Exercise Session 2

Topics:
- Loop shaping (repetition)
- Transfer Functions
- Internal Stability
- Robustness

→ 1.1, 4.3, 7.1, 8.1, 8.6

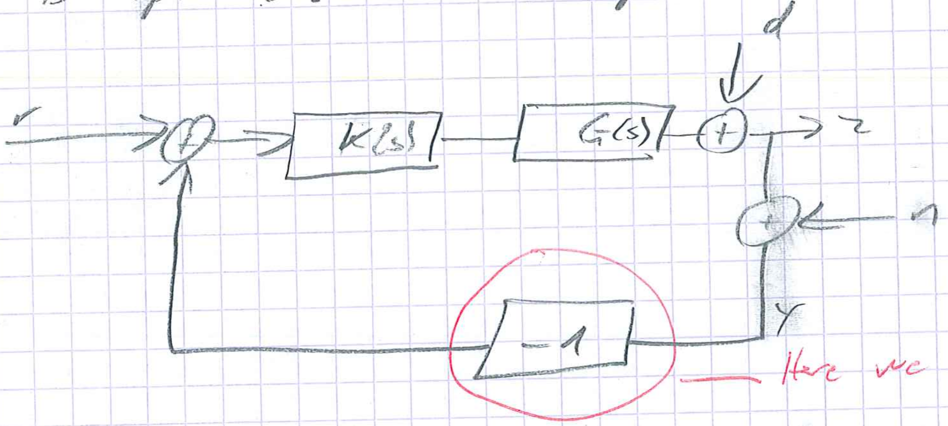
1.1) Consider the system:

$$G(s) = \frac{100}{(s+1)(s+2)(s+10)}$$

// open-loop stable system (= simplified Nyquist criterion can be applied)

We are given <u>closed-loop specification</u>!

→ What is open vs. closed-loop?



Here we close the loop!

Note: S156

Open-loop: $L(s) = G(s) \cdot K(s)$ (no feedback)

closed-loop: $z = \frac{G(s) \cdot K(s)}{1 + G(s)K(s)} r$ (feedback: reference to output)

$= T(s)\, r$

and $z = \frac{1}{1+G(s)K(s)} d$ (feedback: disturbance to output)

$= S(s)\, d$

Note again $S + T = 1$ This is one of the basic limitations!

What we want:

(1) M_s (peak of S) to be smaller than 2

(2) Bandwidth of S to be $\omega_{BS} = 5\ rad/s$

→ What is the bandwidth? ($|S(i\omega_{BS})| = \frac{1}{\sqrt{2}}$)

(3) A static error from d to z of zero!

Translate this into open-loop specifications

(1) We have the relation (with $M_s = 2$)

Proofs: // see Skogestad p. 36

$$\phi_m \geq 2 \arcsin\left(\frac{1}{2M_s}\right) = 29° \geq \frac{1}{M_s}$$

ϕ_m: phase margin

This is only a necessary condition, hence we put some margin and select $\phi_m = 50°$.

(2) We approximately say $\omega_c = \omega_{BS} = 5\ rad/s$

ω_c: crossover frequency

Derivation:
$e_\infty = \lim_{s \to 0} s\, S(s) \frac{1}{s} = 0$
$\Leftrightarrow S(0) = 0$

(3) From final value theorem (and basic control course) we know that for a step input and $e_\infty = 0$ we need $S(0) = 0$, which leads to

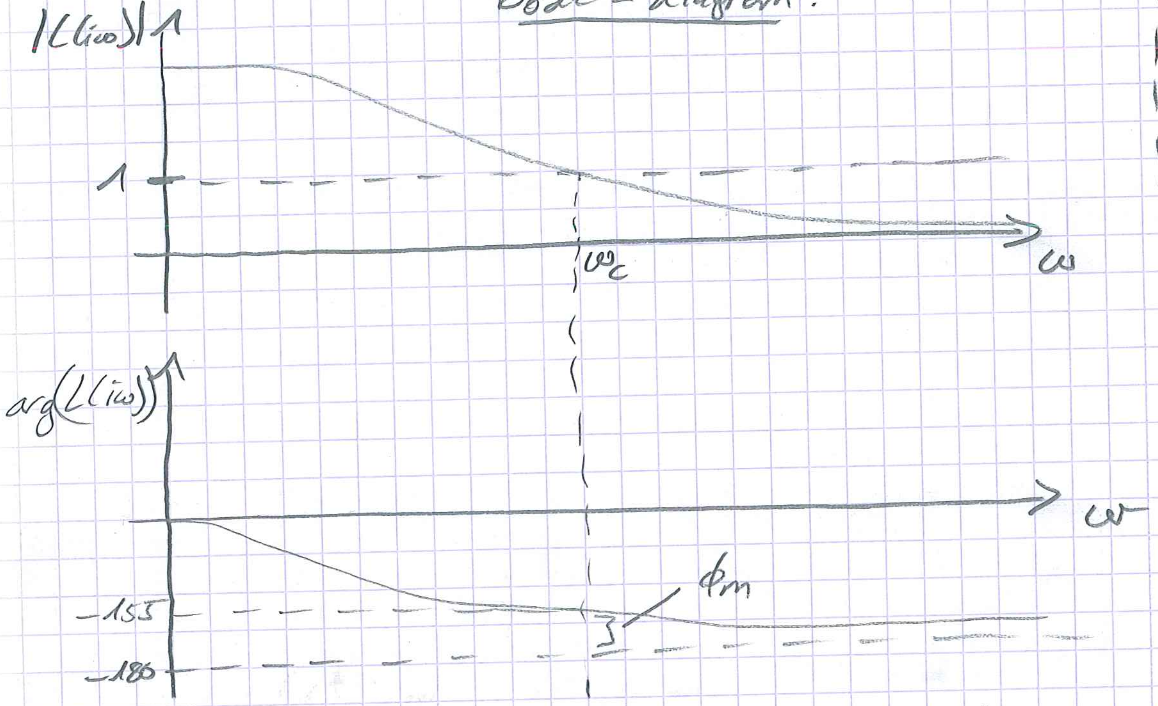
$$S(0) = \frac{1}{1 + L(0)} = 0 \Rightarrow L(0) \to \infty$$

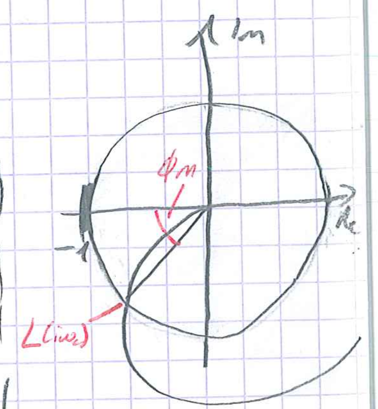
In other words, $L(s)$ needs to contain an integrator.

→ Important: Repeat ω_c and ϕ_m if you're not familiar with it

Given $L(s)$ and the amplitude- and phase plots

Bode-diagram: | Nyquist plot





$\boxed{\phi_m > 0 \text{ guarantees stability!}}$

"Nyquist criterion"

Control design: Loop shaping (here: Sensitivity Shaping)

Given the above $G(s)$ and with $K(s) = 1$, we have

$\phi_m = 41.8°$ and $\omega_c = 2.7$ rad/s (Matlab calculation)

$\Longrightarrow$ Specifications not satisfied. Pick $K(s) = K_1(s)\,K_2(s)\,K_3(s)$

1. Design a lead-element K_1 to adjust phase (elevate phase at $\omega_c = 5$ rad/s)

Recall $\omega_c = 5$ rad/s is desired!

We have $\arg(G(i\omega_c)) = -173{,}45°$

Hence, to achieve a desired $\phi_m = 50°$, we need to increase the phase by

$$50 - (180 - 173.45) = 43.45°$$

We, however, increase by 68 (to account for the lag element in step 3)

Pick the element

$$K_1(s) = \frac{\tau_0 s + 1}{s+10}$$

$$K_1(i\cdot 5) = \frac{\tau_0 5i + 1}{(5i+10)} \cdot \frac{(10-5i)}{(10-5i)}$$

$$= \frac{50\tau_0 i + 25\tau_0 + 10 - 5i}{25+100}$$

$$= \underbrace{\frac{25\tau_0 + 10}{125}}_{real} + \underbrace{\frac{50\tau_0 - 5}{125}}_{Imag} i$$

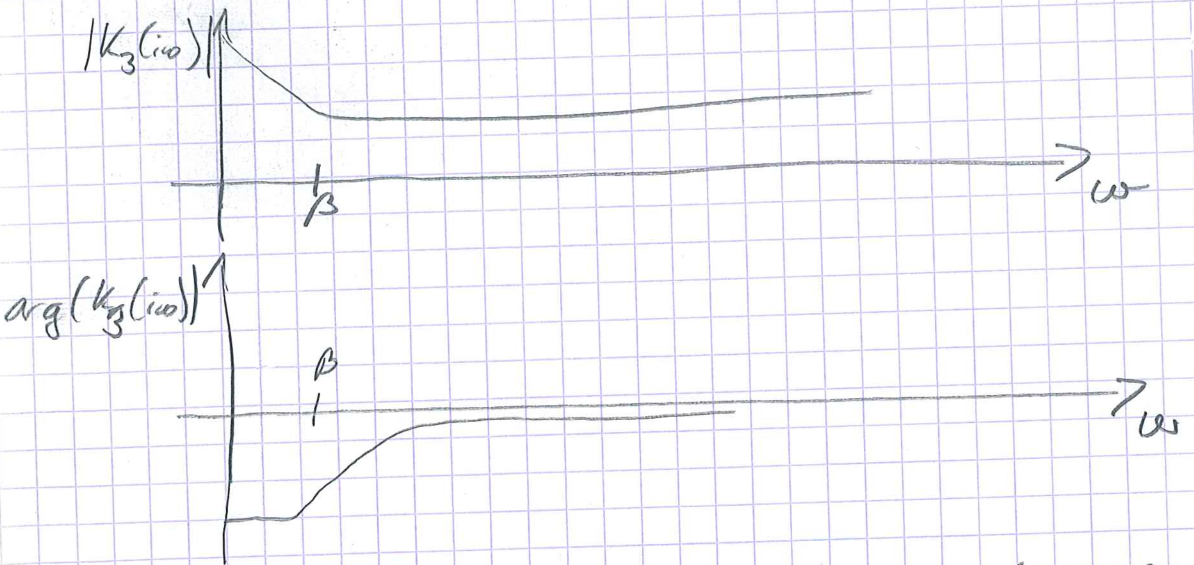
important

$$\arg(K_1(i5)) = \arctan\left(\frac{imag}{real}\right) = \arctan\left(\frac{50\tau_0 - 5}{25\tau_0 + 10}\right) = 48$$

solving for $\tau_0 = 0.7245$

3 Insert an integrator

$K_3(s) = \frac{s+\beta}{s}$ which looks like



Pick a small β to not affect the phase at $\omega_c = 5\,rad/s$, which has been adjusted in 1

$\beta = 0.01$

② Adjust ω_c by $K_2(s)$ which is simply

$$K_2(s) = \frac{1}{|K_1(is) \cdot K_3(is) G(is)|}$$

to achieve

$$|K_1(is) K_2(is) K_3(is) G(is)| = 1$$

The final control law is

$$K(s) = K_1(s) K_2(s) K_3(s)$$

Important to repeat:

- How to draw phase and amplitude plots (Bode diagram)
- Lead Lag design
- poles and zeros + step response

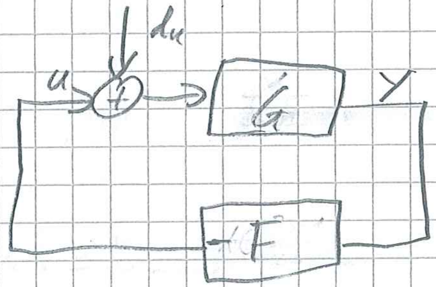
4.3) Assume the system

$$y = \frac{1}{2s+1}(u + d_u)$$

and the control law

$$u = -10\,\frac{2s+1}{2s}\,y$$

a) Draw a block-diagram



$$G(s) = \frac{1}{2s+1}$$

$$F(s) = 10\,\frac{2s+1}{2s}$$

TF from d_u to u and y

$d_u \to u$ $\quad u = -F \cdot G\,(d_u + u)$

$$\Leftrightarrow u + FG\,u = -FG\,d_u$$

$$\Leftrightarrow u = \frac{-FG}{1+FG}\,d_u = -\frac{\frac{5}{s}}{1+\frac{5}{s}}\,d_u = -\frac{5}{s+5}\,d_u$$

$d_u \to y$ $\quad y = G\,(d_u - Fy)$

$$y = \frac{G}{1+GF}\,d_u$$

$$= \frac{\frac{1}{2s+1}}{1+\frac{5}{s}}$$

$$= \frac{1}{2s+1+\frac{10s+5}{s}}$$

$$= \frac{s}{2s^2+s+10s+5}$$

$$= \frac{s}{(2s+1)(s+5)}$$

b) Maximum amplification of a disturbance d_u in the output y?

Look at the <u>system gain</u>!

$$\|G\|_{\infty} = \sup_{\omega} \frac{|\omega|}{|(2i\omega+1)(i\omega+5)|}$$

$$= \sup_{\omega} \frac{|\omega|}{|-2\omega^2+10i\omega+i\omega+5|}$$

$$= \sup_{\omega} \frac{|\omega|}{\sqrt{(5-2\omega^2)^2+11\omega^2}}$$

⋮ → practice calculations by yourself

Matlab <u>hinfnorm</u>!

$= 0.0909$

at $\omega = 1.5811 \frac{rad}{s}$

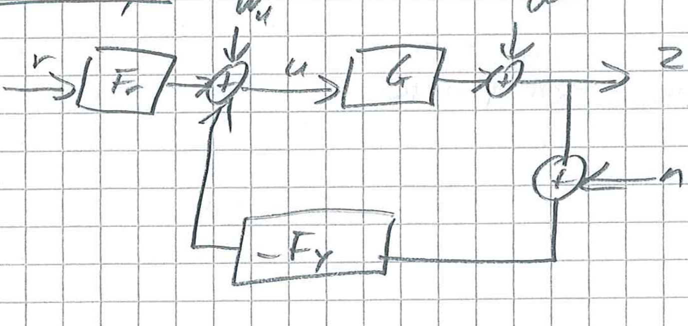
alt: needs to be vanishing over t ("finite energy")

Hence $u = a\sin(\omega t)$ is the worst case disturbance (exciting exactly the frequency $\omega = 1.5811 \frac{rad}{s}$)

$$\left[\text{calc} \quad \frac{d}{dw} \frac{w}{\sqrt{(5-2w^2)^2+11w^2}} = 0 \right]$$

<u>Internal Stability</u>:

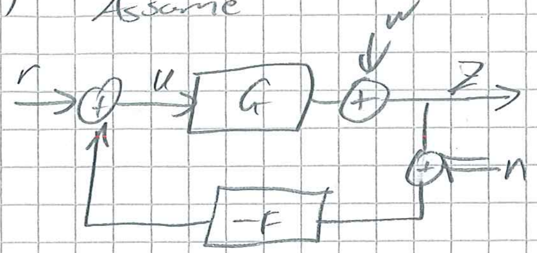
Consider



When G SISO, the closed-loop system is internally stable iff S, SG, SF_y and F_r are stable.

This is the "gang of four"!

7.1) Assume



$G = \frac{s-1}{s+1}$ ⟸ possible zero at $s=1$

$F = \frac{s+2}{s-1}$

⇒ May this zero be a problem?!

Note: If given a G, always inspect for delays, RHP zeros or RHP poles!

$r \to z$

$$z = G(r - Fz)$$

$$\Leftrightarrow \quad z = \frac{G}{1+FG} r \qquad -SG$$

$$G_c = \frac{\frac{s-1}{s+1}}{1 + \frac{s+2}{\cancel{s-1}} \frac{\cancel{s-1}}{s+1}} = \frac{s-1}{s+1+s+2} = \frac{1}{2} \frac{s-1}{s+\frac{3}{2}}$$

(stable)

$w \to z \qquad S = \frac{1}{1+GF} = \frac{1}{1 + \frac{s+2}{\cancel{s-1}} \frac{\cancel{s-1}}{s+1}} = \frac{1}{2} \frac{s+1}{s+\frac{3}{2}}$ (stable)

$n \to z \qquad T = \frac{GF}{1+GF} = \frac{1}{2} \frac{s+2}{s+\frac{3}{2}}$ (stable)

Internal stability: Check the remaining transfer function as mentioned on the previous page!

$F_r = 1$ (stable).

$n \to u \qquad U(s) = -F(N(s) + G\,U(s)) \qquad SF$

$$U(s) = -\frac{F}{1+FG} = \frac{\frac{s+2}{s-1}}{1 + \frac{s+2}{\cancel{s-1}} \frac{\cancel{s-1}}{s+1}} = \frac{s+2}{(s-1)} \frac{s+1}{(2s+3)}$$

not stable

⇒ system is not internally stable.

Alternatively to checking all these 4 transfer function, we know that <u>RHP</u> pole-zero cancellations (between F and G) lead to internal instability.
(this also holds for MIMO systems)

We know that G has a RHP zero at 1, but F has a RHP pole at 1

$$\Rightarrow F(s) \cdot G(s) = \frac{\cancel{s-1}}{s+1} \cdot \frac{s+2}{\cancel{s-1}} \Rightarrow \text{internally not stable}$$

cancellation

sufficient condition

<u>Rule:</u>

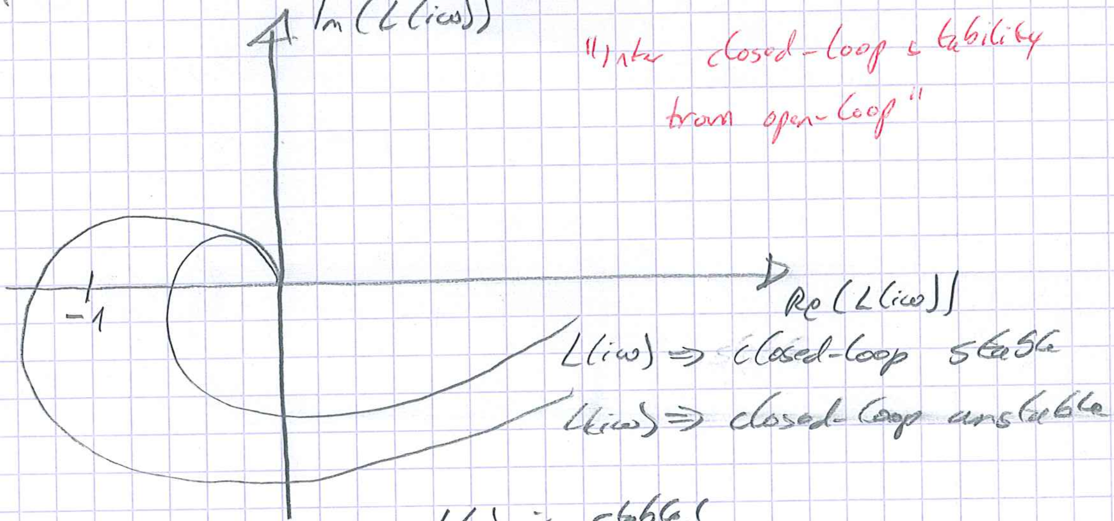
A RHP pole of G at s=p leads to a RHP zero of S at s=p
$\Rightarrow$ If not, internally unstable

A RHP zero of G at s=z leads to a RHP zero of T at s=z
$\Rightarrow$ If not, internally unstable

<u>Important</u>

8.1) Robustness: geometrical derivation

Assume a given F and G such that the open-loop and closed-loop systems are stable.
This condition can conveniently be checked using the Nyquist criterion (repeat from basic control course)



"Infer closed-loop stability from open-loop"

since $L(s)$ is stable!

Nyquist criterion (simplified): The closed-loop is stable if $L(i\omega)$ does not encircle the critical point -1.

Now we consider robustness w.r.t. modelling errors:

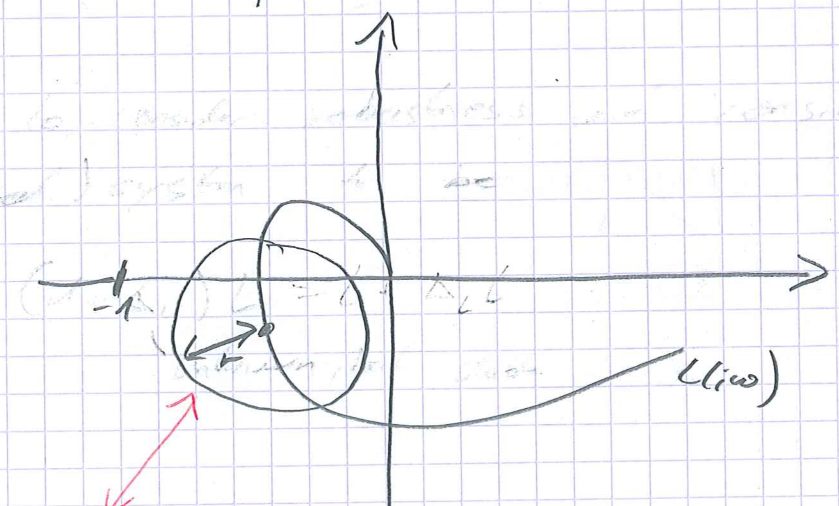
$$L_p(i\omega) = (1 + \Delta_L(i\omega))\, L(i\omega) = L(i\omega) + \Delta_L(i\omega)\, L(i\omega)$$

$L_p(i\omega)$: real (physical) system

$L(i\omega)$: modelled system

$\Delta_L(i\omega)$: unknown, but stable transfer function

Let us illustrate this:



Circle encodes $\Delta_L(i\omega)\, L(i\omega)$ (the modelling error)

Idea: Keep $|\Delta_L L| < r$, then L_p will be contained in a circle of radius r with center L.

If this circle never contains -1, we are stable!

Note: Distance from L to -1 at frequency ω is

$$|L(i\omega) - (-1)| = |L(i\omega) + 1| = |S(i\omega)|^{-1}$$

We need: $|L(i\omega) + 1| > r > |\Delta_L(i\omega) L(i\omega)| \quad \forall \omega$

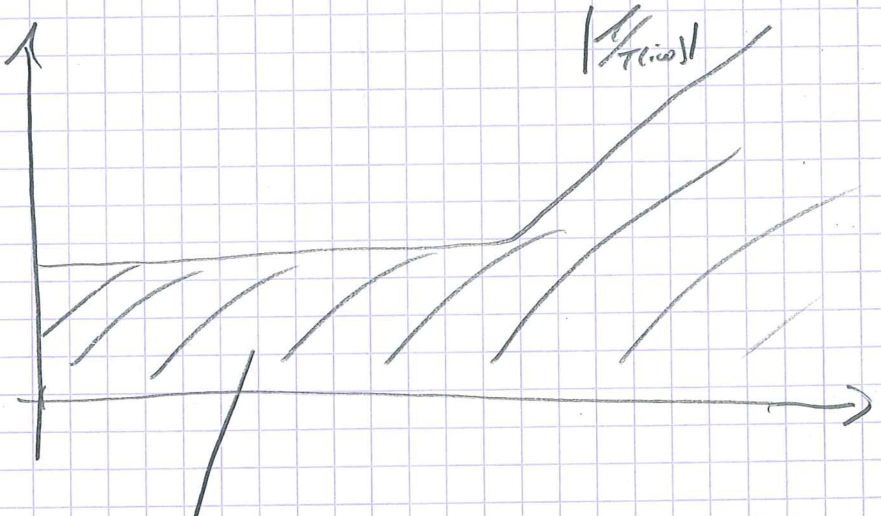
Note: scalar system!

$$\Leftrightarrow \quad \frac{|L(i\omega)+1|}{|L(i\omega)|} > |\Delta_L(i\omega)| \quad \forall \omega$$

$$\Leftrightarrow \quad \frac{1}{|T(i\omega)|} > |\Delta_L(i\omega)|$$

$\rightarrow$ this is what we already know from the basic control course!

Illustration:



$|\Delta_L(i\omega)|$ must lie below the line defined by $|1/T(i\omega)|$.

$\Rightarrow$ We will later derive these conditions (also for MIMO systems) using the small gain theorem.

$\rightarrow$ course compendium lecture 2 page 6-7!

Problem ~~7.6~~ 8.6

Modelling for small-gain analysis.

In this problem we study the use of the small gain theorem with "nonlinear" uncertainty (model error).

Given a system $G(s) = \dfrac{1}{s+\varepsilon}$ with $\varepsilon > 0$ small.

Use a P-controller with gain K. The control signal is limited to the bounded interval $[-1, 1]$. This is represented by a (nonlinear) saturation block. The closed-loop system is as shown in the block diagram.

(this is one possibility)

a) look at  as uncertainty in the model.

Show that

$$\text{sat}(x) = \begin{cases} 1 & x \geq 1 \\ x & -1 \leq x \leq 1 \\ -1 & x \leq -1 \end{cases}$$

has gain 1 and use the small gain theorem to find a condition for K that gaurantees stability of the closed-loop system. What is the largest allowed K?

b) represent the saturation as an additive uncertainty.

Find an expression for Δ in terms of u and show that its gain is equal to 1.

Use the same analysis (using small gain theorem) on the new model. What are the values of K for closed-loop stability?

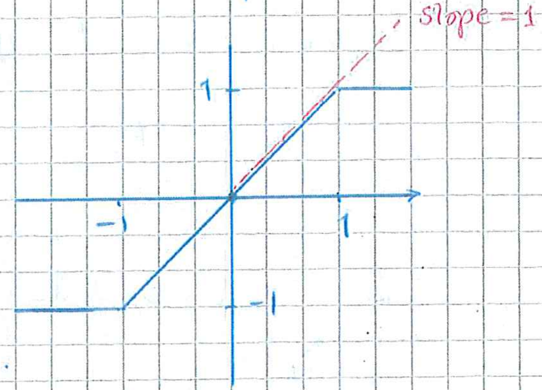
Compare (a) and (b).

Solution

a) If the systems are stable, we can apply the small gain theorem. Since $\varepsilon > 0$, we are ok!

$$sat(x) = \begin{cases} 1 & x \geq 1 \\ x & |x| < 1 \\ -1 & x \leq -1 \end{cases}$$

The plot of the nonlinearity



$$\|Sat\| := \sup_x \frac{\|Sat(x)\|_2}{\|x\|}$$

The sat. is bounded by 1.

We have:

$$|Sat(x)| \leq 1 \cdot |x| \quad \text{for all } x.$$

$\Rightarrow$ take $|x| = 1$ we achieve equality.

$\Rightarrow$ $\|Sat\| = 1$. & for any other $|x| < 1$ $|Sat(x)| < 1$

The controller

$\|K\| = |K|$ always stable. (just a gain).

$\therefore$ the small gain theorem condition is

$$\|K\| \|Sat\| \|G\| < 1.$$

but $\|G\| = \sup_\omega |G(j\omega)| = \sup_\omega \frac{1}{\sqrt{\omega^2 + \varepsilon^2}} = \frac{1}{\varepsilon}$.

Therefore, the condition is

$$|K| \frac{1}{\varepsilon} < 1 \iff |K| < \varepsilon$$

This means that the small gain theorem requires the P-controller to have a negligible gain (arbitrarily small ≈ 0) for the closed loop system to be stable.

Quite conservative!

follow ~~Q6~~ 8.6

b) We want to model the saturation as an additive uncertainty. (and therefor depends on the input).

We want to go from

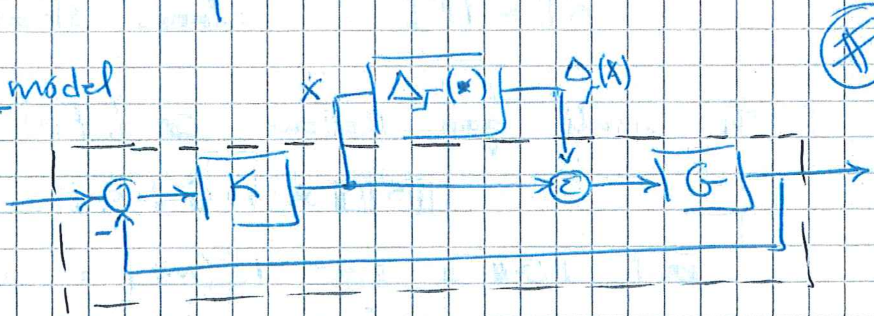
to

we need to solve for $\Delta_r(x)$:

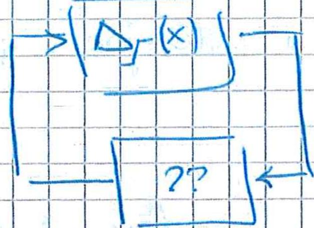
$$sat(x) = \Delta_r(x) + x$$

$$\therefore \quad \Delta_r(x) = \begin{cases} 1-x & x \geq 1 \\ 0 & |x| < 1 \\ -1-x & x \leq -1 \end{cases}$$

The new model



T.F from $\Delta_r(x)$ to X



$$X = -KG\,(X + \Delta_r(s)X)$$

$$\therefore \quad X = \frac{-KG}{1+KG}\Delta_r$$

what is the gain of Δ_r ?

$$\|\Delta_r\| = \sup_x \frac{|\Delta_r(x)|}{|x|}$$ $\Longrightarrow$ find x that maximize this quantity.

$$= \sup_x \frac{|1-x|}{|x|} = 1 \quad \text{at } x \to 0$$

For small gain theorem, we need to check the stability of $\dfrac{-KG}{1+KG}$.

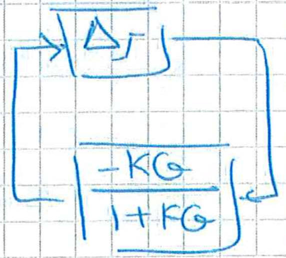
Find roots of $1+KG$; $\quad 1+K\dfrac{1}{s+\varepsilon}=0$

$\Leftrightarrow \quad s+K+\varepsilon=0$

This T.F is stable iff $K>-\varepsilon$.

The condition of the small gain theorem.

$$\left\|\frac{-KG}{1+KG}\right\|\cdot\|\Delta_r\|<1$$



$$\left\|\frac{-KG}{1+KG}\right\|=\sup_{\omega}\left|\frac{-K/s+\epsilon}{1+K/s+\epsilon}\right|$$

$$=\sup_{\omega}\left|\frac{-K}{s+\epsilon+K}\right| = \left|\frac{K}{\varepsilon+K}\right|.$$

Therefore, the closed-loop system is stable according to the small gain theorem if (see * in previous page)

$$\left|\frac{K}{K+\varepsilon}\right|\cdot 1<1 \quad \Leftrightarrow \quad |K|<|\varepsilon+K|$$

$\therefore$ we need $K>-\varepsilon$

① and with positive K we have.

$K<\varepsilon+K$ (because $\varepsilon>0$)

This is always true. this means that for $K>0$ condition automatically holds.

② for $-ve$ $\underline{K}$ $\Rightarrow$ $|K|<\varepsilon$ (Because $K>-\varepsilon$)

But also the condition says $|K|<|\varepsilon+\underset{+ve}{\overset{-ve}{K}}|$

$-K<\varepsilon+K$

$\Leftrightarrow \quad K>-\dfrac{\varepsilon}{2}$

$\therefore \quad \boxed{-\dfrac{\varepsilon}{2}<K<\infty}$

This means that K can have "any" positive value (unbounded above).

The system is stable in closed loop for all $K > 0$

Why different conclusions?

(i) in the first case, we did not consider the model of the uncertainity as additive error
so $\|sat\|$ was in the loop gain. [we only used the gain information]
we had to gaurantee that $\|G\|$ is such that the condition $\|G\| \, \|sat\| \, \|K\| < 1$.
So this had to hold for all G

(ii) in the second case, the saturation is outside the loop. and we only need to check that
The "controlled" model $\left\| \frac{-KG}{1+KG} \right\|$ is ok for the condition

$$\left\| \frac{-KG}{1+KG} \right\| \cdot \|sat\| < 1 \quad \text{to hold.}$$

So here, the feedback action of the P-controller makes that possible.

Here we used the exact description to write the T.F of the uncertainity (we used more information).