www.top-one-percent.com | info@top-one-percent.com | +91-97395-61394

By Sandeep Gupta | GMAT 800/800, Harvard Final Admit

Quant Session: Statistics + Numbers

Part 1: Statistics

Mean (Average)

1. Average or mean or AM = Sum of n quantities (or numbers) / number of them (n) OR Arithmetic Mean (A.M.) is given by $\overline{X} = \Sigma x / N$.
2. Mean of the Combined Series If N_1 and N_2 are the sizes and M_1 and M_2 are the respective means of two series then the mean M of the combined series is given by $M = \dfrac{M_1 N_1 + M_2 N_2}{N_1 + N_2}$ or we can write:

   $\dfrac{N_1}{N_2} = \dfrac{M_2 - M}{M - M_1} = \dfrac{D_2}{D_1}$ ... **This is the most important result in Mean.**

3. If a man (or train or boat or bus) covers some journey from A to B at X km/hr (or m/sec) and returns to A at a uniform speed for Y km/hr, then the average speed during the whole journey is [2XY / (X + Y)] km/hr. **TIP**: The average speed in such a case will be a bit less than the simple average.
4. The sum of first "n" natural numbers is given by n (n + 1)/2.
5. For consecutive integers or for equally spaced numbers (AP), Mean = (First term + Last term) / 2.
6. If the average of a few consecutive integers is 0, then there will be an odd number of integers.
7. The average of an odd number of consecutive integers is an integer and the average of an even number of consecutive integers is a non–integer.
8. If in a set of numbers, the average = the highest or the lowest number, all the numbers will have to be equal.

Median:

- Median is the middle value or the average of two middle values when the values are arranged in an order, either ascending or descending.
- If there are odd number of observations, median is directly the middle number.
- If there is an even number of observations, median is the average of the two middle numbers.

- For consecutive integers or for equally spaced numbers (AP),
  Median = (First term + Last term) / 2. So, Median = Mean in this case.
- Median is the 50th percentile.

Median of a continuous series

$$M_m = l + \left(\frac{\frac{n}{2} - cf}{f}\right)h$$

Where
l = lower limit of median class,
n = number of observations,
cf = cumulative frequency of class preceding the median class,
f = frequency of median class,
h = class size (assuming class size to be equal)

Example:

The number of students absent in a school was recorded every day for 147 days and the raw data was presented in the form of the following frequency table.

No. of students absent :	5	6	7	8	9	10	11	12	13	15	18	20
No. of days :	1	5	11	14	16	13	10	70	4	1	1	1

Obtain the median and describe what information it conveys.

Explanation:

x_i	f_i	CF
5	1	1
6	5	6
7	11	17
8	14	31
9	16	47
10	13	60
11	10	70
12	70	140
13	4	144
15	1	145
18	1	146
20	1	147

We have N = 147. So N/2 = 147/2 = 73.5. The cumulative frequency just greater than N/2 is 140 and the corresponding value of x is 12. Hence, median = 12. This means that for about half the number of days, more than 12 students were absent.

Range + Standard Deviation

- Range is defined as the difference between the two extreme observations of the distribution.
- Range = $X_{max} - X_{min}$. If Range = 0, all the observations are equal. Range ≥ 0 always; it is never negative.
- Standard deviation is defined as positive square root of the A.M. of the squares of the deviations of the given observations from their A.M.
- If $X_1, X_2 \ldots X_N$ is a set of N observations then its standard deviation is given by Standard Deviation $= \sqrt{\frac{\sum(x-\bar{x})^2}{N}}$
- It is a measure of how much each value varies from the mean of all the values.
- Less SD implies more consistency, less variation, less spread, more compactness AND vice versa.
- If SD = 0, all the observations are equal.
- Range is always greater than SD, except when all observations are equal, when both are equal to 0.
    - To be precise $\boldsymbol{SD \leq \frac{Range}{2}}$
- The square of SD is called Variance.

Change in respective statistical parameters:

	Addition	**Subtraction**	**Sign Change**	**Multiplication**	**Division**
Mean	Change	Change	Change	Change	Change
Median	Change	Change	Change	Change	Change
Range	NO Change	NO Change	NO Change	Change	Change
SD	NO Change	NO Change	NO Change	Change	Change

Part 2: Numbers

For the purpose of the GMAT, all numbers are real.

REAL numbers are basically of two types:

1. **Rational numbers:** A rational number can always be represented by a fraction of the form p/q where p and q are integers and q ≠ 0. Examples: finite decimal numbers, infinite repeating decimals, whole numbers, integers, fractions i.e. 3/5, 16/9, 2, 0.666.... ∞ = 2/3 etc.

2. **Irrational numbers:** Any number which cannot be represented in the form p/q where p and q are integers and q ≠ 0 is an irrational number. AN INFINITE NON–RECURRING DECIMAL IS AN IRRATIONAL NUMBER. Examples – √2, π, √5, √7.

INTEGERS: The set of Integers I = {0, ±1, ±2, ±3, … ∞}

EVEN NUMBERS: The numbers divisible by 2 are even numbers. E.g., 0, ±2, ±4, ±6, ±8, ±10....... Even numbers are expressible in the form 2n where n is an integer. **Thus –2, –6 etc. are also even numbers.** *Remember that '0' is an even number.*

ODD NUMBERS: The numbers not divisible by 2 are odd numbers e.g. ±1, ±3, ±5, ±7, ±9.... Odd numbers are expressible in the form $(2n + 1)$ where n is an integer other than zero (not necessarily prime). Thus, –1, –3, –9 etc. are all odd numbers.

You must remember:

Even ± Even = Even	Even ± Odd = Odd	Odd ± Odd = Even	Odd ± Even = Odd
Even × Even = Even	Even × Odd = Even	Odd × Odd = Odd	

POSITIVE INTEGERS: The numbers 1, 2, 3, 4, 5.... are known as positive integers.

- 0 is neither positive nor negative.
- 0 is an even number.
- 0 is not a factor of any integer.
- 0 is a multiple of all integers.

Prime numbers: A natural number which has no other factors besides itself and unity is a prime number. Examples: 2, 3, 5, 7, 11, 13, 17, 19 ......

- If a number has no factor equal to or less than its square root, then the number is prime. This is a test to judge whether a number is prime or not.
- The only even prime number is 2
- 1 is neither prime nor composite (by definition)
- The smallest composite number is 4.

Composite numbers: A composite number has other factors besides itself and unity, e.g., 8, 72, 39 etc. Alternatively, we might say that a natural number greater than 1 that is not prime is a composite number.

FACTORS / HCF (GCD / GCF) & LCM OF NUMBERS

Prime factors:

A composite number can be uniquely expressed as a product of prime factors.

Ex. $12 = 2 \times 6 = 2 \times 2 \times 3 = 2^2 \times 3^1$ $\quad$ $20 = 4 \times 5 = 2 \times 2 \times 5 = 2^2 \times 5^1$

$124 = 2 \times 62 = 2 \times 2 \times 31 = 2^2 \times 31$ etc.

If k and n are both integers greater than 1 and if k is a factor of n, k cannot be a factor of (n + 1).

NOTE:

The number of divisors (factors) of a given number N (including one and the number itself) where $N = a^m \times b^n \times c^p$ .... where a, b, c are prime numbers is given by $(m + 1)(n + 1)(p + 1)$ ......

e.g. (1) $90 = 2 \times 3 \times 3 \times 5 = 2^1 \times 3^2 \times 5^1$

Hence here a = 2 b = 3 c = 5, m = 1 n = 2 p = 1

Number of divisors = $(m + 1)(n + 1)(p + 1)$ .... $= 2 \times 3 \times 2 = 12$

Number of factors of 90 = 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90 = 12

HCF: It is the greatest factor common to two or more given numbers. It is also called GCF OR GCD (greatest common factor or greatest common divisor); e.g. HCF of 10 & 15 = 5, HCF of 55 & 200 = 5, HCF of 64 & 36 = 4

To find the HCF of given numbers, resolve the numbers into their prime factors and then pick the common term(s) from them and multiply them. This is the required HCF.

LCM: Lowest common multiple of two or more numbers is the smallest number which is exactly divisible by all of them.

E.g. LCM of 5, 7, 10 = 70, LCM of 2, 4, 5 = 20, LCM of 11, 10, 3 = 330

To find the LCM resolve all the numbers into their prime factors and then pick all the quantities (prime factors) but not more than once and multiply them. This is the LCM.

NOTE:

1. LCM × HCF = Product of two numbers (valid only for "two")
2. HCF of fractions = HCF of numerators ÷ LCM of denominators
3. LCM of fractions = LCM of numerators ÷ HCF of denominators

Q. Find the LCM of 25 and 35 if their HCF is 5. LCM = 25 x 35/5 = 175

Calculating LCM: After expressing the numbers in terms of prime factors, the LCM is the product of highest powers of all factors.

Q. Find the LCM of 40, 120, and 380.
$40 = 4 \times 10 = 2 \times 2 \times 2 \times 5 = 2^3 \times 5^1$,
$120 = 4 \times 30 = 2 \times 2 \times 2 \times 5 \times 3 = 2^3 \times 5^1 \times 3^1$
$380 = 2 \times 190 = 2 \times 2 \times 95 = 2 \times 2 \times 5 \times 19 = 2^2 \times 5^1 \times 19^1$
Required LCM = $2^3 \times 5^1 \times 3^1 \times 19^1 = 2280$.

Calculating HCF: After expressing the numbers in term of the prime factors, the HCF is product of COMMON factors.

Ex. Find HCF of 88, 24, and 124
$88 = 2 \times 44 = 2 \times 2 \times 22 = 2 \times 2 \times 2 \times 11 = 2^3 \times 11^1$
$24 = 2 \times 12 = 2 \times 2 \times 6 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3^1$
$124 = 2 \times 62 = 2 \times 2^1 \times 31^1 = 2^2 \times 31^1$ HCF = 2^2

Divisibility / Remainders

TESTS FOR DIVISIBILITY:

1. A number is divisible by 2 if its unit's digit is even or zero e.g. 128, 146, 34 etc.
2. A number is divisible by 3 if the sum of its digits is divisible by 3 e.g. 102, 192, 99 etc.
3. A number is divisible by 4 when the number formed by last two right hand digits is divisible by '4' e.g. 576, 328, 144 etc.
4. A number is divisible by 5 when its unit's digit is either five or zero: e.g. 1111535, 3970, 145 etc.
5. A number is divisible by 6 when it's divisible by 2 and 3 both. e.g. 714, 509796, 1728 etc.
6. A number is divisible by 8 when the number formed by the last three right hand digits is divisible by '8'. e.g. 512, 4096, 1304 etc.
7. A number is divisible by 9 when the sum of its digits is divisible by 9 e.g. 1287, 11583, 2304 etc.
8. A number is divisible by 10 when its unit's digit is zero. e.g. 100, 170, 10590 etc.
9. A number is divisible by 11 when the difference between the sums of digits in the odd and even places is either zero or a multiple of 11. e.g. 17259, 62468252, 12221 etc. For the number 17259:

Sum of digits in even places = 7 + 5 = 12, Sum of digits in the odd places = 1 + 2 + 9 = 12 Hence 12 – 12 = 0.

10. A number is divisible by 12 when it is divisible by 3 & 4 both. e.g. 672, 8064 etc.
11. A number is divisible by 25 when the number formed by the last two Right hand digits is divisible by 25, e.g., 1025, 3475, 55550 etc.

<u>Power of a Prime Number in a Factorial:</u> If we have to find the power of a prime number p in n!, it is found using a general rule, which is $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\ldots\ldots$, where $\left[\frac{n}{p}\right]$ denotes the greatest integer ≤ to $\left[\frac{n}{p}\right]$ etc.

For example, power of 3 in 100! = $\left[\frac{100}{3}\right]+\left[\frac{100}{3^2}\right]+\left[\frac{100}{3^3}\right]+\left[\frac{100}{3^4}\right]+\left[\frac{100}{3^5}\right]+\ldots$ = 33 + 11 + 3 + 1 + 0 = 48.

For example, power of 5 in 200! = $\left[\frac{200}{5}\right]+\left[\frac{200}{5^2}\right]+\left[\frac{200}{5^3}\right]+\ldots$ = 40 + 8 + 1 + 0 = 49.

<u>Number of Zeroes at the end of a Factorial:</u> It is given by the power of 5 in the number.

Actually, the number of zeroes will be decided by the power of 10, but 10 is not a prime number, we have 10 = 5 × 2, and hence we check power of 5.

For example, the number of zeroes at the end of 100! = = 20 + 4 = 24.

The number of zeroes at the end of 500! = = 100 + 20 + 4 = 124.

The number of zeroes at the end of 1000! = = 200 + 40 +8 + 1 = 249.

Unit's digits in powers: Every digit has a cyclicity of 4. The fifth power of any single digit number has the same right-hand digit as the number itself.

Example: What will be the unit's digit in 128^{96}?

In all such questions, divide the power by 4 and check the remainder.

If the remainder is 1, 2 or 3, then convert the question to LAST DIGIT RAISED TO REMAINDER.

If the remainder is 0, convert the question to LAST DIGIT RAISED TO FOUR.

In this question, 96/4 = 0, so the question converts to $8^4 = 8^2 \times 8^2$ = 64 x 64 = 4 x 4 = 16 = 6

DECIMALS and FRACTIONS

Recurring Decimals (Conversion to a Rational Number): If in a decimal fraction a figure or a set of figures is repeated continually, then such a number is called a recurring decimal.

(i) 2/3 = 0.6666.... (ii) 22/7 = 3.142857142857 …

Rule: Write the recurring figures only one in the numerator and take as many nines in the denominator as the number of repeating figures.

Ex. (1) 0.666666666 … = 6/9 = 2/3 (2) 0.234234234234 … = 234/999

Rounding Off

Number	Nearest tenth	Nearest hundredth	Nearest thousandth
1.2346	1.2	1.23	1.235
31.6479	31.6	31.65	31.648
9.7462	9.7	9.75	9.746

Whether a fraction will result in a terminating decimal or not? To determine this, express the fraction in the lowest form and then express the denominator in terms of Prime Factors. If the denominator contains powers of only 2 and 5, it is terminating. If the denominator contains any power of any other prime number, it is non-terminating.

Factor Theorem: If f(x) is completely divisible by (x – a), then f(a) = 0. So, (x – a) **is a factor of** f(x), then f(a) = 0

Check whether (x + 1) is a factor of f(x) = $4x^2 + 3x - 1$. Putting x + 1 = 0, i.e., x = –1 in the given expression we get f(–1) = 0. So, (x + 1) is a factor of f(x).

Remainder Theorem: If an expression f(x) is divided by (x – a), then the remainder is f(a).

Let f(x) = $x^3 + 3x^2 - 5x + 4$ be divided by (x – 1). Find the remainder.
Remainder = f(1) = $1^3 + 3 \times 1^2 - 5 \times 1 + 4 = 3$.

Some properties of square numbers:
- A square number always has odd number of factors.
- A square number cannot end with 2, 3, 7, 8 or an odd number of zeroes.
- Every square number is a multiple of 3, or exceeds a multiple of 3 by unity.
- Every square number is a multiple of 4 or exceeds a multiple of 4 by unity.
- If a square number ends in 9, the preceding digit is even.