Summer Project Notes

Sohom Roy

18 July 2016

Contents

Chapter 1

First Chapter

1.1 Form of the metric for a homogeneous, isotropic universe

Initially, Newton had assumed that space is static and Euclidean. Then the distance between two points assumes the very simple form

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1.1}$$

if we are using a Cartesian coordinate system. In spherical coordinates, this can be expressed as

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{1.2}$$
$$\Rightarrow ds^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.3}$$
$$\Rightarrow ds^2 = dr^2 + r^2 d\Omega^2 \tag{1.4}$$

But now, from Einstein's theory of general relativity we see that space can have curvature. That is, space can either be positively curved, flat(Euclidean), or negatively curved.

The three possible metrics for a homogeneous, isotropic, three-dimensional space can be compactly written down as

$$ds^2 = dr^2 + S_\kappa(r)^2 d\Omega^2 \tag{1.5}$$

where

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{1.6}$$

and

$$S_\kappa(r) = \begin{cases} R \, \sin\left(\dfrac{r}{R}\right), & (\kappa = +1) \\ r, & (\kappa = 0) \\ R \, \sinh\left(\dfrac{r}{R}\right), & (\kappa = -1) \end{cases} \tag{1.7}$$

The Minkowski metric is given by

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2 \tag{1.8}$$

Thsi metric applies for a flat or Euclidean space-time. This metric deals with the special case when space-time is not curved by the presence of mass and energy.

1.1.1 Robertson-Walker metric

This metric was introduced to describe a univere which was spatially homogeneous and isotropic, and where distances are allowed to expand or contract as a function of time. It is given by

$$ds^2 = -c^2dt^2 + a(t)^2 \left[\frac{dx^2}{1 - \dfrac{\kappa x^2}{R_0^2}} + x^2d\Omega^2 \right] \tag{1.9}$$

Here $a(t)$ is the scale factor, and the expression in square brackets is the metric for a uniformly curved space of radius R_0.

This equation can be rewritten as

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + S_\kappa(r)^2d\Omega^2] \tag{1.10}$$

(r, θ, ϕ) are called the comoving coordinates. If the expansion of the universe is perfectly homogeneous and isotropic, then the comoving coordinates are constant with time.

The scale factor $a(t)$ is normalized so that $a(t_0) = 1$.

1.2 Proper distance

The distance to a galaxy situated at a comoving coordinate (r, θ, ϕ) is given by the length of the spatial geodesic between these points when the scale factor is fixed at $a(t)$.

So, the distance is given by the Robertson-Walker metric at a fixed time t.

$$ds^2 = a(t)^2[dr^2 + S_\kappa(r)^2d\Omega^2] \tag{1.11}$$

Along the spatial geodesic joining the observer and the galaxy, θ and ϕ are constant. So, the metric becomes

$$ds^2 = a(t)^2dr^2 \tag{1.12}$$
$$\Rightarrow ds = a(t)dr \tag{1.13}$$
$$\tag{1.14}$$

$\therefore$ Proper distance $=$

$$d_p(t) = a(t) \int_0^r dr = a(t).r$$

Rate of change of proper distance:

$$\dot{d}_p = \dot{a}r = \frac{\dot{a}}{a}d_p \tag{1.15}$$

$$\Rightarrow \frac{\dot{d}_p}{d_p} = \frac{\dot{a}}{a} \tag{1.16}$$

So, we can see that there is a linear relation between the proper distance to a galaxy and its recession speed:

$$v_p(t_0) = H_0 d_p(t_0) \tag{1.17}$$

where $v_p(t_0) = \dot{d}_p(t_0)$ and $H_0 = \left(\dfrac{\dot{a}}{a}\right)_{t=t_0}$.

The angular position of a galaxy can easily be known, but not its proper distance, or its comoving distance r.

Light emitted from the galaxy at time t_e is observed by us at a time t_0. During its travel, light travelled along a null geodesic($ds = 0$). We have θ and ϕ constant.

$$c^2 dt^2 = a(t)^2 dr^2 \tag{1.18}$$

$$\Rightarrow c\frac{dt}{a(t)} = dr \tag{1.19}$$

For a single wave crest emitted at time t_e and observed at time t_0, we have

$$c\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r \tag{1.20}$$

If wavelength of emitted light is given by λ_e and that of the observed light is given by λ_o (in general, $\lambda_o \neq \lambda_e$), then we can say that the next wave crest of light is emitted at time $t_e + \dfrac{\lambda_e}{c}$ and observed at time $t_o + \dfrac{\lambda_o}{c}$.

Thus, we can write

$$c\int_{t_e+\frac{\lambda_e}{c}}^{t_o+\frac{\lambda_o}{c}} = \int_0^r dr = r. \tag{1.21}$$

Comparing these equations, we see that

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\frac{\lambda_e}{c}}^{t_o+\frac{\lambda_o}{c}} \frac{dt}{a(t)} \tag{1.22}$$

Subtracting $\int_{t_e+\frac{\lambda_e}{c}}^{t_0} \frac{dt}{a(t)}$ from both sides, we get

$$\int_{t_e}^{t_e+\frac{\lambda_e}{c}} \frac{dt}{a(t)} = \int_{t_0}^{t_o+\frac{\lambda_o}{c}} \frac{dt}{a(t)} \tag{1.23}$$

4

We can also consider the scale factor to remain constant, since the universe does not expand by a significant amount during the time between two crests of the wave.

So, we have

$$\frac{1}{a(t_e)} \int_{t_e}^{t_e + \frac{\lambda_e}{c}} dt = \frac{1}{a(t_o)} \int_{t_o}^{t_o + \frac{\lambda_o}{c}} dt \tag{1.24}$$

$$\Rightarrow \frac{1}{a(t_e)} \cdot \frac{\lambda_e}{c} = \frac{1}{a(t_o)} \cdot \frac{\lambda_o}{c} \tag{1.25}$$

$$\Rightarrow \frac{1}{a(t_e)} \cdot \lambda_e = \frac{1}{a(t_o)} \cdot \lambda_o \tag{1.26}$$

$$\Rightarrow \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)} \tag{1.27}$$

$$\text{Now,} z = \frac{\lambda_o - \lambda_e}{\lambda_e} \tag{1.28}$$

$$\Rightarrow z = \frac{\lambda_o}{\lambda_e} - 1 \tag{1.29}$$

$$\Rightarrow 1 + z = \frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)} = \frac{1}{a(t_e)} \tag{1.30}$$

1.3 The equations governing the expansion of the universe

In cosmology, we come across three important equations which are central to our understanding of how the universe evolves with time.

The first of these is the Friedmann equation, which relates the scale factor $a(t)$, the curvature κ of the universe, the energy density, ε and tells us how the scale factor evolves with time.

The equation is as given below:

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \varepsilon - \frac{\kappa c^2}{R_0^2 a^2} \tag{1.31}$$

κ can be either equal to 0, $+1$, -1, corresponding respectively to a flat, positively curved and a negatively curved universe. The left hand side is the square of the Hubble constant $H(t)$.

The second equation is the fluid equation, which is derived from thermodynamics, and is given by

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0 \tag{1.32}$$

The third equation is the acceleration equation, which is not really an independent equation, but is derived from the above two equations.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) \tag{1.33}$$

The last equation is the equation of state, which gives a relation between the pressure P and energy density ε.

$$P = w\varepsilon \tag{1.34}$$

For non-relativistic matter, $w = 0$. For relativistic matter(radiation), $w = \frac{1}{3}$.

There is another component of the universe, the cosmological constant, Λ, which was introduced by Einstein to account for a static universe containing only matter and radiation. It has $w = -1$.

In the Friedmann equation, putting the curvature constant κ equal to zero, gives us a value for the energy density, which we call the critical density.

$$H^2(t) = \frac{8\pi G}{3c^2}\varepsilon_c \tag{1.35}$$

$$\Rightarrow \varepsilon_c = \frac{3c^2}{8\pi G}H^2(t) \tag{1.36}$$

If we evaluate this at the current time, then we have

$$\varepsilon_{c,0} = \frac{3c^2 H_0^2}{8\pi G} \tag{1.37}$$

If energy density is greater than this, then universe is positively curved, else, it is negatively curved.

The currently estimated value of the critical density is

$$\varepsilon_{c,0} = 5200 \pm 1000 MeV m^{-3} \tag{1.38}$$

The equivalent mass density is given by

$$\rho_{c,0} = \frac{\varepsilon_{c,0}}{c^2} = (1.1 \pm 0.3) \times 10^{11} M_\odot Mpc^{-3} \tag{1.39}$$

We define the density parameter to be

$$\Omega \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)} \tag{1.40}$$

In terms of the density parameter, the Friedmann equation becomes

$$1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2} \tag{1.41}$$

Chapter 2

Second Chapter

So far, we have discussed the basic quantities and equations describing the universe, and the relations between them.

In a universe with more than one component, the energy density in the equations need to be replaced by the total energy density, which is given by:

$$\varepsilon_{total} = \sum_w \varepsilon_w \tag{2.1}$$

and the total pressure needs to be replaced by the total pressure, given by:

$$P_{total} = \sum_w P_w = \sum_w w\varepsilon_w \tag{2.2}$$

For a single component with equation-of-state parameter w, we have

$$\dot{\varepsilon}_w + 3\frac{\dot{a}}{a}(\varepsilon_w + P_w) = 0 \tag{2.3}$$

$$\Rightarrow \dot{\varepsilon}_w + 3\frac{\dot{a}}{a}(1+w)\varepsilon_w = 0 \tag{2.4}$$

$$\Rightarrow \dot{\varepsilon}_w = -3\frac{\dot{a}}{a}(1+w)\varepsilon_w \tag{2.5}$$

$$\Rightarrow \int \frac{d\varepsilon_w}{w} = -3(1+w)\int \frac{da}{a} \tag{2.6}$$

$$\Rightarrow \ln \varepsilon_w = 3(1+w)\ln a + \ln \varepsilon_{w,0} \tag{2.7}$$

$$\Rightarrow \varepsilon_w = \varepsilon_{w,0} a^{-3(1+w)} \tag{2.8}$$

For non-relativistic matter, $w = 0$.

$$\therefore \boxed{\varepsilon_m(a) = \frac{\varepsilon_{m,0}}{a^3}} \tag{2.9}$$

and for radiation, $w = \frac{1}{3}$.

$$\therefore \boxed{\varepsilon_r(a) = \frac{\varepsilon_{r,0}}{a^4}} \tag{2.10}$$

Chapter 3

Cosmological perturbation theory

3.1 Perturbed fluid equations

Let us consider a non-relativistic fluid with mass density ρ, pressure $P << \rho$ and velocity $\vec{u}$, and denote its position vector by $\vec{r}$ and time by t.
Then the fluid equations are

Continuity Equation:

$$\partial_t \rho = -\overrightarrow{\nabla}_r.(\rho \vec{u}) \tag{3.1}$$

Euler's equation:

$$(\partial_t + \vec{u}.\overrightarrow{\nabla}_r)\vec{u} = -\frac{\overrightarrow{\nabla}_r P}{\rho} - \overrightarrow{\nabla}_r \Phi \tag{3.2}$$

The gravitational potential is determined by **Poisson's equation**

$$\overrightarrow{\nabla}_r^2 \Phi = 4\pi G \rho \tag{3.3}$$

Now, we introduce perturbations in the various quantities by decomposing each quantity into background quantities and perturbations. For example,

$$\rho(t, \vec{r}) = \overline{\rho}(t) + \delta\rho(t, \vec{r}) \tag{3.4}$$

3.2 Static space without gravity

In this case we consider the potential to be zero, i.e. $\Phi = 0$.
A solution for this case is given by

$$\overline{\rho} = \text{constant}, \overline{P} = \text{constant, and } \overline{\vec{u}} = 0. \tag{3.5}$$

The linearised evolution equations for the fluctuations are then given by

$$\partial_t \delta\rho = -\overrightarrow{\nabla}_r.(\overline{\rho}\vec{u}) \tag{3.6}$$

$$\overline{\rho}\delta_t \vec{u} = -\overrightarrow{\nabla}_r \delta P \tag{3.7}$$

From (3.6) and (3.7), we get

$$\partial_t^2 \delta\rho - \nabla_r^2 \delta P = 0 \tag{3.8}$$

For adiabatic fluctuations, $\delta P = c_s^2 \delta\rho$, where c_s is the speed of sound.

$$(\partial_t^2 - c_s^2 \nabla^2)\delta\rho = 0. \tag{3.9}$$

The solution is given by a plane wave:

$$\delta\rho = A \exp i(\omega t - \vec{k}.\vec{r}) \tag{3.10}$$

where $\omega = c_s k$.

$$\boxed{\text{Fluctuations oscillate with constant amplitude.}}$$

3.3 Static space with gravity

This case is similar to the last one, but we have a potential source here, which needs to be included.

$$(\partial_t^2 - c_s^2 \nabla_r^2)\delta\rho = 4\pi G \delta\rho \tag{3.11}$$

where we have used the perturbed Poisson equation

$$\nabla^2 \delta\Phi = 4\pi G \delta\rho \tag{3.12}$$

This is still solved by

$$\delta\rho = A \exp i(\omega t - \vec{k}.\vec{r}) \tag{3.13}$$

but with

$$\omega^2 = c_s^2 k^2 - 4\pi G \overline{\rho} \tag{3.14}$$

The critical wavenumber is found by setting $\omega = 0$.

$$k_J = \frac{\sqrt{4\pi G \overline{\rho}}}{c_s} \tag{3.15}$$

$$\boxed{\text{For small scales, pressure dominates and the oscillations occur as before.}}$$

$$\boxed{\text{For large scales, gravity dominates and the fluctuations increase exponentially with time.}}$$

The crossover happens at the **Jeans' length**

$$\boxed{\lambda_J = \frac{2\pi}{k_J} = c_s \sqrt{\frac{\pi}{G\overline{\rho}}}} \tag{3.16}$$

3.4   Expanding space

Relationship between physical coordinates $\vec{r}$ and comoving coordinates $\vec{x}$ is given by,

$$\vec{r}(t) = a(t)\vec{x} \qquad (3.17)$$

The velocity is given by

$$\vec{u}(t) = \dot{\vec{r}} = \dot{a}\vec{x} + a\dot{\vec{x}} = \frac{\dot{a}}{a}.a(t)\vec{x} + a\dot{\vec{x}} \qquad (3.18)$$

$$= H\vec{r} + a\dot{\vec{x}} \qquad (3.19)$$

In the static case, the time and space derivatives defined from t and $\vec{r}$ were independent. But this is not the case anymore.

It is convenient to use comoving coordinates $\vec{x}$ instead. Using the relation,

$$\vec{r}(t) = a(t)\vec{x} \ , \qquad (3.20)$$

we can write

$$\overrightarrow{\nabla}_r = a^{-1}\overrightarrow{\nabla}_x \qquad (3.21)$$

Relationship between time derivatives at fixed $\vec{r}$ and fixed $\vec{x}$ is,

$$\left(\frac{\partial}{\partial t}\right)_r = \left(\frac{\partial}{\partial t}\right)_x + \left(\frac{\partial \vec{x}}{\partial t}\right)_r . \overrightarrow{\nabla}_x$$

$$= \left(\frac{\partial}{\partial t}\right)_x + \left(\frac{\partial a^{-1}\vec{r}}{\partial t}\right)_r . \overrightarrow{\nabla}_x$$

$$= \left(\frac{\partial}{\partial t}\right)_x - \frac{\dot{a}}{a^2}.\vec{r}.\overrightarrow{\nabla}_x$$

$$= \left(\frac{\partial}{\partial t}\right)_x - H\vec{x}.\overrightarrow{\nabla}_x \qquad (3.22)$$

Substituting these quantities into the continuity equation, we get

Continuity equation:

$$\left[\frac{\partial}{\partial t} - H\vec{x}.\overrightarrow{\nabla}\right].[\overline{\rho}(1+\delta)] + \frac{1}{a}.\overrightarrow{\nabla}.[\overline{\rho}(1+\delta)(Ha\vec{x} + \vec{v})] = 0 \qquad (3.23)$$

Here $\delta \equiv \dfrac{\delta\rho}{\overline{\rho}}$ is the **density contrast**.

For the zeroth order in fluctuations,(dropping perturbations δ and $\vec{v}$), we have

$$\frac{\partial\overline{\rho}}{\partial t} + 3H\overline{\rho} = 0 \qquad (\because \overrightarrow{\nabla}_x.\vec{x} = 3) \qquad (3.24)$$

10

For the first order in fluctuations, we have

$$\left[\frac{\partial}{\partial t} - H\vec{x}.\vec{\nabla}\right][\overline{\rho}\delta] + \frac{1}{a}\vec{\nabla}.[\overline{\rho}Ha\vec{x}\delta + \overline{\rho}\vec{v}] = 0$$

$$\Rightarrow \delta\frac{\partial\overline{\rho}}{\partial t} + \overline{\rho}\frac{\partial\delta}{\partial t} - \overline{\rho}H\vec{x}.\vec{\nabla}\delta + \overline{H}.3.\delta + \overline{\rho}H\vec{x}.\vec{\nabla}\delta + \frac{\overline{\rho}}{a}\vec{\nabla}.\vec{v} = 0$$

$$\Rightarrow \left(\frac{\partial\overline{\rho}}{\partial t} + 3H\overline{\rho}\right)\delta + \overline{\rho}\frac{\partial\delta}{\partial t} + \frac{\overline{\rho}}{a}\vec{v} = 0$$

From the zeroth order in fluctuations(3.24), we know that the coefficient of δ is zero.

$$\frac{\partial\delta}{\partial t} = -\frac{1}{a}\vec{\nabla}.\vec{v} \Rightarrow \dot{\delta} = -\frac{1}{a}\vec{\nabla}.\vec{v} \tag{3.25}$$

Euler equation:

$$\dot{\vec{v}} + H\vec{v} = -\frac{1}{a\overline{\rho}}\vec{\nabla}\delta P - \frac{1}{a}\vec{\nabla}\delta\Phi \tag{3.26}$$

In the absence of gravitational and pressure perturbations, we simply get $\vec{v} \propto a^{-1}$.

Poisson's equation:

$$\vec{\nabla}^2\delta\Phi = 4\pi Ga^2\overline{\rho}\delta \tag{3.27}$$

$\partial_t.(3.25)$ gives

$$\frac{\partial}{\partial t}.\frac{\partial\delta}{\partial t} = \frac{\partial}{\partial t}.\left(-\frac{1}{a}\vec{\nabla}.\vec{v}\right) \tag{3.28}$$

$$\Rightarrow \frac{\partial^2\delta}{\partial t^2} = -\frac{1}{a}.\frac{\partial}{\partial t}(\vec{\nabla}.\vec{v}) \tag{3.29}$$

$$\Rightarrow \vec{\nabla}.\dot{\vec{v}} = -a\ddot{\delta} \tag{3.30}$$

$\vec{\nabla}.(3.26)$ gives

$$\vec{\nabla}.\dot{\vec{v}} + H\vec{\nabla}.\vec{v} = -\frac{1}{a\overline{\rho}}\vec{\nabla}^2\delta P - \frac{1}{a}\vec{\nabla}^2\delta\Phi \tag{3.31}$$

Combining (3.25), (3.26), (3.27), we get

$$-a\ddot{\delta} + H\vec{\nabla}.\vec{v} = -\frac{1}{a\overline{\rho}}\vec{\nabla}^2\delta P - 4\pi Ga\overline{\rho}\delta$$

$$\Rightarrow -a\ddot{\delta} + H.(-a\dot{\delta}) = -\frac{1}{a\overline{\rho}}c_s^2\vec{\nabla}^2.\delta\rho - 4\pi Ga\overline{\rho}\delta$$

$$\Rightarrow \ddot{\delta} + H\dot{\delta} = \frac{c_s^2}{a^2}\vec{\nabla}^2\delta + 4\pi Ga\overline{\rho}\delta$$

$$\boxed{\Rightarrow \ddot{\delta} + H\dot{\delta} - \frac{c_s^2}{a^2}\vec{\nabla}^2\delta = 4\pi Ga\overline{\rho}\delta} \tag{3.32}$$

During matter-dominated era, (3.32) reads

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0 \tag{3.33}$$

Since $a \propto t^{\frac{2}{3}}$, $H = \dfrac{3}{2}t$. So we have

$$\ddot{\delta}_m + \frac{4}{3t}\dot{\delta}_m - \frac{2}{3t^2}\delta_m = 0 \tag{3.34}$$

Here we have used the fact that in the matter dominated era, $4\pi G\bar{\rho} = \dfrac{3}{2}H^2$.

To solve this equation, we assume a solution of the form $\delta_m = kt^p$.

Then, $\dot{\delta}_m = pkt^{p-1}$ and $\ddot{\delta}_m = p(p-1)kt^{p-2}$.

Substituting, we get

$$p(p-1)kt^{p-2} + \frac{4}{3}pkt^{p-2} - \frac{2}{3}kt^{p-2} = 0$$
$$\Rightarrow p(p-1) + \frac{4p}{3} - \frac{2}{3} = 0$$
$$\Rightarrow 3p^2 - 3p + 4p - 2 = 0$$
$$\Rightarrow 3p^2 + p - 2 = 0$$
$$\Rightarrow p = -1 \text{ or } p = \frac{2}{3}$$

$$\therefore \delta_m \propto \begin{cases} t^{-1}, & \propto a^{-\frac{3}{2}} \\ t^{\frac{2}{3}}, & \propto a \end{cases} \tag{3.35}$$

Hence, the growing mode of fluctuations grows like the scale factor during the matter-dominated era.

Chapter 4

The Lyman-alpha forest

4.1 Studying the intergalactic medium

We can study various properties of stars by studying their spectral absorption lines and thus gain knowledge about the interstellar medium.

We can extend this method to gas outside our galaxy, that is, the intergalactic medium. The light source that we use to observe the intergalactic medium are quasars, which are extremely luminous and are believed to be the light emitted from supermassive black holes lying at the center of galaxies. Quasars can help us study the properties of the gas present in between the galaxies. If there are intervening galaxies or neutral gas present in the path of light from the quasar to the observer, then the gas might absorb some of the light, which can be observed.

Also, since the universe is constantly expanding, the spectrum of the quasars get redshifted. The wavelength at which the light gets absorbed will depend upon the distance between the quasar and the gas cloud. The spectral lines can thus give us information about when and at what distance the absorption took place.

Since hydrogen is the most prevalent element in the universe, most of the light outside the Milky Way is absorbed by hydrogen. The primary absorption line due to hydrogen is called the **Lyman-alpha line**.

The **Lyman-alpha forest** is a series of absorption lines in the spectra of distant galaxies and quasars arising from the Lyman-alpha electron transition of the hydrogen atom.

4.2 Gunn-Peterson trough

Most of the intergalactic medium is made up of plasma which has temperatures of around tens of thousands celsius. At such temperatures, the electrons are separated from their host atoms.

So, instead of having neutral hydrogen atoms, the protons and electrons are separated and behave as free particles. Hydrogen is said to be ionized in this case.

Previously, it was assumed that the intergalactic medium consists of neutral gas. But when astrophysicists **James E. Gunn** and **Bruce A. Peterson** calculated the expected

level of Lyman-alpha absorption in the quasar spectra, they found that not all of the light was absorbed in the quasar spectra.

To explain this observation, they concluded that this effect is possible only if most of the hydrogen in the intergalactic medium is not neutral, but actually ionized. Ionized hydrogen cannot absorb the light coming from the quasar. So the light from the quasar can pass through the intergalactic medium without being severely absorbed.

According to modern research, the universe was once neutral. The intergalactic medium became ionized after more and more stars formed. The emission from these stars heated the gas and ionized the intergalactic medium.

As calculated by Gunn and Peterson, a decent amount of neutral gas will cause complete absorption in the quasar spectra. By studying at what distance there is complete absorption in the Lyman-alpha forest, we should be able to tell the distance at which most of the atoms in the gas are neutral.

Since looking at distant objects is the same as looking back in time, we can calculate the time at which the universe started to become ionized. This region of complete absorption in the Lyman-alpha forest, which corresponds to the time when the universe was completely neutral, is named the **Gunn-Peterson trough**.

Chapter 5

Fourier transform and its applications

5.1 Basic definitions

The Fourier Transform of $f(x)$ is defined as

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i2\pi\omega t}dt \tag{5.1}$$

The inverse Fourier transform is defined as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{-i2\pi\omega t}d\omega \tag{5.2}$$

In three-dimensional space, this becomes

$$F(\vec{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int f(\vec{r})e^{-i\vec{k}.\vec{r}}d^3r \tag{5.3}$$

$$f(\vec{r}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int F(\vec{k})e^{i\vec{k}.\vec{r}}d^3r \tag{5.4}$$

5.2 Convolution

Let us consider two functions $f(x)$ and $g(x)$ having Fourier transforms $F(\omega)$ and $G(\omega)$ respectively, then the convolution of f and g is defined by

$$f * g \equiv \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} g(y)f(x-y)dy \tag{5.5}$$

In three dimensions, this becomes

$$(f * g)(\vec{r}) \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \int g(\vec{r'})f(\vec{r}-\vec{r'})d^3r \tag{5.6}$$

The convolution theorem states that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the individual functions.

5.3 Discrete Fourier transform

We take a set of N equally spaced points x_k on the interval $(0, 2\pi)$:

$$x_k = \frac{2\pi k}{N}, k = 0, 1, 2, ..., N-1 \tag{5.7}$$

The Discrete Fourier Transform(DFT) is defined as

$$g_p = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i k p}{N}} f_k \tag{5.8}$$

where $f_k \equiv f(x_k)$.
The inverse transformation is defined by

$$f_j = \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} e^{\frac{-2\pi i j p}{N}} g_p \tag{5.9}$$

5.4 The FFTW library

For all computations involving Fourier transforms, we will be using the FFTW(*Fastest Fourier Transform in the West*) library.

The basic things to know about this library are:

The datatype `fftw_complex` is used to store complex numbers, which is basically a `double` type array of size 2.

Memory allocation and de-allocation are done using the commands:

```
void *fftw_malloc(size_t n);
void fftw_free(void *p);
```

All the information for doing the Fourier transforms are stored in plans. Plans are of the type `fftw_plan`.

These plans can be executed later by calling the function `fftw_execute()`.

The multi-dimensional arrays passed to `fftw_plan_dft` are expected to be stored in row-major order.

Let us consider an array of rank d whose dimensions are $n_0 \times n_1 \times n_2 \times ... \times n_{d-1}$. If we specify a location in the array by a tuple of indices, one for each dimension, say: $(i_0, i_1, i_2, i_3, ..., i_{d-1})$. Then the position of the element is given by $i_{d-1} + n_{d-1}(i_{d-2} + n_{d-2}(... + n_1 i_0))$.

After the plans have been executed, they are to be destroyed with the `fftw_destroy_plan` command.

Some common plans which are available in the library are:

- `fftw_plan fftw_plan_dft_1d(int n0, fftw_complex *in, fftw_complex *out, int sign, unsigned flags);`

- `fftw_plan fftw_plan_dft_2d(int n0, int n1, fftw_complex *in, fftw_complex *out, int sign, unsigned flags);`

- `fftw_plan fftw_plan_dft_3d(int n0, int n1, int n2, fftw_complex *in, fftw_complex *out, int sign, unsigned flags);`

- `fftw_plan fftw_plan_dft(int rank, const int *n, fftw_complex *in, fftw_complex *out, int sign, unsigned flags);`

A sample program to compute the Fourier transform of a Gaussian input signal is given below:

```
#include<iostream>
#include<fftw3.h>
#include<complex.h>
#include<fstream>
#include<cmath>
#include <malloc.h>

using namespace std;
const float pi = 3.1415923;

void arrange(fftw_complex *arr, int n)
{
for(int i=0;i<n/2;i++)
{
float temp0 = arr[i][0];
float temp1 = arr[i][1];
arr[i][0]=arr[n/2+i][0];
arr[i][1]=arr[n/2+i][1];
arr[n/2+i][0]=temp0;
arr[n/2+i][1]=temp1;
}
}
double max(double *arr, int n)
{
double m=arr[0];
for(int i=1;i<n;i++)
{
if(arr[i]>m)
{
m=arr[i];
}
}
return m;
}
```

```cpp
int deviate(double *arr, int n)
{
int pos=-1;
double hmax=0.5*max(arr,n);
for(int i=n/2;i<n;i++)
{
if(arr[i]<hmax)
{
pos=i;
break;
}
}
int dev=abs(n-2*pos);
return dev;
}
int main()
{
ofstream ofile,ofilep;
ofstream ifile;
ofile.open("output.txt");
ofilep.open("power.txt");
ifile.open("input.txt");
fftw_complex *in, *out;
double *out_abs,*p;
fftw_plan p1;
int n;
char c;
float beg,end,step,mean,stdev,range;
cout<<"Enter the number of elements: ";
cin>>n;
cout<<"Enter starting element: ";
cin>>beg;
cout<<"Enter ending element: ";
cin>>end;
cout<<"Enter mean: ";
cin>>mean;
cout<<"Enter standard deviation: ";
cin>>stdev;
in=(fftw_complex *) fftw_malloc(sizeof(fftw_complex)*n);
out=(fftw_complex *) fftw_malloc(sizeof(fftw_complex)*n);
out_abs=(double *) malloc(sizeof(double)*n);
p=(double *) malloc(sizeof(double)*n);
range=end-beg;
step=range/float(n);
for(int i=0; i<n; i++)
```

```cpp
{
float elt=beg+step*i;
in[i][0] = exp((-1)*pow(elt-mean,2)/(2*pow(stdev,2)))*(1/sqrt(n));
in[i][1] =0;
}
p1=fftw_plan_dft_1d(n, in, out, FFTW_FORWARD, FFTW_ESTIMATE);
fftw_execute(p1);
cout<<"Input array: ";
for(int i=0;i<n;i++)
{
cout<<i<<" "<<in[i][0]<<" "<<in[i][1]<<endl;
ifile<<i<<" "<<in[i][0]<<endl;
}
fftw_free(in);
cout<<endl;
cin>>c;

arrange(out,n);
cout<<"Output array: ";
for(int i=0;i<n;i++)
{
double norm = sqrt(pow(out[i][0],2)+pow(out[i][1],2));
out_abs[i]=norm;
cout<<i<<" "<<out[i][0]<<" "<<out[i][1]<<" "<<norm<<endl;
ofile<<i<<" "<<out[i][0]<<" "<<out[i][1]<<" "<<norm<<endl;
}

cout<<"Absolute values: ";
for(int i=0;i<n;i++)
{
cout<<i<<" "<<out_abs[i]<<endl;
}

cout<<endl;
int hmax=deviate(out_abs,n);
cout<<hmax<<endl;
for(int i=0;i<n;i++)
{
p[i] = pow(out_abs[i],2)+pow(out_abs[n-1-i],2);
ofilep<<i<<" "<<p[i]<<endl;
}
ofile.close();
ifile.close();
ofilep.close();
fftw_destroy_plan(p1);
```

```
fftw_free(out);
delete out_abs;
delete p;
return 0;
}
```