We use $d \mid n$ to mean d divides n

i.e. $n = d \cdot n'$ where n' is a natural number.

or $\frac{n}{d}$ is a natural number

1) Define LCM of a, b

A number l s.t $a \mid l$, $b \mid l$

and for all l' s.t $a \mid l'$ and $b \mid l'$

we have $l \leq l'$

2) Proving $l = \frac{ab}{g}$ $\quad l = \text{lcm}(a,b)$, $g = \gcd(a,b)$

(Don't read the whole solution at once. Try and complete yourself after reading each line.)

We will prove $lg = ab$ in two parts $l \cdot g \leq ab$ and $lg \geq ab$

$lg \leq ab$ can be proved as

$$l \leq \frac{ab}{g} = \left(\frac{a}{g}\right)b = a\left(\frac{b}{g}\right)$$

Since $g|a$ and $g|b$, the above line shows $a \mid \frac{ab}{g}$ and $b \mid \frac{ab}{g}$ $\therefore$ $l \leq \frac{ab}{g}$

Now for $lg \geq ab$, we prove $g \geq \frac{ab}{l}$

Note $\frac{a}{(ab/l)} = \frac{l}{b}$. Since $b \mid l$ this is a natural.

So $\frac{ab}{l} \mid a$. Similarly $\frac{ab}{l} \mid b$

$\therefore g \geq \frac{ab}{l}$

Note I chose to prove $\leq$ with lcm on lhs and $\geq$ with gcd on lhs to use definitins.

3) $a \cdot b$ could overflow.

We can compute as $\operatorname{lcm}(a,b) = \frac{a}{g} \cdot b$

Note: $\frac{a}{g}$ is a natural always since $g \mid a$.

This causes fewer overflows.

Q 5 Write Programs to implement your ideas. Measure running time on randomly generated i/ps. How does running time compare to repeatedly using gcd(a,b)

6) <u>Base cases</u>

$F_0 = 0 \leq 1.7^0 = 1$
$F_1 = 1 \leq 1.7^1 = 1.7$ } Have to consider two base cases.

<u>Induction</u>

Hypothesis: $F_i \leq 1.7^i$
for all $i \leq n+1$

$$F_{n+2} = F_{n+1} + F_n \underbrace{\leq}_{\text{apply Hypothesis}} 1.7^{n+1} + 1.7^n = 1.7^n(2.7)$$

$$\leq 1.7^n \cdot (1.7)^2$$

$$= 1.7^{n+2}$$

7) Statement is not true for all n.

$$F_0 = 0 \not\geq \frac{1}{2}(1.6)^0 = \frac{1}{2}$$

But for $n \geq 10$ statement is true

Take base cases

$n = 10$ $\quad F_{10} = 55 \geq \frac{1}{2} \cdot (1.6)^{10} = 54.97\ldots$

$n = 11$ $\quad F_{11} = 89 \geq \frac{1}{2}(1.6)^{11} = 87.96\ldots$

Por Induction Hypothesis

$F_i \geq \frac{1}{2}(1.6)^i$ for $10 \leq i < n+2$

$n \geq 12$ $\quad F_{n+2} = F_{n+1} + F_n \geq \frac{1}{2}(1.6)^{n+1} + \frac{1}{2}(1.6)^n$

$= \frac{1}{2}(1.6)^n(2.6) \geq \frac{1}{2}(1.6)^{n+2}$ $\quad 1.6^2 = 2.56$

So for "large" n $\quad \frac{1}{2}(1.6)^n \leq F_n \leq (1.7)^n$

What is the right base c s.t.

$$F_n \sim c^n \quad ?$$

d) $t_1(n) = t_1(n-1) + 3$

$t_1(0) = 5$

"unroll" and observe

$t_1(1) = t_1(0) + 3 = 5 + 3$

$t_1(2) = t_1(1) + 3 = (5 + 3) + 3$

$t_1(3) = t_1(2) + 3 = ((5+3) + 3) + 3$

$\vdots$

Conjecture from observing

$$t_1(n) = 5 + 3 \cdot n$$

Prove by Induction on n

$t_1(0) = 5 + 3 \cdot 0 = 5$ ✓

$t_1(n) = t_1(n-1) + 3 \underbrace{=}_{\text{By hypothesis}} 5 + 3(n-1) + 3 = 5 + 3n$ ✓

9) Similar to 8.

Need base cases for $t_2(i)$ $0 \leq i < a$ for t_2 to be well-defined.

10) Try with $\beta = 1$ $\alpha = 1/2$

and $\beta = 1$. $\alpha = 1/3$

Generalize.

Base cases needed will depend on α but not β.

11) Similar to 10.

For questions 9-11, try to visualize for various parameters. What is common?.