CS52002 Practice Problems
Final Exam

Instructions (for final exam):

1. The exam is closed book and closed notes. You are allowed a double sided cheat-sheet. You may also use a calculator but it isn't required.
2. *Show your work for credit.* Unless otherwise specified, you may leave your answer in the form of a fraction or mathematical expression that clearly shows your thought process. For credit you must show your scratch work and clearly explain how you got to your final answer. If this isn't clear and obvious you won't get full credit for your response.
3. The exam is to be taken in-class during our regular class time. Once started, you have 3 hours and 15 mins to complete *and submit* the exam.
4. You will submit your exam on GradeScope just as you would a regular homework. Make sure you give yourself 15-20 minutes to correctly submit your exam. It is recommended that you hand-write your solutions to use exam time efficiently.
5. When submitting your exam you must specify, for each problem, the page or pages where the solution to that problem is to be found. If we can't see your solution you will not receive credit for the problem.
6. **IMPORTANT:** You may *not* look up answers on the Internet. You may *not* discuss the exam with other students. Even comparing a single answer will be regarded as cheating and you will be given a zero for the entire exam and possibly an automatic F for the course. Please take us seriously on this point. Giving *or* receiving help on the midterm of any kind will have very unfortunate consequences for everyone involved.
7. Do your best and best of luck!

PRINT FULL NAME: ______________________ STUDENT ID: _________

I have read the instructions above and understand that I may not discuss the exam with other students at any time. SIGN HERE: ______________________

Section1: Counting

1. A grocery store has 5 apples, 4 bananas, and 3 pears on the shelf. Assume the apples are indistinguishable as are the bananas and pears. How many ways are there to purchase exactly 4 pieces of fruit? (For example, you might purchase 2 apples, 1 pear, and 1 banana, or 1 apple and 3 pears, etc.)

2. Now lets assume that every piece of fruit is in fact distinguishable. There are still exactly 5 apples, 3 pears, and 4 bananas available for purchase. You can buy any number of pieces of fruit that you wish. For example, you could choose to buy $apple_1, apple_4, pear_2, pear_3$, and $banana_4$. Or you could choose to buy no fruit at all. How many choices do you have altogether?

3. The pieces of fruit are still distinguishable and as before you can buy as few or as many of the pieces of fruit that you want. The available quantities are the same. However the number of pears that you buy must be equal to the number of bananas that you buy. Now how many choices do you have?

4. How many ways are there to arrange the six digits 012345 so that no two adjacent digits add up to 5?

Section2: Probability

1. A sock drawer contains 4 white socks and 4 black socks and 4 tan socks.

   i. What is the probability that if two socks are randomly drawn (without replacement) they will form a matching pair? Reduce your answer to a simple fraction.

   ii. What is the minimum number of socks that one would have to draw (without replacement) to be certain of having at least two matching pairs? Explain your answer and state the principle underlying your argument.

2. A jar contains 3 red balls and 7 blue balls.

   i. If a ball is picked at random, what is the probability that this ball is blue?

   ii. If two balls are taken out of the jar at random (without replacement), what is the probability that they are of different colors?

   iii. If three balls are taken out at random (without replacement), what is the probability that they are all of the same color?

3. What is the probability, if a die is rolled five times, that only two different values appear?

4. Which is more likely, rolling an 8 when two dice are rolled, or rolling an 8 when three dice are rolled?

5. Let $W(x)$ be the number of 1's in the binary representation of x. For example, $W(5) = W(00101_2) = 2$ because there are 2 1's in the binary representation of 5. This is sometimes called the *weight* of the binary number. A deck of 32 cards has numbers 0 to 31_{10} written in 5-bit binary ($00000_2 ... 11111_2$).

   i. What is the probability that the weight of a randomly chosen card is exactly 3?

   ii. What is the probability that the weight of the card is 3 and the number on the card is odd, *i.e.*, $P(W = 3 \cap Odd)$?

   iii. Calculate $P(Odd|W = 3)$, the probability that the card represents an odd number given that the weight of the number is 3.

   iv. You are now dealt 3 random cards. What is the expected value for the total weight of your three-card hand?

   v. What is the probability that the total weight of the three cards you were dealt is equal to 13? You may leave your answer as a simple expression.

Section3: Conditional Probability

1. We are given 5 cards. 3 of the cards are black and they are numbered $1, 2, 3$. The other two cards are red and they are numbered $1, 2$.
   We pick 2 random cards.

   i. What is the probability that both cards are red?

   ii. What is the probability that both cards are red, if we know that at least one of them is red?

   iii. What is the probability that both cards are red, if we know that one of them is red card number 1?

2. Punxsutawney Phil is a weather predicting groundhog. On February 2nd, Phil makes a prediction about how soon spring comes (when weather warms up) by either observing his shadow or not each. Let's make the following assumptions:

   i. If flowers bloom in April, Phil observes his shadow 30% of the time.

   ii. if flowers do not bloom in April, Phil observes his shadow 80% of the time.

   iii. Flowers bloom in April only 15% of the time, whether or not Phil's shadow is observed.

   Given that Phil has not seen his Shadow this year, whats the probability that the flowers will bloom?

Section4: Expectation

A standard 52-card deck of cards contains 4 suits (Hearts, Diamonds, Clubs, and Spades). Each suit contains 13 ranks (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen,King). The Jack, Queen, and King are face cards. Two such decks are combined and shuffled together forming a 104-card deck from which you are dealt exactly five cards. How many cards would you expect to receive that are Clubs, or face cards, or both?

Section5: Algorithms

1. For the first two parts, your answer should be a single integer.

   i. How many steps (compares) does it take, in the worst case, to search for a given element in an **unordered array** of length 512?

   ii. How many steps (compares) does Binary Search take, in the worst case, to search for a given element in an **ordered array** of length 512?

   iii. If algorithms $A_1, A_2, A_3, A_4, A_5, A_6$ run on a list of length n in times $\log_2 n, n, (log_2 n)^3, n^3, 2^n$, and $n!$, miliseconds respectively, what is the **length of the largest list** that each of them could complete a run on in 1 second (1000 miliseconds)?

   Your answers should be integers or integer powers of 2 or 10.

Algorithm	**Run Time**	**Length of List**
A_1	$\log_2 n$	
A_2	n	
A_3	$(\log_2 n)^3$	
A_4	n^3	
A_5	2^n	
A_6	$n!$	

2. You are given n sorted lists $L_1, L_2, \ldots, L_n$, each of length m, and your goal is to create a single sorted list that is the union of the elements in all the lists. Consider the following iterative algorithm: starting with a null (empty) list, merge it with L_1, then merge the resulting list with L_2, and so on up to merging with L_n, where at the ith step, the result from the previous $i - 1$ steps is merged with L_i. Note that the resulting list after each merge is sorted.

   (a) In the worst case, what is the total number of comparisons in terms of n and m?

   (b) What is the asymptotic running time (Big O) of this algorithm in terms of n and m?

Section6: Mathematical Induction

1. Prove by mathematical induction:

$$\forall n \geq 2 : \sum_{i=2}^{n} (i-1) \cdot i = \frac{n \cdot (n-1) \cdot (n+1)}{3}$$

2. $S_n = 1 * 2 + 2 * 2^2 + 3 * 2^3 + 4 * 2^4 + ... + n * 2^n = 2 + (n-1)2^{n+1}$.

Section7: Sequences and Recurrences

i. Using the iterative, substitution method, find a closed-form formula for the recurrence: $T(n) = T(\frac{n}{2}) + n; T(1) = 1$. You may assume that n is a power of 2.

ii. Using the iterative / substitution method, find a closed form formula for the recurrence: $T(n) = 2T(n-1) + 1; T(0) = 0$.

iii. Prove your formula for the second recurrence by induction for all integers $n \geq 1$.

iv. Use two methods to find a closed form formula for the sequence: $a_n = -42, -39, -34, -27, -18, ...$ The first term is a_1.

v. Using the iterative / substitution method, find a closed form formula for the recurrence: $M(n) = M(n-1) + 2n + 1$ and $M(0) = 0$. Give the asymptotic run time of the recurrence you solve. Then, prove that your closed-form formula is correct using Mathematical Induction.

Problem 10: Graphs Consider the graph with vertex set $\{a, b, c, d, e, f, g\}$ and *adjacency lists:*

$$\begin{array}{|c|lllllllll} \hline a & \rightarrow & g & \rightarrow & d & \rightarrow & e & & \\ \hline b & \rightarrow & g & \rightarrow & d & \rightarrow & e & \rightarrow & c \\ \hline c & \rightarrow & f & \rightarrow & b & & & & \\ \hline d & \rightarrow & a & \rightarrow & f & \rightarrow & b & & \\ \hline e & \rightarrow & a & \rightarrow & g & \rightarrow & f & \rightarrow & b \\ \hline f & \rightarrow & d & \rightarrow & e & \rightarrow & c & & \\ \hline g & \rightarrow & a & \rightarrow & e & \rightarrow & b & & \\ \hline \end{array}$$

1. Draw this graph in the plane so that no two edges intersect, except at common end vertices.

2. Starting at vertex a traverse the graph by *depth-first-search*, processing the neighbors of each vertex in the order they appear on the adjacency list of that vertex (thus not in the alphabetical order). In your drawing, color the tree edges in red and next to each vertex write its place in the order in which the vertices are first visited (with a as 1).

3. Starting at vertex a traverse the graph by *breadth-first-search*, processing the neighbors of each vertex in the order they appear on the adjacency list of that vertex (thus not in the alphabetical order). In a fresh copy of your drawing of the graph, color the tree edges in blue and next to each vertex write its place in the order in which the vertices are first visited (with a as 1).

4. Give a spanning tree of the graph.