Summer Project Notes

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Chapter 1

First Chapter

1.1 Form of the metric for a homogeneous, isotropic universe

Initially, Newton had assumed that space is Euclidean. Then the distance between two points assumes the very simple form

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1.1}$$

if we are using a Cartesian coordinate system. In spherical coordinates, this can be expressed as

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{1.2}$$
$$\Rightarrow ds^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.3}$$
$$\Rightarrow ds^2 = dr^2 + r^2 d\Omega^2 \tag{1.4}$$

But now, from Einstein's theory of general relativity we see that space can have curvature. That is, space can either be positively curved, flat(Euclidean), or negatively curved.

The three possible metrics for a homogeneous, isotropic, three-dimensional space can be compactly written down as

$$ds^2 = dr^2 + S_\kappa(r)^2 d\Omega^2 \tag{1.5}$$

where

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{1.6}$$

and

$$S_\kappa(r) = \begin{cases} R\ \sin\left(\frac{r}{R}\right), & (\kappa = +1) \\ r, & (\kappa = 0) \\ R\ \sinh\left(\frac{r}{R}\right), & (\kappa = -1) \end{cases} \tag{1.7}$$

Thw Minkowski metric is given by

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2 \tag{1.8}$$

Thsi metric applies for a flat or Euclidean space-time. This metric deals with the special case when space-time is not curved by the presence of mass and energy.

1.1.1 Robertson-Walker metric

This metric was introduced to describe a univere which was spatially homogeneous and isotropic, and where distances are allowed to expand or contract as a function of time. It is given by

$$ds^2 = -c^2dt^2 + a(t)^2 \left[\frac{dx^2}{1 - \frac{\kappa x^2}{R_0^2}} + x^2d\Omega^2 \right] \tag{1.9}$$

Here $a(t)$ is the scale factor, and the expression in square brackets is the metric for a uniformly curved space of radius R_0.

1.2 The equations governing the expansion of the universe

In cosmology, we come across three important equations which are central to our understanding of how the universe evolves with time.

The first of these is the Friedmann equation, which relates the scale factor $a(t)$, the curvature κ of the universe, the energy density, ε and tells us how the scale factor evolves with time.

The equation is as given below:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2a^2} \tag{1.10}$$

κ can be either equal to 0, +1, −1, corresponding respectively to a flat, positively curved and a negatively curved universe. The left hand side is the square of the Hubble constant $H(t)$.

The second equation is the fluid equation, which is derived from thermodynamics, and is given by

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0 \tag{1.11}$$

The third equation is the acceleration equation, which is not really an independent equation, but is derived from the above two equations.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) \tag{1.12}$$

The last equation is the equation of state, which gives a relation between the pressure P and energy density ε.

$$P = w\varepsilon \tag{1.13}$$

For non-relativistic matter, $w = 0$. For relativistic matter(radiation), $w = \frac{1}{3}$.

There is another component of the universe, the cosmological constant, Λ, which was introduced by Einstein to account for a static universe containing only matter and radiation. It has $w = -1$.

In the Friedmann equation, putting the curvature constant κ equal to zero, gives us a value for the energy density, which we call the critical density.

$$H^2(t) = \frac{8\pi G}{3c^2}\varepsilon_c \tag{1.14}$$

$$\Rightarrow \varepsilon_c = \frac{3c^2}{8\pi G}H^2(t) \tag{1.15}$$

If we evaluate this at the current time, then we have

$$\varepsilon_{c,0} = \frac{3c^2 H_0^2}{8\pi G} \tag{1.16}$$

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