# EL2520
# Control Theory and Practice

## Lecture 8: Brief Recap and LQG

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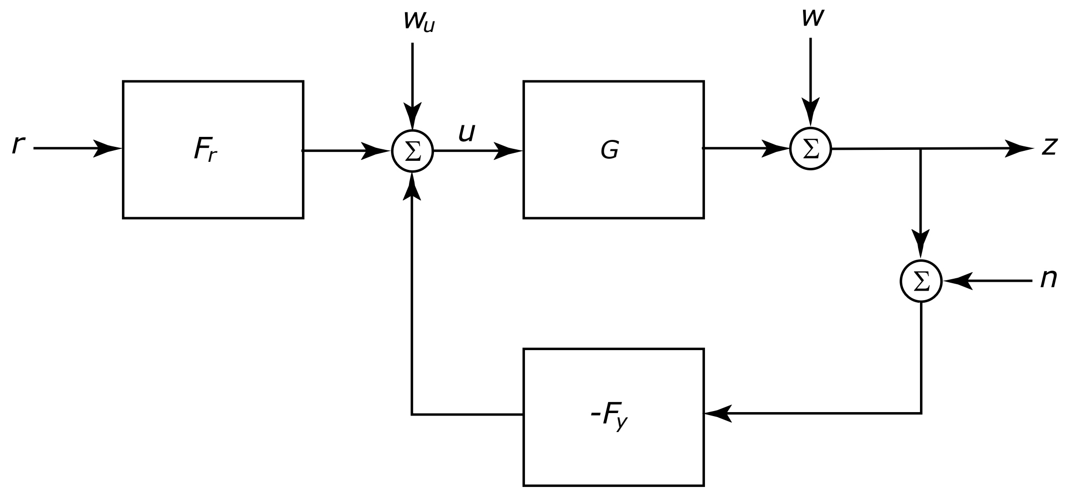
# Lecture 6 and 7




**Aim:** shape closed loop transfer-functions, e.g., $S, T, G_{wu}$ to achieve desired system properties

**How:** introduce weights $W_S, W_T, W_u$ and determine $F_y, F_r$ such that

$$\|W_S S\|_\infty < 1 \quad \|W_T T\|_\infty < 1 \quad \|W_u G_{wu}\|_\infty < 1$$

where we assume $W_S = w_S I$ etc., i.e., scalar weights.

# Selecting Weights

Weights $W_S, W_T, W_u$ should

- reflect our requirements on performance and robustness, e.g., $W_S$ large for frequencies where we need disturbance attenuation, $W_T$ large where we want noise attenuation and where model uncertainty (at output) is large.
- take into account trade-offs and limitations, e.g., S+T=I, RHP poles, RHP zeros and time delays, such that $\|\cdot\|_\infty < 1$ is feasible.

Usually a good idea to scale all signals, such that their expected / allowed magnitude is less than 1, prior to designing weights.

*Example:* if maximum expected disturbance at all frequencies is $|w| < 1$ and maximum available input is $|u| < 1$ then requirement is

$$\bar{\sigma}(G_{wu}(i\omega)) < 1 \; \forall \omega \;\; \Rightarrow \;\; W_u = 1$$

# Controller Design – H $_\infty$

Determine $F_y(s)$ to achieve $\|W_S S\|_\infty < 1$ $\|W_T T\|_\infty < 1$ $\|W_u G_{wu}\|_\infty < 1$

*1.Translate closed-loop bounds into bounds on open-loop and use* $F_y(s)$ *to shape loop-gain* $L = GF_y$

*2.Synthesis,* i.e., solve optimization problem

$$F_{y,opt}(s) = \arg\min_{F_y} \left\| \begin{matrix} W_S S \\ W_T T \\ W_u G_{wu} \end{matrix} \right\|_\infty \qquad (*)$$

Note that the if the ”stacked” objective above is less than 1, then we have achieved the three individual objectives (with some margin)

# Loop Shaping

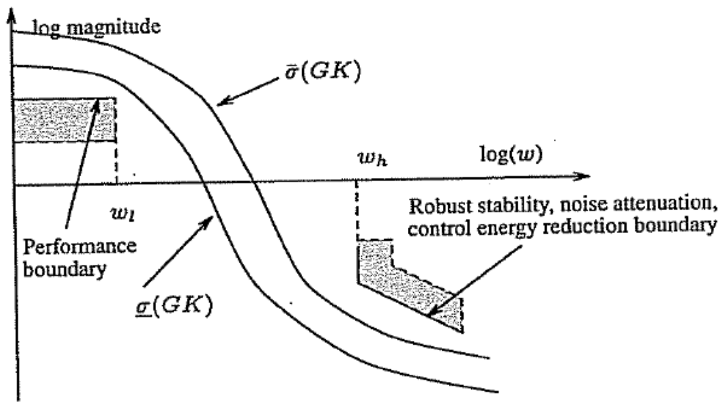
Translate bounds on $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ into bounds on $\sigma_i(L),\ L = GF_y$

•From Fan's Thm: $\underline{\sigma}(L) - 1 \leq \underline{\sigma}(I+L) \leq \underline{\sigma}(L) + 1$ and $\bar{\sigma}(A^{-1}) = 1/\underline{\sigma}(A) \Rightarrow$

$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1$$

•Then, $\underline{\sigma}(L) >> 1 \ \Rightarrow \ \bar{\sigma}(S) \approx 1/\underline{\sigma}(L)$ and we get $\underline{\sigma}(L) > |w_S|$ for frequencies where $|w_S| >> 1$

•Similarily, $\bar{\sigma}(T) \approx \bar{\sigma}(L)$ when $\bar{\sigma}(L) << 1$ and we get $\bar{\sigma}(L) < |w_T^{-1}|$ for frequencies where $|w_T| >> 1$



# $H_\infty$ Synthesis

- Given a state space system $G_0$ on the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Nw_e \\ z_e &= Mx + Du \qquad\qquad (**) \\ y &= Cx + w_e \end{aligned}$$

- Determine if a controller $u = -F_y(s)y$ exists such that for the resulting closed-loop system $G_{ec}$ and given $\gamma$

$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} < \gamma \qquad (1)$$

- Let P>0 be a solution to the Riccati equation

$$A^T P + PA + M^T M + P(\gamma^{-2} N N^T - BB^T)P = 0$$

if $A - BB^T P$ is stable then the controller exists

# The $H_\infty$–optimal controller

- A controller satisfying requirement (1) is then given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + N(y - C\hat{x}) \\ u &= -L_\infty \hat{x} \; ; \quad L_\infty = B^T P \end{aligned}$$

  i.e., state observer combined with state feedback

- The optimal controller found by iterating on $\gamma$ untill $\gamma \approx \gamma_{min}$
- To solve the original problem $(*)$ we note that with $z_e = G_{ec} w_e$

$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} < \gamma \;\Leftrightarrow\; \|G_{ec}\|_\infty < \gamma$$

- Thus, select the output $z_e$ and input $w_e$ such that

$$z_e = G_{ec} w_e = \begin{bmatrix} W_S S \\ W_T T \\ W_u G_{wu} \end{bmatrix} w_e$$

  and determine corresponding open-loop system $G_0$

# Outline of Proof

For a proof, see Lecture notes. Main idea:

- Consider function

$$V(t) = x^T(t)Px(t) + \int_0^t (z_e^T(\tau)z_e(\tau) - \gamma^2 w_e^T(\tau)w_e(\tau))d\tau$$

- If P>0 and V(t)<0 for all t, then the integral term must be negative and it follows that

$$\sup_{w_e \neq 0} \frac{\|z_e\|_2}{\|w_e\|_2} < \gamma$$

- Since V(0)=0 it is sufficient to show that $\dot{V}(t) < 0 \ \forall t$

$$\begin{aligned} \dot{V} \quad &= \quad x^T(A^TP + PA + M^TM - P(BB^T - \gamma^{-2}NN^T)P)x + (u + B^TPx)^T(u + B^TPx) \\ &\quad -\gamma^{-2}(w - \gamma^{-2}N^TPx)^T(w - \gamma^{-2}N^TPx) \end{aligned}$$

- Thus, if P>0 solves Riccati equation and we choose $U = -B^TPx$ we get the result

# Raison d'etre for H $\infty$

- Note that $H_\infty$ in some sense is a worst case approach to control, i.e., we optimize with respect to worst direction and worst frequency
- The main reason is that requirements on robust stability naturally leads to $H_\infty$-bounds since we must guarantee stability even in the worst case
- The introduction of weights still makes it relevant to employ $H_\infty$ also for performance
- Interesting fact: we formulate objectives in input-output (frequency) space, but determine optimal controller in state-space

# Today: LQG

- Formulate optimal control problem in state-space, using classical quadratic cost functions in time domain
- Result:
  - controller with similar structure as in $H_\infty$-optimal control, i.e., observer + state feedback
  - but, quadratic cost function does not allow explicit robustness consideration; robustness must be dealt with indirectly

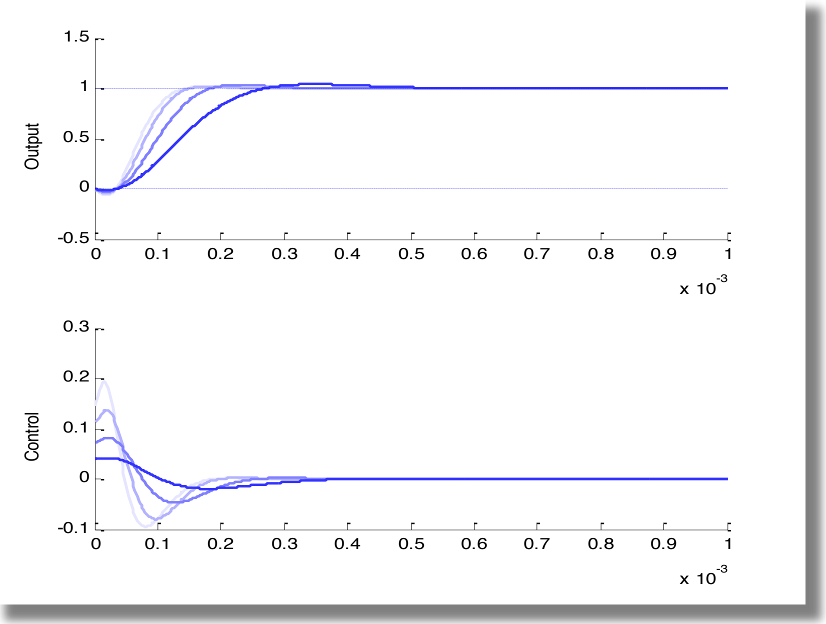
# Linear quadratic control

Compute the controller $F_y(s)$ that minimizes

$$J = \lim_{T\to\infty} \frac{1}{T} \int_0^T [z^T Q_1 z + u^T Q_2 u]\, dt$$

for given (positive definite) weight matrices $Q_1$ and $Q_2$.

Easy to influence control energy/transient performance trade-off



# Linear quadratic control

Challenge: framework developed for stochastic disturbances $v_1, v_2$

$$\begin{aligned} \dot{x} &= Ax + Bu + Nv_1 \\ y &= Cx + v_2 \\ z &= Mx \end{aligned}$$

$$J = \mathbf{E}\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T [z^T Q_1 z + u^T Q_2 u] \, dt \right\}$$

- Need to review stochastic disturbance models
- Have to skip some details
  (continuous-time stochastic processes technically tricky)

# Learning aims

After this lecture, you should be able to

- model disturbances in terms of their spectra
- use spectral factorization to re-write disturbances as filtered white noise
- compute the LQG-optimal controller (observer/controller gains)
- describe how the LQG weights qualitatively affect the time responses

Material: Lecture notes 8, course book 5.1-5.4 + 9.1-9.3 + 9.A

# Outline

- Controllability, Observability
- Modeling disturbances as filtered white noise
- Linear quadratic Gaussian (LQG) control

# State-space form

State space description of multivariable linear system

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where $x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$

- x(t) is called the *state vector*,
- systems on state-space form often written as (A,B,C,D)

Transfer matrix given by

$$G(s) = C(sI - A)^{-1}B + D$$

# Controllability

The *state* $\tilde{x}$ is *controllable* if, given x(0)=0, there exists u(t) such that x(t)= $\tilde{x}$ for some t< $\infty$

The *system* is *controllable* if all $\tilde{x}$ are controllable.

The *controllability matrix*

$$S(A,B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times mn}$$

- Controllable states $\tilde{x}$ can be written as $\tilde{x} = S(A,B)\alpha$ some $\alpha$ in $R^{mn}$
- System is controllable if S(A,B) has full rank
  (i.e., for each x there exists $\alpha$ such that x=S(A,B)$\alpha$)

# Observability

The *state* $\tilde{x} \neq 0$ is *<u>un</u>observable* if $x(0) = \tilde{x}$ and u(t)=0 for t>0 implies that y(t)=0 for t ≥0.
The *system* is *observable* if no states are unobservable

The *observability matrix*

$$O(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{pm \times n}$$

- Unobservable states $\tilde{x}$ are solutions to $O(A, C)\tilde{x} = 0$
- System is observable if O(A,C) has full rank (i.e., only $\tilde{x} = 0$ solves $O(A, C)\tilde{x} = 0$)

# Modifying dynamics via state feedback

Open loop system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

can be controlled using state feedback u(t)=-Lx(t) ➔

$$\frac{d}{dt}x(t) = (A - BL)x(t)$$

**Q:** can we choose L so that A-BL gets arbitrary eigenvalues?

**A:** if and only if the system is controllable.

# Observers

The state vector of the system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

can be estimated using an observer

$$\frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) + Bu(t) + K(y(t) - C\widehat{x}) \Rightarrow$$
$$\frac{d}{dt}\tilde{x}(t) = (A - KC)\tilde{x}(t) \qquad \text{where } \tilde{x}(t) = x(t) - \widehat{x}(t)$$

**Q:** can we choose K so  A-KC gets arbitrary eigenvalues?

**A:** if and only if system is observable

# Stabilizability and detectability

Control objective concerns only outputs z of system, i.e., controllability and observability of all states may not be so important.

**Exception:** must be able to control and observe unstable modes!

A system (A,B) is *stabilizable* if there exists L so that A-BL is stable

A system (A,C) is *detectable* if there exists K so that A-KC is stable

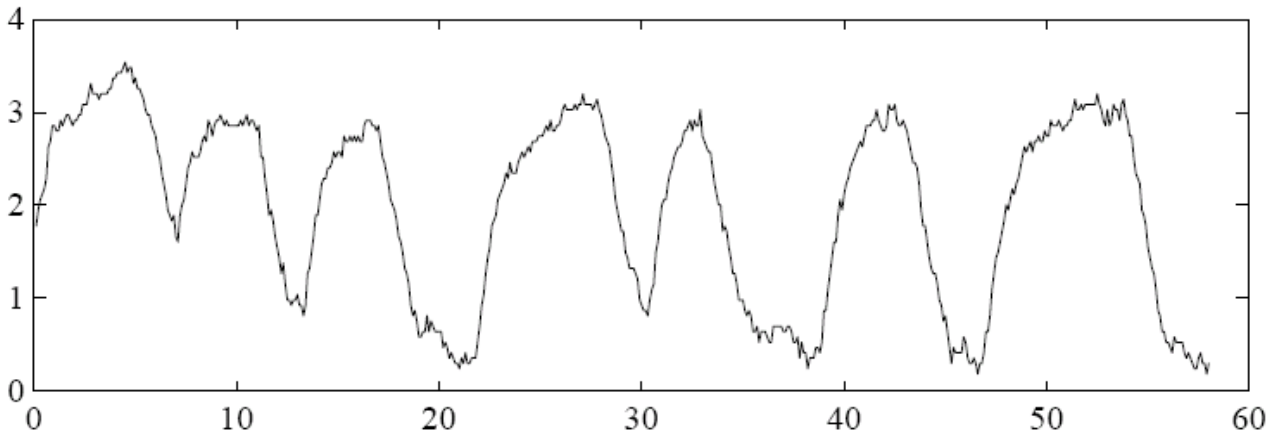
# Today’s lecture

- Controllability, Observability
- Modeling disturbances as filtered white noise
- Linear quadratic Gaussian (LQG) control

# Disturbances

Disturbances model a wide range of phenomena that are not easily described in more detail, e.g.

- load variations, measurement noise, process variations, …




Important to model

- size, frequency content and correlations between disturbances.

# Signal sizes

So far in the course, we have used the 2-norm

$$\|z\|_2^2 = \int_0^\infty |z(t)|^2 \, dt$$

If the integral does not converge, we can use

$$\|z\|_e^2 = \lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} |z(t)|^2 \, dt$$

Only measures size of signal, i.e., not frequency etc

# More informative measure

How is z(t) related to z(t-τ)? One measure is

$$r_z(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T z(t) z(t-\tau)\, dt$$

For ergodic stochastic processes, we have

$$r_z(\tau) = \mathbf{E} z(t) z(t-\tau)$$

(i.e., the covariance function of the signal)

# Vector valued signals

For vector-valued z, coupling between $z_i$ and $z_j$ can be described by

$$r_{ij}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T z_i(t) z_j(t - \tau)\, dt$$

Can be combined into matrix

$$R_z(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T z(t) z^T(t - \tau)\, dt$$

For ergodic stochastic processes, we get

$$R_z(\tau) = \mathbf{E}\, z(t) z^T(t - \tau)$$

i.e., the covariance matrix for z

# Signal spectra

Translating the signal measure to the frequency domain

$$\Phi_z(\omega) = \int_{-\infty}^{\infty} R_z(\tau) e^{-i\omega\tau}\, d\tau$$

$\Phi_z(\omega)$ is called the *spectrum* of z

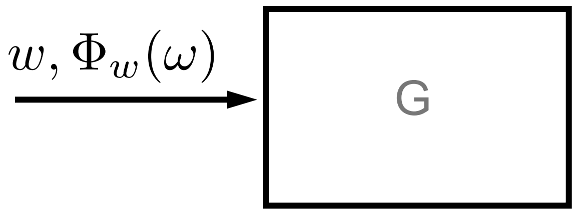
Interpretation

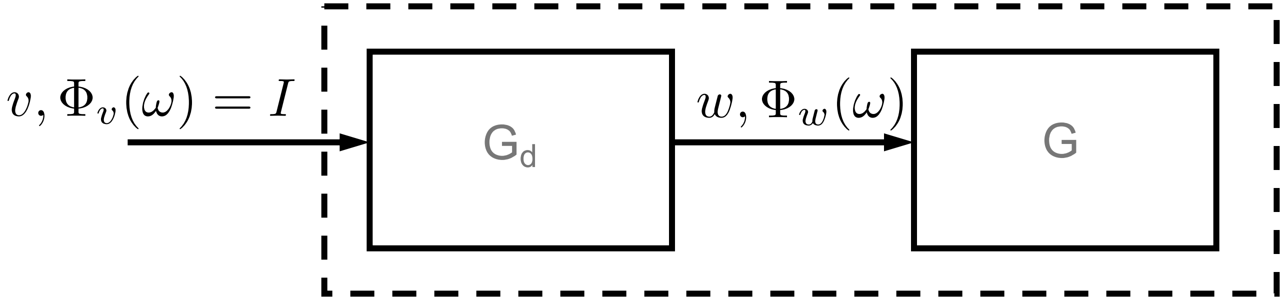
- $[\Phi_z(\omega)]_{ii}$ measures the energy content of $z_i$ at frequency $\omega$
- $[\Phi_z(\omega)]_{ij}$ measures coupling of $z_i$ and $z_j$ at frequency $\omega$
- $[\Phi_z(\omega)]_{ij} = 0$ implies that $z_i$ and $z_j$ are uncorrelated

A signal with $\Phi_z$ constant for all $\omega$ is called *white noise,*
(in this case, we call $\Phi_z$ the *covariance matrix* of z)

# Disturbances as filtered white noise

**Fact (spectral factorization):** any spectrum $\Phi(\omega)$ which is rational in $\omega^2$, can be represented as white noise filtered through a stable non-minimum phase linear system.




(see course book Theorem 5.1 for a precise statement)

# State-space model with disturbances

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nw_1(t)$$
$$y(t) = Cx(t) + Du(t) + w_2(t)$$

If disturbances $w_1$ and $w_2$ are not white, but have spectra that can be obtained via $w_i = G_i v_i$ where $v_i$ is white noise, then we can re-write system as

$$\frac{d}{dt}\overline{x}(t) = \overline{A}\overline{x}(t) + \overline{B}u(t) + \overline{N}v_1(t)$$
$$y(t) = \overline{C}\overline{x}(t) + Du(t) + v_2(t)$$

**Note:** $\overline{x}$ is x augmented with the states from $G_1$, $G_2$;

$\overline{A}, \overline{B}, \ldots$ are A, B,… augmented with state-space descriptions of $G_i$

# Today’s lecture

- Recap: State-space represenation, state feedback and observers
- Recap: Modeling disturbances as filtered white noise
- Linear quadratic Gaussian (LQG) control

# Linear quadratic Gaussian control

Model: linear system with white noise

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nv_1(t)$$
$$y(t) = Cx(t) + v_2(t)$$
$$z(t) = Mx(t)$$

where $v_1$, $v_2$ are white noise with

$$\mathrm{cov}([v_1,\ v_2]) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Objective: minimize effect of v on z, punish control cost

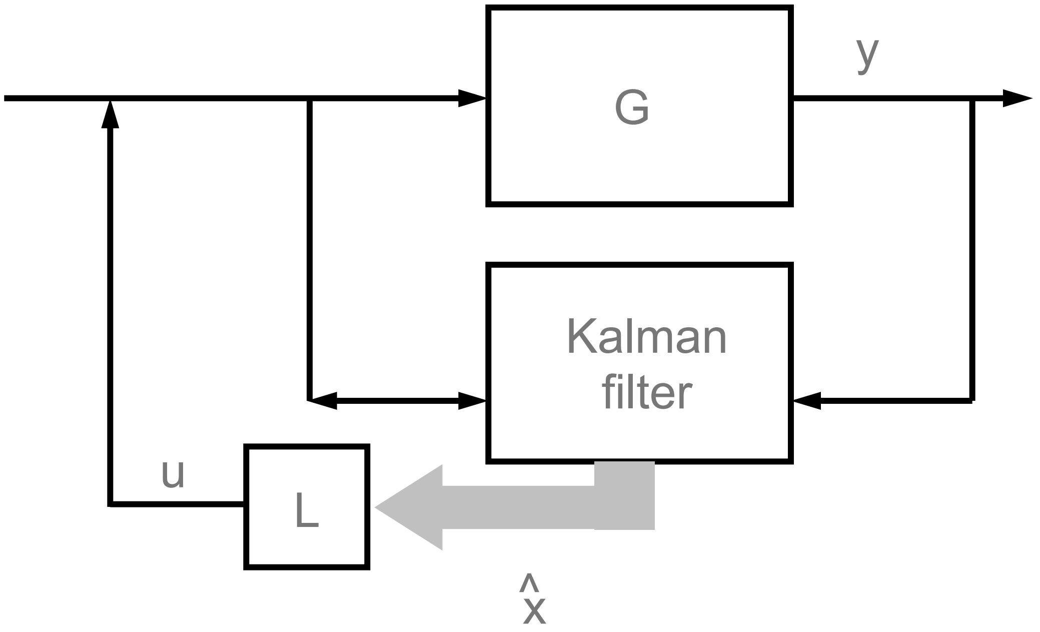
$$J = \mathbf{E}\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T [z^T Q_1 z + u^T Q_2 u]\, dt\right\}$$

LQG: Linear system, Quadratic cost, Gaussian noise

# Solution structure

Optimal solution satisfies *separation principle*, composed of

- Optimal linear state feedback (Linear-quadratic regulator)
- Optimal observer (Kalman filter)



# The LQR Problem

Disturbance and noise free system

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t);\ \ x(0) = x_0 \\ z(t) &= Mx(t) \end{aligned}$$

LQ problem

$$\min_{u} \lim_{T\to\infty} \frac{1}{T} \int_0^T z^T Q_1 z + u^T Q_2 u \ dt$$

Optimal input given by state feedback

$$u(t) = -Lx(t) = -Q_2^{-1} B^T S x(t)$$

where S>0 is the solution to the algebraic Riccati equation

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0$$

# Example: LQR for scalar system

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t), \qquad y(t) = x(t)$$

with cost

$$J = \int_0^\infty [x^2 + \rho u^2] \, dt$$

Riccati equation

$$2as + 1 - s^2/\rho = 0$$

has solutions

$$s = a\rho \pm \sqrt{(\rho a)^2 + \rho}$$

so the optimal feedback law is

$$u = -(s/\rho)x = -(a + \sqrt{a^2 + 1/\rho})x$$

# Example: LQR for scalar system

Closed loop system

$$\dot{x}(t) = -(\sqrt{a^2 + 1/\rho})x(t)$$

- If $\rho \to 0$, closed loop bandwidth is approx. $1/\sqrt{\rho}$
- If $\rho \to \infty$
  - and a<0 $\Rightarrow u = 0 \cdot x$
  - and a>0 $\Rightarrow u = -2ax$ which gives $\dot{x} = -ax$, i.e., pole is mirrored about imaginary axis

# The Kalman Filter

- System with disturbance/noise

$$\begin{aligned} \dot{x} &= Ax + Bu + Nv_1 \ , \quad E\{v_1 v_1^T\} = R_1 \\ y &= Cx + v_2 \ , \quad E\{v_2 v_2^T\} = R_2 \end{aligned}$$

- Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x})$$

- Determine the $K_f$ that minimizes square of estimation error

$$E\{(x - \hat{x})^T (x - \hat{x})\}$$

- Solution: $K_f = PC^T R_2^{-1}$, where P>0 solves Riccati eq.

$$PA^T + AP - PC^T R_2^{-1} CP + NR_1 N^T = 0$$

- The optimal observer is called the Kalman filter

# Example: Scalar system Kalman filter

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t) + v_1(t), \qquad y(t) = x(t) + v_2(t)$$

with covariances $E\{v_1^2\}=R_1$, $E\{v_2^2\}=R_2$, $E\{v_1 v_2\}=0$.

Riccati equation

$$2ap + r_1 - p^2/r_2 = 0$$

gives

$$k = a + \sqrt{a^2 + r_1/r_2}$$

and estimation error dynamics

$$\frac{d}{dt}\tilde{x}(t) = -(\sqrt{a^2 + r_1/r_2})\tilde{x}$$

**Interpretation:** measurements discarded if too noisy.

# The LQG Controller

- The LQG controller that minimizes the objective function

$$J = \mathbf{E}\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T [z^T Q_1 z + u^T Q_2 u]\, dt\right\}$$

is given by the Separation Principle, i.e., LQR + Kalman filter

$$\begin{aligned} u &= -L\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + Bu + K_f(y - C\hat{x}) \end{aligned}$$

# LQG and loop shaping

LQG: simple to trade-off response-time vs. control effort

- but what about sensitivity and robustness?

These aspects can be to some extent be accounted for using the noise models

- Sensitivity function S: transfer matrix $w_u \rightarrow z$
- Complementary sensitvity T: transfer matrix $n \rightarrow z$

**Example:** S forced to be small at low frequencies by letting (some component of) $w_1$ affect the output of the system, and let $w_1$ have large energy at low frequencies,

$$w_1(t) = \frac{1}{p+\delta} v_1(t)$$

(delta small, strictly positive, to ensure stabilizability)

# LQG Control: pros and cons

Pros:

- Simple to trade off response time vs. control effort
- Applies to multivariable systems

Cons:

- Often hard to see connection between weight matrices $Q_1$, $Q_2$, $R_1$, $R_2$ and desired system properties (e.g. sensitivity, robustness, etc)
- In practice, iterative process in which $Q_1$ and $Q_2$ are adjusted until closed loop system behaves as desired
- Poor robustness properties in general

# Summary

- State-space theory recap:
  - Controllability, observability, stabilizability, detectability
  - State feedback and observers
- Modeling disturbances as white noise
  - Mean, covariance, spectrum
  - Spectral factorization: disturbances as filtered white noise
- Linear-quadratic controller
  - Kalman-filter + state feedback
  - Obtained by solving Riccati equations
  - Focuses on time-responses
  - Loop shaping and robustness less direct