Homework #1
FIM/MA 548, Spring '25
Due: **11:00 pm Friday, February 15**

Instructions:

- Please submit (on Moodle) one work per group: one file that consists of your code and one pdf file where you should present/discuss the results/findings.
- NO interactions between groups.
- Each group member must make a substantial contribution to each part of the assignment.
- The work counts for 20% of your final grade.
- You can use any software (Matlab, Python, C/C++, R etc.), but Matlab is preferred. Your code must compile and be well-commented.

Problem #1 (Binomial model)

For each fixed $N \geq 1$, consider an N-step market model $(B^N_{k/N}, S^N_{k/N})_{k=0,1,...,N}$ where

$$B^N_{k/N} = e^{\frac{rk}{N}}, \quad k = 0, ..., N$$

is a bank account with the annual (continuously compounded) risk-free rate $r > -1$, and

$$S^N_0 = s \in (0, \infty), \; S^N_{k/N} = S^N_{(k-1)/N} R^N_k, \quad k = 1, ..., N$$

is a risky stock, defined using a one-step i.i.d. random growth factors $(R^N_k)_{k=1,...,N}$. In particular, if we set $u^N := e^{\mu/N + \sigma\sqrt{1/N}}$ and $d := 1/u$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are fixed parameters, then

$$p_N := \mathbb{P}[R^N_k = u^N] = 1 - \mathbb{P}[R^N_k = d^N] = \frac{e^{r/N} - d}{u - d}, \quad k = 1, ..., N.$$

In what follows set $r = 0.05$, $\sigma = 0.4$, $s = 100$ and $\mu \in \{-0.2, 0, 0.2, (r - \sigma^2/2)\}$.

1. Plot $N \mapsto p_N$ and $N \mapsto p_N(1 - p_N)$, and (numerically) determine $\lim_{N\to\infty} p_N$ and $\lim_{N\to\infty} p_N(1 - p_N)$. How do your results depend on μ?

   Hint: for the limits, replace ∞ by a large N_{max}; e.g., $N_{max} = 10000$.

2. Now investigate the convergence of Part 1. In particular, for both $N \mapsto p_N$ and $N \mapsto p_N(1 - p_N)$ plot the relevant log-log graphs and show that (for large N)

$$p_N \approx (\lim_{N\to\infty} p_N) + (1/N)^{k_1} c_1, \quad p_N(1 - p_N) \approx (\lim_{N\to\infty} p_N(1 - p_N)) + (1/N)^{k_2} c_2,$$

   for some $c_1, c_2 \in \mathbb{R}$ and $k_1, k_2 \in \mathbb{R}_+$.

   Using the generated graphs, determine k_1 and k_2.

3. Use Monte-Carlo (with 1000000 samples) to estimate the expectation and standard deviation of $\ln\left(\frac{S^N_1}{S^N_0}\right)$. Compare your estimates with $(r - \sigma^2/2)$ and σ, respectively. Produce a histogram of your generated values and suggest an appropriate limiting distribution for $\ln\left(\frac{S^N_1}{S^N_0}\right)$. Again, discuss the dependence on μ.

4. Using Parts 1. and 2., give a theoretical basis for your results obtained in Part 3.

5. Use Monte-Carlo (with 1000000 samples) to determine the time-0 no-arbitrage price of a European call with strike $K = 100$ and maturity $T = 1$, written on S^N with $N = 10000$. In particular, compute

$$\mathbb{E}\left[e^{-r}(S^N_1 - K)^+\right].$$

For each value of μ, produce a plot that shows the convergence (as the number of generated random numbers increases). Compare your estimates with the theoretical Black-Scholes price of the same call option (and with the same model parameters r and σ). Discuss how your results depend on μ.

Problem #2 (Confidence intervals)

Let $(Z_i)_{i=1,...N}$ be a sequence of i.i.d. random variables with $\mathbb{E}[Z_i] = \mathbb{E}[Z]$. Then the Central Limit Theorem asserts that, for large $N \geq 1$,

$$\mathbb{P}\left[\bar{Z}_N - 1.96 s_N/\sqrt{N} \leq \mathbb{E}[Z] \leq \bar{X}_N + 1.96 s_N/\sqrt{N}\right] \approx 0.95,$$

where $\bar{Z}_N$ and s_N are the sample mean and sample standard deviation, respectively.

The goal of this exercise is to test this claim using Monte-Carlo.

(Recall how we estimated the value of π; see Lecture 1 and the corresponding Matlab code). Fix $N = 1000$, let (X_i, Y_i) be a a pair of i.i.d. $U[0,1]$ random variables. Set $Z_i = 1$ if $X_i^2 + Y_i^2 \leq 1$ and 0 otherwise. Now test the claim by generating 10000 independent values of $\bar{Z}_N$.

Problem #3 (Pareto distribution)

Use Monte-Carlo (with $N = 1000000$ samples) to estimate $\mathbb{E}[X]$ and $Var[X]$, where X follows Pareto distribution with scale parameter $\beta = 1$ and shape parameter $\alpha \in \{0.5, 1.5, 3\}$:

$$f_X(x) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}}, \quad x \geq \beta.$$

1. Plot the graphs of sample mean $n \mapsto \bar{X}_n$ and sample standard deviation $n \mapsto s_n$ (where $n \in \{1, ..., N\}$). Discuss the observed convergence.
2. Now repeat (independently) Part 1. ten more times. Discuss your findings.

Problem #4 (Value of the game)

You flip a (fair) coin repeatedly until a tail first appears. The pot starts at 1$ and doubles every time a head appears. You win whatever is in the pot the first time you throw tails, and then the game ends. For example: $\{T\}$ (tail on the first toss) wins 1$, $\{H, T\}$ (tail on the second toss): wins 2$, $\{H, H, T\}$ wins 4$ and so on.

Which would you prefer, 20$ for sure, or the right to play this game? To answer this question, use Monte-Carlo (with $N = 1000000$ samples) to estimate the value of this game, which is given by the expectation of the size of the winning pot.

Discuss your findings (and compare them with Problem #3). You could also try to increase N to check if 'better' results can be obtained.

Problem #5(Acceptance-Rejection)

Suppose X has a distribution with pdf

$$f(x) = \frac{1}{6}x^3 e^{-x}, \quad x \geq 0.$$

Apply the acceptance-rejection (AR) method to simulate X using exponential distribution $Exp(\lambda)$ where $\lambda > 0$.

1. For each fixed λ, find the constant $c(\lambda)$ for the AR method.
2. Find the optimal λ^* that makes the AR method most efficient.

Problem #6 (Card deck)

A deck of 100 cards, numbered $1, 2, ..., 100$, are shuffled and then turned over one card at a time. Say that a 'hit' occurs whenever card i is the i-th to be turned over, $i = 1, 2, ..., 100$. Write a simulation program to estimate the expectation and variance of the total number of hits. Run the program with $N = 10^k$, $k = 1, ...6$, Monte-Carlo samples, and present your results. You can use the build-in function in your software that simulates discrete uniform distribution on $\{1, ..., 100\}$.