Homework #3

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Note that the problems are not intended to be realistic models for the corresponding situations but rather dramatically simplified scenarios. This is individual work, and not a group project; do not share solutions. All work turned in must be your own work.

1. The random variable X is given by the following PDF. (9 points each)

$$f(x) = \frac{3}{2}(1 - x^2), \quad 0 \le x \le 1$$

1. **Check that this is a valid PDF**

   The integral of the function should be equal to 1 to be a valid PDF.

   $\int_0^1 \frac{3}{2}(1-x^2)dx = \frac{3}{2}\int_0^1 1 - x^2 dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{3}{2}\left(1 - \frac{1}{3}\right) = \frac{3}{2} \cdot \frac{2}{3} = 1$

   Therefore, the function is valid for PDF.

2. **Calculate expected value of X**

   $\int_0^1 x \cdot \frac{3}{2}(1-x^2)dx = \frac{3}{2}\int_0^1 x - x^3 dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$

3. **Calculate the standard deviation of X**

   To get the standard deviation, the variance is firstly needed.

   $\int_0^1 (x-\mu)^2 \cdot \frac{3}{2}(1-x^2)dx = \int_0^1 x^2 \cdot \frac{3}{2}(1-x^2)dx - \mu^2 = \frac{3}{2}\int_0^1 x^2 - x^4 dx - \mu^2 = \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 - \mu^2 = \frac{3}{2}\left(\frac{1}{3} - \frac{1}{5}\right) - \mu^2$
   $= \frac{3}{2} \cdot \frac{2}{15} - \mu^2 = \frac{1}{5} - \mu^2$

   The μ is the expected value of X, which is derived as $\frac{3}{8}$ in the previous question so that the variance is $\frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320}$

   Hence, the standard deviation is $\sqrt{\frac{19}{320}} \fallingdotseq 0.24367$

2. A specific computer part lasts 10 years on average. The length of time the computer part lasts is exponentially distributed.

1. **What is the probability that a computer part lasts more than 6 years? (8 points)**

   A continuous random variable X is exponentially distributed, and the mean of the distribution is given as $\mu = 10$.

   From $\lambda = \frac{1}{\mu}$, the rate parameter is $\lambda = \frac{1}{10} = 0.1$.

   The PDF of the exponential distribution $f(x) = \lambda e^{-\lambda x}$. Therefore, the probability that a computer part lasts more than 6 years is

   $\int_0^6 0.1e^{-0.1x}dx = -e^{-0.1x}\Big|_0^6 = -e^{-0.1\cdot 6} - (-e^{-0.1\cdot 0}) = (-e^{-0.6}) - (-1) = 1 - e^{-0.6} \fallingdotseq 0.451188$

2. **On the average, how long would 3 computer parts last if they are used one after another? (5 points)**

   Each computer part lasts 10 years on average independently so that the expected value of the sum of three computer parts is

   $10 + 10 + 10 = 30$ years.

3. **What is the probability that a computer part lasts between nine and 11 years? (8 points)**

   The probability that a computer part lasts between 9 and 11 years is

   $\int_9^{11} 0.1e^{-0.1x}dx = -e^{-0.1x}\Big|_9^{11} = (-e^{-1.1}) - (-e^{-0.9}) = e^{-0.9} - e^{-1.1} \fallingdotseq 0.073699$

3.You are given that random variable Y belongs to continuous uniform distribution between 100 and 300 (Y~U(100,300)). Calculate the following : (8 points each)

1. **P(Y>174)**

   The PDF of the continuous uniform distribution is $f(y) = \frac{1}{300-100} = \frac{1}{200}$ for $100 \le y \le 300$.

   Therefore, the probability that Y is greater than 174 is

   $\int_{174}^{300} \frac{1}{200}dy = \frac{1}{200}\int_{174}^{300} 1dy = \frac{1}{200}[y]_{174}^{300} = \frac{1}{200}(300 - 174) = \frac{126}{200} = 0.63$

2. **P(100<Y<226)**

   Aslike the previous question, the probability that Y is between 100 and 226 is

   $\int_{100}^{226} \frac{1}{200}dy = \frac{1}{200}\int_{100}^{226} 1dy = \frac{1}{200}[y]_{100}^{226} = \frac{1}{200}(226 - 100) = \frac{126}{200} = 0.63$

3. **What is the expected value and standard deviation?**

The expected value of the continuous uniform distribution is $\frac{a+b}{2}$ and the standard deviation is $\frac{b-a}{\sqrt{12}}$.

Therefore, the expected value is $\frac{100+300}{2} = 200$ and the standard deviation is $\frac{300-100}{\sqrt{12}} = \frac{200}{\sqrt{12}} \fallingdotseq 57.735$

4. Assume that an average of 30 customers per hour arrive at a store and the time between arrivals is exponentially distributed. (7 points each)

1. On average, how many minutes elapse between two successive arrivals?

A continuous random variable X is exponentially distributed, and the mean of the distribution is given as $\mu = \frac{30 \text{ customers}}{1 \text{ hour}} = 30.$

Converting the mean into the scale of minutes, $\mu = \frac{30 \text{ customers}}{60 \text{ minutes}} = 0.5$ customers per minute.

As a consequence, the average time between two successive arrivals is $\frac{1}{0.5} = 2$ minutes.

2. When the store first opens, how long on average does it take for three customers to arrive?

The average time for three customers to arrive is $2 \times 3 = 6$ minutes.

3. After a customer arrives, find the probability that it takes less than one minute for the next customer to arrive.

The rate parameter of the exponential distribution is $\lambda = \frac{1}{\mu} = \frac{1}{0.5} = 2.$

The probability that it takes less than one minute for the next customer to arrive is

$$P(X < 1) = \int_0^1 2e^{-2x} dx = -e^{-2x}\Big|_0^1 = -e^{-2} - (-e^{-0}) = 1 - e^{-2} \fallingdotseq 0.86466 \quad (x \geq 0)$$

4. After a customer arrives, find the probability that it takes more than five minutes for the next customer to arrive.

The probability that it takes more than five minutes for the next customer to arrive is

$$P(X > 5) = \int_5^\infty 2e^{-2x} dx = -e^{-2x}\Big|_5^\infty = 0 - (-e^{-10}) = e^{-10} \fallingdotseq 0.0000453999$$