Complex Variables & Transforms(Laplace & Fourier) MAT241

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Lecture-6

Cauchy-Riemann Equations:

The necessary and sufficient condition that $\omega = f(z) = u + iv$ be analytic in a region $\mathbb{R}$ is that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied in $\mathbb{R}$ where it is supposed that these partial derivatives are continuous in $\mathbb{R}$.

Note: If $f(z)$ is analytic in $\mathbb{R}$, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied in $\mathbb{R}$.

$\# \omega = f(z)$

$\# u \,\&\, v$

$$\frac{\partial u}{\partial x} \ldots\ldots (1)$$

$$\frac{\partial v}{\partial y} \ldots\ldots (2)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} \ldots\ldots (3)$$

$$\frac{\partial v}{\partial x} \ldots\ldots (4)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Harmonic Functions:

If the second partial derivatives of u and v with respect to x and y exist and are continuous in a region R, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; ; \; \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

It follows that under these conditions, the real and imaginary parts of an analytic function satisfy Laplace's equation denoted by

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

or $\nabla^2 \varphi = 0$ where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The operator ∇^2 is often called the Laplacian.

Function such as u and v which satisfy Laplace's equation in a region R are called harmonic functions.

#v is harmonic or not?

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial x^2} \dots\dots (1)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial y^2} \dots\dots (2)$$

$$(1) + (2) \Rightarrow$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Example: Show that $f(z) = e^{-z}$ is a analytic function in a region $\mathbb{R}$.

Sol^n: Given that,

$$f(z) = e^{-z}$$

Let, $u + iv = f(z) = e^{-z}$

$$\Rightarrow u + iv = e^{-(x+iy)}$$

$$= e^{-x} \cdot e^{-iy}$$

$$= e^{-x}(\cos y - i \sin y)$$;[Using Euler's Formula]

$$= e^{-x}\cos y - i e^{-x} \sin y$$

$\therefore u = e^{-x}\cos y$ and $v = -e^{-x}\sin y$

Now, $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^{-x}\cos y)$

$$= \cos y \frac{\partial}{\partial x}(e^{-x})$$

$$= \cos y(-e^{-x}) = -e^{-x}\cos y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial x}(2x^2 + xy + y^2)$$

$$\frac{\partial}{\partial x}(2x^2) + \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(y^2)$$

$$= 2.2x + y.1 + 0$$
$$= 4x + y$$

And,

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-e^{-x}\sin y)$$

$$= -e^{-x}\frac{\partial}{\partial y}(\sin y)$$

$$= -e^{-x}\cos y = \frac{\partial u}{\partial x}$$

Again,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^{-x}\cos y)$$

$$= e^{-x}\frac{\partial}{\partial y}(\cos y)$$

$$= e^{-x}(-\sin y)$$

$$= -e^{-x}\sin y \; \boxed{= -\frac{\partial v}{\partial x}}$$

And, $$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(-e^{-x}\sin y)$$

$$= -\sin y\,\frac{\partial}{\partial x}(e^{-x})$$

$$= -\sin y\,(-e^{-x})$$

$$= e^{-x}\sin y$$

$$\boxed{\frac{\partial}{\partial x}(e^{ax}) = ae^{ax}}$$

$$\frac{\partial u}{\partial y} = -e^{-x}\sin y = -[e^{-x}\sin y] = -\frac{\partial v}{\partial x}$$

Since, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Therefore, $f(z) = e^{-z}$ is a analytic function.
(Shown)

Show that the real and imaginary part of the following function satisfy the Cauchy-Riemann equations.

$$f(z) = z e^{-z}$$

Sol^n: Given that,

$$f(z) = z e^{-z}$$

Let, $u + iv = f(z) = z e^{-z}$

$$\Rightarrow u + iv = z e^{-z}$$

$$= (x+iy) e^{-(x+iy)}$$

$$= (x+iy) e^{-x} \cdot e^{-iy}$$

$$= e^{-x}(x+iy)(\cos y - i \sin y)$$

$$= e^{-x}(x+iy)(\cos y - i\sin y)$$

$$= e^{-x}(x\cos y - ix\sin y + iy\cos y + y\sin y)$$

$$= e^{-x}\{(x\cos y + y\sin y) + i(y\cos y - x\sin y)\}$$

$$\therefore u = e^{-x}(x\cos y + y\sin y)$$

$$v = e^{-x}(y\cos y - x\sin y)$$

Now, $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\{e^{-x}(x\cos y + y\sin y)\}$

$$= e^{-x}\frac{\partial}{\partial x}(x\cos y + y\sin y) + (x\cos y + y\sin y)\frac{\partial}{\partial x}(e^{-x})$$

$$= e^{-x}(\cos y \cdot 1 + 0) + (x\cos y + y\sin y)(-e^{-x})$$

$$= e^{-x}(\cos y - x\cos y - y\sin y)$$

And, $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\{e^{-x}(y\cos y - x\sin y)\}$

$$= e^{-x}\frac{\partial}{\partial y}(y\cos y - x\sin y) + (y\cos y - x\sin y)\frac{\partial}{\partial y}(e^{-x})$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

$$\frac{\partial}{\partial x}(2x^3) = 2\frac{\partial}{\partial x}(x^3) = 2.3x^2 = 6x^2$$

$$\frac{\partial}{\partial x}(2) = 0$$

$$\frac{\partial}{\partial x}(e^{ax}) = ae^{ax}$$

$$= e^{-x}\left\{\frac{\partial}{\partial y}(y\cos y) - \frac{\partial}{\partial y}(x\sin y)\right\} + (y\cos y - x\sin y)(0)$$

$$= e^{-x}\left\{ y\frac{\partial}{\partial y}(\cos y) + \cos y \frac{\partial}{\partial y}(y) - x\frac{\partial}{\partial y}(\sin y)\right\}$$

$$= e^{-x}(-y\sin y + \cos y - x\cos y)$$

$$= e^{-x}(\cos y - x\cos y - y\sin y)$$

$$= \frac{\partial u}{\partial x}$$

$$\boxed{\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

Similarly, find $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\{e^{-x}(x\cos y + y\sin y)\}$$

$$= e^{-x}\frac{\partial}{\partial y}\{(x\cos y + y\sin y)\} + (x\cos y + y\sin y)\frac{\partial}{\partial y}(e^{-x})$$

$$= e^{-x}\left\{\frac{\partial}{\partial y}(x\cos y) + \frac{\partial}{\partial y}(y\sin y)\right\}$$

$$= e^{-x}\left\{x\frac{\partial}{\partial y}(\cos y) + y\frac{\partial}{\partial y}(\sin y) + \sin y\frac{\partial}{\partial y}(y)\right\}$$

$$= e^{-x}(-x\sin y + y\cos y + \sin y)$$

Again,

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\{e^{-x}(y\cos y - x\sin y)\}$$

$$= e^{-x}\frac{\partial}{\partial x}\{(y\cos y - x\sin y)\} + (y\cos y - x\sin y)\frac{\partial}{\partial x}\{e^{-x}\}$$

$$= e^{-x}\{0 - \sin y . 1\} + (y\cos y - x\sin y)(-e^{-x})$$

$$= -e^{-x}(\sin y + y\cos y - x\sin y)$$

$$= -\frac{\partial u}{\partial y}$$

Prove that $u = e^{-x}(x\sin y - y\cos y)$ is a harmonic function.

$u = e^{-x}(x\cos y + y\sin y)$ is harmonic or not?

Hints. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Given that,

$$u = e^{-x}(xsiny - ycosy)$$

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\{e^{-x}(xsiny - ycosy)\}$$

$$= e^{-x}\frac{\partial}{\partial x}(xsiny - ycosy) + (xsiny - ycosy)\frac{\partial}{\partial x}\{e^{-x}\}$$

$$= e^{-x}(siny - 0) + (xsiny - ycosy)(-e^{-x})$$

$$= e^{-x}(siny - xsiny + ycosy)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^{-x}(siny - xsiny + ycosy)\right)$$

$$= e^{-x}\frac{\partial}{\partial x}\left((siny - xsiny + ycosy)\right) + (siny - xsiny + ycosy)\frac{\partial}{\partial x}(e^{-x})$$

$$= e^{-x}(0 - \mathrm{siny} + 0) + (siny - xsiny + ycosy)(-e^{-x})$$

$$= -e^{-x}(siny + siny - xsiny + ycosy) = -e^{-x}(2siny - xsiny + ycosy)$$

Again ,

$$\frac{\partial^2 u}{\partial y^2}$$