Home work 8

Problem 1

1.1. soltution: It's a arithmetic sequence.

The first term is -5.

The constant is 4.

Formula: $a_n = -5 + 4(n-1)$

$= 4n - 9$

1.2. solution: It's a geometric sequence.

The first term is -2.

The constant is -3.

Formula: $a_n = (-2) \cdot (-3)^{n-1}$

1.3. Solution: It's not arithmetic or geometric.

So we test if it's quadratic.

$a_n = an^2 + bn + c$.

$a_1 = a + b + c = 0$

$a_2 = 4a + 2b + c = 5$

$a_3 = 9a + 3b + c = 16$

The equations can be expressed as:

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 16 \end{pmatrix}$$

We solve the augmented-matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 5 \\ 9 & 3 & 1 & 16 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 5 \\ 0 & -6 & -8 & 16 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Thus, $c = 1$, $b = -4$, $a = 3$

Formula: $a_n = 3n^2 - 4n + 1$

1.4 solution: It's a geometric sequence.

The first term is 6.

The constant is 2.

Formula: $a_n = 6 \cdot 2^{n-1}$

Problem 2

2.1 Solution: $\sum_{k=5}^{15} 8k = 8\sum_{k=5}^{15} k = 8\times\left(\frac{(5+15)\times 11}{2}\right) = 880$

2.2 Solution= $\sum_{k=1}^{n} 8k = 8\times\sum_{k=1}^{n} k = 8\times\frac{(1+n)\times n}{2} = 4n(1+n)$

2.3 Solution= $\sum_{k=5}^{15} a\cdot b^k = a\sum_{k=5}^{15} b^k = a\times(b^5+b^6+b^7+\cdots+b^{15})$

$= a\times b^5(1+b+b^2+\cdots+b^{10}$

$= a\times b^5\times\frac{(1-b^{11})}{1-b}$

Problem 3

3.1 ① Solution:

$A_n = 50000 \cdot (1.05)^{n-1}$

$X_n = 60000 + 2000(n-1)$

In the short term, X's salary is better.
But in the long term, A's salary will exceed X's.

② Solution:

$A_n = X_n$

$5000 \times (1.05)^{n-1} = 6000 + 2000 \times (n-1)$

$\begin{cases} n_1 \approx -21.047 \\ n_2 \approx 9.23 \end{cases}$

We use the positive number, and after $\lceil n_2 \rceil = 10$ years,
A will exceed X.

3.2 Solution:

$a_1 = 1$

$a_2 = 5$

$a_3 = 13$

Let $a_n = an^2 + bn + c$

$a_1 = a + b + c = 1$

$a_2 = 4a + 2b + c = 5$

$a_3 = 9a + 3b + c = 13$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 5 \\ 9 & 3 & 1 & 13 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & 1 \\ 0 & -6 & -8 & 4 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$c = 1 \quad b = -2 \quad a = 2$

$a_n = 2n^2 - 2n + 1$

3.3 Solution:

$$T_k = \frac{k(k+1)}{2}$$

$$C(n) = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{k(k+1)}{2}$$

$$= \frac{1}{2} \sum_{k=1}^{n} (k^2 + k)$$

$$= \frac{1}{2} \left(\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \right)$$

$$= \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right)$$

$$= \frac{n(n+1)(n+2)}{6}$$

$$C(15) = \frac{15 \times 16 \times 17}{6} = 680$$

Problem 4

4.1 Solution:

① base case: $n=1$ $\quad 1 = 1^2$ base case holds.

② $n=k$: $\quad k^2 = 1+3+5+\cdots+(2k-1)$

③ $n=k+1$: $\quad (k+1)^2 = 1+3+5+\cdots+(2k-1)+(2(k+1)-1)$

$$= k^2 + (2k+2-1)$$

$$= k^2+2k+1$$

$$= (k+1)^2$$

So for all $n \geq 1$, the induction holds.

4.2 Solution:

① base case: $n=1$ $\quad 1^3 = \frac{1^2 \times (1+1)^2}{4} = 1$ base case holds

② $n=j$: $\quad \sum_{k=1}^{j} k^3 = \frac{j^2(j+1)^2}{4}$

$$j^4+\cancel{4j^3}+\cancel{4j^2}+\cancel{2j^3}+8j^2+8j+\cancel{j^2}+4j+4$$

③ $n=j+1$: $\quad \sum_{k=1}^{j+1} k^3 = \frac{(j+1)^2(j+2)^2}{4} = \frac{(j^2+2j+1)(j^2+4j+4)}{4} = \frac{j^4+6j^3+13j^2+12j+4}{4}$

$$\sum_{k=1}^{j+1} k^3 = \sum_{k=1}^{j} k^3 + (j+1)^3 = \frac{j^2(j+1)^2}{4} + (j+1)^3 = \frac{j^2(j^2+2j+1)}{4} + j^3+3j^2+3j+1$$

$$= \frac{j^4+6j^3+13j^2+12j+1}{4}$$

So the induction holds.

4.3 Solution:

① $n=1$, $7^1-1=6$, base case holds.

② $n=k$: $\quad 7^k-1=6m$, m is an integer.

③ $n=k+1$: $\quad 7^{k+1}-1 = 7(7^k-1)+6$

$$= 7\times 6m+6$$

$$= 6(7m+1)$$

So the induction holds.