DAY 5

CONDITIONS FOR CONSISITENCY (NATURE OF SOLUTIONS OF LINEAR EQUATIONS):-

In previous sections, we have discussed solutions of pair of linear equations. In this section we shall discuss that without solving how we come to know about their nature $i.e.$ system of linear equations has consistent solution with unique or infinitely solutions **or** inconsistent solution.

Given equations are

$$a_1x + b_1y + c_1 = 0 \ldots\ldots..\text{(i) and} \quad a_2x + b_2y + c_2 = 0 \ldots\ldots\ldots\ldots.\text{(ii)}$$

By cross Multiplication method, we obtain

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\ ;\ y = \frac{a_1c_2 - a_2c_1}{a_2b_1 - a_1b_2}$$

Now the following cases Arise : it is observed that value of x & y depends on value of $a_1b_2 - a_2b_1$

$\therefore$ Two cases arise : -

- When $a_1b_2 - a_2b_1 \neq 0 \quad \Rightarrow a_1b_2 \neq a_2b_1 \quad \Rightarrow \boxed{\boldsymbol{\frac{a_1}{a_2} \neq \frac{b_1}{b_2}}}$

  This condition gives that system of equations has a **unique solution**.

- When $a_1b_2 - a_2b_1 = 0$ then we can not divide the given equations by this to get value of x & y.

  $$\Rightarrow a_1b_2 = a_2b_1 \quad \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

  Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k\ \ (k \neq 0)$ Then $a_1 = ka_2$ & $b_1 = kb_2$

  Now there are two possibilities

  - If $\frac{c_1}{c_2} = k \ \Rightarrow\ c_1 = kc_2$

    First Equation can be written as after replacing values of a_1, b_1, c_1

    $$ka_2x + kb_2y + kc_2 = 0$$

    $$\Rightarrow\ k(a_2x + b_2y + c_2) = 0 \qquad \Rightarrow k(a_2x + b_2y + c_2) = 0$$

    Which is same as second equation. Hence every solution of i) is a solution of ii)

    $\therefore$ The system has infinite number of solutions

    Condition for infinitely many solutions is $\boxed{\boldsymbol{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}}$

  - If $\frac{c_1}{c_2} \neq k \quad \Rightarrow c_1 \neq kc_2$

    i) $\Rightarrow\ \ ka_2x + kb_2y + c_1 = 0 \quad \Rightarrow k(a_2x + b_2y) + c_1 = 0$

    $\Rightarrow\ k(-c_2) + c_1 = 0$ (Using ii)) $\quad \Rightarrow \quad c_1 = kc_2$

    Which contradicts $c_1 \neq kc_2$

  $\therefore$ The system has no solution or inconsistent

  Condition for no solution or inconsistency is $\boxed{\boldsymbol{\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}}}$

SUMMARY

If	*System has*	*Graphical Representation*
$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	*Consistent with unique sol.*	*A pair of intersecting lines*
$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	*Consistent with infinitely many sol.*	*Coincident lines*
$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	*Inconsistent (No solution)*	*Parallel lines*

1. Verify whether the following pair of linear equations has a unique solution no solution or infinitely many solutions.

i) $\boldsymbol{5x - 2y = 7}$ **and** $\boldsymbol{3x + 4y = 6}$

ii) $\boldsymbol{2x + 3y = 5}$ **and** $\boldsymbol{4x + 6y = 8}$

iii) $\boldsymbol{3x - 5y - 2 = 0}$ **and** $\boldsymbol{9x - 15y - 6 = 0}$

iv) $\boldsymbol{2x + 5y = 17}$ **and** $\boldsymbol{5x + 3y = 14}$

v) $\boldsymbol{3x - 2y - 2 = 0}$ **and** $\boldsymbol{6x - 4y + 5 = 0}$

Sol:

i) $5x - 2y = 7$ and $3x + 4y = 6$

Compare these with $a_1x + b_1y = c_1$ & $a_2x + b_2y = c_2$

$a_1 = 5, b_1 = -2, c_1 = 7$ & $a_2 = 3, b_2 = 4, c_2 = 6$

$$\frac{a_1}{a_2} = \frac{5}{3} \text{ and } \frac{b_1}{b_2} = \frac{-2}{4} = \frac{-1}{2} \qquad \therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

$\therefore$ Given pair of equations has **a unique solution**.

ii) Given equations are $2x + 3y = 5$ & $4x + 6y = 8$

Compare with $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

$a_1 = 2, b_1 = 3, c_1 = 5$ & $a_2 = 4, b_2 = 6, c_2 = 8$

$$\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2}; \frac{b_1}{b_2} = \frac{3}{6} = \frac{1}{2}; \frac{c_1}{c_2} = \frac{5}{8}$$

Clearly $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

$\therefore$ The system has **no solution.**

iii) Given equations are $3x - 5y - 2 = 0$ & $9x - 15y - 6 = 0$

Compare with $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

$a_1 = 3, b_1 = -5, c_1 = -2$ & $a_2 = 9, b_2 = -15, c_2 = -6$

$$\frac{a_1}{a_2} = \frac{3}{9} = \frac{1}{3}; \frac{b_1}{b_2} = \frac{-5}{-15} = \frac{1}{3}; \frac{c_1}{c_2} = \frac{-2}{-6} = \frac{1}{3}$$

Clearly $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\therefore$ The system has **infinitely many solutions**.

iv) Given equations are $2x + 5y = 17$ or $2x + 5y - 17 = 0$

$\&\quad 5x + 3y = 14 \quad$ or $\quad 5x + 3y - 14 = 0$

Compare these with $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

$a_1 = 2\,, b_1 = 5, c_1 = -17$ & $a_2 = 5, b_2 = 3, c_2 = -14$

$$\frac{a_1}{a_2} = \frac{2}{5}\ ;\ \frac{b_1}{b_2} = \frac{5}{3} \qquad \therefore\ \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

$\therefore$ Given pair of equations has **a unique solution.**

v) Given equations are $3x - 2y - 2 = 0$ & $6x - 4y + 5 = 0$

Compare with $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

$a_1 = 3\,, b_1 = -2, c_1 = -2$ & $a_2 = 6, b_2 = -4, c_2 = 5$

$$\frac{a_1}{a_2} = \frac{3}{6}\ ;\ \frac{b_1}{b_2} = \frac{-2}{-4}\ ;\ \frac{c_1}{c_2} = \frac{-2}{5}$$

Clearly $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

$\therefore$ The system has **no solution**

2. **For what value of p, the following system of equations has a unique solution**
   $\boldsymbol{3x + py = 5\ \&\ 2x + 4y = 7}$

Sol: Given equations are $3x + py = 5$ & $2x + 4y = 7$

Compare With $a_1x + b_1y = c_1$ & $a_2x + b_2y = c_2$

$a_1 = 3\,, b_1 = p, c_1 = 5$ & $a_2 = 2, b_2 = 4, c_2 = 7$

$$\frac{a_1}{a_2} = \frac{1}{3}\ ;\ \frac{b_1}{b_2} = \frac{-k}{2}\ ;\ \frac{c_1}{c_2} = \frac{-2}{5}$$

For Unique solution $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} \quad \Rightarrow \dfrac{3}{2} \neq \dfrac{p}{4} \quad$ i.e. $p \neq 6$

3. **Find the value of k for which the following system has infinitely many solutions**
   $\boldsymbol{kx + 3y = k - 3}$ **and** $\boldsymbol{12x + ky = k}$

Sol: Given equations are $kx + 3y = k - 3$ and $12x + ky = k$

Compare with $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

$a_1 = k\,, b_1 = 3, c_1 = k - 3,\ a_2 = 12, b_2 = k, c_2 = k$

For Infinite many solutions: $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \quad \Rightarrow \dfrac{k}{12} = \dfrac{3}{k} = \dfrac{k-3}{k}$

Taking 1st & 2nd $\dfrac{k}{12} = \dfrac{3}{k} \quad \Rightarrow k^2 = 36 \quad \Rightarrow \boldsymbol{k = \pm 6}$

$\Rightarrow 2k - 2 = k + 1 \quad \Rightarrow k = 3$

Taking 2nd & last $\dfrac{3}{k} = \dfrac{k-3}{k} \quad \Rightarrow 3k = k^2 - 3k \quad \Rightarrow k^2 - 6k = 0$

$\Rightarrow k(k - 6) = 0 \quad \Rightarrow \boldsymbol{k = 0, 6}$

In both conditions, $\boldsymbol{k = 6}$ is common solution. So required solution $k = 6$.

4. **For what values of a & b, the following system has infinite number of solutions.**
   $\boldsymbol{2x + 3y = 7}$ **and** $\boldsymbol{(a - b)x + (a + b)y = 3a + b - 2}$

Sol :- Given equations are $2x + 3y = 7$ & $(a - b)x + (a + b)y = (3a + b - 2)$

compare with $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

$a_1 = 2, b_1 = 3, c_1 = 7$ & $a_2 = a - b, b_2 = a + b, c_2 = (3a + b - 2)$

For infinitely many solutions: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{2}{a-b} = \frac{3}{a+b} = \frac{7}{(3a+b-2)}$

Take first and second $\frac{2}{a-b} = \frac{3}{a+b} \Rightarrow 2a + 2b = 3a - 3b$

$\Rightarrow 3a - 2a = 3b + 2b \Rightarrow \boldsymbol{a = 5b} \ldots\ldots \mathbf{i})$

Take second and last $\frac{3}{a+b} = \frac{7}{(3a+b-2)} \Rightarrow 9a + 3b - 6 = 7a + 7b$

$\Rightarrow 2a - 4b - 6 = 0 \Rightarrow 2(5b) - 4b - 6 = 0$ {by i) $\boldsymbol{a = 5b}$}

$10b - 4b - 6 = 0 \Rightarrow 6b - 6 = 0 \Rightarrow b = 1$

Replace in i), we get $\Rightarrow a = 5 \times 1 = 5$

$\therefore \boldsymbol{a = 5, b = 1}$ is required solution.

EXERCISE

1. Verify whether the following pair of linear equations has a unique solution no solution or infinitely many solutions.

i) $x - 3y = 3$ and $3x - 9y = 2$

ii) $2x + y = 5$ and $3x + 2y = 8$

iii) $3x - 5y - 20 = 0$ and $6x - 10y - 40 = 0$

iv) $4x + 3y - 5 = 0$ and $8x - 6y - 10 = 0$

v) $3x - 2y = 6$ and $12x - 8y = 24$

2. For what value of p, the following system has unique solution:

i) $4x + py + 8 = 0$ and $2x + 4y + 2 = 0$

ii) $3x - 5y = 2$ and $px + 2y = -3$

3. For what values of k, the following system has no solution

$3x + y = 1$ and $(2k - 1)x + (k - 1)y = 2k + 1$

4. **Exercise 3.2, Q no. 2,3**