On novel generalized hyperbolic radial basis functions for scattered data approximation

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Abstract

This study introduces a new class of radial basis functions, known as Generalized Hyperbolic Radial Basis Functions (GH-RBFs), for approximating scattered data. We show that the GH-RBFs are conditionally positive definite when raised to non-negative integer exponents, similar to Generalized Multi-Quadric RBFs. Furthermore, we provide a formal definition for GH-RBFs and validate their performance on two numerical examples with Runge's phenomenon. The results from the numerical experiments show that GH-RBFs with negative exponents outperformed the Gaussian and GMQ-RBFs with negative exponents in terms of accuracy.

Keywords: Generalized Hyperbolic Radial Basis Functions (GH-RBFs), Conditionally positive definite functions, Scattered data approximation

1. Introduction

The Radial Basis Functions (RBFs) are primarily used for approximating unknown functions from discrete data [1]. Furthermore, the RBF approximation is an advanced mesh-free method for interpolating scattered data. Moreover, it can be interpreted as a three-layered neural network with RBFs placed in the hidden layer. As a result, RBFs are also widely employed in machine learning tasks, such as classification, regression, image processing, etc. [2]. There are many types of RBFs introduced for various applications, viz. the Gaussian, multi-quadrics, inverse multi-quadrics, thin plate splines, etc. For a function to be radial, it has to follow the definition [1, 3],

Definition 1. *A function* $\Phi : \mathbb{R}^s \to \mathbb{R}$ *is said to be radial provided there exists a univariate function,* $\phi : [0, \infty) \to \mathbb{R}$ *such that*

$$\Phi(\mathbf{x}) = \phi(r)$$

where $r = \|\mathbf{x}\|$, and $\|.\|$ is some norm on $\mathbb{R}^s$, usually the euclidean norm.

Although many functions fall into this category, not all are suitable for the RBF approximation due to the requirement of a positive definite interpolation matrix [1]. For example, the hyperbolic secant function with any real exponent can be used directly for RBF approximation. However, the hyperbolic cosine function with non-negative integer exponents cannot be used because of its singular interpolation matrix. To the best of our knowledge, there are no cases in the literature that address this behavior despite the potential of hyperbolic RBFs in machine learning and approximation tasks [4, 5, 6]. The goal of this paper is to formally define the Generalized Hyperbolic Radial Basis Functions (GH-RBFs) to increase their applicability in various fields. In the process, we prove the conditional positive definiteness of GH-RBFs and explain their resemblance to the GMQ-RBFs. Furthermore, we demonstrate the efficacy of GH-RBF interpolation through numerical experiments. Section 2 presents the preliminaries for RBF interpolation and theoretical background for the GH-RBFs; Section 3 illustrates the numerical experiment. Finally, section 4 concludes the paper.

2. Preliminaries and theoretical background

2.1. Interpolation using RBFs

The approximation or interpolation problem can be stated as follows [3],

Given the function values $f_1,\ f_2,\ ...,\ f_N \in \mathbb{R}$ for the distinct data sites $\mathbf{x_1},\ \mathbf{x_2},\ ...,\ \mathbf{x_N} \in \mathbb{R}^n$, we are interested in finding a continuous function $s : \mathbb{R}^n \to \mathbb{R}$ with

$$s(\mathbf{x_i}) = f_i, \qquad 1 \le i \le N \tag{1}$$

In 1971, Hardy [7] introduced the early RBF interpolation using multiquadric functions to solve the above problem. In RBF interpolation, the approximant s is the linear combination of the RBFs whose centers shifted from origin to c_i, which can be defined as follows,

$$s(\mathbf{x}) = \sum_{i=1}^{N} \lambda_i \phi(\|\mathbf{x} - c_i\|), \qquad \mathbf{x} \in \mathbb{R}^s,\ s > 0 \tag{2}$$

where N is the number of data sites, and λ_i are the real coefficients. Now there are $2N$ number of unknowns (c_i and λ_i) with N equations. By considering the data sites as the RBF centers ($c_i = x_i$), the number of unknowns is reduced to N. Substituting equation 2 in 1 gives the following system of linear equations,

$$\begin{bmatrix} \phi(\|\mathbf{x_1}-\mathbf{x_1}\|) & \phi(\|\mathbf{x_1}-\mathbf{x_2}\|) & \dots & \phi(\|\mathbf{x_1}-\mathbf{x_N}\|) \\ \phi(\|\mathbf{x_2}-\mathbf{x_1}\|) & \phi(\|\mathbf{x_2}-\mathbf{x_2}\|) & \dots & \phi(\|\mathbf{x_2}-\mathbf{x_N}\|) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \phi(\|\mathbf{x_N}-\mathbf{x_1}\|) & \phi(\|\mathbf{x_N}-\mathbf{x_2}\|) & \dots & \phi(\|\mathbf{x_N}-\mathbf{x_N}\|) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \dots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ \dots \\ f_N \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} \phi_1(0) & \phi_2(\mathbf{x_1}) & \dots & \phi_m(\mathbf{x_1}) \\ \phi_1(\mathbf{x_2}) & \phi_2(0) & \dots & \phi_m(\mathbf{x_2}) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \phi_1(\mathbf{x_m}) & \phi_2(\mathbf{x_m}) & \dots & \phi_m(0) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \dots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ \dots \\ f_N \end{bmatrix} \tag{4}$$

$$\mathbf{A}_{N\times N}\lambda_{N\times 1} = \mathbf{F}_{N\times 1} \tag{5}$$

where the real coefficients $\lambda_{N\times 1}$ are obtained by inverting the interpolation matrix $\mathbf{A}_{N\times N}$. Thus, the invertibility of the interpolation matrix plays a major role in RBF interpolation.

2.2. Theoretical background

We use the gamma function and the generalized Fourier transform to prove the positive definiteness of the GH-RBFs.

Theorem 2.1. *The function $\Phi(x) = sech^{\beta}(x/c)$, $x \in \mathbb{R}^1$ with $c > 0$ and $\beta > 0$ is strictly positive definite.*

Proof. From the definition of Γ- function for $\beta > 0$ we see that

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1}e^{-t}dt \tag{6}$$

Put $t = su$ with $s > 0$. we get

$$\Gamma(\beta) = s^\beta \int_0^\infty u^{\beta-1}e^{-su}du \tag{7}$$

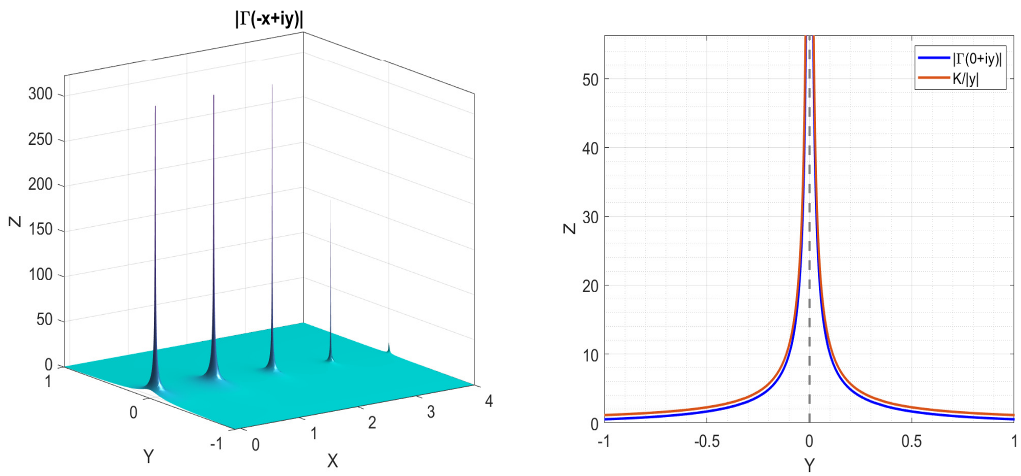

Figure 1: Left: Absolute gamma function $|\Gamma(-x+iy)|$, and Right: comparison of absolute gamma function and $K/|y|$ for $x=0$.

By substituting $s = cosh(x/c)$, we get

$$sech^{\beta}\left(\frac{x}{c}\right) = \frac{1}{\Gamma(\beta)}\int_0^{\infty} u^{\beta-1}e^{-ucosh(x/c)}du \tag{8}$$

Inserting this into the Fourier Transform leads to

$$\begin{aligned}\hat{\Phi}(\omega) &= (2\pi)^{-1/2}\int_{\mathbb{R}^1}\Phi(x)e^{-ix\omega}dx \\ &= (2\pi)^{-1/2}\frac{1}{\Gamma(\beta)}\int_{\mathbb{R}^1}\int_0^{\infty} u^{\beta-1}e^{-ucosh(x/c)}due^{-ix\omega}dx \\ &= (2\pi)^{-1/2}\frac{1}{\Gamma(\beta)}\int_0^{\infty} u^{\beta-1}\left[\int_{\mathbb{R}^1} e^{-ucosh(x/c)}e^{-ix\omega}dx\right]du\end{aligned} \tag{9}$$

We know that from the definition of *Modified Bessel Function of Third kind of the Imaginary Order* [8].

$$2K_{i\nu}(u) = \int_{-\infty}^{\infty} e^{-ucosh(x)-i\nu x}dx,\ \ Re(u) > 0 \tag{10}$$

By substituting $p = x/c$ and using equations (9) and (10), the fourier transform leads to

$$\hat{\Phi}(\omega) = (2\pi)^{-1/2}\frac{2c}{\Gamma(\beta)}\int_0^{\infty} u^{\beta-1}K_{ic\omega}(u)du \tag{11}$$

Here $\beta > 0$ and $Re(ic\omega) = 0$. We know that from integrals of modified bessel function of third kind of imaginary order [9].

$$\int_0^\infty u^{\beta-1} K_{ic\omega}(u) du = 2^{\beta-2} \Gamma\left(\frac{\beta + ic\omega}{2}\right) \Gamma\left(\frac{\beta - ic\omega}{2}\right) \tag{12}$$

Put $z = (\beta + ic\omega)/2$. We know that $\Gamma(z)\Gamma(\overline{z}) = |\Gamma(z)|^2$, the fourier transform becomes,

$$\hat{\Phi}(\omega) = (2\pi)^{-1/2} \frac{2^{\beta-1} c}{\Gamma(\beta)} \left| \Gamma\left(\frac{\beta}{2} + \frac{ic\omega}{2}\right) \right|^2 \tag{13}$$

For $\beta > 0$ and $\omega \in \mathbb{R}$ the Fourier transform is non-negative and non-vanishing. From Theorem 8.12 in [1], the function $\Phi(x) = sech^\beta(x/c)$ is strictly positive definite. $\square$

Lemma 2.2. *For* $x \geq 0$ $(x \in \mathbb{R}_+ \cup \{0\})$ *and* y $(|y| << 1/\pi) \to 0$ *the absolute gamma function has the following bound*

$$|\Gamma(-x + iy)| \leq \frac{K}{|y|}$$

where $K = max\{1/\Gamma(x)\}$ *for* $x \geq 0$

Proof. We know that from the definition of Gamma function

$$\begin{aligned} \Gamma(z) &= \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^z}{\left(1 + \frac{z}{n}\right)} \\ &= \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^z \left(1 - \frac{z}{n}\right)}{\left(1 - \frac{z^2}{n^2}\right)} \\ &= \frac{1}{z} \frac{\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^z \left(1 - \frac{z}{n}\right)}{\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)} \end{aligned} \tag{14}$$

We know from the definition of the sine function, $sin(\pi z) = \pi z \prod_{n=1}^{\infty} (1 - z^2/n^2)$,

$$\Gamma(z) = \frac{\pi}{sin(\pi z)} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^z \left(1 - \frac{z}{n}\right) \tag{15}$$

Substituting $z = -x + iy$ in the equation leads to,

$$\Gamma(-x + iy) = \frac{\pi}{sin(-\pi x + i\pi y)} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-x+iy} \left(1 + \frac{x}{n} - \frac{iy}{n}\right) \tag{16}$$

Here as $y \to 0$, $|(1+x/n-iy/n)| \approx |(1+x/n)|$ and $\left|(1+1/n)^{-x+iy}\right| = \left|(1+1/n)^{-x}\right|$, the equation now becomes

$$|\Gamma(-x+iy)| \approx \left|\frac{\pi}{sin(-\pi x+i\pi y)}\right| \prod_{n=1}^{\infty} \left| \left(1+\frac{1}{n}\right)^{-x} \left(1+\frac{x}{n}\right) \right| \tag{17}$$

From the definition of $|1/\Gamma(x+1)|$, the equation leads to

$$|\Gamma(-x+iy)| \approx \left|\frac{\pi}{sin(-\pi x+i\pi y)}\right| \left|\frac{1}{\Gamma(x+1)}\right| \tag{18}$$

Now, using the equation (18) we see the following cases,
Case (i) As y ($|y| << 1/\pi$) $\to 0$ and $\forall\ x \in \mathbb{N}_0$, where the reciprocal gamma function is less than or equal to one, $1/\Gamma(x+1) \leq 1$, the inequality now becomes,

$$\begin{aligned} |\Gamma(-x+iy)| &\leq \left|\frac{\pi}{sin(i\pi y)}\right| \\ &\leq \frac{1}{|y|} \end{aligned} \tag{19}$$

Case (ii) As y ($|y| << 1/\pi$) $\to 0$ and $\forall\ x \in (0,1)$, the reciprocal gamma function is greater than one, $\dfrac{1}{\Gamma(x+1)} > 1$.
Now as y ($|y| << 1/\pi$) $\to 0$,

$$\begin{aligned} &sin(-\pi x+i\pi y) \approx -sin(\pi x) + icos(\pi x)\pi y \\ &|-sin(\pi x) + icos(\pi x)\pi y| \geq \pi|y| \\ &\left|\frac{1}{-sin(\pi x) + icos(\pi x)\pi y}\right| \leq \frac{1}{\pi|y|} \end{aligned} \tag{20}$$

Let $K = max\,\{1/\Gamma(x+1)\}$, and $|1/\Gamma(x+1)| \leq K$, $\forall\ x \in (0,\ 1)$

$$\begin{aligned} |\Gamma(-x+iy)| &\leq \left|\frac{K\pi}{sin(-\pi x+i\pi y)}\right| \\ &\leq \left|\frac{K\pi}{\pi y}\right| \\ &\leq \frac{K}{|y|} \end{aligned} \tag{21}$$

Case (iii) As y ($|y| << 1/\pi) \to 0$ and $\forall x \in \mathbb{R} \setminus \{\mathbb{N}_0 \cup\ (0,\ 1)\}$ the reciprocal gamma function $1/\Gamma(x+1) \leq 1$ and from the inequality (20)

$$\begin{aligned} |\Gamma(-x+iy)| &\leq \left|\frac{\pi}{\pi y}\right| \\ &\leq \frac{1}{|y|} \end{aligned} \tag{22}$$

From cases (i), (ii) and (iii) the absolute gamma function has the bound for $x\ \geq 0$ ($x \in \mathbb{R}$) and y ($|y| << 1/\pi) \to 0$.

$$|\Gamma(-x+iy)| \leq \frac{K}{|y|} \tag{23}$$

□

The absolute gamma function and comparison of $K/|y|$ and $|\Gamma(-x+iy)|$ for $x = 0$ are plotted and shown in figure 1.

Theorem 2.3. *The function $\Phi(x) = cosh^{\beta}(x/c)$, $x \in \mathbb{R}^1$ with $c > 0$ and $\beta \in \mathbb{R} \setminus \mathbb{N}_0$ possesses the generalized-fourier transform of order m.*

$$\hat{\Phi}(\omega) = (2\pi)^{-1/2} \frac{2^{-\beta-1}c}{\Gamma(-\beta)} \left|\Gamma\left(\frac{-\beta}{2} + \frac{ic\omega}{2}\right)\right|^2, \quad \omega \neq 0 \tag{24}$$

where $m = max(0, \lceil\beta\rceil)$

Proof. We prove the theorem the same way as Theorem 8.15 from reference [1] using the analytic continuation. Define $G = \{\lambda \in \mathbb{C} : Re(\lambda) < m\}$ and let us denote the right-hand side of the equation (24) by $\phi_\beta(\omega)$ and the function $cosh^{\lambda}(\omega/c)$ by $\Phi_\lambda(\omega)$. Now, from the Definition 8.9 in [1], we are going to show that for all $\lambda \in G$,

$$\int_{\mathbb{R}} \Phi_\lambda(\omega)\hat{\gamma}(\omega)d\omega = \int_{\mathbb{R}} \phi_\lambda(\omega)\gamma(\omega)d\omega,\ \gamma \in S_{2m} \tag{25}$$

Note that the above equation holds for $\lambda < 0$ (from Theorem 2.1) and for $\lambda = 0\ ,1\ ,2\ ,...,\ m-1$ for $m > 0$ (since $1/\Gamma(-\lambda) = 0$). From Theorem 8.15 [1], it is now sufficient to show that the integrand in the right-hand side of equation (25) is bounded uniformly on a closed curve C in G.

Let us investigate the asymptotic characteristics of the integrand near the origin, say for $|\omega| << min\{1/c,\ 1/\pi\}$. Let $a = Re(\lambda)$ and using the Lemma. 2.2 the integrand has the following behaviour,

$$|\phi_\lambda(\omega)\gamma(\omega)| \leq C_\gamma (2\pi)^{-1/2} \frac{2^{-a-1}}{|\Gamma(-\lambda)|} \frac{4K^2}{c} |\omega|^{2m-2} \tag{26}$$

Here C is compact and $1/\Gamma(-\lambda)$ is analytic, now for all $\lambda \in C$ and $|\omega| < min\{1/c,\ 1/\pi\}$, the inequality can be written as,

$$|\phi_\lambda(\omega)\gamma(\omega)| \leq C_{\gamma,m,c,C} |\omega|^{2m-2} \tag{27}$$

Now for large arguments, $\omega \to \infty$, using the asymptotic expansions of gamma function [9]

$$|\phi_\lambda(\omega)\gamma(\omega)| \leq C_\gamma \frac{2^{-a-1}c}{|\Gamma(-\lambda)|} |\omega|^{-(a+1)/2} e^{-\pi c|\omega|/4} \tag{28}$$

Here $\gamma \in S$ is bounded and since C is compact the integrand can be bounded independent of $\lambda \in C$ by,

$$|\phi_\lambda(\omega)\gamma(\omega)| \leq C_{\gamma,m,c,C} e^{-\pi c|\omega|/4} \tag{29}$$

The proof is complete. □

Theorem 2.4. *The function $\Phi(x) = (-1)^m cosh^\beta(x/c)$, $x \in \mathbb{R}^1$ with $c > 0$, $\beta \in \mathbb{R} \setminus \mathbb{N}_0$ and $m = max(0, \lceil \beta \rceil)$ is conditionally positive definite of order m.*

Proof. The proof is a direct consequence of Theorems. 2.1 and 2.3. □

With this, we define the Generalized Hyperbolic-Radial Basis Function (GH-RBF) as follows,

Definition 2. *For $n \in \mathbb{R} \setminus \mathbb{N}_0$ and $c > 0$, the Generalized Hyperbolic Radial Basis Function (GH-RBF) is defined as*

$$\phi(x) = cosh^n \left(\tfrac{r}{c}\right)$$

here $r = |x - x_i|$, $x \in \mathbb{R}^1$, and x_i are the data sites.

3. Numerical experiment

For the numerical experiment, we considered approximating two different functions with Runge's phenomenon, using gridded data and scattered data to validate the theoretical results and the performance of the GH-RBFs.

3.1. Case-1: Approximation with gridded data

$$f(x) = exp(xcos(3\pi x)),\ x \in [0,\ 1] \tag{30}$$

The interval is divided into fifteen equally spaced points as data sites for interpolation with shape parameter c as $1/3\sqrt{2}$. The shape parameter is considered same for all the cases and the interpolation weights are computed using the equation (5) for various values of n. The results are then compared with the approximations using GMQ and Gaussian RBFs. Figure 2 shows the interpolation plots and error plots of GH-RBFs and GMQ-RBFs for different values of n.

3.2. Case-2: Approximation with scattered data

$$f(x) = \frac{1}{(1+25x^2)},\ x \in [-1,\ 1] \tag{31}$$

A total of fifteen data sites are generated from the interval, $x \in [-1,\ 1]$ following a uniform random distribution for the interpolation. The shape parameter is considered the same ($c = 1/5$) for all cases and the interpolation weights are computed using the equation (5) for various values of n. The results are then compared with the approximations using GMQ and Gaussian RBFs. The interpolation plots and error plots are shown in figure 2 for different values of n.

Table 1: Mean Square Error (MSE) for positive integer exponents

	Case-1: Gridded data points		Case-2: Scattered data points	
Exponent (n)	GH-RBF	GMQ-RBF	GH-RBF	GMQ-RBF
1	10.5	3.3	3805.1	11.0253
2	6673.4	61.3	1.066×10^8	63.7901
3	940.8	5.8	9.898×10^8	111.797

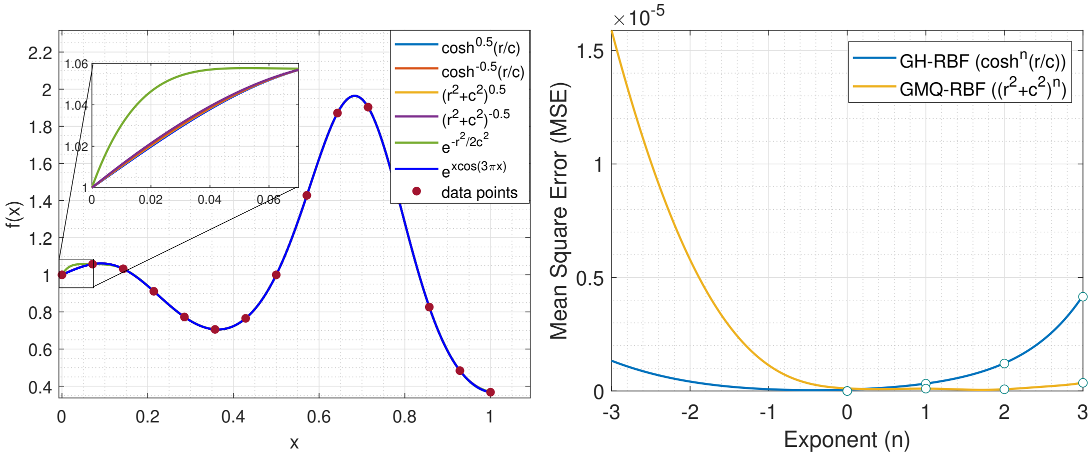

Figure 2: Case-1: Interpolation with gridded data points (Left: Interpolation using GH-RBFs, GMQ-RBFs, and Gaussian RBF; Right: Variation of Mean Square Error (MSE) with exponent (n))

Note that from the interpolation plots in figures 2 and 3, both the GH-RBF and the GMQ-RBF produced a more accurate interpolation than the Gaussian RBF interpolation. For $0 < n < 1$, the MSE of GH-RBF and GMQ-RBF is comparable in both cases and for $n > 1$, the MSE of GH-RBF increased at a higher rate than the GMQ-RBFs. However, for $n < 0$ the MSE of GMQ-RBFs increased at a larger rate than the GH-RBFs in both cases. The GH-RBFs with negative real exponents approximate the function better than those of the GMQ-RBFs. Table 1 reports the MSE of GH-RBFs and GMQ-RBFs for non-negative integer exponents ($n = \{1, 2, 3\}$) (shown as discontinuities in figure 2 and 3). The MSE values are several orders of magnitude higher because the RBFs are not strictly positive definite and the interpolation matrix is singular.

4. Conclusions

The current paper introduces a novel class of RBFs known as the Generalized Hyperbolic Radial Basis Functions (GH-RBFs). The major outcomes of this study are listed as,

- We formally defined the GH-RBFs for $x \in \mathbb{R}^1$ and proven the conditional positive definiteness of the GH-RBFs for the non-negative integer exponents.

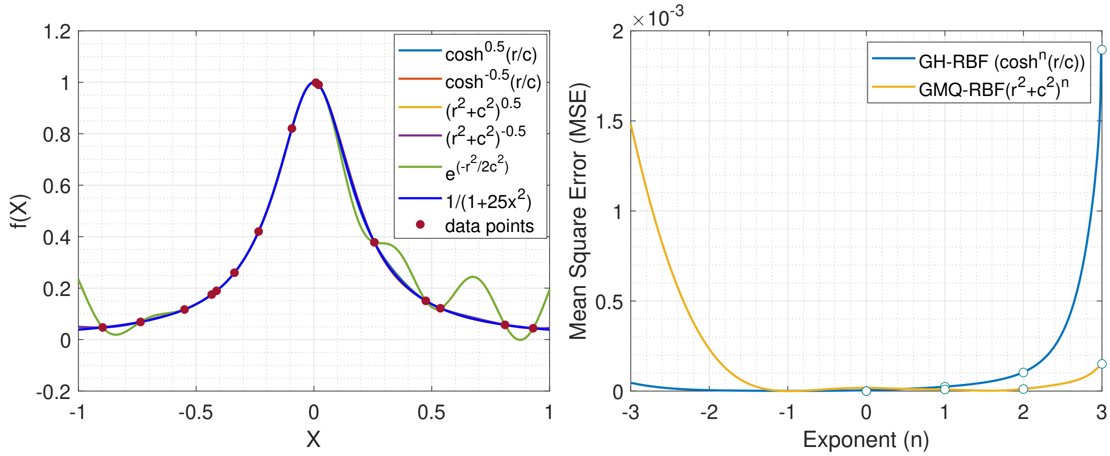

Figure 3: Case-2: Interpolation with scattered data points (Left: Interpolation using GH-RBFs, GMQ-RBFs, and Gaussian RBF; Right: Variation of Mean Square Error (MSE) with exponent (n))

- We establish an upper bound for the absolute gamma function with the negative real part when the imaginary part tends to zero.
- The results of numerical experiments using gridded data and scattered data show that GH-RBFs with negative exponents produced a more accurate approximation compared to the GMQ-RBFs and the Gaussian RBF, while the accuracy is comparable to GMQ-RBFs for $0 < n < 1$.

Acknowledgement

The work is supported by the Vikram Sarabhai Space Center (VSSC), ISRO.

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