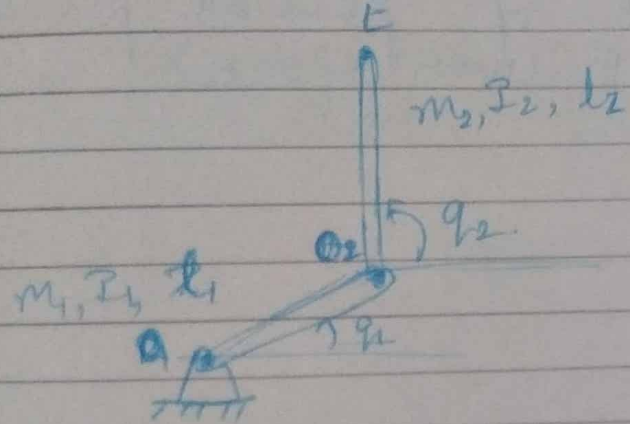
R - revolute
E - end ~~of~~ effector.

2-R Manipulator (Elbow Manipulator)



$q_1, q_2 \rightarrow$ absolute angles

→ Motors connected at both joints.
→ We can control either torques (τ_1 & τ_2) or angles (q_1 and q_2)

Problem Statement

→ Consider 4 tasks

Task 1) Given arbitary trajectory of E. (give x, y as functⁿ of time) make it follow the trajectory

Task 2) Given locatⁿ of wall, make the robot touch the wall & apply const. force against it.

Task 3) Make the robot behave like a virtual spring connected to a fictitious point. (x_0, y_0)

Task 4) Given any mechanical constraints on the angles, determine range of possible positions of E (workspace).

Task 1)

$x = l_1 \cos q_1 + l_2 \cos q_2$
$y = l_1 \sin q_1 + l_2 \sin q_2$

$\dot{x} = -l_1 \sin q_1 \dot{q}_1 - l_2 \sin q_2 \cdot \dot{q}_2$
$\dot{y} = l_1 \cos q_1 \dot{q}_1 + l_2 \cos q_2 \dot{q}_2$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin q_1 & -l_2 \sin q_2 \\ l_1 \cos q_1 & l_2 \cos q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

→ Reverse relations also required.

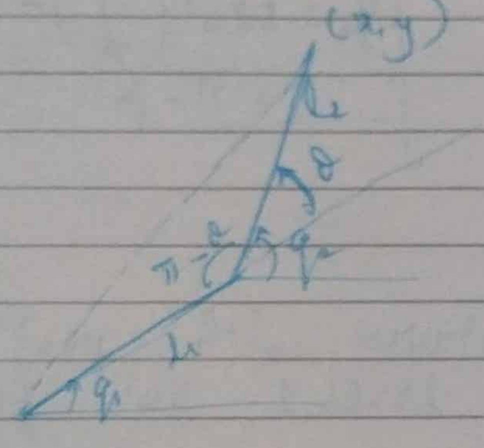
$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{to}} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

→ 2 options.

1) Solve algebrically (require computat^n al power)

2) Derive close form ~~eqn~~ expression

→ hard in general
→ multiple sol^ns

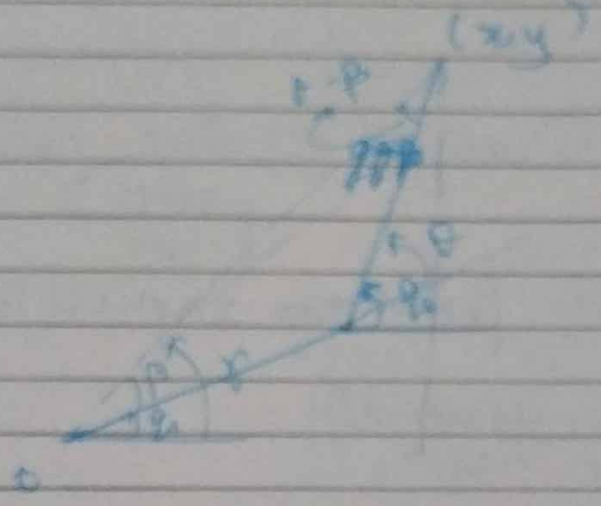


$$\cancel{\cos(\pi - \theta) = \frac{l_1^2 + l_2^2 - (x^2 + y^2)}{2 l_1 l_2}}$$

$$\cancel{\theta = \pi - \cos^{-1}\left(\frac{l_1^2 + l_2^2 - (x^2 + y^2)}{2 l_1 l_2}\right)}$$

$$\frac{(x^2 + y^2) - l_1^2 - l_2^2}{2 l_1 l_2} = + \cos(\theta) \quad \text{(cosine rule)}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)}$$



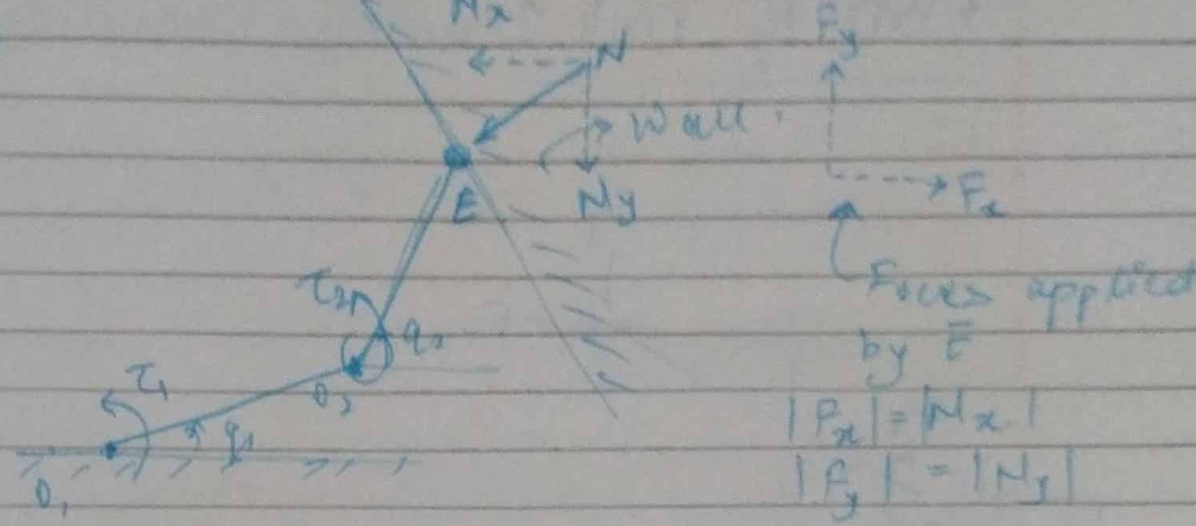
$$\tan\gamma = \frac{y}{x}$$

$$q_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left[\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right] = \gamma - \beta$$

$$q_2 = q_1 + \theta$$

→ Control the motors with position control and you get level 1 soln for task 1.

Task 2



$|F_x| = |N_x|$

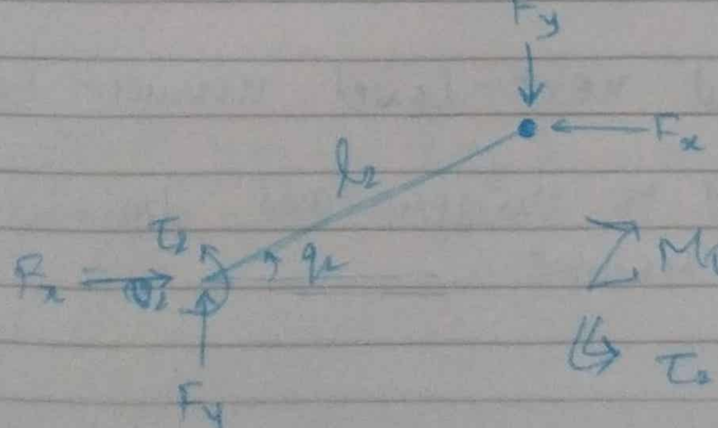
$|F_y| = |N_y|$

Static Equilibrium Eq"s

$\sum M_{O_1} = 0 \qquad \sum M_{O_2}$

~~FBD of~~

FBD of link 2

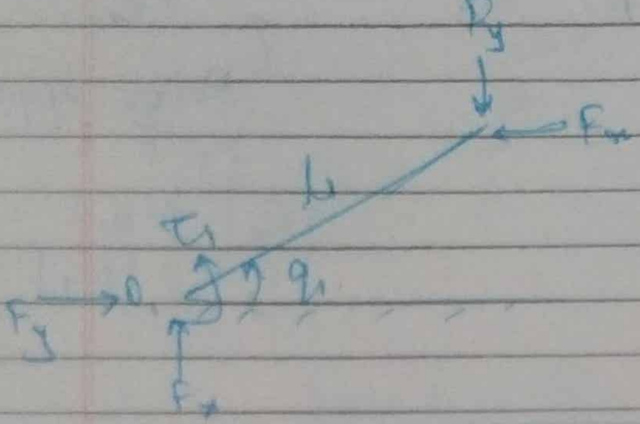


$\sum M_{O_2} = 0$

$\Rightarrow \tau_2 + F_x l_2 \sin q_2 - F_y l_2 \cos q_2 = 0$

$$\boxed{\tau_2 = F_y l_2 \cos q_2 - F_x l_2 \sin q_2} \quad -(3)$$

FBD of link 1



$$\boxed{\tau_1 = F_y l_1 \sin q_1 - F_x l_1 \cos q_1} \quad (4)$$

→ Using Torque controlled motors, τ_1, τ_2 answer task 2.

→ Reach the wall using positn control, use torque control to apply force

⟹ For T_3 and next-level answer to T_1

↳ need to understand dynamics of robot.

→ In the above derivation, we assumed the forces applied by the end effector along +ve axes.

→ However, to get a more general code (where ~~forces~~ external forces on the end effector are taken as input), we can derive the same relations (but with reversed sign of F_x and F_y

→ Thus, we get

$$\left.\begin{aligned} \tau_2 &= + F_x l_2 \sin q_2 - F_y l_2 \cos q_2 \\ \cancel{\tau_1} &\cancel{= + F_x l_1 \cos} \\ \tau_1 &= + F_x l_1 \sin q_1 - F_y l_1 \cos q_1 . \end{aligned}\right\} (3), (4)$$

→ These are the 2 equations used in the code.

Langragian Eqtions

$$\mathcal{L} = K - V$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i'}$$

Q_i' → are generalised forces derived using principle of virtual work.

$$K = \frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{q}_1^2 + \frac{1}{2} m_2 v_{C_2}^2$$

$$+ \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right)\dot{q}_2^2$$

$$v_{C_2}^2 = (l_1 \dot{q}_1)^2 + \left(\frac{l_2}{2}\dot{q}_2\right)^2 + 2 l_1 \dot{q}_1 \frac{l_2}{2}\dot{q}_2 \cos(q_2 - q_1)$$

$$V = m_1 g \frac{l_1}{2} s q_1 + m_2 g \left(l_1 \sin q_1 + \frac{l_2}{2} \sin q_2\right)$$

$$U = m_1 g \frac{l_1}{2} \sin q_1 + m_2 g \left(l_1 \sin q_1 + \frac{l_2}{2} \sin q_2 \right)$$

$$U = \frac{1}{2} m_1 g l_1 \sin q_1 + m_2 g l_1 \sin q_1 + \frac{1}{2} m_2 g l_2 \sin q_2$$

$$\mathcal{L} = K - U$$

$$= \frac{1}{6} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{6} m_2 l_2^2 \dot{q}_2^2 + \frac{1}{2} m_2 l_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1) - \frac{1}{2} m_1 g l_1 \sin q_1 - m_2 g l_1 \sin q_1 - \frac{1}{2} m_2 g l_2 \sin q_2$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = + \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1) - \frac{1}{2} m_1 g l_1 \cos q_1 - m_2 g l_1 \cos q_1$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = - \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1) - \frac{1}{2} m_2 g l_2 \cos q_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{1}{3} m_1 l_1^2 \dot{q}_1 + m_2 l_1^2 \dot{q}_1 + \frac{1}{2} m_2 l_1 l_2 \dot{q}_2 \cos(q_2 - q_1)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \frac{1}{3} m_2 l_2^2 \dot{q}_2 + \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \cos(q_2 - q_1)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) = \frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + \frac{1}{2} m_2 l_1 l_2 \cos(q_2 - q_1) \ddot{q}_2 - \frac{1}{2} m_2 l_1 l_2 \dot{q}_2 \sin(q_2 - q_1)(\dot{q}_2 - \dot{q}_1)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) = \frac{1}{3} m_2 l_2^2 \ddot{q}_2 + \frac{1}{2} m_2 l_1 l_2 \cos(q_2 - q_1) \ddot{q}_1 - \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \sin(q_2 - q_1)(\dot{q}_2 - \dot{q}_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \tau_1$$

$$\frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1$$

$$+ \frac{1}{2} m_2 l_1 l_2 \cos(q_1 - q_1)\, \ddot{q}_2$$

$$- \frac{1}{2} m_2 l_1 l_2 \dot{q}_2 \sin(q_2 - q_1)(\dot{q}_2 - \dot{q}_1)$$

$$- \Big(\frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1)$$

$$- \frac{1}{2} m_1 g l_1 \cos q_1 - m_2 g l_1 \cos q_1 \Big) = \tau_1$$

$$\boxed{\begin{aligned} &\frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + \frac{1}{2} m_2 l_1 l_2 \cos(q_2 - q_1)\, \ddot{q}_2 \\ &- \frac{1}{2} m_2 l_1 l_2 \sin(q_2 - q_1)\, \dot{q}_2^2 \\ &+ \frac{1}{2} m_1 g l_1 \cos q_1 + m_2 g l_1 \cos q_1 = \tau_1 \end{aligned}}$$

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$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = \tau_2$$

$$\frac{1}{3} m_2 l_2^2 \ddot{q}_2 + \frac{1}{2} m_2 l_1 l_2 \cos(q_2 - q_1)\ddot{q}_1$$

$$- \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \sin(q_2 - q_1)(\dot{q}_2 - \dot{q}_1)$$

$$- \Big(- \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1)$$

$$- \frac{1}{2} m_2 g l_2 \cos q_2 \Big) = \tau_2$$

$$\boxed{\frac{1}{3} m_2 l_2^2 \ddot{q}_2 + \frac{1}{2} m_2 l_1 l_2 \cos(q_2 - q_1)\ddot{q}_1 + \frac{1}{2} m_2 l_1 l_2 \sin(q_2 - q_1)\dot{q}_1^2 + \frac{1}{2} m_2 g l_2 \cos q_2 = \tau_2} \quad (7)$$

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→ Note that (3) & (4) is true for any F_x, F_y

For task (3),

~~$F_x = k($~~

$F_x = kx$
$F_y = ky$
$\xrightarrow[\text{generally}]{\text{more}}$
$F_x = k(x - x_0)$
$F_y = k(y - y_0)$

Substituting x & y,

$F_x = k(l_1 cq_1 + l_2 cq_2)$
$F_y = k(l_1 sq_1 + l_2 sq_2)$

From (3) & (4),

$$k(l_1 sq_1 + l_2 sq_2) l_2 cq_2 - k(l_1 cq_1 + l_2 cq_2) l_2 sq_2 = \tau_{2s}$$
$$k(l_1 sq_1 + l_2 sq_2) l_1 cq_1 - k(l_1 cq_1 + l_2 cq_2) l_1 sq_1 = \tau_{1s}$$

(8)

Set the motor torque to be

$\tau_1 + \tau_{1s}$ & $\tau_2 + \tau_{2s}$,

Answer to task (3)

SHOT ON MI A2