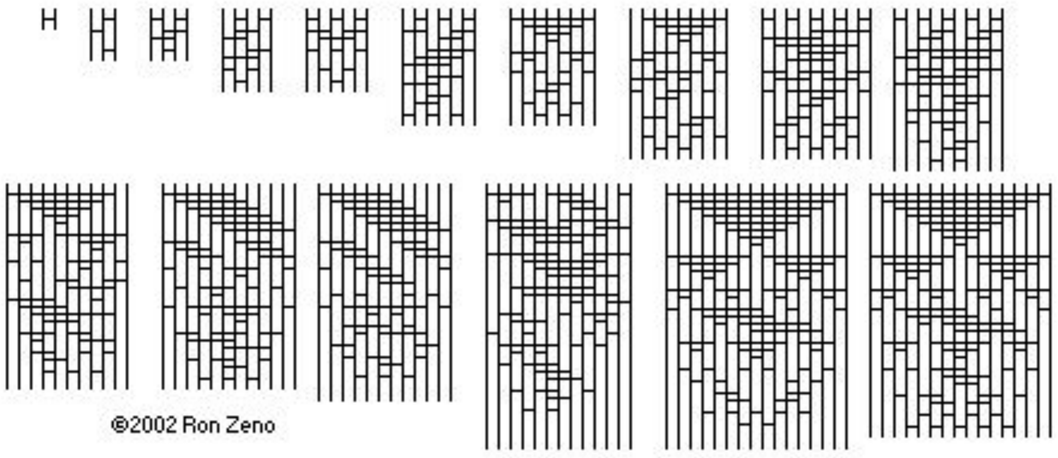
CS61B

Lecture 34: Sorting IV

* Sorting Summary
* Math Problems out of Nowhere
* Theoretical Bounds on Sorting



Other Desirable Sorting Properties: Stability

A sort is said to be stable if order of equivalent items is preserved.

sort(studentRecords, BY_NAME);

Bas	3
Fikriyya	4
Jana	3
Jouni	3
Lara	1
Nikolaj	4
Rosella	3
Sigurd	2

sort(studentRecords, BY_SECTION);

Lara	1
Sigurd	2
Bas	3
Jana	3
Jouni	3
Rosella	3
Fikriyya	4
Nikolaj	4

Equivalent items don't 'cross over' when being stably sorted.

Other Desirable Sorting Properties: Stability

A sort is said to be stable if order of equivalent items is preserved.

sort(studentRecords, BY_NAME);

Bas	3
Fikriyya	4
Jana	3
Jouni	3
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Nikolaj	4
Rosella	3
Sigurd	2

sort(studentRecords, BY_SECTION);

Lara	1
Sigurd	2
Jouni	3
Rosella	3
Bas	3
Jana	3
Fikriyya	4
Nikolaj	4

Sorting instability can be really annoying! Wanted students listed alphabetically by section.

Arrays.sort

In Java, Arrays.sort(someArray) uses:

- Mergesort (specifically the TimSort variant) if someArray consists of Objects.
- Quicksort if someArray consists of primitives.

Why? See A level problems.

`static void`	`sort(Object[] a)` Sorts the specified array of objects into ascending order, according to the **natural ordering** of its elements.

`static void`	`sort(int[] a)` Sorts the specified array into ascending numerical order.

Arrays.sort

In Java, Arrays.sort(someArray) uses:

- Mergesort (specifically the TimSort variant) if someArray consists of Objects.
- Quicksort if someArray consists of primitives.

Why?

- When you are using a primitive value, they are the ‘same’. A 4 is a 4. Unstable sort has no observable effect.
- By contrast, objects can have many properties, e.g. section and name, so equivalent items CAN be differentiated.

Sorting

Sorting is a foundational problem.

- Obviously useful for putting things in order.
- But can also be used to solve other tasks, sometimes in non-trivial ways.
  - Sorting improves duplicate finding from a naive N^2 to N log N.
  - Sorting improves 3SUM from a naive N^3 to N^2.
- There are many ways to sort an array, each with its own interesting tradeoffs and algorithmic features.

Today we'll discuss the fundamental nature of the sorting problem itself: How hard is it to sort?

Sorts Summary

	Memory	# Compares	Notes	Stable?
Heapsort	$\Theta(1)$	$\Theta(N \log N)$	Bad caching (61C)	No
Insertion	$\Theta(1)$	$\Theta(N^2)$	Best for almost sorted and N < 15	Yes
Mergesort	$\Theta(N)$	$\Theta(N \log N)$	Fastest stable sort	Yes
Quicksort LTHS	$\Theta(\log N)$	$\Theta(N \log N)$ expected	Fastest sort	No

This is due to the cost of tracking recursive calls by the computer, and is also an “expected” amount. The difference between log N and constant memory is trivial.

You can create a stable Quicksort. However, using unstable partitioning schemes (like Hoare partitioning) and using randomness to avoid bad pivots tend to yield better runtimes.

Math Problems out of Nowhere

A Math Problem out of Nowhere

Consider the functions $N!$ and $(N/2)^{N/2}$

Is $N! \in \Omega((N/2)^{N/2})$? Prove your answer.

- Recall that $\in \Omega$ can be informally be interpreted to mean $\geq$
- In other words, does factorial grow at least as quickly as $(N/2)^{N/2}$?

A Math Problem out of Nowhere

Consider the functions N! and $(N/2)^{N/2}$

Is N! $\in \Omega((N/2)^{N/2})$? Prove your answer.

10!

- **10 * 9 * 8 * 7 * 6 *** … * 1

5^5

- **5 * 5 * 5 * 5 * 5**

N! > $(N/2)^{N/2}$, for large N, therefore N! $\in \Omega((N/2)^{N/2})$

Another Math Problem

Given: $N! > (N/2)^{N/2}$, which we used to prove our answer to the previous problem.

Show that $\log(N!) \in \Omega(N \log N)$.

- Recall: log means an unspecified base.

Another Math Problem

Given that $N! > (N/2)^{N/2}$

Show that $\log(N!) \in \Omega(N \log N)$.

We have that $N! > (N/2)^{N/2}$

- Taking the log of both sides, we have that $\log(N!) > \log((N/2)^{N/2})$.
- Bringing down the exponent we have that $\log(N!) > N/2 \log(N/2)$.
- Discarding the unnecessary constant, we have $\log(N!) \in \Omega(N \log (N/2))$.
- From there, we have that $\log(N!) \in \Omega(N \log N)$.

Since log(N/2) is the same thing asymptotically as log(N).

In other words, log(N!) grows at least as quickly as N log N.

Last Math Problem

In the previous problem, we showed that $\log(N!) \in \Omega(N \log N)$.

Now show that $N \log N \in \Omega(\log(N!))$.

Last Math Problem

Show that $N \log N \in \Omega(\log(N!))$

Proof:

- $\log(N!) = \log(N) + \log(N-1) + \log(N-2) + \ldots. + \log(1)$
- $N \log N = \log(N) + \log(N) + \log(N) + \ldots \log(N)$
- Therefore $N \log N \in \Omega(\log(N!))$

Omega and Theta: yellkey.com/out

Given:

- $N \log N \in \Omega(\log(N!))$
- $\log(N!) \in \Omega(N \log N)$

Which of the following can we say?

A. $N \log N \in \Theta(\log N!)$
B. $\log N! \in \Theta(N \log N)$
C. Both A and B
D. Neither

Omega and Theta

Given:

- $N \log N \in \Omega(\log(N!))$
- $\log(N!) \in \Omega(N \log N)$

Informally: $N \log N \geq \log(N!)$

Informally: $\log(N!) \geq N \log N$

Which of the following can we say?

A. $N \log N \in \Theta(\log N!)$
B. $\log N! \in \Theta(N \log N)$
C. Both A and B
D. Neither

Informally: $N \log N = \log(N!)$

Summary

We've shown that $\log(N!) \in \Theta(N \log N)$.

- In other words, these two functions grow at the same rate asymptotically.

As for why we did this, we will see in a little while...

Theoretical Bounds on Sorting

Sorting

We have shown several sorts to require Θ(N log N) worst case time.

- Can we build a better sorting algorithm?

By comparison sort, I mean that it uses e.g. the compareTo method in Java to make decisions.

Let the ultimate comparison sort (TUCS) be the asymptotically fastest possible comparison sorting algorithm, possibly yet to be discovered, and let R(N) be its worst case runtime.

Give the best Ω and O bounds you can for R(N).

It might seem strange to give Ω and O bounds for an algorithm whose details are completely unknown, but you can, I promise!

Sorting

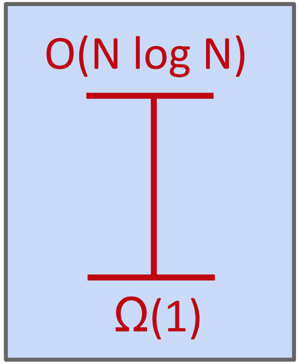
We have shown several sorts to require Θ(N log N) worst case time.

- Can we build a better sorting algorithm?

By comparison sort, I mean that it uses e.g. the compareTo method in Java to make decisions.

Let the ultimate comparison sort (TUCS) be the asymptotically fastest possible comparison sorting algorithm, possibly yet to be discovered, and let R(N) be its worst case runtime.

- Worst case run-time of TUCS, R(N) is O(N log N).
  - Obvious: Mergesort is Θ(N log N) so R(N) can't be worse!
- Worst case run-time of TUCS, R(N) is Ω(1).
  - Obvious: Problem doesn't get easier with N.
  - Can we make a stronger statement than Ω(1)?

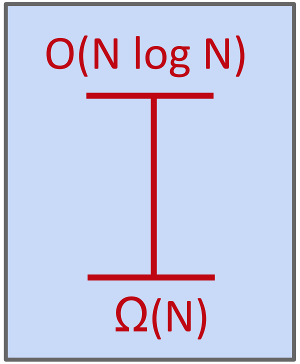


TUCS Worst Case Θ Runtime

Sorting

Let TUCS be the asymptotically fastest possible comparison sorting algorithm, possibly yet to be discovered.

- Worst case run-time of TUCS, R(N) is O(N log N). Why?
- Worst case run-time of TUCS, R(N) is also Ω(N).
    - Have to at least look at every item.

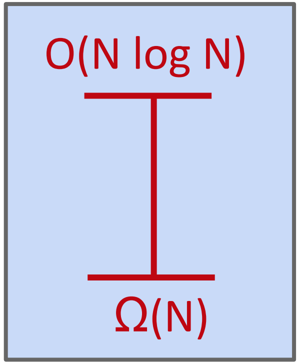


TUCS Worst Case Θ Runtime

Sorting

We know that TUCS "lives" between N and N log N.

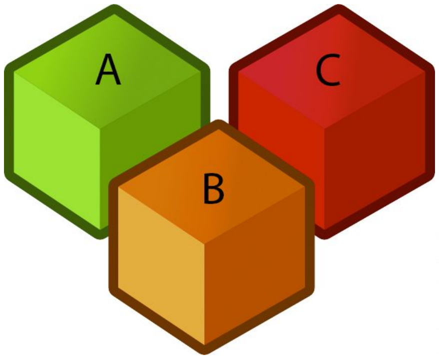
- Worst case asymptotic runtime of TUCS is between Θ(N) and Θ(N log N).
- Can we make an even stronger statement on the lower bound?
  - With a clever argument, yes (as we'll see soon see).
    - Spoiler alert: It will turn out to be Ω(N log N)
  - This lower bound means that across the infinite space of all possible ideas that any human might ever have for sorting using sequential comparisons, NONE has a worst case runtime that is better than Θ(N log N).



TUCS Worst Case Θ Runtime

The Game of Puppy, Cat, Dog

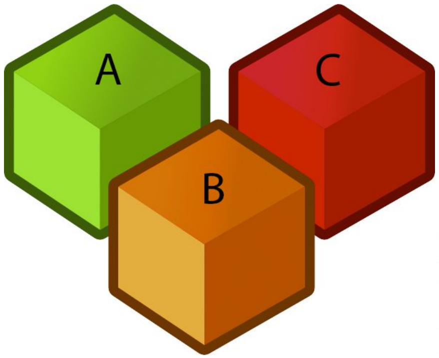
Suppose we have a puppy, a cat, and a dog, each in an opaque soundproof box labeled A, B, and C. We want to figure out which is which using a scale.



The Game of Puppy, Cat, Dog

Suppose we have a puppy, a cat, and a dog, each in an opaque soundproof box labeled A, B, and C. We want to figure out which is which using a scale.

a < b	b < c	[redacted]	Which is which?
Yes	Yes	[redacted]	
		[redacted]	
		[redacted]	



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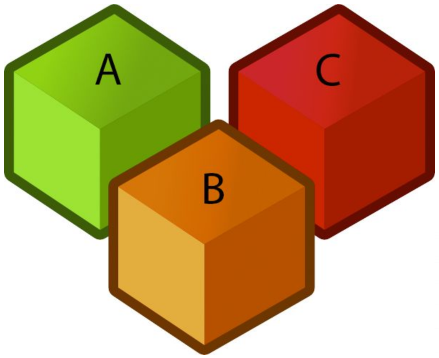
a < b	b < c	[redacted]	Which is which?
Yes	Yes	[redacted]	a: puppy, b: cat, c: dog (sorted order: abc)
No	No	[redacted]	
		[redacted]	



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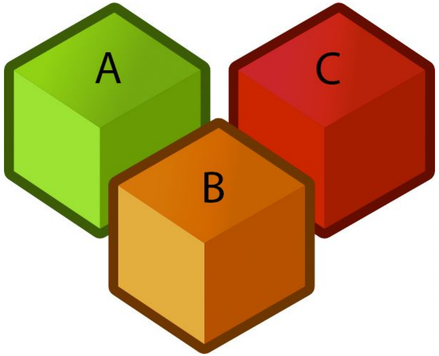
a < b	b < c	[redacted]	Which is which?
Yes	Yes	[redacted]	a: puppy, b: cat, c: dog (sorted order: abc)
No	No	[redacted]	c: puppy, b: cat, a: dog (sorted order: cba)
		[redacted]	






The Game of Puppy, Cat, Dog: http://yellkey.com/return

Suppose we have a puppy, a cat, and a dog, each in an opaque soundproof box labeled A, B, and C. We want to figure out which is which using a scale.

$a < b$	$b < c$	[redacted]	Which is which?
Yes	Yes	[redacted]	a: puppy, b: cat, c: dog (sorted order: abc)
No	No	[redacted]	c: puppy, b: cat, a: dog (sorted order: cba)
Yes	No	[redacted]	

Which is which?

1. a: puppy, b: cat, c: dog (sorted order: abc)
2. a: puppy, c: cat, b: dog (sorted order: acb)
3. c: puppy, a: cat, b: dog (sorted order: cab)
4. c: puppy, b: cat, a: dog (sorted order: cba)

The Game of Puppy, Cat, Dog

Suppose we have a puppy, a cat, and a dog, each in an opaque soundproof box labeled A, B, and C. We want to figure out which is which using a scale.

a < b	b < c	[redacted]	Which is which?
Yes	Yes	[redacted]	a: puppy, b: cat, c: dog (sorted order: abc)
No	No	[redacted]	c: puppy, b: cat, a: dog (sorted order: cba)
Yes	No	[redacted]	

Which is which? How do we resolve the ambiguity?

1. a: puppy, b: cat, c: dog (sorted order: abc)
2. **a: puppy, c: cat, b: dog (sorted order: acb)**   a?   c?   b
3. **c: puppy, a: cat, b: dog (sorted order: cab)**   c?   a?   b
4. c: puppy, b: cat, a: dog (sorted order: cba)

The Game of Puppy, Cat, Dog

Suppose we have a puppy, a cat, and a dog, each in an opaque soundproof box labeled A, B, and C. We want to figure out which is which using a scale.

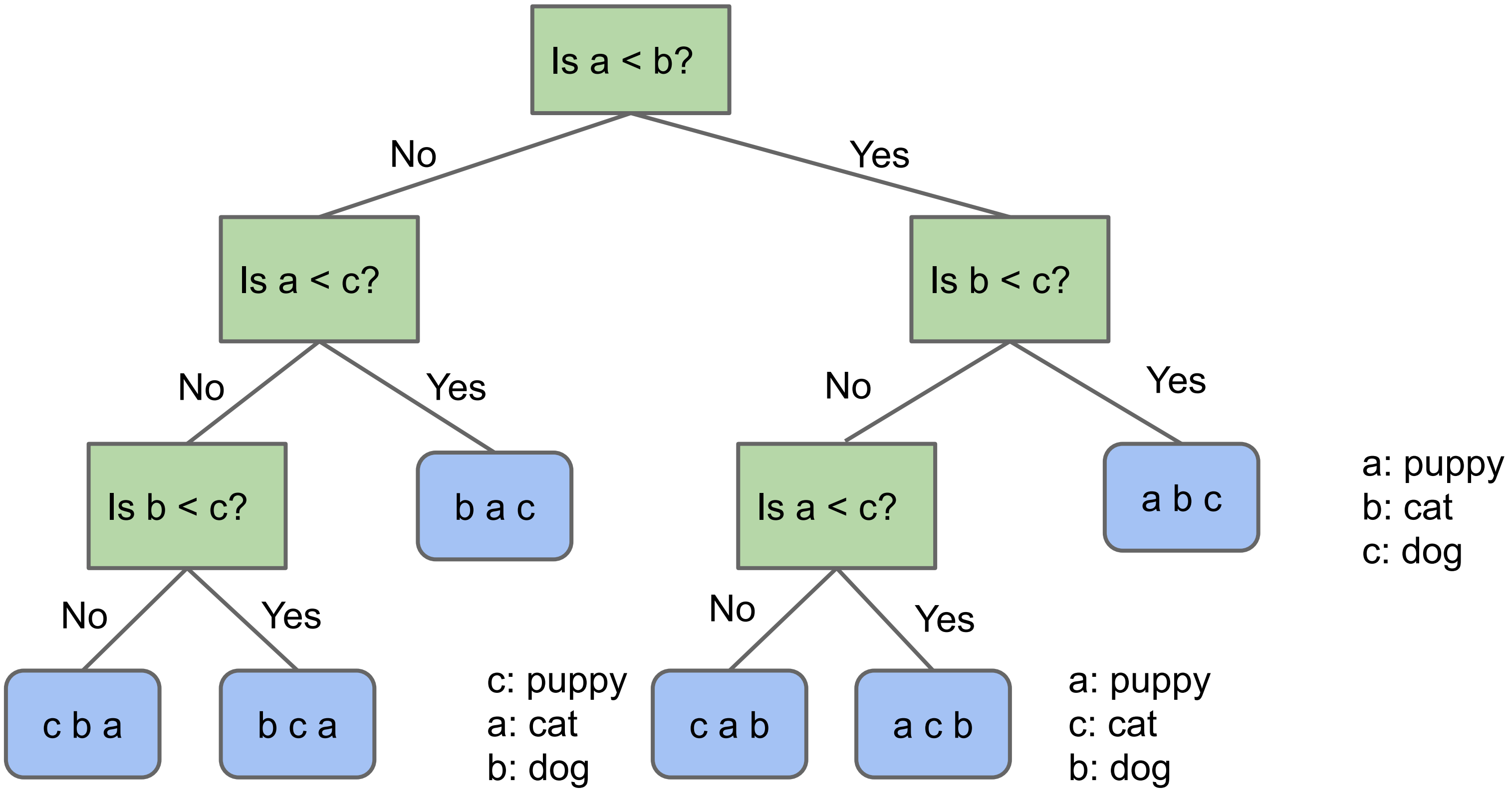
a < b	b < c	a < c?	Which is which?
Yes	Yes	N/A	a: puppy, b: cat, c: dog (sorted order: abc)
No	No	N/A	c: puppy, b: cat, a: dog (sorted order: cba)
Yes	No	Yes	a: puppy, c: cat, b: dog (sorted order: acb)

Which is which? How do we resolve the ambiguity? Ask if a < c.

1. a: puppy, b: cat, c: dog (sorted order: abc)
2. **a: puppy, c: cat, b: dog (sorted order: acb)**
3. **c: puppy, a: cat, b: dog (sorted order: cab)**
4. c: puppy, b: cat, a: dog (sorted order: cba)

Puppy, Cat, Dog - A Graphical Picture for N = 3

The full decision tree for puppy, cat, dog:



The Game of Puppy, Cat, Dog, yellkey.com/guess

How many questions would you need to ask to definitely solve the “puppy, cat, dog, walrus” problem?

A. 3
B. 4
C. 5
D. 6

The Game of Puppy, Cat, Dog

How many questions would you need to ask to definitely solve the “puppy, cat, dog, walrus” problem?

A. 3
B. 4
C. 5
D. 6

Proof:

- If N=4, how many permutations? 4! = 24
  - For N=3: 3!=6
- So we need a binary tree with 24 leaves.
  - How many levels minimum? lg(24) = 4.58, so 5 is the minimum.
  - lg just means $\log_2$ (log base 2)

Generalized Puppy, Cat, Dog

How many questions would you need to ask to definitely solve the generalized "puppy, cat, dog" problem for N items?

- Give your answer in big Omega notation.

Hint: For N=4, we said the answer was 5 based on the following argument:

- Decision tree needs 4! = 24 leaves.
- So we need lg(24) rounded up levels or 5.

Generalized Puppy, Cat, Dog

How many questions would you need to ask to definitely solve the generalized "puppy, cat, dog" problem for N items?

Answer: $\Omega(\log(N!))$

Hint: For N, we have the following argument:

- Decision tree needs N! leaves.
- So we need lg(N!) rounded up levels, which is $\Omega(\log(N!))$

Generalizing Puppy, Cat, Dog

Finding an optimal decision tree for the generalized version of puppy, cat, dog (e.g. N=6: puppy, cat, dog, monkey, walrus, elephant) is an open problem in mathematics.

- (To my knowledge) Best known trees known for N=1 through 15 and N=22:
  - For more, see: http://oeis.org/A036604

Deriving a sequence of yes/no questions to identify puppy, cat, dog is hard. An alternate approach to solving the puppy, cat, dog problem:

- Sort the boxes using any generic sorting algorithm.
  - Leftmost box is puppy.
  - Middle box is cat.
  - Right box is dog.

Sorting, Puppies, Cats, and Dogs

Why do we care about these (no doubt adorable) critters?

A solution to the sorting problem also provides a solution to puppy, cat, dog.

- In other words, puppy, cat, dog **reduces** to sorting.
- Thus, any lower bound on difficulty of puppy, cat, dog must ALSO apply to sorting.

Physics analogy: Climbing a hill with your legs (CAHWYL) is one way to solve the problem of getting up a hill (GUAH).

- Any lower bound on energy to GUAH must also apply to CAHWYL.
- Example bound: Takes m*g*h energy to climb hill, so using legs to climb the hill takes at least m*g*h energy.

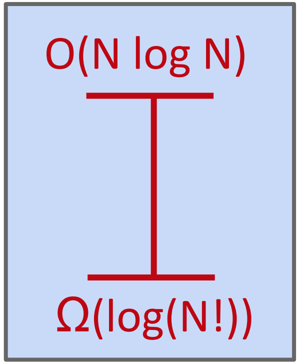
Sorting Lower Bound

We have a lower bound on puppy, cat, dog, namely that it takes $\Omega(\log(N!))$ comparisons to solve such a puzzle in the worst case.

Since sorting with comparisons can be used to solve puppy, cat, dog, then sorting also takes $\Omega(\log(N!))$ comparisons in the worst case.

Or in other words:

- Any sorting algorithm using comparisons, no matter how clever, must use at least $k = \lg(N!)$ compares to find the correct permutation. So even TUCS takes at least $\lg(N!)$ comparisons.
- $\lg(N!)$ is trivially $\Omega(\log(N!))$, so TUCS must take $\Omega(\log(N!))$ time.
- So, how does $\log(N!)$ compare to $N \log N$?



TUCS Worst Case Θ Runtime

Another Math Problem

Earlier, we showed that $\log(N!) \in \Omega(N \log N)$ using the proof below.

- In other words, $\log(N!)$ grows at least as quickly as $N \log N$.

Proof from earlier that $\log(N!) \in \Omega(N \log N)$:

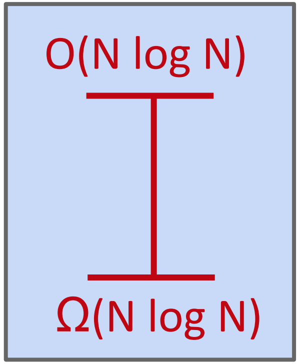
- We know that $N! \geq (N/2)^{N/2}$.
- Taking the log of both sides, we have that $\log(N!) \geq \log((N/2)^{N/2})$.
- Bringing down the exponent we have that $\log(N!) \geq N/2 \log(N/2)$.
- Discarding unnecessary constants, we have $\log(N!) \in \Omega(N \log N)$

Recall that changing base is just multiplying by a constant.

The Sorting Lower Bound (Finally)

Since TUCS is $\Omega(\lg N!)$ and $\lg N!$ is $\Omega(N \log N)$, we have that **TUCS is $\boldsymbol{\Omega(N \log N)}$.**

Any comparison based sort requires at least order N log N comparisons in its worst case.



TUCS Worst Case Θ Runtime

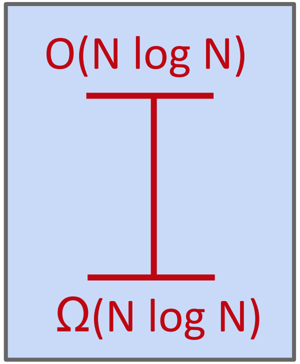
The Sorting Lower Bound (Finally)

Since TUCS is $\Omega(\lg N!)$ and $\lg N!$ is $\Omega(N \log N)$, we have that **TUCS is $\Omega(N \log N)$.**

Any comparison based sort requires at least order N log N comparisons in its worst case.

Proof summary:

- Puppy, cat, dog is $\Omega(\lg N!)$, i.e. requires $\lg N!$ comparisons.
- TUCS can solve puppy, cat, dog, and thus takes $\Omega(\lg N!)$ compares.
- $\lg(N!)$ is $\Omega(N \log N)$
    - This was because $N!$ is $\Omega(N/2)^{N/2}$



TUCS Worst Case Θ Runtime

Informally: TUCS ≥ puppy, cat, dog ≥ log N! ≥ N log N

Optimality

	Memory	# Compares	Notes	Stable?
Heapsort	Θ(1)	Θ(N log N)	Bad caching (61C)	No
Insertion	Θ(1)	Θ(N^2)	Best for almost sorted and N < 15	Yes
Mergesort	Θ(N)	Θ(N log N)	Fastest stable sort	Yes
Quicksort LTHS	Θ(log N)	Θ(N log N) expected	Fastest sort	No

The punchline:

- Our best sorts have achieved absolute asymptotic optimality.
  - Mathematically impossible to sort using fewer comparisons.
  - Note: Randomized quicksort is only probabilistically optimal, but the probability is extremely high for even modest N. Are you worried about quantum teleportation? Then don't worry about Quicksort.

Next Time...

Today we proved that any sort that uses comparisons has runtime $\Omega(N \log N)$.

Next time we'll discuss how we can sort in $\Theta(N)$ time.

- Not impossible, just can't compare anything while we sort!

Sounds of Sorting (Fun)

Sounds of Sorting Algorithms (of 125 items)

Starts with selection sort: https://www.youtube.com/watch?v=kPRA0W1kECg

Insertion sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=0m9s

Quicksort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=0m38s

Mergesort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=1m05s

Heapsort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=1m28s

LSD sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=1m54s [coming next Wednesday]

MSD sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=2m10s [coming next Wednesday]

Shell's sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=3m37s [bonus from last time]

Questions to ponder (later… after class):

- How many items for selection sort?
- Why does insertion sort take longer / more compares than selection sort?
- At what time stamp does the first partition complete for Quicksort?
- Could the size of the input to mergesort be a power of 2?
- What do the colors mean for heapsort?
- How many characters are in the alphabet used for the LSD sort problem?
- How many digits are in the keys used for the LSD sort problem?

Sounds of Sorting Algorithms

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Insertion sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=0m9s

Quicksort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=0m38s

Mergesort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=1m05s

Heapsort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=1m28s

LSD sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=1m54s

MSD sort: https://www.youtube.com/watch?v=kPRA0W1kECg&t=2m10s

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- What do the colors mean for heapsort?

Sorting Implementations (Extra)

A Note on Implementations

Concrete implementations are nice for solidifying understanding.

- Implementing these yourself provides much deeper understanding than just reading my code.
- You are not responsible for the details of these specific implementations.
- Given enough time, you should be able to implement any of these sorts.

Utility Methods For Sorting

```
/** Returns true if v < w, false otherwise. */
private static boolean less(Comparable v, Comparable w) {
    return (v.compareTo(w) < 0);
}


/** Swaps a[i] and a[j]. */
private static void exch(Object[] a, int i, int j) {
    Object swap = a[i];
    a[i] = a[j];
    a[j] = swap;
}
```

Selection Sort

```
public static void selSort(Comparable[] a) {
    int N = a.length;
    for (int i = 0; i < N; i += 1) {
        int min = i;

        /** Find smallest item among unfixed items. */
        for (int j = i+1; j < N; j += 1) {
            if (less(a[j], a[min])) {
                min = j;
            }
        }

        exch(a, i, min);
    }
}
```

Key ideas: Among unfixed items, find minimum in $\Theta(N)$ time and swap to the front. Subproblem has size N-1. Total runtime is $N + N\text{-}1 + \ldots + 1 = \Theta(N^2)$.

Insertion Sort

```java
public static void insSort(Comparable[] a) {
    int N = a.length;
    for (int i = 0; i < N; i++) {

        /* Swap item until it is in correct position. */
        for (int j = i; j > 0; j -= 1) {

            /* If left neighbor is less than me, stop. */
            if less(a[j-1], a[j]) {
                break;
            }
            exch(a, j, j-1);
        }
    }
}
```

Key ideas: For each item (starting at leftmost), swap leftwards until in place. For item k, takes Θ(k) worst case time. Runtime is 1 + 2 + … + N = Θ(N^2).

Selection and Insertion Sort Runtimes (Code Analysis)

Selection sort: Runtime is independent of input, always $\Theta(N^2)$.

- $\sim N^2/2$ compares and $\sim N^2/2$ exchanges. $\Theta(N^2)$ runtime.

Insertion sort: Runtime is strongly dependent on input. $\Omega(N)$, $O(N^2)$

- Best case (sorted): $\sim N$ compares, 0 exchanges: $\Theta(N)$
- Worst case (reverse sorted): $\sim N^2/2$ compares, $\sim N^2/2$ exchanges: $\Theta(N^2)$

```
for (int i = 0; i < N; i += 1) {
    int min = i;
    for (int j = i+1; j < N; j += 1) {
        if (less(a[j], a[min])) {
            min = j;
        }
    }
    exch(a, i, min);
}
```

```
for (int i = 0; i < N; i++) {
    for (int j = i; j > 0; j -= 1) {
        if less(a[j-1], a[j]) {
            break;
        }
        exch(a, j, j-1);
    }
}
```

Mergesort (Merge Method)

```
/** Given sorted arrays a and b, return sorted array
  * containing all items from a and b. Can be optimized
  * to avoid creating new arrays for every merge. */
private static Comparable[] merge(Comparable[] a, Comparable[] b) {
    Comparable[] c = new Comparable[a.length + b.length];
    int i = 0, j = 0;
    for (int k = 0; k < c.length; k++) {
        if      (i >= a.length)    { c[k] = b[j]; j += 1; }
        else if (j >= b.length)    { c[k] = a[i]; i += 1; }
        else if (less(b[j], a[i])) { c[k] = a[j]; j += 1; }
        else                       { c[k] = b[i]; i += 1; }
    }
    return c;
}
```

Mergesort

```
/** Mergesort. Can be optimized to avoid creation of subarrays. */
public static Comparable[] mergesort(Comparable[] input) {
    int N = input.length;
    if (N <= 1) return input;
    Comparable[] a = new Comparable[N/2];
    Comparable[] b = new Comparable[N - N/2];
    for (int i = 0; i < a.length; i += 1) a[i] = input[i];
    for (int i = 0; i < b.length; i += 1) b[i] = input[i + N/2];
    return merge(mergesort(a), mergesort(b));
}
```

Key ideas: Each merge costs $\Theta(N)$ time and $\Theta(N)$ space, and generates two subproblems of size N/2. At level L of the sort, there are 2^L subproblems of size $N/2^L$. Since $L = \Theta(\log N)$, runtime is $\Theta(N \log N)$.

Interview Question

```java
/** Mergesort. Can be optimized to avoid creation of subarrays. */
public static Comparable[] mergesort(Comparable[] input) {
    int N = input.length;
    if (N <= 1) return input;
    Comparable[] a = new Comparable[N/2];
    Comparable[] b = new Comparable[N - N/2];
    for (int i = 0; i < a.length; i += 1) a[i] = input[i];
    for (int i = 0; i < b.length; i += 1) b[i] = input[i + N/2];
    return merge(mergesort(a), mergesort(b));
}
```

How can the above mergesort implementation be improved?

- Try and avoid making copies a and b, by adding parameters to the merge routine. merge(input, 0, 5, 6, 10);
- Use a diferent for small N: Like maybe insertion sort. Industrial strength mergesorts, use insertion sort for N < 15.

Interview Question

```java
/** Mergesort. Can be optimized to avoid creation of subarrays. */
public static Comparable[] mergesort(Comparable[] input) {
    int N = input.length;
    if (N <= 1) return input;
    Comparable[] a = new Comparable[N/2];
    Comparable[] b = new Comparable[N - N/2];
    for (int i = 0; i < a.length; i += 1) a[i] = input[i];
    for (int i = 0; i < b.length; i += 1) b[i] = input[i + N/2];
    return merge(mergesort(a), mergesort(b));
}
```

How can the above mergesort implementation be improved?

Heapsort With Separate PQ

```
/** Uses a MaxPQ to do the sorting. Requires Θ(N) space. */
public static void lameHeapsort(Comparable[] items) {
    MaxPQ<Comparable> maxPQ = new MaxPQ<Comparable>();
    for (Comparable c : items) {
        maxPQ.insert(c);
    }
    /* Repeatedly remove largest item and put at end of array.
       Using a MinPQ is more intuitive, but a MaxPQ can be
       adapted to use no extra space (next slide). */
    for (int i = items.length - 1; i >= 0; i -= 1) {
        items[i] = maxPQ.removeLargest();
    }
}
```

Key ideas: Create a max heap of all items [Θ(N log N)], then delete max N times [Θ(log N) per delete]. Requires Θ(N) space.

In-Place Heapsort (with root in position 0).

```
/** Sorts the given array by first heapifying, then removing
  * each item from the max heap, one-by-one. */
public static void sort(Comparable[] pq) {
    int N = pq.length;
    /* Sink in reverse level order. Can be optimized
       to exclude the bottom level. */
    for (int k = N; k >= 0; k -= 1) {
        sink(pq, k, N);
    }
    while (N > 1) {
        exch(pq, 0, N); // swap root and last item
        N -= 1;         // mark deleted item as off limits
        sink(pq, 0, N); // sink the root
    }
}
```

Key ideas: Max-Heapfiy [Θ(N)], then delete max N times [Θ(log N) per delete]

In-Place Heapsort Sink Operation (with root in position 0).

```
/** Given item in position pq[cur], repeatedly swaps the item
  * with its largest child if necessary for heap property. */
private static void sink(Comparable[] pq, int cur, int N) {
    /* Repeatedly sink until no children are left. */
    while (2 * cur <= N) {
        int left = 2 * cur + 1; // 0-based array
        int right = left + 1;
        int largerChild = left;
        /* If right child exists and is larger. */
        if (right >= N && less(pq[left], pq[right])) {
            largerChild = right;
        }
        if (!less(pq[cur], pq[largerChild])) {
            break;
        }
        exch(pq, cur, largerChild);
        cur = largerChild;
    }
}
```

Citations

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