address the dynamics of systems subject to arbitrary time-dependent potentials.

Chapter 4

Semi-Classical Relational Dynamics for Light-Matter Interaction

It has been shown in Chapter 2 how one can reduce a TISE to a TDSE for a system in the presence of a classical environment. In this chapter, we will discuss the application of the previous chapter to a simple model of light-matter interaction and enumerating the conditions that must be obtained in order to treat the electromagnetic field classically. The upcoming sections outline the derivations as presented in [Braun 04].

To get the time dependence, we start by considering a time-independent Hamiltonian of an interacting quantum system, with a Hamiltonian given as

$$\hat{H} = \hat{H}_s(\hat{p}, \hat{x}) + \hat{H}_F(\hat{Q}, \hat{P}) + \hat{H}_I(\hat{x}, \hat{Q}). \tag{4.1}$$

where $(\hat{x}, \hat{p})$ and $(\hat{Q}, \hat{P})$ are (position, momentum) operators for the system and Boson field, respectively. Also, $\hat{H}_s$ is the Hamiltonian of the atomic system, $\hat{H}_F$ is the field Hamiltonian, and $\hat{H}_I$ is the interaction Hamiltonian. The Boson field Hamiltonian will be takes a sum over field modes, with Hamiltonian given as

$$\hat{H}_F(\hat{P}, \hat{Q}) = \sum_k \frac{1}{2}\left(\hat{P}_k^2 + \omega_k^2 \hat{Q}_k^2\right) \tag{4.2}$$

We write the above Hamiltonian in terms of creating and annihilation operators

$$\hat{H} = \sum_i \epsilon_i \hat{c}_i^\dagger \hat{c}_i + \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right) + \hbar \sum_k g_{jk}^k \hat{c}_i^\dagger \hat{c}_i \left(\hat{a}_k^\dagger + \hat{a}_k\right). \tag{4.3}$$

where c_i and $\hat{c}_i^\dagger$ are the annihilation and creation atomic operators for the system, and a_k and $a_k^\dagger$ are the annihilation and creation operators for the Boson field.

A special case of the above Hamiltonian is when we take a two-level atom (or a spin 1/2 system) interacting with a single mode of the electromagnetic field.

Eqn. 4.3 then reduces to

$$H = \hbar\omega_0\hat{\sigma}_z + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar g\hat{\sigma}_x\left(\hat{a} + \hat{a}^\dagger\right), \tag{4.4}$$

where $\hat{\sigma}_z$ and $\hat{\sigma}_x$ are the Pauli matrices. This special case is known as the Jaynes-Cummings model.

§ 4.1 Semiclassical Limit of the System

As done in Eqn. 2.2, one can write in general the solution of TISE

$$(H - E)\Psi = 0 \tag{4.5}$$

with H in the form of Eqn. 4.3 as

$$\Psi(x, Q) = \sum_k \chi_k(Q)\phi_k(x, Q). \tag{4.6}$$

We employ this particular form, wherein $\chi_k(Q)$ displays no reliance on x, under the assumption that the system exerts minimal back-reaction on the field. This assumption has been utilized previously in Chapter 2 and will be employed again in forthcoming derivations.

Doing the same exercise as in Eqn. 2.5, we substitute Eqn. 4.6 into Eqn. 4.5 and obtain a set of coupled equations

$$\begin{aligned}&\sum_i \chi_i(Q)\left[H_s + H_I - \left(E - \sum_k \frac{1}{2}\omega_k^2 Q_k^2 + \frac{\hbar^2}{2\chi_i}\frac{\partial^2}{\partial Q_k^2}\chi_i\right)\right] \\ &+ \sum_i \chi_i(Q)\left[\frac{-\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} - \frac{-\hbar^2}{\chi_i}\frac{\partial}{\partial Q_k}\chi_i\frac{\partial}{\partial Q_k}\right]\phi_i(x, Q) = 0.\end{aligned} \tag{4.7}$$

Eqn. 4.7 when projected onto state ϕ_j gives

$$\begin{aligned}&\sum_k\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} + \frac{1}{2}\omega_k^2 Q_k^2\right]\chi_j + \sum_i \langle\phi_j|\, H_s + H_I\,|\phi_i\rangle\,\chi_i \\ &- \sum_{i,k}\left[\langle\phi_j|\,\frac{\hbar^2\partial^2}{2\partial Q_k^2}\,|\phi_i\rangle + \langle\phi_j|\,\hbar^2\frac{\partial}{\partial Q_k}\,|\phi_i\rangle\,\frac{\partial}{\partial Q_k}\right]\chi_i = E\chi_j.\end{aligned} \tag{4.8}$$

The equation above describes the "close-coupled" form for χ_j. The off-diagonal terms account for changes in the state of the Boson field caused by changes in the

system's state. Neglecting these off-diagonal coupling terms simplify the equation, providing the state of χ_j when the system is in state ϕ_j, i.e.,

$$\begin{aligned}\left[\sum_k \left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} + \frac{1}{2}\omega_k^2 Q_k^2\right) + E_j(Q) - E\right]\chi_j \\ = \hbar^2 \sum_k \langle\phi_j| H_s + H_I |\phi_j\rangle \frac{\partial}{\partial Q_k}\chi_j.\end{aligned} \tag{4.9}$$

with

$$E_j(Q) = \langle\phi_j| H_s + H_I - \sum_k \frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} |\phi_i\rangle . \tag{4.10}$$

The diagonal $\langle\phi_j| H_s + H_I |\phi_j\rangle$ terms on the RHS of Eqn. 4.9 are zero for real ϕ_j or else can be eliminated by a (Berry) phase transformation [Braun 04]. In addition, since we are neglecting the first-order derivatives w.r.t. Q_k, it is consistent to neglect the second-order derivatives as well, which gives us the following equation

$$\left[\sum_k \left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} + \frac{1}{2}\omega_k^2 Q_k^2\right) + E_j(Q) - E\right]\chi_j(Q) = 0. \tag{4.11}$$

this is the defining equation for the state of the Boson field when the system is in state ϕ_j.

For complete independence of the field from the system, we need to replace $E_j(Q)$ by some fixed average energy $\bar{E}(Q)$ and corresponding $\chi_j(Q)$ by some "mean"-field state $\chi(Q)$. Hence, we replace

$$\chi_j(Q) = a_j(Q)\chi(Q). \tag{4.12}$$

where we assume $a_j(Q)$ to be slowly varying function of Q. The Eqn. 4.6 reduces to

$$\begin{aligned}\Psi(x,Q) = \chi(Q)\sum_j a_j(Q)\phi_j(x) \\ \Psi(x,Q) = \chi(Q)\psi(x,Q).\end{aligned} \tag{4.13}$$

We've identified the key approximations needed to express the exact wave function (Eqn. 4.6) in the simpler, factorized form (Eqn. 4.13). This step is critical because it assumes that the influence of the atom on the field is minimal while the field has a strong influence on the atom. Now, we can focus on deriving the effective Schrödinger equation for the wave function (denoted by ψ) that represents the quantum system. Note that one can, in general, view Eqn. 4.13 as a

general ansatz for the wave function and find from Eqn. 4.4[1]

$$(H-E)\Psi = \chi(Q)\left[H_S + H_I(x,Q) - \frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial Q^2} + 2\frac{\chi'(Q)}{\chi(Q)}\frac{\partial}{\partial Q}\right)\right]\psi(x,Q)$$
$$+\psi(x,Q)\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q^2} + \frac{\omega^2}{2}Q^2 - E\right]\chi(Q) = 0 \tag{4.14}$$

We'll now focus on a single-field mode to streamline our analysis. This derivation takes a slightly different approach compared to both the methods outlined in Chapter 2 and the one we'll use later for the Jaynes-Cummings model (our toy model universe). We present this alternative approach to lay the groundwork for the new considerations in the next section, which will leverage coherent states.

We split the action of the total Hamiltonian on the field and quantum system such that the wave function $\chi(Q)$ described the field with energy close to total energy E, while the reaming part of the equation describes the quantum system (with negligible energy in comparison). Completely ignoring the influence of the quantum system on the field (back coupling) is analogous to neglecting the terms $\bar{E}(Q)$ in the equations. This simplification allows us to select a wave function for the field, which will ultimately be treated classically. This is equivalent to neglecting $\bar{E}(Q)$ and allows one to choose the wave function of the the field which is to become classical, This wave function must be an eigenstate of the fixed-field Schrödinger equation , i.e.

$$\left[\sum_k \left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} + \frac{1}{2}\omega_k^2 Q_k^2\right) - E\right]\chi(Q) = 0. \tag{4.15}$$

Under the assumption of these large (classical) energies and assuming we're sufficiently far from any classical turning points, we can approximate the actual wave function, denoted by $\chi(Q)$, by its WKB approximation, i.e.,

$$\chi(Q) = \exp\left(\frac{i}{\hbar}\int^Q dQ' P(Q')\right). \tag{4.16}$$

where $P(Q)$ is the momentum of the field, given as

$$P(Q) = \sqrt{2\left(E - \frac{1}{2}\omega^2 Q^2\right)}. \tag{4.17}$$

Using Eqn. 4.14 - Eqn. 4.16 gives

$$\left[\sum_k \left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial Q_k^2} + \frac{1}{2}\omega_k^2 Q_k^2\right) - E\right]\exp\left(\frac{i}{\hbar}\int^Q dQ' P(Q')\right) = 0$$
$$\left[H_s + H_I(x,Q) - \frac{\hbar^2}{2}\frac{\partial^2}{\partial Q^2} - i\hbar P(Q)\frac{\partial}{\partial Q}\right]\psi(x,Q) = 0. \tag{4.18}$$

[1]Here we assumed only single mode of E.M field

The above differential equation describes the dynamics of (atomic part) the quantum system. We simplify this equation by replacing the equation by a parameter t, which parametrizes $Q(t)$ trajectory and define

$$P(Q)\frac{\partial}{\partial Q} \equiv \frac{\partial}{\partial t} \tag{4.19}$$

It's evident that $P(Q)$ derived from Eqn. 4.17 corresponds to the velocity $\dot{Q}$, as dictated by classical equations of motion. These equations state that $\dot{Q} = P$ and $\ddot{P} = -\omega^2 Q$, where $Q(t) = Q_0 \cos(\omega t)$ represents *the solution, demonstrating that the parameter t signifies classical time.*

We define $\psi(x,t) = \psi(x, Q(t))$ and find from Eqn. 4.18

$$i\frac{\partial}{\partial t}\psi(x,t) = \left[H_s + H_I(x,Q(t))\frac{\hbar^2}{2}\left(\frac{\ddot{Q}(t)}{\dot{Q}^3(t)}\frac{\partial}{\partial t} - \frac{1}{\dot{Q}^2(t)}\frac{\partial^2}{\partial t^2}\right)\right]\psi(x,t). \tag{4.20}$$

The derivatives appearing on the right-hand side stem from the second-order derivative $\frac{\partial^2}{\partial Q^2}$ wrt position, when expressed in terms of time derivatives. In the scenario of a substantial ("classical") amount of energy E, predominantly residing in the classical degree of freedom Q, these additional terms become negligible, as we will soon illustrate. Initially, it's worth noting that $\frac{\partial}{\partial t}$ is of the order of energy of the quantum system, $E_s = \langle H_s + H_I \rangle$, implying it is considerably smaller than E. This allows us to disregard the supplementary derivative terms on the right side of Eqn. 4.20. Proceeding with self-consistency, and leveraging $\dot{Q} = P \approx \sqrt{E}$ for the harmonic oscillator away from its turning points, as well as $\ddot{Q} \approx \omega \dot{Q}$, we can estimate their respective orders of magnitude. First,

$$\left\langle \frac{\hbar^2}{2}\frac{\ddot{Q}(t)}{\dot{Q}^3(t)}\frac{\partial}{\partial t} \right\rangle \approx E_s\frac{\hbar\omega}{E}, \text{ and } \left\langle \frac{\hbar^2}{2}\frac{1}{\dot{Q}^2(t)}\frac{\partial^2}{\partial t^2} \right\rangle \approx \frac{E_s^2}{E^2}. \tag{4.21}$$

The additional derivative terms on the right-hand side of Eqn. 4.20 exhibit an order of magnitude of $E_s\left(\frac{E_s+\hbar\omega}{E}\right)$. Hence, in comparison to the remaining terms on the right-hand side, which are of the order E_s, these additional terms are diminished by a factor approximately $\frac{\hbar\omega}{E} \approx \frac{1}{n}$, where n represents the number of photons in the field mode. This factor is significantly small for a classical field. Hence, we can neglect these terms and write

$$i\frac{\partial}{\partial t}\psi(x,t) = \left[H_s + H_I(x,Q(t))\right]\psi(x,t) \tag{4.22}$$

which is the usual form of TDSE for the quantum system interacting with the classical field.

The derivation relies on the Time-Independent Schrödinger Equation (TISE) for both the atom and the field. Time arises as a consequence of classical motion, serving solely as a derived classical parameter. It's important to note that the aforementioned arguments hold true only away from classical turning points where the velocity $\dot{Q}(t)$ is non-zero. But it's evident that the Time-Dependent Schrödinger Equation (TDSE) of (Eqn. 4.22) remains valid for all times. The reason behind this limitation is apparent: it lies in the selection of Q, the position representation for the field mode. While this choice aids in diagonalizing the coupling Hamiltonian H_I, in this representation, the real field quadrature $Q(t)$ undergoes harmonic motion with periodic zeros in its time derivative $\dot{Q}(t)$. Consequently, the position representation fails to offer a global notion of time. The next section will illustrate how this limitation can be addressed by utilizing a coherent state representation of the field state.

§ 4.2 Coherent State Derivation of TDSE

In quantum optics, coherent states are unique as they exhibit classical behavior while having minimal uncertainty in both position (Q) and momentum (P) of the field mode. It's, therefore, unsurprising that coherent states play a crucial role in deriving a time-dependent Schrödinger equation for the quantum system's degrees of freedom. This derivation leverages a Time-independent Schrödinger equation encompassing both the coupled quantum system and the field mode.

We begin by writing the total Hamiltonian for a single mode of E.M field in term of usual creation and annihilation operators as

$$\begin{aligned} \hat{H}_F &= \hbar\omega \left(\hat{a}^\dagger \hat{a} + 1/2\right), \\ \hat{H}_I &= \hbar \hat{S}(\hat{a} + \hat{a}^\dagger) \end{aligned} \tag{4.23}$$

with $\hat{S} = \sum_{ij}^{k} g_{ij} \hat{c}_i^\dagger \hat{c}_i$ as in Eqn. 4.3.

Coherent state, defined as, $|\alpha\rangle = \exp\left(|\alpha|^2/2 + \alpha \hat{a}^\dagger\right)|0\rangle$ where α is a complex number, is an eigenstate of the annihilation operator, i.e., $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$. Some crucial properties that will be of use to us are

$$\begin{aligned} \langle\alpha|\,\hat{a}^\dagger &= \alpha^* \langle\alpha|\,, \\ \langle\alpha|\,\hat{a} &= \left(\frac{\partial}{\partial \alpha^*} + \frac{\alpha}{2}\right) \langle\alpha|\,. \end{aligned} \tag{4.24}$$

Coherent states form an overcomplete basis, i.e., any state $|\psi\rangle$ can be expanded as

$$|\psi\rangle = \int \frac{d^2\alpha}{\pi} |\alpha\rangle \, \langle\alpha|\psi\rangle \equiv \int \frac{d^2\alpha}{\pi} |\alpha\rangle \, \tilde{\chi}(\alpha, \alpha^*). \tag{4.25}$$

where $\tilde{\chi}(\alpha, \alpha^*) = \langle \alpha | \psi \rangle$. Note that we can always write

$$\tilde{\chi}(\alpha, \alpha^*) = \chi(\alpha^*) \exp\left(-|\alpha|/2\right)$$

where $\chi(\alpha, \alpha^*)$ is a complex function of α. Instead of utilizing the position Q representation as previously done, we leverage the properties of coherent states mentioned above to express the Time-Independent Schrödinger Equation (TISE) within the coherent state representation relative to the classical degree of freedom. We write the total wave function

$$\langle \alpha | \Psi \rangle = \exp\left(-|\alpha|^2/2\right) \chi(\alpha^*) \psi(\alpha^*). \tag{4.26}$$

As before, we will consider $\exp\left(-|\alpha|^2/2\right) \chi(\alpha^*)$ to describe the classical degree of freedom, while $\psi(\alpha^*)$ will represent the quantum system without back-reaction to the classical degree of freedom.

Another significant property of coherent states is their overlap with photon number states, expressed as $\langle n | \alpha \rangle = \exp\left(-|\alpha|/2\right) (\alpha^*)^n / \sqrt{n!}$. When considering a large photon number n, it becomes evident that only coherent states $|\alpha\rangle$ satisfying the condition:

$$|\alpha|^2 = n \tag{4.27}$$

contribute significantly to the corresponding number state $|n\rangle$. With Eqn. 4.26 in mind, the TISE with H_F and H_I from Eqn. 4.25 now reads

$$\begin{aligned} \langle \alpha | (H - E) | \Psi \rangle = \chi(\alpha^*) \left[H_s + \hbar g \left(\alpha^* + \frac{\chi'(\alpha^*)}{\chi(\alpha^*)} + \frac{\partial}{\partial \alpha^*} \right) \sigma_x + \hbar \omega \alpha^* \frac{\partial}{\partial \alpha^*} \right] |\psi(\alpha^*)\rangle \\ + |\psi(\alpha^*)\rangle \left[\hbar \omega \left(\alpha^* \frac{\partial}{\partial \alpha^*} + \frac{1}{2} \right) - E \right] \chi(\alpha^*) = 0. \end{aligned} \tag{4.28}$$

The above equation corresponds to Eqn. 4.14 of the position space approach. Just as we did in Section 4.1, we chose χ to be an eigenstate of the field Hamiltonian, i.e., to be a number state $|n\rangle$ with energy $E = \hbar\omega\,[n + 1/2]$. In the coherent state representation,

$$\chi(\alpha^*) = (\alpha^*)^n / \sqrt{n!}. \tag{4.29}$$

This ensures that the second part of Eqn. 4.28 disappears and one notices that

$$\frac{\chi'(\alpha^*)}{\chi(\alpha^*)} = \frac{n}{\alpha^*} = \frac{\alpha^* \alpha}{\alpha^*} = \alpha. \tag{4.30}$$

Using Eqn. 4.29 and Eqn. 4.30 in Eqn. 4.28 gives

$$\left[H_s + \hbar S \left(\alpha^* + \alpha + \frac{\partial}{\partial \alpha^*} \right) + \hbar \omega \alpha^* \frac{\partial}{\partial \alpha^*} \right] |\psi(\alpha^*)\rangle = 0. \tag{4.31}$$

We now introduce a new parameter t, which replaces α analogous to what we did in position space representation. The parameter t is defined through a complex trajectory $\alpha^*(t)$ for the coherent state field amplitude, such that

$$\hbar\omega\alpha^*\frac{\partial}{\partial\alpha^*} \equiv -i\hbar\frac{\partial}{\partial t}. \tag{4.32}$$

Hence, time is determined by the classical motion of the field amplitude, which, in this case, is

$$\alpha(t) = \alpha_0 e^{-i\omega t}. \tag{4.33}$$

Importantly, whereas the position space expression (Eqn. 4.19) becomes problematic near classical turning points due to $P(Q) = 0$, the coherent state expression (Eqn. 4.32) maintains finiteness throughout all times. Furthermore, it's noteworthy that the coherent state equation (Eqn. 4.31) exclusively involves first-order derivatives. This characteristic resembles the first-order time-dependent Schrödinger equation, whereas the position space expression (Eqn. 4.20) is of second order.

We see that the field part of the interaction Hamiltonian becomes,

$$\left[\alpha^* + \alpha - \alpha\left(\hbar\omega|\alpha|^2\right)^{-1}(i\hbar\partial/\partial t)\right] \tag{4.34}$$

In large photon number limit $1/|\alpha|^2 \approx 1/n$, vanishes. This can be compared to the similar discussion after Eqn. 4.20. Therefore, we find that in the limit of large photon number, the TDSE for state $|\psi(t)\rangle$ is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \left[\hat{H}_s + \hbar\hat{S}\left(\alpha_0 e^{-i\omega t} + \alpha_0 e^{i\omega t}\right)\right]|\psi(t)\rangle\,. \tag{4.35}$$

Under the assumption of a valid two-level approximation, the system can be represented by the operator $\hat{S} = g\hat{\sigma}_x$. Crucially, unlike the equivalent result obtained in position space representation, the derivation of Eqn. 4.35 using coherent states is valid for all times.

Furthermore, this equation for a two-level system interacting with a single electromagnetic field mode is important as we will leverage it to compare the system's dynamics obtained through this approach to those obtained from a fully *quantum relational dynamics* approach in the upcoming section.

Chapter 5

Quantum Relational Dynamics of JC Model

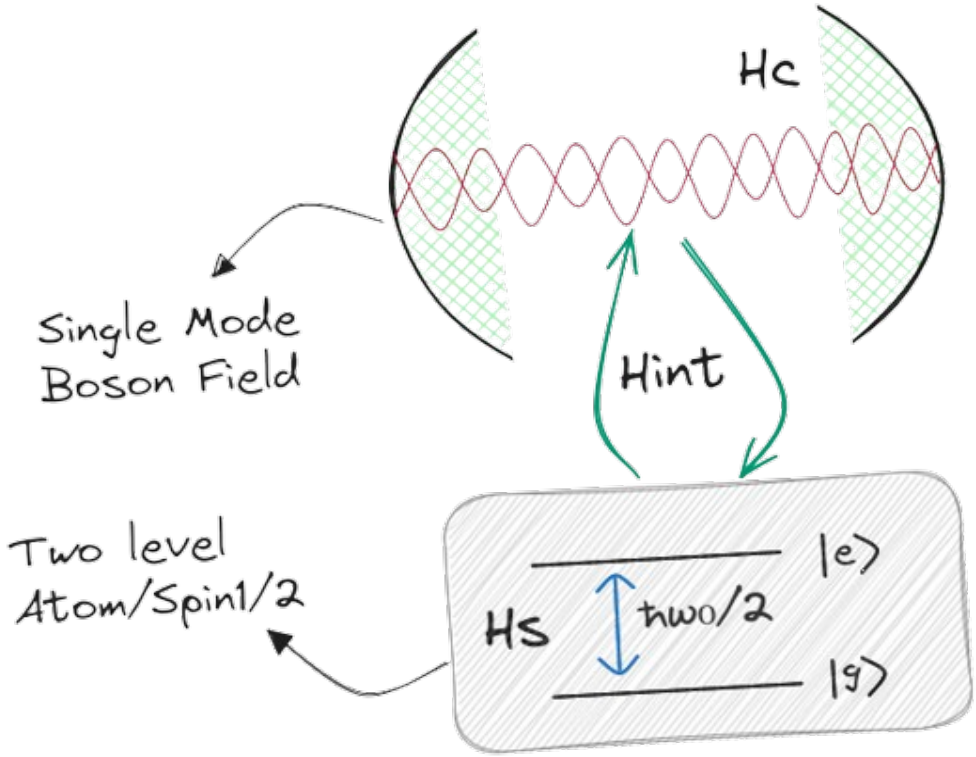

Figure 5.1: Illustration of the Jaynes-Cummings model. A two-level atom is coupled to an optical cavity (single field mode), shown on top. The energy levels of the atom that couple to the field mode within the cavity are shown in the box below.

In Chapter 4 we discussed the semi-classical approach to derive the Time Dependent Schrödinger Equation (TDSE), governing the Jaynes-Cummmings model.

In Section 4.1, we derived the TDSE in the position space representation, by assuming a semiclassical (WKB-like) wave function for the field mode and Section 4.2 presented an alternative derivation, i.e., by making use of coherent state representation. However, both these approaches inherit the concept of "time" as a purely classical parameter, compromising its deeper and more fundamental nature.

On the contrary, in Chapter 3, we presented a purely quantum-mechanical framework to describe the dynamics of the general interacting quantum system from the relational point of view. Here, "time" emerges as a symmetry parameter stemming from the principle of global invariance of the global energy eigenstate of the total Hamiltonian.

A natural question arises: Are both of these different approaches to describe the dynamics of the system (relationally) equivalent? If so, then in what limit? In this chapter, we aim to address these questions. We begin by adopting the Jaynes-Cummings Hamiltonian as our model system. We then explore the Jaynes-Cummings model from the perspective of Quantum Relational Dynamics. We aim to examine the transition from a purely quantum environment to a classical environment, as previously discussed in Chapter 4, and to juxtapose the dynamics as characterized by these two distinct approaches. We demonstrate that the two approaches yield identical results under appropriate limits (consistent with the semiclassical treatment in previous chapters).

We assume our Hamiltonian to be of the form,

$$\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar g\hat{\sigma}_x\left(\hat{a} + \hat{a}^\dagger\right). \tag{5.1}$$

Under dipole-approximation and rotating-wave approximation [Bina 12, Chap 2] the above Hamiltonian (Eqn. 5.1) can be written as

$$\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar g\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right). \tag{5.2}$$

§ 5.1 Exact Energy Eigenstate of JC-Hamiltonian (resonant case)

In the specific scenario of the Hamiltonian in Equation 5.2, when $\omega = \omega_0$, known as the "resonance case," the system becomes exactly solvable. This model is char-

acterized by the Hamiltonian operator[1]

$$\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right). \tag{5.3}$$

It portrays the resonant interaction between two energy levels of an atom and a single Boson field mode. To understand how this system evolves over time, we initially aim to compute the unitary evolution operator $U(t) = e^{-itH/\hbar}$ through the diagonalization of H. Once we achieve an eigenvalue decomposition $H = \sum_n \epsilon_n|\epsilon_n\rangle\langle\epsilon_n|$, it follows that:

$$U(t) = \sum_n e^{-it\epsilon_n/\hbar}|\epsilon_n\rangle\langle\epsilon_n| \tag{5.4}$$

Finding such an eigenvalue decomposition is not trivial, particularly in this context, given the infinite-dimensional nature of the underlying Hilbert space. However, the JC-Hamiltonian possesses a notably simple structure as it is block-diagonal with respect to the $|\pm, n\rangle$-basis. The diagonalization of the JC-Hamiltonian in Eqn. 5.3 is given in Appendix D . Once you diagonalize the Hamiltonian, the eigenvalues and eigenvectors of the JC-Hamiltonian are given by

$$E_{\pm,n} = \hbar\omega\left(n - \frac{1}{2}\right) \pm \hbar g\sqrt{n+1}, \tag{5.5}$$

and the corresponding eigenvectors are

$$|\psi_{n,\pm}\rangle := \frac{1}{\sqrt{2}}\left(|+, n-1\rangle \pm |-, n\rangle\right). \tag{5.6}$$

In addition, the exact analytical form of the unitary evolution operator is [refer to Appendix D]

$$\begin{aligned}
U(t) &= e^{\frac{1}{2}it\omega}\,|-,0\rangle\,\langle-,0| + \sum_{n=1}^{\infty} e^{-it\omega(n-\frac{1}{2})}e^{-itg\sqrt{n}}\,|\psi_{n,+}\rangle\,\langle\psi_{n,+}| \\
&+ \sum_{n=1}^{\infty} e^{-it\omega(n-\frac{1}{2})}e^{itg\sqrt{n}}\,|\psi_{n,-}\rangle\,\langle\psi_{n,-}| \\
&= e^{\frac{1}{2}it\omega}\,|-,0\rangle\,\langle-,0| \\
&+ \sum_{n=1}^{\infty} e^{-it\omega(n-\frac{1}{2})}\Big[\cos\big(tg\sqrt{n}\big)\left(|+,n-1\rangle\,\langle+,n-1| + |-,n\rangle\,\langle-,n|\right) \\
&- i\sin\big(tg\sqrt{n}\big)\left(|+,n-1\rangle\,\langle-,n| + |-,n\rangle\,\langle+,n-1|\right)\Big].
\end{aligned} \tag{5.7}$$

We discussed the coherent state-based semiclassical approximation in Chapter 4, where the final result simply involved replacing the field operators, i.e., $\hat{a}$ and $\hat{a}^\dagger$,

[1]Here, we dropped the $1/2$ term from H_c, simplifying calculations. This adjustment doesn't affect our analysis.

with their classical expectation values. Following a similar approach, we could assume the semi-classical form of the Hamiltonian in Equation 5.3 as

$$\begin{aligned}\hat{H}_{\text{semi}} &= \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\omega\langle\hat{a}^\dagger\hat{a}\rangle + \hbar g\left(\hat{\sigma}_+\langle\hat{a}\rangle + \hat{\sigma}_-\langle\hat{a}^\dagger\rangle\right), \\ &= \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\omega N + \hbar g\alpha\left(\hat{\sigma}_+ e^{-it\omega} + \hat{\sigma}_- e^{it\omega}\right),\end{aligned} \tag{5.8}$$

where $\langle a(t)\rangle = \alpha e^{-i\omega t}$. This leads to the evolution operator being expressed as

$$\begin{aligned}\tilde{U}(t) = &\cos(\Omega t)\left(|+\rangle\langle+| + |-\rangle\langle-|\right) \\ &- i\sin(\Omega t)\left(|+\rangle\langle-| + |-\rangle\langle+|\right)\end{aligned} \tag{5.9}$$

where $\Omega = \frac{g\alpha}{\hbar}$. We will see in the next section (Section 5.2) that the semiclassical limit can be made more rigorous under the limits discussed Chapter 4, i.e., in the limit of large photon number and weak "back-coupling" from the quantum system.

§ 5.2 Semiclassical JC-Hamiltonian in Large Amplitude and Weak Interaction Limit

It would be a reasonable assumption to anticipate the emergence of semi-classicality when the fields are in a state of coherence. However, recovering semi-classical behavior is not straightforward by simply substituting the quantum degree of freedom with their classical expectation values (as was done in Eqn. 5.8).

Building upon this observation, we posit that the emergence of semi-classical behavior is contingent upon specific, well-defined conditions. This assertion is further supported by the findings presented in [Irish 22]:

- **Large Amplitude of Coherent State**: When the amplitude of the coherent state is substantial, the quantum fluctuations (i.e., the influence of atom onto the field in our model!) become less significant, resembling classical behavior.
- **Weak Atom-Mode Coupling**: If the coupling between the atom and the mode (field) is weak, the quantum effects become less pronounced, favoring semiclassical behavior.

In this section, we will explore an interesting limit of Eqn. 5.1, corresponding to the aforementioned conditions. We will consider what happens when the factor, $g = \frac{\Omega}{|\alpha|} \to 0$, as $|\alpha| \to \infty$. Here, Ω represents a fixed constant[2]. While we won't

[2] Ω sets the energy scale of the problem in consideration

delve into rigorous proofs, the following discussion will argue that, under these conditions, the dynamics of the atomic system can still be described by unitary evolution. To begin, let us recall the form of coherent states.

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{5.10}$$

A unique feature of these states is that the particle number has a Poisson distribution. In other words, the probability of finding the system in photon number state, $|n\rangle$ is given by:

$$\mathrm{p}_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{(|\alpha|^2)^n}{n!} \tag{5.11}$$

Our objective is to understand the evolution of the atom when it interacts with a coherent state. Therefore, we aim to compute the reduced density operator of the atom, which is given by:

$$\begin{aligned} \hat{\rho}_{\text{atom}}(t) &= \mathrm{Tr}_{\text{mode}} \left(U(t)[\hat{\rho} \otimes |\alpha\rangle \langle\alpha|] U^\dagger(t) \right) \\ &= \sum_n e^{-\gamma} \gamma^n \frac{V_g(n) \hat{\rho} V_g^\dagger(n)}{n!}, \end{aligned} \tag{5.12}$$

where $\gamma = |\alpha|$ and[3]:

$$V_g(n) = e^{-it\omega\frac{1}{2}} \cos\left(tg\sqrt{n+1}\right) |+\rangle \langle +| \tag{5.13}$$

$$+ e^{it\omega\frac{1}{2}} \cos\left(tg\sqrt{n}\right) |-\rangle \langle -| \tag{5.14}$$

$$- ie^{-it\omega\frac{1}{2}} \sin\left(tg\sqrt{n+1}\right) \frac{\alpha}{\sqrt{n+1}} |+\rangle \langle -| \tag{5.15}$$

$$- ie^{-t\omega\frac{1}{2}} \sin\left(tg\sqrt{n}\right) \frac{\sqrt{n}}{\alpha} |-\rangle \langle +| . \tag{5.16}$$

These square roots and α are quite bothersome. If we could eliminate them, we would achieve a unitary evolution. To facilitate this, let us make a few observations. First, as x increases, the function $\sqrt{x}$ becomes flatter (its derivative approaches zero). Second, as the mean increases, the Poisson distribution increasingly peaks and concentrates around the mean, as illustrated in Figure 5.2, the Poisson distribution is approximately equivalent to the normal distribution (in large mean limit),

$$\mathfrak{N}_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{5.17}$$

In the context of coherent states, the approximate normal distribution has a mean of $|\alpha|^2$ and a standard deviation of $|\alpha|$.

[3] [Refer to Appendix D for explicit form of $U(t)$]

This implies that the width of the distribution increases at a slower rate than the mean. Consequently, for large $|\alpha|$, it is plausible that the square root remains nearly constant within the region where the Poisson/normal distribution is significant. On considering these observations, the subsequent reasoning for the approximation might be easier to grasp.

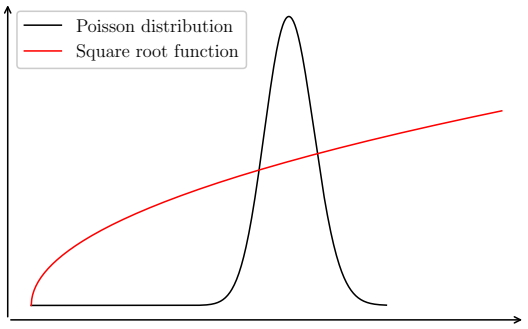

Figure 5.2: The plot compares the Poisson distribution for large mean values with a square root function. As the mean increases, the Poisson distribution increasingly peaks and concentrates around the mean, while the square root function remains nearly flat for large values.

$$\hat{\rho}_{\text{atom}}(t) = \sum_n e^{-\gamma}\gamma^n \frac{V_g(n)\hat{\rho}V_g^\dagger(n)}{n!} \tag{5.18}$$

By approximating the weighted sum over the Poisson distribution with the integral weighted by the normal distribution $\mathfrak{N}_{\mu,\sigma}(x)$, having a mean ($\mu = |\alpha|^2$) and standard deviation ($\sigma = |\alpha|$), we obtain

$$\hat{\rho}_{\text{atom}}(t) \approx \int \mathfrak{N}_{\mu,\sigma}(x) V(t,x)\hat{\rho}V(t,x)^\dagger dx \tag{5.19}$$

We cut away the tails of the integral at (s) standard deviations

$$\hat{\rho}_{\text{atom}}(t) \approx \int_{|\alpha|^2-s|\alpha|}^{|\alpha|^2+s|\alpha|} \mathfrak{N}_{\mu,\sigma}(x) V(t,x)\hat{\rho}V(t,x)^\dagger dx \tag{5.20}$$

A change of variables gives ($x \to x|\alpha| + |\alpha|^2$)

$$\hat{\rho}_{\text{atom}}(t) = \int_{-s}^{s} \mathfrak{N}_{0,1}(x) V(x|\alpha| + |\alpha|^2)\hat{\rho}V(x|\alpha| + |\alpha|^2)^\dagger dx \tag{5.21}$$

Now we put in our assumption that $g = \hbar\Omega/|\alpha|$. Let's assume, for simplicity, that $\alpha = |\alpha|$, i.e., it's real. Then Eqn. 5.13 becomes

$$\begin{aligned}
V_{\Omega/|\alpha|}(x|\alpha| + |\alpha|^2) = {} & e^{-it\omega\frac{1}{2}} \cos\left(t\Omega\sqrt{1 + \frac{x}{|\alpha|} + \frac{1}{|\alpha|^2}}\right) |+\rangle\langle+| \\
& + e^{it\omega\frac{1}{2}} \cos\left(t\Omega\sqrt{1 + \frac{x}{|\alpha|}}\right) |-\rangle\langle-| \\
& - ie^{-it\omega\frac{1}{2}} \sin\left(t\Omega\sqrt{1 + \frac{x}{|\alpha|} + \frac{1}{|\alpha|^2}}\right) \frac{1}{\sqrt{1 + \frac{x}{|\alpha|} + 1}} |+\rangle\langle-| \\
& - ie^{it\omega\frac{1}{2}} \sin\left(t\Omega\sqrt{1 + \frac{x}{|\alpha|}}\right) \sqrt{1 + \frac{x}{|\alpha|}} |-\rangle\langle+| .
\end{aligned}$$

Since, $-s < x < s$, with s constant, in the limit of $|\alpha| \to \infty$,

$$\begin{aligned}\tilde{U}(t) &:= \lim_{|\alpha|^2\to\infty} V_{\Omega/|\alpha|}(x|\alpha| + |\alpha|^2) \\ &= \cos(\Omega t)\left(e^{-it\omega\frac{1}{2}}\,|+\rangle\langle+| + e^{it\omega\frac{1}{2}}\,|-\rangle\langle-|\right) \\ &- i\sin(\Omega t)\left(e^{-it\omega\frac{1}{2}}\,|+\rangle\langle-| + e^{it\omega\frac{1}{2}}\,|-\rangle\langle+|\right)\end{aligned} \tag{5.22}$$

Inserting this limit back into Eqn. 5.21, we obtain

$$\begin{aligned}\hat{\rho}_{\text{atom}}(t) &\approx \int_{-s}^{s} \mathfrak{N}_{0,1}(x)\tilde{U}(t)\hat{\rho}\tilde{U}(t)^{\dagger}dx \\ &\approx \tilde{U}(t)\hat{\rho}\tilde{U}(t)^{\dagger}.\end{aligned} \tag{5.23}$$

Hence, in the limit of large α and weak interactions, we get the semiclassical limit. The final form of the evolution operator is given in Eqn. 5.22, i.e.,

$$\begin{aligned}\tilde{U}(t) = \cos(\Omega t)&\left(e^{-it\omega\frac{1}{2}}\,|+\rangle\langle+| + e^{it\omega\frac{1}{2}}\,|-\rangle\langle-|\right) \\ - i\sin(\Omega t)&\left(e^{-it\omega\frac{1}{2}}\,|+\rangle\langle-| + e^{it\omega\frac{1}{2}}\,|-\rangle\langle+|\right)\end{aligned} \tag{5.24}$$

Eqn. 5.24 describes the unitary progression of a two-level atom within the JC Hamiltonian, under the conditions of a high amplitude coherent state and a weak interaction limit. This will be employed in the upcoming sections to derive the precise semi-classical dynamics of our model system.

§ 5.3 Numerical Results: Comparing Quantum Relational Dynamics with Semiclassical Dynamics

The structure of this section is as follows: It presents the results of numerical simulations that investigate the relational dynamics of the Jaynes-Cummings (JC) model under both quantum mechanical and semiclassical paradigms. The focus will be on exploring the system's time evolution within the contexts of the rotating wave approximation (RWA) and the original, non-rotating wave approximation (NRWA). A detailed analysis of these numerical findings will be provided in Section 5.4.

JC-Model: Non Rotating Wave Approximation

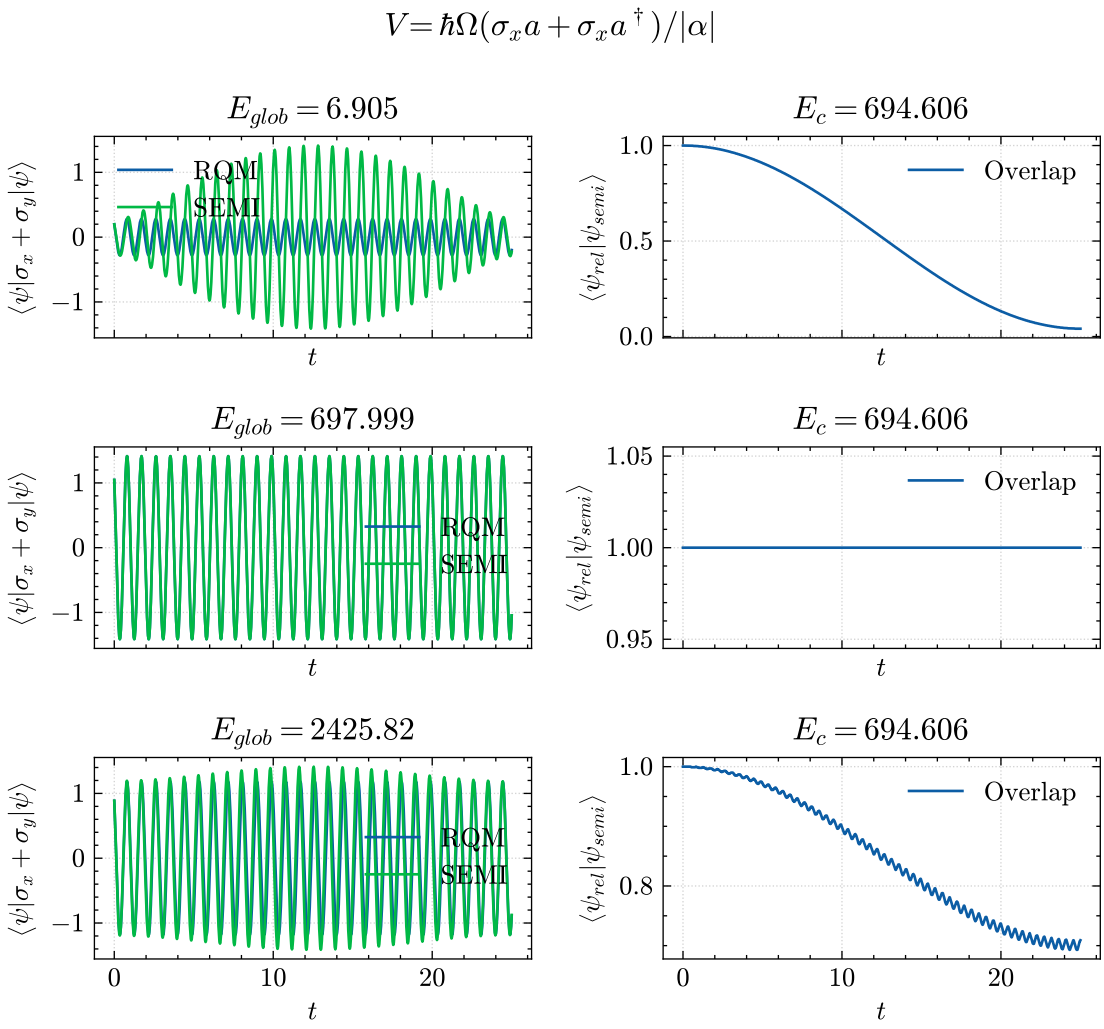

Figure 5.3: Expectation value $\langle\psi|\,\hat{\sigma}_x + \hat{\sigma}_y\,|\psi\rangle$ v/s Time (t) for Semi-Classical and Quantum Relational Dynamics and $|\langle\psi_{\text{rel}}|\psi_{\text{semi}}\rangle|$ v/s Time (t). The Global eigenstates are calculated from the Hamiltonian as given in Eqn. 5.25 and the semi-classical dynamics given by Eqn. 5.26. The parameters are given in Table 5.1

Figure 5.3 depict the expectation value of the operator $\hat{\sigma}_x + \hat{\sigma}_y$, computed using the relational evolution described by the Hamiltonian in Eqn. 5.1., i.e.,

$$\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\frac{g}{|\alpha|}\hat{\sigma}_x\left(\hat{a} + \hat{a}^\dagger\right). \tag{5.25}$$

and the the semi-classical Hamiltonain as given in Eqn. 4.35, i.e.,

$$\hat{H}_{\text{semi}} = \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\frac{g}{|\alpha|}\hat{\sigma}_x\left(\alpha e^{-it\omega} + \alpha e^{it\omega}\right). \tag{5.26}$$

The clock (environment) quantum state, onto which the global eigenstate is projected to extract the quantum dynamics, is taken to be the coherent states $|\alpha(t)\rangle$,

$$V = \hbar\Omega(\sigma_x a + \sigma_x a^\dagger)/|\alpha|$$

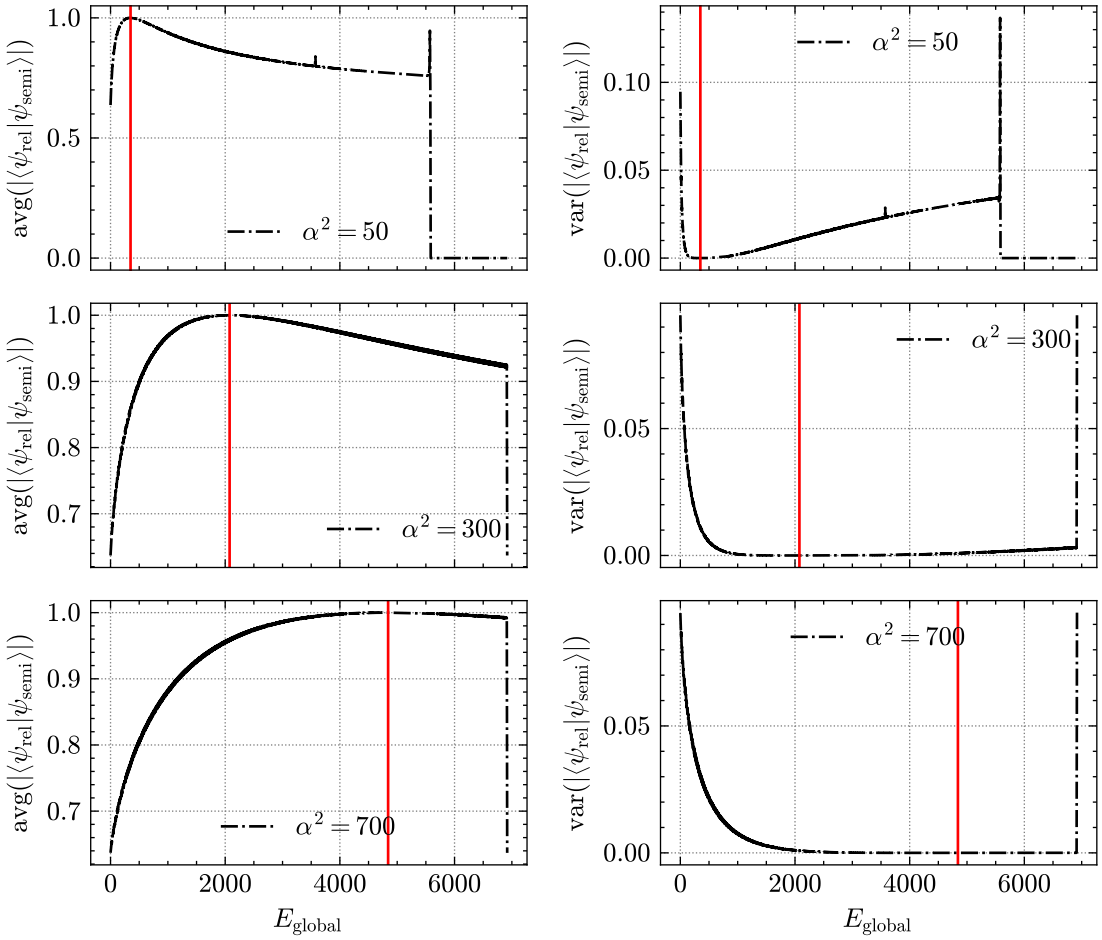

Figure 5.4: The plots shows the average $|\langle\psi_{rel}|\psi_{semi}\rangle|^2$ and variance $|\langle\psi_{rel}|\psi_{semi}\rangle|^2$ over $[0, 4\pi/\omega]$ Time domain v/s corresponding global eigenenergy. The red lines represent E_c. Global eigenstates are calculated from Hamiltonian as given in Eqn. 5.25 and the semiclassical dynamics given by Eqn. 5.26. The parameters are given in Table 5.1

where $\alpha(t) = \alpha e^{-i\omega t}$. The dynamics of the quantum state of the system is then obtained according to Eqn. 3.5, i.e.,

$$|\psi_{rel}(t)\rangle = \mathcal{N}\langle\alpha(t)|\Psi_{\text{glob}}\rangle\rangle, \mathcal{N} = \frac{1}{\sqrt{||\langle\alpha(t)|\Psi_{\text{glob}}\rangle\rangle||^2}} \tag{5.27}$$

where $|\Psi_{E_{glob}}\rangle$ is an eigenstate of Eqn. 5.25.

The initial state is taken as $|\psi_{semi}(0)\rangle = N\left\langle\alpha(0)|\Psi_{E_{glob}}\right\rangle\rangle$, where N denotes normalization for the semiclassical time evolution. The plots are shown for three different global energy eigenstates and a fixed $E_c(\alpha^2) = \hbar\omega|\alpha|^2$. The overlap $|\langle\psi_{rel}|\psi_{semi}\rangle|$ for the identical time evolution is depicted in Figure 5.3.

Figure 5.4 depicts the mean and variance of the absolute value of overlap $|\langle\psi_{rel}|\psi_{semi}\rangle|$ across two time intervals, for three distinct values of $E_c(\alpha^2)$ (repre-

sented as red vertical lines in the diagrams).

The parameters utilized for the numerical calculations are listed below in Table 5.1.

Parameter	g	ω	N_{cutoff}	$\hbar$
Value	$2\pi/100$	2.2π	1000	1

Table 5.1: Parameters used for Numerical Results in Figure 5.4 and Figure 5.3

JC-Model: Rotating Wave Approximation

$$V = \hbar\Omega(\sigma_+ a + \sigma_- a^\dagger)/|\alpha|$$

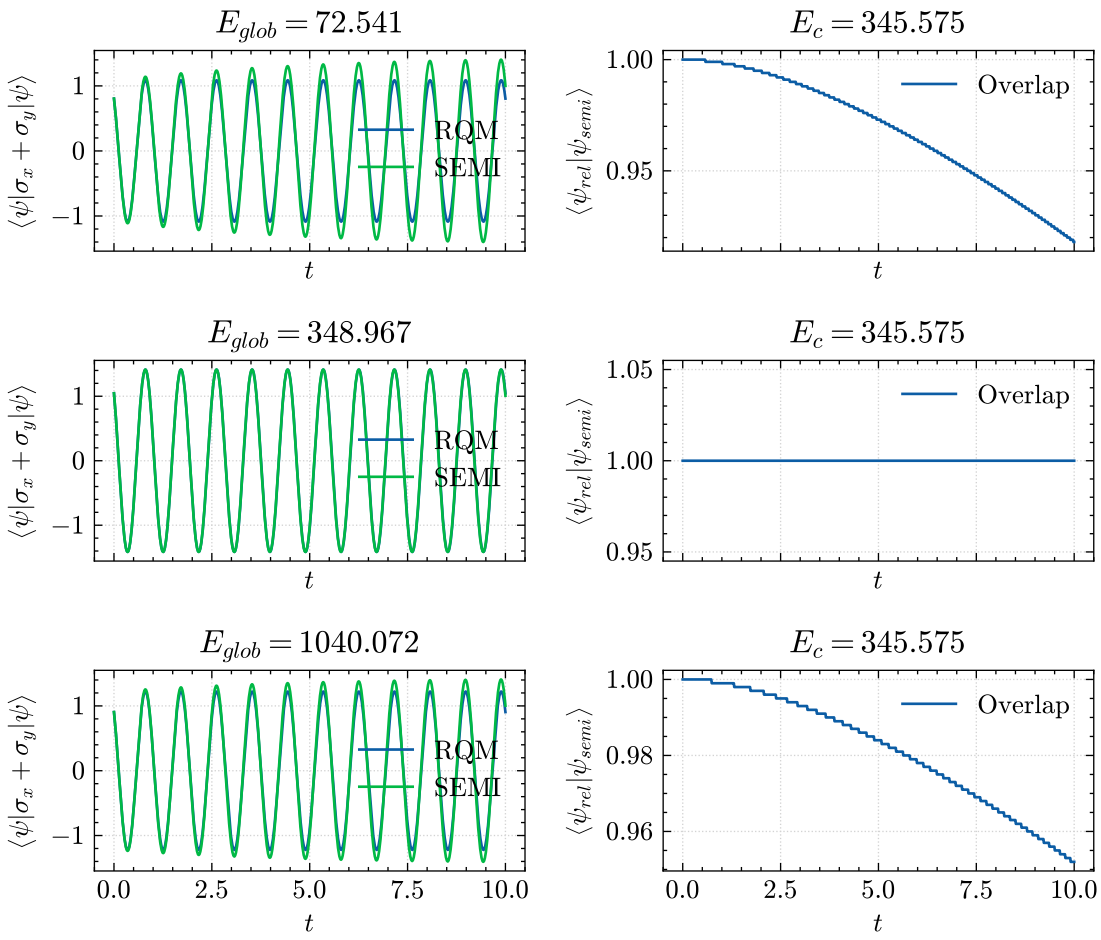

Figure 5.5: Expectation value $\langle\psi|\,\hat{\sigma}_x + \hat{\sigma}_y\,|\psi\rangle$ v/s Time (t) for Semi-Classical and Quantum Relational Dynamics and $|\langle\psi_{\text{rel}}|\psi_{\text{semi}}\rangle|$ v/s Time (t). Global eigenstates are calculated from Hamiltonian as given in Eqn. 5.28 and the semiclassical dynamics given by the unitary operator in Eqn. 5.29. The parameters are given in Table 5.2

Similar to the non-rotating wave approximation, we compare the expectation values of the operator $\hat{\sigma}_x + \hat{\sigma}_y$ in Figure 5.5, as described by the Hamiltonain in Eqn. 5.3, i.e.,

$$\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\frac{g}{|\alpha|}\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right). \tag{5.28}$$

and the semi-classical unitary evolution operator derived as in Eqn. 5.24, i.e.,

$$V = \hbar\Omega(\sigma_+ a + \sigma_- a^\dagger)/|\alpha|$$

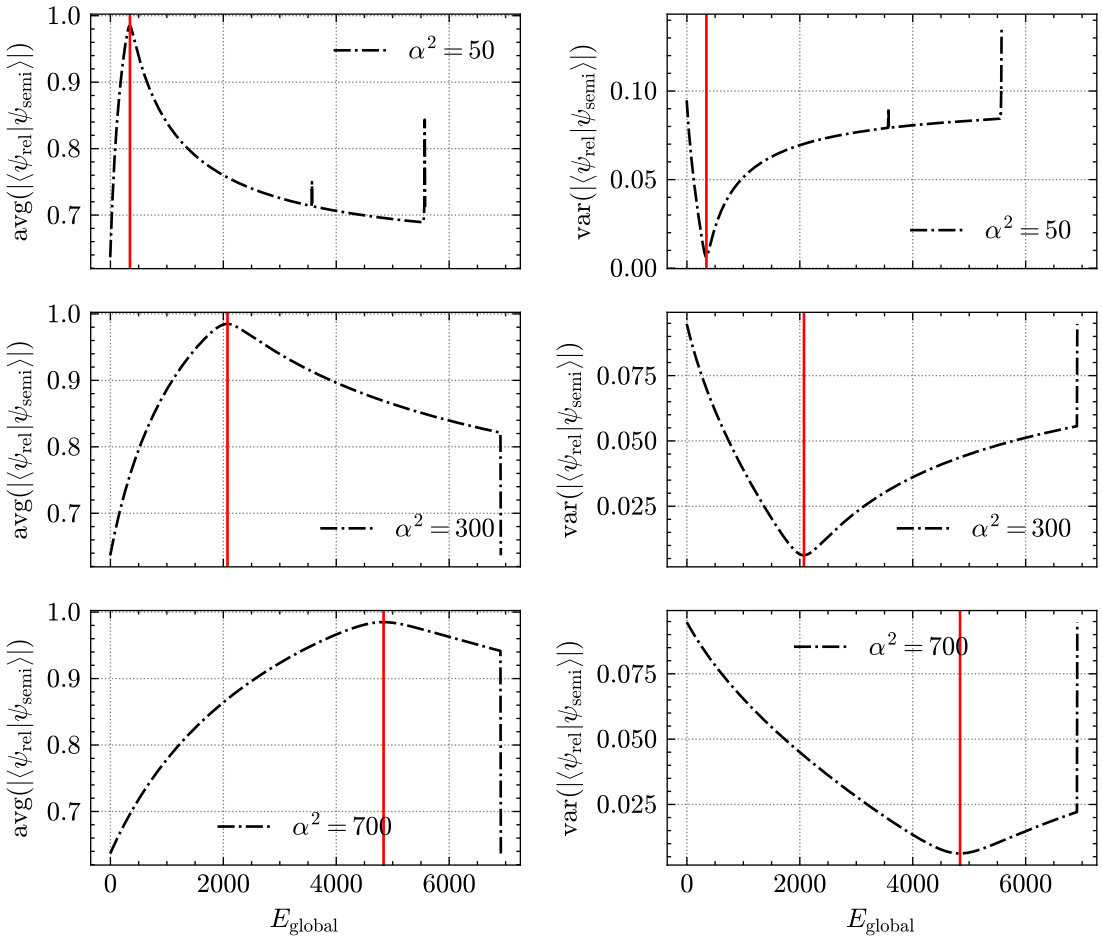

Figure 5.6: The plots shows the average $|\langle\psi_{rel}|\psi_{semi}\rangle|^2$ and variance $|\langle\psi_{rel}|\psi_{semi}\rangle|^2$ over $[0, 4\pi/\Omega]$ Time domain v/s corresponding global eigenenergy. The red lines represent E_c. The Global eigenstates are calculated from Hamiltonian as given in Eqn. 5.28 and the semiclassical dynamics given by the unitary operator in Eqn. 5.29. The parameters are given in Table 5.2

$$\begin{aligned}\tilde{U}(t) \quad &= \cos(\Omega t)\left(e^{-it\frac{\omega}{2\hbar}}|+\rangle\langle+| + e^{it\frac{\omega}{2\hbar}}|-\rangle\langle-|\right) \\ &-i\sin(\Omega t)\left(e^{-it\frac{\omega}{2\hbar}}|+\rangle\langle-| + e^{it\frac{\omega}{2\hbar}}|-\rangle\langle+|\right)\end{aligned} \tag{5.29}$$

Analogous to the non-rotating wave approximation scenario, we calculate the mean and the variance of the absolute value of overlap $|\langle\psi_{rel}|\psi_{semi}\rangle|$ over a span

of two time periods, for three distinct values of $E_c(\alpha^2)$ (illustrated as red vertical lines in the diagrams) in Figure 5.6.

The parameters utilized for the numerical calculations are listed below in Table 5.2.

Parameter	$\Omega = g$	ω	N_{cutoff}	$\hbar$
Value	$2\pi/100$	2.2π	1000	1

Table 5.2: Parameters used for Numerical Results in Figure 5.5 and Figure 5.6

The numerical implementation of the aforementioned results can be found at [Dev 24] or by contacting the author at adityadev.iisermohali@gmail.com.

§ 5.4 Discussion

The numerical results presented in Section 5.3 show a good agreement between the classical limit of quantum relational theory and the semi-classical theories discussed in Chapter 4 and Section 5.2, under appropriate limits. The agreement is more pronounced for the non-rotating wave approximation than for the rotating wave approximation. This is expected as the rotating wave approximation is a more stringent approximation, and the agreement is expected to be better for the non-rotating wave approximation. The agreement is close when the $E_c \approx E_{\text{glob}}$, i.e., the energy carried by the environment state is near the energy of the global state.

Additionally, the above analysis reveals a key trend; as we increase the average number of photons (α^2), and while the energy of the global eigenstate is close to the energy of the conditional state of the environment, the average overlap curve flattens, indicating better agreement even for larger fluctuations of E_c around E_{glob}. This implies that the weak back-coupling limit maintains the semiclassical behavior even for a larger deviation of E_c about the global system's energy, which can be seen both in Figure 5.4 and in Figure 5.6.

Our numerical results align with the conditions outlined in previous chapters for a classical treatment of the environment. These conditions stipulate that the environment should possess most of the global system's energy content and since, for our case, the coupling $g \propto \frac{1}{|\alpha|}$. Hence, for $|\alpha| \to \infty \implies g \to 0$. This signifies a weak interaction limit, where the "back coupling", or the influence of the quantum system on the environment, becomes negligible.

Coherent states as clock states: Our work suggests that coherent states are the most suitable candidates for "clock states" in the quantum relational framework, particularly when considering the classical limit of the theory. In this context, the classical limit of the quantum approach aligns seamlessly with established semiclassical theories. As discussed in [Braun 04], coherent states offer a significant advantage in semi-classical theory. They elegantly overcome the challenge of defining time derivatives near classical turning points, leading to a more robust treatment of the problem.

Another intriguing aspect of coherent states is their potential connection to decoherence. Theoretical studies have conclusively demonstrated that when a single-mode Boson field interacts with its environment and undergoes decoherence – the process by which quantum systems lose their superposition and entan-

glement properties – the field transitions naturally to a coherent state. Remarkably, the only solution of the "master equation" that remains pure in the presence of decoherence are coherent states (Gaussian states) [Dutra 98, Zurek 93].

Potential Connection with Entanglement measure: As discussed in previous chapters, *disentangling* the environment from the quantum subsystem is crucial for extracting the subsystem dynamics in semi-classical theory. In a simplified sense, this *disentanglement* can be achieved by substituting the degrees of freedom of the environment with their classical trajectories (replacing operators such as $\hat{Q}$ with $Q(t)$ within the interaction potential. In other words, $\hat{V}(\hat{x}, \hat{Q})$ becomes $\hat{V}(\hat{x}, Q(t))$. . By comparing the resulting "emergent potential" (Eqn. 3.13) with this "semi-classical" potential, a measure of entanglement could potentially be established.

Chapter 6

Conclusion and Outlook

Our work investigated the Quantum Relational approach to extract the dynamics within the Jaynes-Cummings model. We employ both the rotating wave approximation (RWA) and the non-rotating wave approximation (non-RWA) for our analysis, and we successfully identified the conditions necessary to treat the environment as classical within a relational framework.

Chapter 4 detailed the derivation of the Time-Dependent Schrödinger Equation utilizing semi-classical techniques based on the WKB approximation. The coherent state-based semiclassical approach was further discussed as an alternative method for deriving the TDSE. This approach was then applied to obtain the semi-classical Hamiltonian for the JC-Hamiltonian without the rotating wave approximation.

In Chapter 3, we focused on the emergence of time from symmetry arguments applied to the global entangled energy eigenstate. It then elaborated on incorporating generic interacting quantum systems within the P&W formalism and provided an illustrative example using a simple model. This established a general framework for extending the formalism to arbitrary interactions between subsystems within a global Hilbert space. Chapter 5 addressed the JC-Hamiltonian in the RWA context, diagonalizing it for the resonant case.

Furthermore, Section 5.2 of Chapter 5 explored the semiclassical limit of the JC-Hamiltonian in the large amplitude and weak interaction regime. We were able to derive the exact analytical form of the unitary-time evolution operator under those conditions.Section 5.3 presented the numerical results obtained for the dynamics of the two-level quantum subsystem using both relational and semiclassical approaches. These results were compared for both RWA and non-RWA cases. As anticipated and theoretically derived in previous chapters, our findings

demonstrate that the dynamics described by the Quantum Relational theory converge to those predicted by semi-classical approaches when the environment can be treated classically.

In addition to providing a fundamental interpretation of "time" based on symmetry arguments applied to the global energy eigenstate and the entanglement within the subsystems, the relational formalism presents a powerful framework for obtaining exact analytical results for subsystem dynamics in the presence of complex potentials. This holds true if the conditional clock states are chosen in a way that generates the desired emergent potential, as described by Eqn. 3.14. This has significant implications for both analytical and numerical methods in physics.

However, several open questions remain in our analysis, which require further investigation.

- Thus far, we have only considered dividing the global Hilbert space into two subspaces, one representing the environment and the other the system of interest; future work could explore the definition of the semi-classical limit for scenarios involving more than two subspaces.

- Additionally, a critical question remains: what types of environmental states within a generic quantum system would serve as the best candidates for the conditional clock states, particularly in the context of semi-classical theories?

- This framework highlights the crucial role of entanglement in extracting dynamics and underpins the emergence of time. However, the precise dependence of our findings on the nature of dynamics and entanglement remains an open question.

Appendix A

The Problem of Time

In physics, the action principle is a powerful approach for formulating laws, including general relativity, which essentially describes how a system's properties change with different paths it might take. Interestingly, certain continuous transformations on variables that don't alter this action hold a hidden key. The significance of continuous symmetries lies in their ability to create conservation laws, as per the well-known "Noether's Theorem". Given the essential role of conservation laws in physics, it is both meaningful and beneficial to explore the symmetries of the action.

It is useful to distinguish between two types of symmetries:

- *Dynamical symmetries* corresponding to some inherent property of matter or spacetime evolution (e.g., the Lagrangian being independent of a coordinate, leading to a conserved conjugate momentum)
- *Non-Dynamical symmetries* arise because of the way in which we formulate the action (e.g., the gauge symmetries). Dynamical symmetries constrain the solutions of the equations of motion, while non-dynamical symmetries give rise to special laws called identities. They are distinct from conservation laws because they hold regardless of whether or not one has extremized the action.

§ A.1 Parameterization-Invariance and Hamiltonian Constrain

Consider a system with n degrees of freedom - the generalized coordinates q_i - with a parameter t giving the evolution of the trajectory in configuration space.

We will remove the superscript on q_i when it is clear from the context. Let the action of this system be:

$$\mathcal{S} = \int L_s \left(q, \frac{dq}{dt} \right) dt \tag{A.1}$$

Now consider a new integration parameter τ, which now parameterizes the trajectory and promotes $t \to t(\tau)$ i.e to a dynamical variable. In terms of τ the action (Eqn. A.1) can be expressed as:

$$\mathcal{S} = \int L_s \left(q, \frac{\dot{q}}{\dot{t}} \right) \dot{t} d\tau = \int L \left(q, \dot{q}, \dot{t} \right) d\tau \tag{A.2}$$

where $\dot{a} \equiv \dfrac{da}{d\tau}$ and $L\left(q, \dot{q}, \dot{t}\right) = \dot{t} L_s \left(q, \frac{\dot{q}}{\dot{t}}\right)$. The Hamiltonian for the modified Lagrangian is then obtained by taking the Legendre transformation w.r.t. both $\dot{q}$ and $\dot{t}$ [Deriglazov 11]:

$$\begin{aligned} H &= p_t \dot{t} + p_q \dot{q} - L \\ H &= p_t \dot{t} + \dot{t} p_q (\dot{q}/\dot{t}) - \dot{t} L_s \\ H &= \dot{t} \left(p_t + p_q q' - L_s \right) \end{aligned} \tag{A.3}$$

where $q' = \dfrac{dq}{dt} = (\dot{q}/\dot{t})$.

Let's calculate the conjugate momenta:

$$\begin{aligned} p_q := \frac{\partial L}{\partial \dot{q}} = \dot{t} \frac{\partial L_s}{\partial \dot{q}} = \frac{\partial L_s}{\partial (\dot{q}/\dot{t})} \\ p_q = \frac{\partial L_s}{\partial q'} \end{aligned} \tag{A.4}$$

which coincides with the momentum conjugate to q defined by $L_s(q, q)$. Hence, (Eqn. A.3) get's modified as:

$$H = \dot{t} \left(p_t + H_s \right) \tag{A.5}$$

where $H_s = p_q q' - L_s$ i.e Hamiltonian conjugate to L_s. We have:

$$\begin{aligned} p_t := \frac{\partial L}{\partial \dot{t}} \\ p_t = L_s + \dot{t} \frac{\partial L_s(q, q')}{\partial \dot{t}} \\ p_t = L_s + \dot{t} \left(\frac{\partial L_s(q, q')}{\partial q} \cancelto{0}{\frac{dq}{d\dot{t}}} + \frac{\partial L_s(q, q')}{\partial q'} \frac{d(\dot{q}/\dot{t})}{d\dot{t}} \right) \\ p_t = L_s - q' p_s = -H_s \end{aligned} \tag{A.6}$$

Using (Eqn. A.6) in (Eqn. A.5), the equation reduces to:

$$\boxed{H = 0} \tag{A.7}$$

Therefore, for parameterization-invariant theory, the Hamiltonian function is identically zero. It is crucial to note that this *derivation makes no assumptions about the extremality of the action or the satisfaction of the Euler-Lagrange equations* by the (q's). Consequently, Eqn. A.7 represents a non-dynamical symmetry.

Identity $H = 0$ is very different from conservation law $H = \text{constant}$ arising from a time-independent Lagrangian. The conservation law holds only for solutions of the equations of motion; in contrast, when the action is parameterization-invariant, $H = 0$ holds for any trajectory. The non-dynamical symmetry, therefore, does not constrain the motion.

Appendix B

Super-Selection Rule

In quantum theory, physically measurable quantities of a microscopic system are represented by self-adjoint operators. However, not all of the self-adjoint operators correspond to measurable quantities. The super-selection rule is a criterion to distinguish measurable self-adjoint operator from the un-measurable ones, i.e., any measurable quantity must obey the superselection rules. By contraposition, any quantity which does not obey the superselection rules cannot be measured. It should be noted that such a statement implies that the set of (physically realizable) observables is strictly smaller than the set of all self-adjoint operators on Hilbert space. The notion of superselection rule (henceforth abbreviated SSR) was first introduced in 1952 by Wick, Wightman, and Wigner [Wick 52] in connection with the problem of consistently assigning intrinsic parity to elementary particles.

In the context of quantum field theory, the electric current $J^{\mu} = \overline{\zeta}\gamma^{\mu}\zeta$ is defined in terms of the Dirac spinor field operator ζ for electrons. The electric current J^{μ} is self-adjoint and measurable. However, the operators

$$\frac{1}{2}\left(\zeta + \zeta^{\dagger}\right) \qquad \frac{1}{2i}\left(\zeta - \zeta^{\dagger}\right) \tag{B.1}$$

are also self-adjoint but they are not measurable even via indirect methods.

A SSR is stated as follows: There is an operator $\hat{J}$, which we call the superselection charge. If a self-adjoint operator $\hat{A}$ represents a measurable quantity, it must satisfy commutativity [Tanimura 11]

$$\left[\hat{J}, \hat{A}\right] = 0 \tag{B.2}$$

This is a SSR, which is a necessary condition for the measurability of $\hat{A}$. The SSR can be compared with a conservation law. The conservation of $\hat{J}$ is formulated as

$$\left[\hat{J}, \hat{H}\right] = 0 \tag{B.3}$$

where $\hat{H}$ is the Hamiltonian $\hat{H}$ of the system. The conservation law (Eqn. B.3) requires that $\hat{J}$ commutes with the Hamiltonian $\hat{H}$ while the superselection rule (Eqn. B.2) requires that $\hat{J}$ commutes with all of the measurable quantities. Thus, the superselection rule is a stronger requirement for $\hat{J}$ than the conservation law. It can be said that the superselection rule is an extreme form of conservation laws.

Notice that the SSR (Eq Eqn. B.2) implies that, for all physically measurable observables $\hat{A}$ and any eigenvector $|\zeta_i\rangle$ of $\hat{J}$ with charge q_i[1].

$$\begin{aligned} \langle\zeta_i|\,\hat{J}\hat{A}\,|\zeta_j\rangle - \langle\zeta_i|\,\hat{A}\hat{J}\,|\zeta_j\rangle &= 0 \\ (q_i - q_j)\,\langle\zeta_i|\,\hat{A}\,|\zeta_j\rangle &= 0 \\ \langle\zeta_i|\,\hat{A}\,|\zeta_j\rangle &= 0 \quad (\text{for } i \neq j) \end{aligned} \tag{B.4}$$

So, if we have a quantum state in a coherent superposition of eigenstates of superselection charge $\hat{J}$ i.e. $|\zeta_+\rangle = \frac{|\zeta_1\rangle + |\zeta_2\rangle}{\sqrt{2}}$. One can check that:

$$\begin{aligned} \langle\zeta_+|\,\hat{A}\,|\zeta_+\rangle &= \frac{\langle\zeta_1|\,\hat{A}\,|\zeta_1\rangle + \langle\zeta_2|\,\hat{A}\,|\zeta_2\rangle + \overbrace{2\mathrm{Re}[\langle\zeta_1|\,\hat{A}\,|\zeta_2\rangle]}^{=0}}{2} \\ \langle\zeta_+|\,\hat{A}\,|\zeta_+\rangle &= \frac{\langle\zeta_1|\,\hat{A}\,|\zeta_1\rangle + \langle\zeta_2|\,\hat{A}\,|\zeta_2\rangle}{2} = \mathrm{Tr}\Big\{\rho\hat{A}\Big\} \end{aligned} \tag{B.5}$$

where

$$\rho = \frac{|\zeta_1\rangle\langle\zeta_1| + |\zeta_2\rangle\langle\zeta_2|}{2} \tag{B.6}$$

i.e. any relative phase between $|\zeta_1\rangle$ and $|\zeta_2\rangle$ is not measurable and that coherent superposition of $|\zeta_1\rangle$ and $|\zeta_2\rangle$ cannot be verified (or prepared). For an observer, the state $|\zeta_+\rangle$ will not be any different from a mixed state ρ.

[1] we assume $\hat{J}$ is non degenerate

Appendix C

Detailed Calculation of Two coupled Two-level systems

For $\hat{H}_{\mathrm{S}} = 0$, the global Hamiltonian reads

$$\hat{H} = \underbrace{\hat{\sigma}_{\mathrm{C},x}}_{=1_S \otimes \hat{H}_{\mathrm{C}}} + \underbrace{(\hat{\sigma}_{\mathrm{S},x} + \hat{\sigma}_{\mathrm{S},x}) \otimes \hat{\sigma}_{\mathrm{C},x}}_{=\hat{V}} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \tag{C.1}$$

and has eigenvalues $E = \{-\sqrt{3}, \sqrt{3}\}$, both with multiplicity two. The Eigenvectors corresponding to them are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ -(1+\sqrt{3}) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1-\sqrt{3} \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ \sqrt{3}-1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1+\sqrt{3} \\ -1 \end{pmatrix} \tag{C.2}$$

for components $\{|\uparrow_S\uparrow_C\rangle, |\uparrow_S\downarrow_C\rangle, |\downarrow_S\uparrow_C\rangle, |\downarrow_S\downarrow_C\rangle\}$.

One can check for $|\Psi\rangle = |\uparrow_S\uparrow_{\mathrm{C}}\rangle - |\downarrow_{\mathrm{S}}\uparrow_{\mathrm{C}}\rangle - (1+\sqrt{3})\,|\downarrow_{\mathrm{s}}\downarrow_{\mathrm{C}}\rangle$, which we have takes as the global state in Section 3.1, yields

$$\begin{aligned} \hat{H}|\Psi\rangle = &(|\uparrow_{\mathrm{s}}\uparrow_{\mathrm{C}}\rangle + |\uparrow_{\mathrm{s}}\downarrow_{\mathrm{C}}\rangle + |\downarrow_{\mathrm{s}}\downarrow_{\mathrm{C}}\rangle) \\ &- (|\downarrow_{\mathrm{s}}\uparrow_{\mathrm{C}}\rangle + |\uparrow_{\mathrm{s}}\downarrow_{\mathrm{C}}\rangle - |\downarrow_{\mathrm{s}}\downarrow_{\mathrm{C}}\rangle) \\ &- (\sqrt{3}+1)\,(-|\downarrow_{\mathrm{S}}\downarrow_{\mathrm{C}}\rangle + |\uparrow_{\mathrm{s}}\uparrow_{\mathrm{C}}\rangle - |\downarrow_{\mathrm{s}}\uparrow_{\mathrm{C}}\rangle) \\ = &-\sqrt{3}\,|\uparrow_{\mathrm{s}}\uparrow_{\mathrm{C}}\rangle + \sqrt{3}\,|\downarrow_{\mathrm{s}}\uparrow_{\mathrm{C}}\rangle + (\sqrt{3}+3)\,|\downarrow_{\mathrm{s}}\downarrow_{\mathrm{C}}\rangle \\ = &-\sqrt{3}|\Psi\rangle. \end{aligned} \tag{C.3}$$

We define $c \equiv 1 + \sqrt{3}$ and take the clock state to be

$$|\chi(\lambda)\rangle_{\mathrm{C}} = e^{iE\lambda}\left[a_0 e^{-ig\lambda}\,|\uparrow_{\mathrm{C}}\rangle_{\mathrm{C}} + a_1 e^{ig\lambda}\,|\downarrow_{\mathrm{C}}\rangle_{\mathrm{C}}\right] \tag{C.4}$$

Then, the unnormalized system state becomes

$$\begin{aligned}|\phi\rangle_{\mathrm{S}} =&\langle\chi(\lambda) \mid \Psi\rangle_{\mathrm{C}} \\ =&e^{-iE\lambda}\left[a_0^* e^{ig\lambda}\,|{\uparrow_{\mathrm{S}}}\rangle_{\mathrm{S}} - \left(ca_1^* e^{-ig\lambda} + a_0^* e^{ig\lambda}\right)|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}}\right].\end{aligned} \tag{C.5}$$

For simplicity, we use $a_0 = a_1 = 1$ with $\langle\chi \mid \chi\rangle_{\mathrm{C}} = 2$. The λ-dependent unnormalized state is

$$\begin{aligned}|\phi(\lambda)\rangle_{\mathrm{S}} &= \langle\chi(\lambda) \mid \Psi\rangle_{\mathrm{C}} \\ &= e^{i\sqrt{3}\lambda}\left[e^{i\lambda}\,|{\uparrow_{\mathrm{S}}}\rangle_{\mathrm{S}} - \left(ce^{-i\lambda} + e^{i\lambda}\right)|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}}\right] \\ &= e^{ic\lambda}\left[|{\uparrow_{\mathrm{S}}}\rangle_{\mathrm{S}} - \left(ce^{-2i\lambda} + 1\right)|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}}\right]\end{aligned} \tag{C.6}$$

with norm

$$\begin{aligned}\langle\phi \mid \phi\rangle_{\mathrm{S}} &= \langle{\uparrow_{\mathrm{S}}}|{\uparrow_{\mathrm{S}}}\rangle_{\mathrm{S}} + \left|c + e^{2i\lambda}\right|^2 \langle{\downarrow_{\mathrm{S}}}|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}} \\ &= 1 + \underbrace{c^2}_{=2(2+\sqrt{3})} + 1 + c\underbrace{\left(e^{2i\lambda} + e^{-2i\lambda}\right)}_{=2\cos(2\lambda)} \\ &= 2[3 + \sqrt{3} + c\cos(2\lambda)] \\ &= 2[2 + c(1 + \cos(2\lambda))] \\ &= 4\left[1 + c\cos^2(\lambda)\right]\end{aligned} \tag{C.7}$$

and

$$\sqrt{\langle\phi \mid \phi\rangle_{\mathrm{S}}} = 2\sqrt{1 + c\cos^2(\lambda)}. \tag{C.8}$$

For $\lambda = 0$. the initial state reads

$$|\phi(0)\rangle_{\mathrm{S}} = |{\uparrow_{\mathrm{S}}}\rangle_{\mathrm{S}} - (\sqrt{3} + 2)\,|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}}\,. \tag{C.9}$$

With the expression

$$\begin{aligned}\hat{V}|\Psi\rangle =& (|{\downarrow_S}{\downarrow_C}\rangle + |{\uparrow_S}{\downarrow_C}\rangle) - (|{\uparrow_S}{\downarrow_C}\rangle - |{\downarrow_S}{\downarrow_C}\rangle) \\ &- c\,(|{\uparrow_s}{\uparrow_C}\rangle - |{\downarrow_s}{\uparrow_C}\rangle) \\ =& - c\,|{\uparrow_s}{\uparrow_C}\rangle + c\,|{\downarrow_s}{\uparrow_C}\rangle + 2\,|{\downarrow_S}{\downarrow_C}\rangle\,,\end{aligned} \tag{C.10}$$

we determine

$$\begin{aligned}\langle\chi(\lambda)|\hat{V}|\Psi\rangle_{\mathrm{C}} =&e^{-iE\lambda}\left[-ce^{i\lambda}\,|{\uparrow_{\mathrm{S}}}\rangle_{\mathrm{S}} + ce^{i\lambda}\,|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}}\right. \\ &\left.+2e^{-i\lambda}\,|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}}\right] \\ =&e^{-iE\lambda}\left[-ce^{i\lambda}\,|{\uparrow_{\mathrm{S}}}\rangle_{\mathrm{S}} + \left(ce^{i\lambda} + 2e^{-i\lambda}\right)|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}}\right] \\ =&e^{-iE\lambda}e^{i\lambda}\left[-c\,|{\uparrow_{\mathrm{S}}}\rangle_{\mathrm{S}} + \left(c + 2e^{-2i\lambda}\right)|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}}\right]\end{aligned} \tag{C.11}$$

Using also

$$\langle\,\phi|_{\mathrm{S}} = e^{iE\lambda}e^{-i\lambda}\left[|{\uparrow_{\mathrm{S}}}\rangle_{\mathrm{S}} - \left(ce^{2i\lambda} + 1\right)|{\downarrow_{\mathrm{S}}}\rangle_{\mathrm{S}}\right], \tag{C.12}$$

allows us to express

$$\begin{aligned}\left\langle \chi \left| \hat{V}\hat{P}_\Psi \right| \chi \right\rangle_{\mathrm{C}} =& \left[-c\,|{\uparrow_{\mathrm{s}}}\rangle_{\mathrm{s}} + \left(c + 2e^{-2i\lambda}\right)|{\downarrow_{\mathrm{s}}}\rangle_{\mathrm{s}}\right] \\ & \otimes \left[\langle{\uparrow_{\mathrm{s}}}|_{\mathrm{s}} - \left(ce^{2i\lambda}+1\right)\langle{\downarrow_{\mathrm{s}}}|_{\mathrm{s}}\right] \\ =& -c\,|{\uparrow_{\mathrm{s}}}\rangle\,{\uparrow_{\mathrm{s}}}|_{\mathrm{s}} + c\left(ce^{2i\lambda}+1\right)|{\uparrow_{\mathrm{s}}}\rangle\langle{\downarrow_{\mathrm{s}}}|_{\mathrm{s}} \\ & + \left(c + 2e^{-2i\lambda}\right)|{\downarrow_{\mathrm{S}}}\rangle\,{\uparrow_{\mathrm{s}}}\big|_{\mathrm{s}} \\ & - \left(ce^{2i\lambda}+1\right)\left(c + 2e^{-2i\lambda}\right)|{\downarrow_{\mathrm{S}}}\rangle\,{\downarrow_{\mathrm{S}}}\big|_{\mathrm{s}}\end{aligned} \tag{C.13}$$

and

$$\begin{aligned}&\left\langle \chi \left| \hat{P}_W\hat{V} \right| \chi \right\rangle_{\mathrm{C}} = -c\,|{\uparrow_{\mathrm{S}}}\rangle\langle{\uparrow_{\mathrm{S}}}|_{\mathrm{S}} + c\left(ce^{-2i\lambda}+1\right)|{\downarrow_{\mathrm{S}}}\rangle\langle{\uparrow_{\mathrm{S}}}|_{\mathrm{S}} \\ &+ \left(c + 2e^{2i\lambda}\right)|{\uparrow_s}\rangle\langle{\downarrow_s}|_s \\ &- \left(ce^{-2i\lambda}+1\right)\left(c + 2e^{2i\lambda}\right)|{\downarrow_S}\;K\;{\downarrow_S}|_s\end{aligned} \tag{C.14}$$

With the definition

$$f(\lambda) = \cos(2\lambda) + c\cos^2(\lambda), \tag{C.15}$$

the individual components read

$$\begin{aligned}\left\langle {\uparrow_{\mathrm{S}}}\,\chi \left| \left\{\hat{P}_\Psi, \hat{V}\right\} \right| {\uparrow_{\mathrm{S}}}\,\chi \right\rangle_{\mathrm{C}} &= -2c \\ \left\langle {\uparrow_{\mathrm{S}}}\,\chi \left| \left\{\hat{P}_\Psi, \hat{V}\right\} \right| {\downarrow_{\mathrm{S}}}\,\chi \right\rangle_{\mathrm{C}} &= c\left(ce^{2i\lambda}+1\right) + \left(c + 2e^{2i\lambda}\right) \\ &= 2\left[2\cos(2\lambda) + c\left(1 + e^{2i\lambda}\right)\right] \\ &= 4\left[\cos(2\lambda) + c\cos^2(\lambda)\right] + 2ic\sin(2\lambda) \\ &= 4f(\lambda) + 2ic\sin(2\lambda)\end{aligned} \tag{C.16}$$

$$\left\langle {\downarrow_{\mathrm{s}}}\,\chi \left| \left\{\hat{P}_\Psi, \hat{V}\right\} \right| {\uparrow_{\mathrm{s}}}\,\chi \right\rangle_{\mathrm{C}} = 4\left[\cos(2\lambda) + c\cos^2(\lambda)\right] - 2ic\sin(2\lambda) \quad \text{and}$$

$$= 4f(\lambda) - 2ic\sin(2\lambda) \tag{C.17}$$

$$\left\langle {\downarrow_{\mathrm{s}}}\,\chi \left| \left\{\hat{P}_\Psi, \hat{V}\right\} \right| {\downarrow_{\mathrm{s}}}\,\chi \right\rangle_{\mathrm{C}} = -2\,\mathrm{Re}\left(ce^{2i\lambda}+1\right)\cdot\left(c + 2e^{-2i\lambda}\right) \tag{C.18}$$

$$= -2\,\mathrm{Re}\left[2\left(e^{2i\lambda} + e^{-2i\lambda}\right) + 2c\left(1 + e^{2i\lambda}\right) + c\right] \tag{C.19}$$

$$= -2[4\cos(2\lambda) + 2c(1 + \cos(2\lambda)) + c] \tag{C.20}$$

$$= -2\left[4\cos(2\lambda) + 4c\cos^2(\lambda) + c\right] \tag{C.21}$$

$$\begin{aligned}&= -8\left[\cos(2\lambda) + c\cos^2(\lambda)\right] - 2c \\ &= -8f(\lambda) - 2c.\end{aligned} \tag{C.22}$$

Thus, the main part of the effective system potential is

$$\begin{aligned}\left\langle \chi \left| \left\{ \hat{V}, \hat{P}_{\psi} \right\} \right| \chi \right\rangle_{\mathrm{C}} = &- 2c \sin(2\lambda) \hat{\sigma}_{\mathrm{s},y} + 4f(\lambda) \left(\hat{\sigma}_{\mathrm{S},x} + \hat{\sigma}_{\mathrm{S},x} \right) \\ &- 2[c + 2f(\lambda)] \mathrm{i}_{\mathrm{S}}.\end{aligned} \tag{C.23}$$

in terms of Pauli operators. In addition,

$$\begin{aligned}\left\langle \Psi \left| \hat{V} \hat{P}_{\chi} \right| \Psi \right\rangle =& \left[-c \left\langle \uparrow_{\mathrm{S}} |_{\mathrm{S}} + \left(c + 2e^{2i\lambda} \right) \left\langle \downarrow_{\mathrm{S}} |_{\mathrm{S}} \right] \right. \\ & \left. \cdot \left[|\uparrow_{\mathrm{S}} \rangle_{\mathrm{S}} - \left(ce^{-2i\lambda} + 1 \right) |\downarrow_{\mathrm{S}} \rangle_{\mathrm{S}} \right] \right. \\ =& -c - \left(c + 2e^{2i\lambda} \right) \left(ce^{-2i\lambda} + 1 \right) \\ =& -[2c + 2e^{2i\lambda} + \underbrace{c^2}_{=2(1+c)} e^{-2i\lambda} + 2c] \\ =& -2 \left[2c + e^{2i\lambda} + (1+c) e^{-2i\lambda} \right] \\ =& -2 \left[2c + e^{2i\lambda} + e^{-2i\lambda} + ce^{-2i\lambda} \right] \\ =& -2[2c + 2\cos(2\lambda) + c\cos(2\lambda) - i\sin(2\lambda)] \\ =& 2i\sin(2\lambda) - 2[2c + (2+c)\cos(2\lambda)] \\ =& 2i\sin(2\lambda) - 2 \left[2c + (2+c) \left(2\cos^2(\lambda) - 1 \right) \right]\end{aligned} \tag{C.24}$$

$$\begin{aligned}\mathrm{Re} \left\langle \Psi \left| \hat{V} \hat{P}_{\chi} \right| \Psi \right\rangle &= -2[2c + (2+c)\cos(2\lambda)] \\ &= -2 \left[c + 2 \left(\cos(2\lambda) + c\cos^2(\lambda) \right) \right] \\ &= -2[c + 2f(\lambda)]\end{aligned} \tag{C.25}$$

Hence, and

$$\hat{V}_{\mathrm{S}} = \frac{\left\langle \chi \left| \left\{ \hat{V}, \hat{P}_{\Psi} \right\} \right| \chi \right\rangle_{\mathrm{C}} - \mathrm{Re} \left\langle \Psi \left| \hat{V} \hat{P}_{\chi} \right| \Psi \right\rangle \hat{1}_{\mathrm{S}}}{\left\langle \Psi \left| \hat{P}_{\chi} \right| \Psi \right\rangle} \tag{C.26}$$

$$= \frac{-c\sin(2\lambda)}{2\left[1 + c\cos^2(\lambda)\right]} \hat{\sigma}_{\mathrm{S},y} + \frac{\cos(2\lambda) + c\cos^2(\lambda)}{\left[1 + c\cos^2(\lambda)\right]} \left(\hat{\sigma}_{\mathrm{S},x} + \hat{\sigma}_{\mathrm{S},z} \right) \tag{C.27}$$

$$\equiv \boldsymbol{V}_s(\lambda) \cdot \hat{\sigma} \tag{C.28}$$

Appendix D

Diagonalization of JC-model

Our initial task is to achieve the block-diagonalization of 5.3, followed by the diagonalization of these blocks. A quick check shows that the ground state of the atom and the vacuum state of the mode constitute eigenstates of the Hamiltonian, represented as

$$\hat{H}\left|-,0\right\rangle = -\frac{\hbar\omega}{2}\left|-,0\right\rangle . \tag{D.1}$$

This results in a 1×1 block in the decomposition. Furthermore, we observe that

$$\begin{aligned} \hat{H}\left|+,n-1\right\rangle &= \left(\frac{\hbar\omega}{2} + \hbar\omega(n-1)\right)\left|+,n-1\right\rangle + \hbar g\sqrt{n}\left|-,n\right\rangle , \\ \hat{H}\left|-,n\right\rangle &= \left(-\frac{\hbar\omega}{2} + \hbar\omega n\right)\left|-,n\right\rangle + \hbar g\sqrt{n}\left|+,n-1\right\rangle . \end{aligned} \tag{D.2}$$

This implies that $\hat{H}$ maps all elements within the subspace spanned by $\{\left|+,n-1\right\rangle, \left|-,n\right\rangle\}$ back to that same subspace. In essence, $\hat{H}$ block-diagonalizes concerning these subspaces. Using Eqn. D.2, we can derive the matrix elements of the corresponding block matrices, i.e.,

$$\begin{bmatrix} \langle +,n-1|\hat{H}|+,n-1\rangle & \langle +,n-1|\hat{H}|-,n\rangle \\ \langle -,n|\hat{H}|+,n-1\rangle & \langle -,n|\hat{H}|-,n\rangle \end{bmatrix} = \begin{bmatrix} 0 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & 0 \end{bmatrix} + \hbar\omega\left(n-\frac{1}{2}\right)\hat{I} \tag{D.3}$$

The eigenvalues and eigenvectors of this matrix can be expressed as

$$E_{n,\pm} = \pm\hbar g\sqrt{n} + \hbar\omega\left(n-\frac{1}{2}\right),\ \left|\psi_{n,\pm}\right\rangle := \frac{1}{\sqrt{2}}\left(\left|+,n-1\right\rangle \pm \left|-,n\right\rangle\right) \tag{D.4}$$

and finally, the uniraty evolution operator can be expressed as

$$
\begin{aligned}
U(t) = e^{-itH/\hbar} &= e^{\frac{1}{2}it\omega} \left|-,0\right\rangle \left\langle -,0\right| + \sum_{n=1}^{\infty} e^{-it\omega(n-\frac{1}{2})} e^{-itg\sqrt{n}} \left|\psi_{n,+}\right\rangle \left\langle \psi_{n,+}\right| \\
&+ \sum_{n=1}^{\infty} e^{-it\omega(n-\frac{1}{2})} e^{itg\sqrt{n}} \left|\psi_{n,-}\right\rangle \left\langle \psi_{n,-}\right| \\
&= e^{\frac{1}{2}it\omega} \left|-,0\right\rangle \left\langle -,0\right| \\
&+ \sum_{n=1}^{\infty} e^{-it\omega(n-\frac{1}{2})} \Big[\cos\big(tg\sqrt{n}\big) \left(\left|+,n-1\right\rangle \left\langle +,n-1\right| + \left|-,n\right\rangle \left\langle -,n\right| \right) \\
&- i \sin\big(tg\sqrt{n}\big) \left(\left|+,n-1\right\rangle \left\langle -,n\right| + \left|-,n\right\rangle \left\langle +,n-1\right| \right) \Big]. \qquad \text{(D.5)}
\end{aligned}
$$

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