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Chapter 1

of

Fractions, Decimals, & Percents

FDPs

In This Chapter…

Chapter 1

FDPs

FDPs stands for Fractions, Decimals, and Percents, the title of this book. The three forms are grouped into one book because they are different ways to represent the same number. In fact, the GMAT often mixes fractions, decimals, and percents in one problem. In order to achieve success with FDP problems, you are going to need to shift amongst the three accurately and quickly.

A **fraction** consists of a numerator and a denominator:	$\frac{1}{2}$
A **decimal** uses place values:	0.5
A **percent** expresses a relationship between a number and 100:	50%

$$\frac{1}{2} = 0.5 = 50\%$$

Each of these representations equals the same number but in a different form. Certain kinds of math operations are easier to do in percent or decimal form than in fraction form and vice versa. Try this problem:

> Three sisters split a sum of money between them. The first sister receives $\frac{1}{2}$ of the total, the second receives $\frac{1}{4}$ of the total, and the third receives the remaining \$10. How many dollars do the three sisters split?
>
> (A) \$10
>
> (B) \$20
>
> (C) \$30
>
> (D) \$40
>
> (E) \$50

To solve, you have to figure out what proportion of the money the first two sisters get, so that you know what proportion the third sister's $10 represents. It's not too difficult to add up the relatively simple fractions $\frac{1}{2}$ and $\frac{1}{4}$, but harder fractions would make the work a lot more cumbersome. In general, adding fractions is annoying because you have to find a common denominator.

On this problem, it's easier to convert to percentages. The first sister receives 50% of the money and the second receives 25%, leaving 25% for the third sister. That 25% represents $10, so 100% is $40. The correct answer is (D).

In order to do this kind of math quickly and easily, you'll need to know how to convert among fractions, decimals, and percents. Luckily, certain common conversions are used repeatedly throughout the GMAT. If you memorize these conversions, you'll get to skip the calculations. The next two sections cover these topics.

Common FDP Equivalents

Save yourself time and trouble by memorizing the following common equivalents:

Fraction	Decimal	Percent
1/1	1	100%
1/2 = 2/4 = 3/6 = 4/8 = 5/10	0.5	50%
3/2	1.5	150%

1/4 = 2/8	0.25	25%
3/4 = 6/8	0.75	75%
5/4	1.25	125%
7/4	1.75	175%

1/8	0.125	12.5%
3/8	0.375	37.5%
5/8	0.625	62.5%

7/8	0.875	87.5%

1/5 = 2/10	0.2	20%
2/5 = 4/10	0.4	40%
3/5 = 6/10	0.6	60%
4/5 = 8/10	0.8	80%

1/10	0.10	10%
3/10	0.3	30%
7/10	0.7	70%
9/10	0.9	90%

1/3 = 2/6	$0.\overline{3} \approx 0.333$	≈ 33.3%
2/3 = 4/6	$0.\overline{6} \approx 0.666$	≈ 66.7%
4/3	$1.\overline{3} \approx 1.33$	133%

1/6	$0.1\overline{6} \approx 0.167$	≈ 16.7%
5/6	$0.8\overline{3} \approx 0.833$	≈ 83.3%
1/9	$0.1\overline{1} \approx 0.111$	≈ 11.1%

1/100	0.01	1%
1/50	0.02	2%
1/25	0.04	4%
1/20	0.05	5%

Converting Among Fractions, Decimals, and Percents

The chart below summarizes various methods to convert among fractions, decimals, and percents (for any conversions that you haven't memorized!).

FROM ↓ TO →	Fraction $\frac{3}{8}$	Decimal 0.375	Percent 37.5%
Fraction $\frac{3}{8}$		Divide the numerator by the denominator: $3 \div 8 = 0.375$ Alternatively, multiply the top and bottom to get the denominator to equal 100: $\frac{3}{8} \times \frac{12.5}{12.5} = \frac{37.5}{100} = 0.375$	Divide the numerator by the denominator and move the decimal two places to the right: $3 \div 8 = 0.375 \rightarrow 37.5\%$
Decimal 0.375	Use the place value of the last digit in the decimal as the denominator, and put the decimal's digits in the numerator. Then simplify: $\frac{375}{1{,}000} = \frac{3}{8}$		Move the decimal point two places to the right: $0.375 \rightarrow 37.5\%$
Percent 37.5%	Use the digits of the percent for the numerator and 100 for the denominator. Then simplify: $\frac{37.5}{100} = \frac{3}{8}$	Find the percent's decimal point and move it two places to the left: $37.5\% \rightarrow 0.375$	

You'll get plenty of practice with these skills throughout this book, but if you'd like some more, see the FDPs section in the *Foundations of GMAT Math Strategy Guide*.

When to Use Which Form

As you saw in the "three sisters" problem, percentages (or decimals) are easier to add and subtract. Fractions, on the other hand, work very well with multiplication and division.

If you have already memorized the given fraction, decimal, and percent conversions, you can move among the forms quickly. If not, you may have to decide between taking the time to convert from one form to the other and working the problem using the less convenient form (e.g., dividing fractions to produce decimals or expressing those fractions with a common denominator in order to add).

Try this problem:

> What is 37.5% of 240?

If you convert the percent to a decimal and multiply, you will have to do a fair bit of arithmetic:

$$\begin{array}{r} 0.375 \\ \times\ \ 240 \\ \hline 0 \\ 15000 \\ 75000 \\ \hline 90.000 \end{array}$$

> Alternatively, recognize that $0.375 = \frac{3}{8}$.
>
> $(0.375)(240) = \left(\frac{3}{8}\right)(\cancel{240}^{30}) = 3(30) = 90$.
>
> This is much faster!

Try something a bit harder:

> A dress is marked up $16\frac{2}{3}\%$ to a final price of \$140. What is the original price of the dress?

$16\frac{2}{3}\%$ is on the memorization list; it is equal to $\frac{1}{6}$. Adding $\frac{1}{6}$ of a number to itself is the same thing as multiplying by $1 + \frac{1}{6} = \frac{7}{6}$. Call the original price *x* and set up an equation to solve.

$$x + \frac{1}{6}x = 140 \qquad \frac{7}{6}x = 140 \qquad\qquad x = \left(\frac{6}{7}\right)140 = \left(\frac{6}{\cancel{7}}\right)\cancel{140}^{20} = 120.$$

Therefore, the original price is \$120.

As you've seen, decimals and percents work very well with addition and subtraction: you don't have to find common denominators! For this same reason, decimals and percents are also preferred when you want to compare numbers or perform certain estimations. For example, what is $\frac{3}{5} - \frac{1}{4}$?

You can find common denominators, but both fractions are on your “conversions to memorize” list:

$$\frac{3}{5} = 60\% \qquad \frac{1}{4} = 25\%$$

$$60\% - 25\% = 35\%$$

If the answers are in fraction form, convert back:

$$35\% = \frac{35}{100} = \frac{7}{20}$$

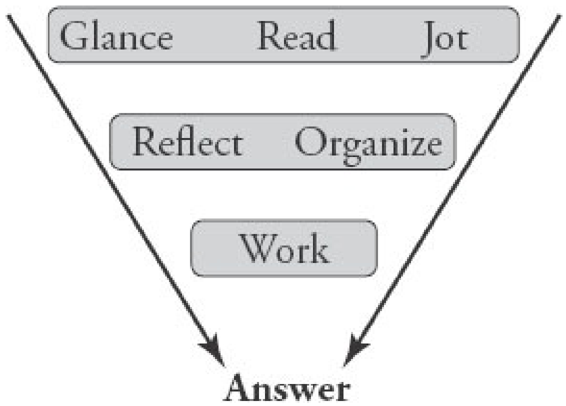
In some cases, you may decide to stick with the given form rather than convert to another form. If you do have numbers that are easy to convert, though, then use fractions for multiplication and division and use percents or decimals for addition and subtraction, as well as for estimating or comparing numbers.

Introduction to Estimation

FDP conversions can sometimes help you to estimate your way to an answer.

Try this problem:

> 65% of the students at a particular school take language classes. Of those students, 40% have studied more than one language. If there are 300 students at the school, how many have studied more than one language?
>
> (A) 78
>
> (B) 102
>
> (C) 120



Step 1: Glance, Read, Jot: What's going on?

Glance at the problem: is it Problem Solving or Data Sufficiency? If it's Problem Solving, glance at the answers. Are they numerical or do they contain variables? Are they “easy” numbers or hard ones? Close together or far apart? If they're far apart, you can estimate!

As you read, jot down any obvious information:

$65\% = L \rightarrow 40\%$ OF $L > 1$ lang
$300 = T$

Step 2: Reflect, Organize: What's my plan?

Okay, 300 is the starting point, but 65% is a bit annoying. You *can* figure out that number. Do you want to take the time to do so?

If you've noticed that the answers are decently far apart, you know you can estimate. Since 65% is very close to $\frac{2}{3}$, it is a far easier number to use (especially with 300 as the starting point!).

Step 3: Work: Solve!

$\frac{2}{3}$ of 300 is 200

Note that you rounded up, so your answer will be a little higher than the official number.

To calculate 40% of that number, use one of two methods:

Method 1: For multiplication, convert to fractions:

$$\left(\frac{2}{5}\right)(200)=(2)(40)=80$$

Method 2: Find 10% of the number, then multiply by 4 to get 40%:

$$10\% \text{ of } 200 = 20, \text{ so } 40\% = 20 \times 4 = 80$$

Approximately 80 students have studied more than one language. The correct answer is **(A)**.

This book will teach you how to perform proper calculations (and you do need to learn how!), but you should also keep an eye out for opportunities to estimate on GMAT problems. You'll learn multiple strategies when you get to Chapter 8, "Strategy: Estimation."

Problem Set

1. Express the following as fractions and simplify:	2.45	0.008
2. Express the following as fractions and simplify:	420%	8%
3. Express the following as decimals:	$\frac{9}{2}$	$\frac{3,000}{10,000}$
4. Express the following as decimals:	$1\frac{27}{4}$	$12\frac{8}{3}$
5. Express the following as percents:	$\frac{1,000}{10}$	$\frac{25}{8}$
6. Express the following as percents:	80.4	0.0007

7. Order from least to greatest:

$\frac{8}{18}$ 0.8 40%

8. 200 is 16% of what number?

9. What number is 62.5% of 192?

Solutions

1. To convert a decimal to a fraction, write it over the appropriate power of 10 and simplify:

$$2.45 = 2\frac{45}{100} = \mathbf{2\frac{9}{20}} \text{ (mixed)} = \mathbf{\frac{49}{20}} \text{ (improper)}$$

$$0.008 = \frac{8}{1{,}000} = \mathbf{\frac{1}{125}}$$

2. To convert a percent to a fraction, write it over a denominator of 100 and simplify:

$$420\% = \frac{420}{100} = \mathbf{\frac{21}{5}} \text{ (improper)} = \mathbf{4\frac{1}{5}} \text{ (mixed)}$$

$$8\% = \frac{8}{100} = \mathbf{\frac{2}{25}}$$

3. To convert a fraction to a decimal, divide the numerator by the denominator:

$$\frac{9}{2} = 9 \div 2 = \mathbf{4.5}$$

It often helps to simplify the fraction *before* you divide:

$$\frac{3{,}000}{10{,}000} = \frac{3}{10} = \mathbf{0.3}$$

4. To convert a mixed number to a decimal, simplify the mixed number first, if needed:

$$1\frac{27}{4} = 1 + 6\frac{3}{4} = \mathbf{7.75}$$

$$12\frac{8}{3} = 12 + 2\frac{2}{3} = 14\frac{2}{3} = \mathbf{14.\overline{6}}$$

5. To convert a fraction to a percent, rewrite the fraction with a denominator of 100:

$$\frac{1{,}000}{10} = \frac{10{,}000}{100} = \mathbf{10{,}000\%}$$

Or, convert the fraction to a decimal and shift the decimal point two places to the right:

$$\frac{25}{8} = 25 \div 8 = 3\frac{1}{8} = 3.125 = 312.5\%$$

6. To convert a decimal to a percent, shift the decimal point two places to the right:

$80.4 = \mathbf{8{,}040\%}$
$0.0007 = \mathbf{0.07\%}$

7. $\mathbf{40\% < \frac{8}{18} < 0.8}$**:** To order from least to greatest, express all the terms in the same form:

$$\frac{8}{18} = \frac{4}{9} = 0.4444\ldots = 0.\overline{4}$$

$0.8 = 0.8$
$40\% = 0.4$
$0.4 < 0.\overline{4} < 0.8$

8. **1,250:** This is a percent vs. decimal conversion problem. If you simply recognize that $16\% = 0.16 = \frac{16}{100} = \frac{4}{25}$, this problem will be a lot easier:

$\frac{4}{25}x = 200$, so $x = 200 \times \frac{25}{4} = 50 \times 25 = 1{,}250$. Dividing out 200 ÷ 0.16 will probably take longer to complete.

9. **120:** This is a percent vs. decimal conversion problem. If you simply recognize that $62.5\% = 0.625 = \frac{5}{8}$, this problem will be a lot easier: $\frac{5}{8} \times 192 = \frac{5}{1} \times 24 = 120$. Multiplying 0.625 × 240 will take much longer to complete.

Chapter 2 of

Fractions, Decimals, & Percents

Digits & Decimals

In This Chapter…

Chapter 2

Digits & Decimals

Digits

Every number is composed of digits. There are only ten digits in our number system: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The term digit refers to one building block of a number; it does not refer to a number itself. For example, 356 is a number composed of three digits: 3, 5, and 6.

Integers can be classified by the number of digits they contain. For example:

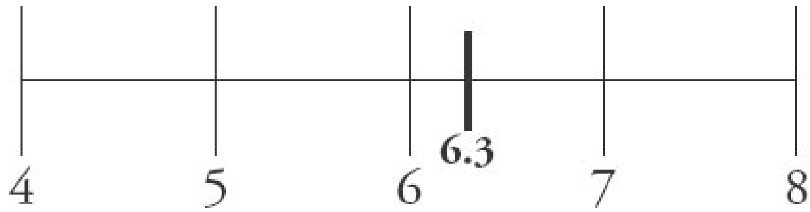
> 2, 7, and −8 are each single-digit numbers (they are each composed of one digit).
> 43, 63, and −14 are each double-digit numbers (composed of two digits).
> 500,000 and −468,024 are each six-digit numbers (composed of six digits).
> 789,526,622 is a nine-digit number (composed of nine digits).

Non-integers are not generally classified by the number of digits they contain, since you can always add any number of zeroes at the end, on the right side of the decimal point:

> 9.1 = 9.10 = 9.100

Decimals

GMAT math goes beyond an understanding of the properties of integers (which include the counting numbers, such as 1, 2, 3, their negative counterparts, such as −1, −2, −3, and the number 0). The GMAT also tests your ability to understand the numbers that fall in between the integers: decimals. For example, the decimal 6.3 falls between the integers 6 and 7:



Some useful groupings of decimals include:

Group	**Examples**
Decimals less than −1:	−3.65, −12.01, −145.9
Decimals between −1 and 0:	−0.65, −0.8912, −0.076
Decimals between 0 and 1:	0.65, 0.8912, 0.076
Decimals greater than 1:	3.65, 12.01, 145.9

Note that an integer can be expressed as a decimal by adding the decimal point and the digit 0. For example:

$8 = 8.0$ $\qquad -123 = -123.0$ $\qquad 400 = 400.0$

Place Value

Every digit in a number has a particular place value depending on its location within the number. For example, in the number 452, the digit 2 is in the ones (or “units”) place, the digit 5 is in the tens place, and the digit 4 is in the hundreds place. The name of each location corresponds to the value of that place. Thus:

The 2 is worth two ones, or 2 (i.e., 2×1);
The 5 is worth five tens, or 50 (i.e., 5×10); and
The 4 is worth four hundreds, or 400 (i.e., 4×100).

You can now write the number 452 as the *sum* of these products:

$$452 = 4 \times 100 + 5 \times 10 + 2 \times 1$$

2	5	6	7	8	9	1	0	2	3	.	8	3	4	7
	HUNDRED	TEN		HUNDRED	TEN		HUNDREDS	TENS	UNITS OR		TENTHS	HUNDREDTHS	THOUSANDTHS	TEN THOUSANDTHS
BILLIONS	MILLIONS	MILLIONS	MILLIONS	THOUSANDS	THOUSANDS	THOUSANDS			ONES					

The chart to the left analyzes the place value of all the digits in the number **2,567,891,023.8347**.

Notice that all of the place values that end in "ths" are to the right of the decimal; these are all fractional values.

Analyze just the decimal portion of the number: **0.8347:**

8 is in the tenths place, giving it a value of 8 tenths, or $\frac{8}{10}$.

3 is in the hundredths place, giving it a value of 3 hundredths, or $\frac{3}{100}$

.

4 is in the thousandths place, giving it a value of 4 thousandths, or $\frac{4}{1,000}$.

7 is in the ten-thousandths place, giving it a value of 7 ten thousandths, or $\frac{7}{10,000}$.

To use a concrete example, 0.8 might mean eight tenths of one dollar, which would be 80 cents. Additionally, 0.03 might mean three hundredths of one dollar, or 3 cents.

Rounding to the Nearest Place Value

The GMAT occasionally requires you to round a number to a specific place value. For example:

> What is 3.681 rounded to the nearest tenth?

First, find the digit located in the specified place value. The digit 6 is in the tenths place.

Second, look at the right-digit-neighbor (the digit immediately to the right) of the digit in question. In this case, 8 is the right-digit-neighbor of 6. If the right-digit-neighbor is 5 or greater, round the digit in question UP. Otherwise, leave the digit alone. In this case, since 8 is greater than 5, the digit in question, 6 must be rounded up to 7. Thus, 3.681 rounded to the nearest tenth equals 3.7. Note that all the digits to the right of the right-digit-neighbor are irrelevant when rounding.

Rounding appears on the GMAT in the form of questions such as this:

> If x is the decimal 8.1d5, with d as an unknown digit, and x rounded to the nearest tenth is equal to 8.1, which digits could not be the value of d?

In order for x to be 8.1 when rounded to the nearest tenth, the right-digit-neighbor, d, must be less than 5. Therefore, d cannot be 5, 6, 7, 8 or 9.

Powers of 10: Shifting the Decimal

What are the patterns in the below table?

In words	thousands	hundreds	tens	ones	tenths	hundredths	thousandths
In numbers	1,000	100	10	1	0.1	0.01	0.001
In powers of ten	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

The place values continually decrease from left to right by powers of 10. Understanding this can help you understand the following shortcuts for multiplication and division.

When you multiply any number by a positive power of 10, move the decimal to the right the specified number of places. This makes positive numbers larger:

$3.9742 \times 10^3 = 3{,}974.2$	Move the decimal to the right 3 spaces.
$89.507 \times 10 = 895.07$	Move the decimal to the right 1 space.

When you divide any number by a positive power of 10, move the decimal to the left the specified number of places. This makes positive numbers smaller:

$4{,}169.2 \div 10^2 = 41.692$	Move the decimal to the left 2 spaces.
$89.507 \div 10 = 8.9507$	Move the decimal to the left 1 space.

Sometimes, you will need to add zeroes in order to shift a decimal:

$2.57 \times 10^6 = 2{,}570{,}000$	Add 4 zeroes at the end.
$14.29 \div 10^5 = 0.0001429$	Add 3 zeroes at the beginning.

Finally, note that negative powers of 10 reverse the regular process. Multiplication makes the number smaller and division makes the number larger:

$6{,}782.01 \times 10^{-3} = 6.78201$	$53.0447 \div 10^{-2} = 5{,}304.47$

You can think about these processes as trading decimal places for powers of 10.

For instance, all of the following numbers equal 110,700:

110.7	×	10^3
11.07	×	10^4
1.107	×	10^5
0.1107	×	10^6
0.01107	×	10^7

The number in the first column gets smaller by a factor of 10 as you move the decimal one place to the left, but the number in the second column gets bigger by a factor of 10 to compensate, so the overall number still equals 110,700.

Decimal Operations

Addition & Subtraction

To add or subtract decimals, first line up the decimal points. Then add zeroes to make the right sides of the decimals the same length:

4.319 + 221.8

Line up the decimal points and add zeroes.

$$\begin{array}{r} 4.319 \\ +\ 221.800 \\ \hline 226.119 \end{array}$$

10 − 0.063

Line up the decimal points and add zeroes.

$$\begin{array}{r} 10.000 \\ -\ \ 0.063 \\ \hline 9.937 \end{array}$$

Addition and subtraction: Line up the decimal points!

Multiplication

To multiply decimals, ignore the decimal point until the end. Just multiply the numbers as you would if they were whole numbers. Then count the total number of digits to the right of the decimal point in the starting numbers. The product should have the same number of digits to the right of the decimal point.

	Count the digits to the right of the decimal:	Multiply normally:	Move the decimal 3 places to the left:
0.02 × 1.4	3	$\begin{array}{r} 14 \\ \times\ 2 \\ \hline 28 \end{array}$	28 → 0.028

If the product ends with 0, that 0 still counts as a place value. For example: 0.8 × 0.5 = 0.40, since 8 × 5 = 40.

> **Multiplication:** Count all the digits to the right of the decimal point—then multiply normally, ignoring the decimals. Finally, put the same number of decimal places in the product.

If you are multiplying a very large number and a very small number, the following trick works to simplify the calculation: move the decimals the same number of places, but *in the opposite direction*.

0.0003 × 40,000 = ?

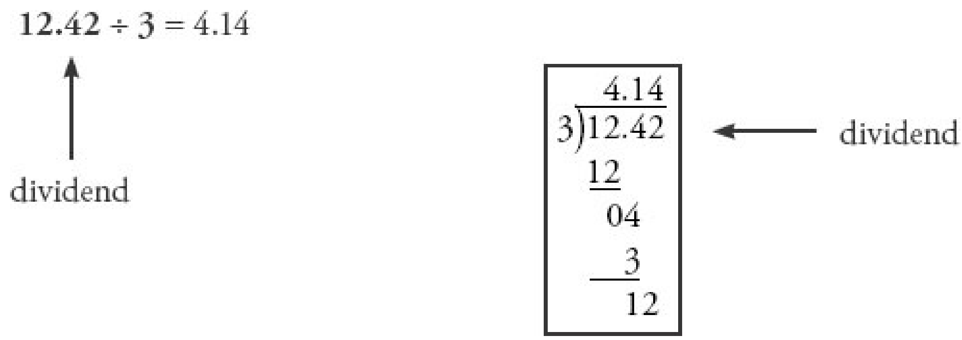
Move the decimal point *right* four places on the 0.0003 ⟶ 3
Move the decimal point *left* four places on the 40,000 ⟶ 4

0.0003 × 40,000 = 3 × 4 = 12

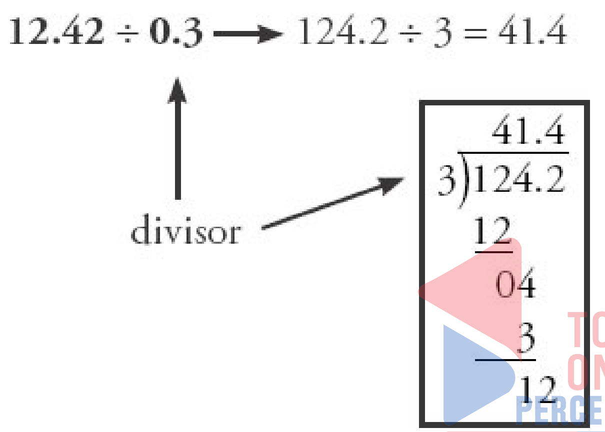
This technique works because you are multiplying and then dividing by the same power of 10. In other words, you are trading decimal places in one number for decimal places in another number. This is just like trading decimal places for powers of 10, as you saw earlier.

Division

If there is a decimal point in the dividend (the number under the division sign) only, you can simply bring the decimal point straight up to the answer and divide normally:



However, if there is a decimal point in the divisor (the outer number), shift the decimal point in both the divisor and the dividend to make the *divisor* a whole number. Then, bring the decimal point up and divide:



Move the decimal one space to the right to make 0.3 a whole number. Then, move the decimal one space to the right in 12.42 to make it 124.2.

Division: Divide by whole numbers! Move the decimal in both numbers so that the divisor is a whole number.

You can always simplify division problems that involve decimals by shifting the decimal point *in the same direction* in both the divisor and the dividend, even when the division problem is expressed as a fraction:

$$\frac{0.0045}{0.09} = \frac{45}{900}$$

Move the decimal 4 spaces to the right to make both the numerator and the denominator whole numbers.

Note that this is essentially the same process as simplifying a fraction. You multiply the numerator and denominator of the fraction by a power of 10—in this case, 10^4, or 10,000.

Keep track of how you move the decimal point! To simplify multiplication, you can move decimals in *opposite* directions. But to simplify division, move decimals in the *same* direction.

Problem Set

Solve each problem, applying the concepts and rules you learned in this section.

1. In the decimal, 2.4d7, d represents a digit from 0 to 9. If the value of the decimal rounded to the nearest tenth is less than 2.5, what are the possible values of d?

2. Simplify: $\frac{0.00081}{0.09}$

3. Which integer values of b would give the number $2002 \div 10^{-b}$ a value between 1 and 100?

4. Simplify: $(4 \times 10^{-2}) - (2.5 \times 10^{-3})$

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

5. If k is an integer, and if 0.02468×10^{k} is greater than 10,000, what is the least possible value of k?

6. What is $4{,}563{,}021 \div 10^{5}$, rounded to the nearest whole number?

7. Which integer values of j would give the number $-37{,}129 \times 10^{j}$ a value between −100 and −1?

Solutions

1. {**0, 1, 2, 3, 4**}: If d is 5 or greater, the decimal rounded to the nearest tenth will be 2.5.

2. **0.009:** Shift the decimal point 2 spaces to eliminate the decimal point in the denominator:

$$\frac{0.00081}{0.09} = \frac{0.081}{9}$$

Now divide. First, drop the 3 decimal places: 81 ÷ 9 = 9. Then put the 3 decimal places back: 0.009.

3. {**−2, −3**}: In order to give 2002 a value between 1 and 100, you must shift the decimal point to change the number to 2.002 or 20.02. This requires a shift of either two or three places to the left. Remember that while multiplication shifts the decimal point to the right, division shifts it to the left. To shift the decimal point 2 places to the left, you would divide by 10^2. To shift it 3 places to the left, you would divide by 10^3. Therefore, the exponent $-b$ is equal to {2, 3}, and b is equal to {-2, -3}.

4. **0.0375:** First, rewrite the numbers in standard notation by shifting the decimal point. Then, add zeroes, line up the decimal points, and subtract:

$$\begin{array}{r} 0.0400 \\ -\ 0.0025 \\ \hline 0.0375 \end{array}$$

5. **6:** Multiplying 0.02468 by a positive power of 10 will shift the decimal point to the right. Simply shift the decimal point to the right until the result is greater than 10,000. Keep track of how many times you shift the decimal point. Shifting the decimal point 5 times results in 2,468. This is still less than 10,000. Shifting one more place yields 24,680, which is greater than 10,000.

6. **46:** To divide by a positive power of 10, shift the decimal point to the left. This yields 45.63021. To round to the nearest whole number, look at the tenths place. The digit in the tenths place, 6, is more than 5. Therefore, the number is closest to 46.

7. {−**3**, −**4**}: In order to give −37,129 a value between −100 and −1, you must shift the decimal point to change the number to −37.129 or −3.7129. This requires a shift of either 3 or 4 places to the left. Remember that multiplication by a positive power of 10 shifts the decimal point to the right. To shift the decimal point 3 places to the left, you would multiply by 10^{-3}. To shift it 4 places to the left, you would multiply by 10^{-4}. Therefore, the exponent j is equal to {−3, −4}.

Chapter 3

Fractions, Decimals, & Percents

Strategy: Test Cases

In This Chapter…

Chapter 3

Strategy: Test Cases

Certain problems allow for multiple possible scenarios, or cases. When you **test cases**, you try different numbers in a problem to see whether you have the same outcome or different outcomes.

The strategy plays out a bit differently for Data Sufficiency (DS) compared to Problem Solving. This chapter will focus on DS problems; if you have not yet studied DS, please see Appendix A of this guide. For a full treatment of Problem Solving, see the Strategy: Test Cases chapter in the *Number Properties GMAT Strategy Guide.*

Try this problem, using any solution method you like:

> If x is a positive integer, what is the units digit of x?
>
> (1) The units digit of $\frac{x}{10}$ is 4.
>
> (2) The tens digit of $10x$ is 5.

How to Test Cases

Here's how to test cases to solve the above problem:

Step 1: What possible cases are allowed?

The problem doesn't seem to give you much: the number x is a positive integer. You do know one more thing, though: the units digit can consist of only a single digit. By definition, then, the units digit of x has to be one of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. (Some problems could limit your options further by, for example, indicating that x is even.)

Step 2: Choose numbers that work for the statement.

Before you dive into the work, remember this crucial rule:

> *When choosing numbers to test cases, ONLY choose numbers that are allowed by that statement.*

If you inadvertently choose numbers that make the statement false, discard that case and try again.

Step 3: Try to prove the statement *insufficient.*
Here's how:

(1) The units digit of $\frac{x}{10}$ is 4.

What numbers would make this statement true?

Case 1: $x = 45$:

Statement true? (units digit $\frac{x}{10} = 4$)	Units digit of x?
$\frac{45}{10} = 4.5$ ✓	5

First, ensure that the value you've chosen to test does make the statement true. In this case, the units digit of $\frac{45}{10}$ is 4, so $x = 45$ is a valid number to test. If you had chosen, say, 54, then the units digit of $\frac{54}{10}$ would be 5, not 4, so you would discard that case.

Second, answer the question asked. If $x = 45$, then the units digit of x is 5.

Next, ask yourself: Is there another possible case that would give you a *different* outcome?

Case 2: $x = 46$:

Statement true?	Units digit of x?

(units digit $\frac{x}{10} = 4$)	
$\frac{46}{10} = 4.6$ ✓	6

Because there are at least two possible values for the units digit, this statement is not sufficient; $\frac{\cancel{\text{AD}}}{\text{BCE}}$ cross off answers (A) and (D) on your answer grid.

Try statement (2) next:

(2) The tens digit of 10*x* is 5.

Case 1: $x = 45$:

Statement true? (tens digit $10x = 5$)	Units digit of *x*?
$10x = 450$ ✓	5

Is there another possible case that would give you a different outcome?

Case 2: $x = 46$:

Statement true? (tens digit $10x = 5$)	Units digit of *x*?
$10x = 460$ ✗	

Careful! The tens digit of 460 is *not* 5. You have to pick a value that makes statement (2) true. Discard this case. (Literally cross it off on your scrap paper.)

Case 3: $x = 65$.

Statement true? (tens digit $10x = 5$)	Units digit of *x*?
$10x = 650$ ✓	5

The units digit of *x* is 5, once again. Hmm.

It turns out that, no matter how many cases you try for statement (2), the units digit of x will always be 5. Why?

When you multiply x by 10, what used to be the units digit becomes the tens digit. If you know that the tens digit of the new number is 5, then the units digit of the original number also has to be 5. This statement is sufficient.

The correct answer is **(B)**.

When you test cases in Data Sufficiency, your ultimate goal is to try to prove the statement insufficient, if you can. The first case you try will give you one outcome. For the next case, think about what numbers would be likely to give a *different* outcome.

As soon as you do find two different outcomes, as in statement (1) above, you know the statement is not sufficient, and you can cross off some answer choices and move on.

If you have tried several times to prove the statement insufficient but you keep getting the same outcome, then that statement is probably sufficient. You may be able to prove to yourself why you will always get the same outcome, as in statement (2) above. However, if you can't do that in a reasonable amount of time, you may need to assume you've done enough and move on. When it's time to review your work, take the time to try to understand why the result was always the same.

Try another problem:

> If $a = 2.4d7$, and d represents a digit from 0 to 9, is d greater than 4?
>
> (1) If a were rounded to the nearest hundredth, the new number would be greater than a.
>
> (2) If a were rounded to the nearest tenth, the new number would be greater than a.

Step 1: What possible cases are allowed?

The variable d represents a digit, so it could be any number from 0 to 9. There are no additional constraints to begin with, but you do have one more thing to consider.

The question is different this time: it doesn't ask for the value of *d*, it just asks whether *d* is greater than 4. When you have a yes/no question, make sure you understand (before you begin!) what would be sufficient and what would not be sufficient.

In this case, if you know that *d* is 4 or less, then the answer to the question is no and the statement is sufficient. If you know that *d* is greater than 4, then the answer to the question is yes and the statement is sufficient. This is true even if you do not know exactly what *d* is.

If the possible values cross the barrier of 4 (e.g., *d* could be 4 or 5), then the statement is not sufficient.

Step 2: Choose numbers that work for the statement.

The statements are pretty complicated; it would be easy to make a mistake with this. Remind yourself to separate your evaluation into two parts. First, have you chosen numbers that do make this statement true? Second, is the answer to the question yes or no based on this one case?

Step 3: Try to prove the statement *insufficient.*

(1) If *a* were rounded to the nearest hundredth, the new number would be greater than *a*.

The hundredth digit of $a = 2.4d7$ is the variable *d*. If $d = 5$, then $a = 2.457$. Rounding to the nearest hundredth produces 2.46, which is indeed greater than 2.457. It's acceptable, then, to choose $d = 5$.

Next, is *d* greater than 4? Yes, in this case, it is.

Can you think of another case that would give the opposite answer, a no?

Try $d = 3$. In this case $a = 2.437$. Rounding to the nearest hundredth produces 2.44, which is indeed greater than 2.437. It's acceptable, then, to choose $d = 3$, and in this case, the answer to the question is no, *d* is not greater than 4.

Because you're getting Sometimes Yes, Sometimes No, this statement is not sufficient to answer the question. Cross off answers (A) and (D). Now look at statement (2):

(2) If *a* were rounded to the nearest tenth, the new number would be

greater than *a*.

The tenths digit of $a = 2.4d7$ is 4. Find a value of *d* that will make this statement true. If $d = 9$, then $a = 2.497$. Rounding to the nearest tenth produces 2.5, which is greater than 2.497, so 9 is an acceptable number to choose.

In this case, yes, *d* is greater than 4.

Try to find another acceptable number that will give you the opposite answer, no. If $d = 3$, then $a = 2.437$, and the rounded number is 2.4. Wait a second! 2.4 is not larger than 2.437. You can't pick $d = 3$.

What about 4? Then $a = 2.447$, which still rounds down to 2.4. In fact, any number below 5 will cause *a* to round down to 2.4, which contradicts the statement. The only acceptable values for *d* are 5, 6, 7, 8, and 9.

Is *d* greater than 4? Yes, always, so statement (2) is sufficient. The correct answer is (B).

In sum, when you are asked to test cases, follow three main steps:

Step 1: What possible cases are allowed?

Before you start solving, make sure you know what restrictions have been placed on the basic problem in the question stem. You may be told to use the 10 digits, or that the particular number is positive, or odd, and so on. Follow these restrictions when choosing numbers to try later in your work.

Step 2: Choose numbers that work for the statement.

Pause for a moment to remind yourself that you are only allowed to choose numbers for each statement that make that particular statement true. With enough practice, this will begin to become second nature. If you answer a Testing Cases problem incorrectly but aren't sure why, see whether you accidentally tested cases that weren't allowed because they didn't make the statement true.

Step 3: Try to prove the statement *insufficient*.

<u>Value</u>

Sufficient: single numerical answer

Not Sufficient: two or more possible answers

Yes/No

Sufficient: Always Yes *or* Always No

Not Sufficient: Maybe or Sometimes Yes, Sometimes No

When to Test Cases

You can test cases whenever a Data Sufficiency problem allows multiple possible starting points. In that case, try some of the different possibilities allowed in order to see whether different scenarios, or cases, result in different answers or in the same answer.

All problems will have one thing in common: your initial starting point is every possible number on the number line. The problem then may give you certain restrictions that narrow the possible values. As you saw above, the digit constraint (0, 1, 2, 3, 4, 5, 6, 7, 8, or 9) is one possible restriction.

Other common restrictions include classes of numbers that react differently to certain mathematical operations. For instance, positive and negative numbers have different properties, as do odds and evens. Integers and fractions can also have different properties, particularly proper fractions (those between 0 and 1). You'll learn more about proper fractions in the next chapter of this book.

How to Get Better at Testing Cases

First, try to problems associated with this chapter in your online *Official Guide* problem sets. Work each problem using the three-step process for testing cases. If you mess up any part of the process, try the problem again, making sure to write out all of your work.

Afterwards, review the problem. In particular, when a statement is sufficient because it produces the same answer in each case, see whether you can articulate the reason (as the solutions to the earlier problems did). Could you explain to a fellow student who is confused? If so, then you are starting to learn both the

Chapter 4

of

Fractions, Decimals, & Percents

Fractions

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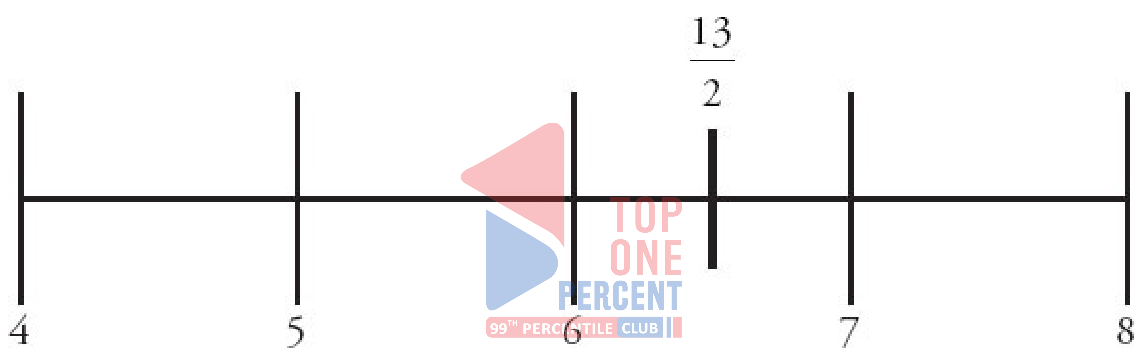
Chapter 4

Fractions

Decimals are one way of expressing the numbers that fall in between the integers. Another way of expressing these numbers is fractions.

For example, the fraction $\frac{13}{2}$, which equals 6.5, falls between the integers 6 and 7:



Proper fractions are those that fall between 0 and 1. In proper fractions, the numerator is always smaller than the denominator. For example:

$$\frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \frac{7}{10}$$

Improper fractions are those that are greater than 1. In improper fractions, the numerator is greater than the denominator. For example:

$$\frac{5}{4}, \frac{13}{2}, \frac{11}{3}, \frac{101}{10}$$

Improper fractions can be rewritten as **mixed numbers**. A mixed number is an integer and a proper fraction. For example:

$$\frac{5}{4}=1\frac{1}{4} \qquad \frac{13}{2}=6\frac{1}{2} \qquad \frac{11}{3}=3\frac{2}{3} \qquad \frac{101}{10}=10\frac{1}{10}$$

Although the preceding examples use positive fractions, note that fractions and mixed numbers can be negative as well.

Numerator and Denominator Rules

Certain key rules govern the relationship between the **numerator** (the top number) and the **denominator** (the bottom number) of proper fractions. These rules apply only to positive fractions.

If you increase the numerator of a fraction, while holding the denominator constant, the fraction increases in value:

$$\frac{1}{8}<\frac{2}{8}<\frac{3}{8}<\frac{4}{8}<\frac{5}{8}<\frac{6}{8}<\frac{7}{8}<\frac{8}{8}<\frac{9}{8}<\frac{10}{8}<\ldots$$

If you increase the denominator of a fraction, while holding the numerator constant, the fraction decreases in value as it approaches 0:

$$\frac{3}{2}>\frac{3}{3}>\frac{3}{4}>\frac{3}{5}>\frac{3}{6}\ldots>\frac{3}{1{,}000}\ldots\rightarrow\ 0$$

Adding the same number to *both* the numerator and the denominator brings the fraction *closer* to 1, regardless of the fraction's value.

If the fraction is originally smaller than 1, the fraction *increases* in value as it approaches 1:

$$\frac{1}{2}<\frac{1+1}{2+1}=\frac{2}{3}<\frac{2+9}{3+9}=\frac{11}{12}<\frac{11+1{,}000}{12+1{,}000}=\frac{1{,}011}{1{,}012}$$

Thus: $\frac{1}{2}<\frac{2}{3}<\frac{11}{12}<\frac{1{,}011}{1{,}012}\ldots\rightarrow\ 1$

Conversely, if the fraction is originally larger than 1, the fraction *decreases* in value as it approaches 1:

$$\frac{3}{2} > \frac{3+1}{2+1} = \frac{4}{3} > \frac{4+9}{3+9} = \frac{13}{12} > \frac{13+1,000}{12+1,000} = \frac{1,013}{1,012}$$

Thus: $\frac{3}{2} > \frac{4}{3} > \frac{13}{12} > \frac{1,013}{1,012} \ldots \rightarrow 1$

Simplifying Fractions

Simplifying is a way to express a fraction in its lowest terms. Fractional answers on the GMAT will always be presented in fully simplified form. The process of simplifying is governed by one simple rule: multiplying or dividing both the numerator and the denominator by the same number does not change the value of the fraction:

$$\frac{24}{30} = \frac{24 \div 6}{30 \div 6} = \frac{4}{5} \qquad \frac{4}{5} = \frac{4(3)}{5(3)} = \frac{12}{15} = \frac{12(2)}{15(2)} = \frac{24}{30}$$

Simplifying a fraction means dividing both the numerator and the denominator by a common factor until no common factors remain:

$\frac{40}{30} = \frac{40 \div 5}{30 \div 5} = \frac{8}{6} = \frac{8 \div 2}{6 \div 2} = \frac{4}{3}$ or in one step: $\frac{40}{30} = \frac{40 \div 10}{30 \div 10} = \frac{4}{3}$

Converting Mixed Numbers to Improper Fractions

In order to convert a mixed number into an improper fraction (something you need to do in order to multiply or divide mixed numbers), use the following procedure:

$2\frac{1}{4}$	Multiply the whole number (2) by the denominator (4) and add the numerator (1).
$2 \times 4 + 1 = 9$	Now place the number 9 over the original denominator (4): $\frac{9}{4}$.

Alternatively, since $2\frac{1}{4} = 2 + \frac{1}{4}$, just split the mixed fraction into its two parts and rewrite the whole number using a common denominator:

$$2\frac{1}{4} = 2 + \frac{1}{4} = \frac{8}{4} + \frac{1}{4} = \frac{9}{4}$$

Simplify Before You Multiply

When multiplying fractions, you could first multiply the numerators together, then multiply the denominators together, and finally simplify the resulting product. For example:

$$\frac{\mathbf{8}}{\mathbf{15}} \times \frac{\mathbf{35}}{\mathbf{72}} = \frac{8(35)}{15(72)} = \frac{280}{1{,}080} = \frac{28}{108} = \frac{7}{27}$$

This is pretty painful without a calculator, though. Instead, simplify the fractions before you multiply: cancel similar terms from the top and bottom of the fractions.

Notice that the **8** in the numerator and the **72** in the denominator both have 8 as a factor. Thus, they can be simplified from $\frac{8}{72}$ to $\frac{1}{9}$. It doesn't matter that the numbers appear in two different fractions. When multiplying fractions together, you can treat all of the numerators as one group, and all of the denominators as another. You can cancel anything in the top group with anything in the bottom.

Notice also that **35** in the numerator and **15** in the denominator both have 5 as a factor. Thus, they can be simplified from $\frac{35}{15}$ to $\frac{7}{3}$.

Now the multiplication will be easier and no further simplification will be necessary:

$$\frac{{}^{1}\cancel{8}}{{}_{3}\cancel{15}} \times \frac{\cancel{35}^{\,7}}{\cancel{72}_{\,9}} = \frac{1(7)}{3(9)} = \frac{7}{27}$$

Always try to cancel factors before multiplying fractions!

In order to multiply mixed numbers, first convert each mixed number into an improper fraction:

$$2\frac{1}{3}\times 6\frac{3}{5}=\frac{7}{3}\times\frac{33}{5}$$

Then simplify before you multiply:

$$\frac{7}{{}_{1}\cancel{3}}\times\frac{\cancel{33}^{11}}{5}=\frac{7(11)}{1(5)}=\frac{77}{5}$$

Add and Subtract: Use a Common Denominator

In order to add or subtract fractions, follow these steps:

1. Find a common denominator.
2. Change each fraction so that it is expressed using this common denominator.
3. Add up the numerators only.

Here's an example:

$$\frac{3}{8}+\frac{7}{12}$$

$$\frac{9}{24}+\frac{14}{24}$$

A common denominator is 24. Thus, $\frac{3}{8}=\frac{9}{24}$ and $\frac{7}{12}=\frac{14}{24}$.

$$\frac{9}{24}+\frac{14}{24}=\frac{23}{24}$$

Finally, add the numerators to find the answer.

In this example, you have to simplify the fraction at the end:

$\mathbf{\frac{11}{15} - \frac{7}{30}}$

$\frac{22}{30} - \frac{7}{30}$ A common denominator is 30. $\frac{11}{15} = \frac{22}{30}$ and $\frac{7}{30}$ stays the same.

$\frac{22}{30} - \frac{7}{30} = \frac{15}{30}$ Subtract the numerators.

$\frac{15}{30} = \frac{1}{2}$ Simplify $\frac{15}{30}$ to find the answer: $\frac{1}{2}$.

In order to add or subtract mixed numbers, first convert to improper fractions and then solve as shown above.

Dividing Fractions: Use the Reciprocal

In order to divide fractions, use the reciprocal. You can think of the reciprocal as the fraction flipped upside down:

The reciprocal of $\mathbf{\frac{3}{4}}$ is $\frac{4}{3}$. The reciprocal of $\mathbf{\frac{2}{9}}$ is $\frac{9}{2}$.

What is the reciprocal of an integer? Think of an integer as a fraction with a denominator of 1.

Thus, the integer 5 is really just $\frac{5}{1}$. To find the reciprocal, just flip it:

The reciprocal of **5**, or $\frac{5}{1}$, is $\frac{1}{5}$. The reciprocal of **8** is $\frac{1}{8}$.

In order to divide fractions, follow these steps:

1. Change the divisor into its reciprocal (the divisor is the second number).
2. Multiply the fractions.

For example:

$$\frac{1}{2} \div \frac{3}{4}$$

First, change the divisor $\frac{3}{4}$ into its reciprocal $\frac{4}{3}$.

$$\frac{1}{\cancel{2}_1} \times \frac{\cancel{4}^2}{3} = \frac{2}{3}$$

Then simplify if needed and multiply to find the answer.

Split Up Double-Decker Fractions

The division of fractions can be shown by using the division sign, or by putting the fractions themselves into a fraction. Consider one of the previous examples:

$\frac{1}{2} \div \frac{3}{4}$ can also be written as a “double-decker” fraction this way: $\dfrac{\frac{1}{2}}{\frac{3}{4}}$

You can rewrite this as the top fraction divided by the bottom fraction, and solve normally by using the reciprocal of the bottom fraction and then multiplying:

$$\frac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2} \div \frac{3}{4} = \frac{1}{\cancel{2}} \times \frac{\cancel{4}}{3} = \frac{2}{3}$$

In addition, you can often simplify quickly by multiplying both top and bottom by a common denominator:

$$\frac{\frac{1}{2}}{\frac{3}{4}} = \frac{\frac{1}{2} \times 4}{\frac{3}{4} \times 4} = \frac{2}{3}$$

Comparing Fractions: Cross-Multiply

Which fraction is greater, $\frac{7}{9}$ or $\frac{4}{5}$?

The traditional method of comparing fractions involves finding a common denominator and comparing the two fractions. The common denominator of 9 and 5 is 45.

Thus, $\frac{7}{9}=\frac{35}{45}$ and $\frac{4}{5}=\frac{36}{45}$. In this case, $\frac{4}{5}$ is slightly greater than $\frac{7}{9}$.

However, there is a shortcut to comparing fractions: cross-multiplication. This is a process that involves multiplying the numerator of one fraction by the denominator of the other fraction, and vice versa:

$(7 \times 5) = 35 \quad (4 \times 9) = 36$	
$\frac{7}{9} \qquad \frac{4}{5}$	Set up the fractions next to each other.
	Cross-multiply the fractions and put each answer by the corresponding numerator (*not* the denominator!)
$\frac{7}{9} \quad < \quad \frac{4}{5}$	Since 35 is less than 36, the first fraction must be less than the second one.

Essentially, you have done the same thing as before—you just didn't bother to write the common denominator. This process can save you a lot of time when comparing fractions on the GMAT.

Complex Fractions: Don't Split the Denominator

A complex fraction is a fraction in which there is a sum or a difference in the numerator or the denominator. Three examples of complex fractions are:

(a) $\frac{15+10}{5}$ (b) $\frac{5}{15+10}$ (c) $\frac{15+10}{5+2}$

In example (a), the numerator is expressed as a sum.

In example (b), the denominator is expressed as a sum.
In example (c), both the numerator and the denominator are expressed as sums.

When simplifying fractions that incorporate sums or differences, remember this rule: You may split up the terms of the numerator, but you may *never* split the terms of the denominator.

For example, the terms in example (a) may be split into two fractions:

$$\frac{15+10}{5} = \frac{15}{5} + \frac{10}{5} = 3 + 2 = 5$$

But the terms in example (b) may not be split:

$$\frac{5}{15+10} \neq \frac{5}{15} + \frac{5}{10} \quad \textbf{NO!}$$

Instead, simplify the denominator first:

$$\frac{5}{15+10} = \frac{5}{25} = \frac{1}{5}$$

The terms in example (c) may not be split either:

$$\frac{15+10}{5+2} \neq \frac{15}{5} + \frac{10}{2} \quad \textbf{NO!}$$

Instead, simplify both parts of the fraction:

$$\frac{15+10}{5+2} = \frac{25}{7} = 3\frac{4}{7}$$

Often, GMAT problems will involve complex fractions with variables. On these problems, it is tempting to split the denominator. Do not fall for it!

$$\frac{5x-2y}{x-y} \neq \frac{5x}{x} - \frac{2y}{y}$$

The reality is that $\frac{5x-2y}{x-y}$ cannot be simplified further, because neither of the terms in the numerator shares a factor with the entire denominator.

On the other hand, the expression $\frac{6x-15y}{10}$ can be simplified by splitting the numerator. Both terms in the numerator share a factor with the denominator, and by splitting into two fractions, you can write each part in simplified form:

$$\frac{6x-15y}{10}=\frac{6x}{10}-\frac{15y}{10}=\frac{3x}{5}-\frac{3y}{2}$$

Problem Set

For problems #1−5, decide whether the given operation will cause the original value to **increase**, **decrease**, or **stay the same**.

1. Multiply the numerator of a positive, proper fraction by $\frac{3}{2}$.

2. Add 1 to the numerator of a positive, proper fraction and subtract 1 from its denominator.

3. Multiply both the numerator and denominator of a positive, proper fraction by $3\frac{1}{2}$.

4. Multiply a positive, proper fraction by $\frac{3}{8}$.

5. Divide a positive, proper fraction by $\frac{3}{13}$.

Solve problems 6–15.

6. Simplify: $\frac{10x}{5+x}$

7. Simplify: $\frac{8(3)(x)^2(3)}{6x}$

8. Simplify: $\dfrac{\dfrac{3}{5}+\dfrac{1}{3}}{\dfrac{2}{3}+\dfrac{2}{5}}$

9. Simplify: $\dfrac{12ab^3-6a^2b}{3ab}$ (given that $ab \neq 0$)

10. Are $\dfrac{\sqrt{3}}{2}$ and $\dfrac{2\sqrt{3}}{3}$ reciprocals?

Solutions

1. **Increase:** Multiplying the numerator of a positive fraction by a number greater than 1 increases the numerator. As the numerator of a positive, proper fraction increases, its value increases.

2. **Increase:** As the numerator of a positive, proper fraction increases, the value of the fraction increases. As the denominator of a positive, proper fraction decreases, the value of the fraction also increases. Both actions will work to increase the value of the fraction.

3. **Stay the same:** Multiplying or dividing the numerator and denominator of a fraction by the same number will not change the value of the fraction.

4. **Decrease:** Multiplying a positive number by a proper fraction decreases the number.

5. **Increase:** Dividing a positive number by a positive, proper fraction increases the number.

6. **Cannot simplify:** There is no way to simplify this fraction; it is already in simplest form. Remember, you cannot split the denominator!

7. **12*x***: First, cancel terms in both the numerator and the denominator. Then combine terms:

$$\frac{8(3)(x)^2(3)}{6x}=\frac{8(\cancel{3})(x)^2(3)}{\cancel{6}2x}=\frac{\cancel{8}4(x)^2(3)}{\cancel{2}x}=\frac{4(x)^{\cancel{2}}(3)}{\cancel{x}}=4(x)(3)=12x$$

8. $\mathbf{\frac{7}{8}}$**:** First, add the fractions in the numerator and denominator:

$$\frac{\frac{14}{15}}{\frac{16}{15}}=\frac{\overset{7}{\cancel{14}}}{\underset{1}{\cancel{15}}}\times\frac{\overset{1}{\cancel{15}}}{\underset{8}{\cancel{16}}}=\frac{7}{8}$$

Alternatively, to save time, multiply each of the small fractions by 15, which is the common denominator of all the fractions in the problem. Because you are multiplying the numerator *and* the denominator of the whole complex fraction by 15, you are not changing its value:

$$\frac{9+5}{10+6}=\frac{14}{16}=\frac{7}{8}$$

9. $\mathbf{2(2b^2 - a)}$ **or** $\mathbf{4b^2 - 2a}$**:** First, factor out common terms in the numerator. Then, cancel terms in both the numerator and denominator:

$$\frac{6ab(2b^2-a)}{3ab}=2(2b^2-a) \text{ or } 4b^2-2a$$

10. **Yes:** The product of a number and its reciprocal must equal 1. To test whether or not two numbers are reciprocals, multiply them. If the product is 1, they are reciprocals; if it is not, they are not:

$$\frac{\sqrt{3}}{2}\times\frac{2\sqrt{3}}{3}=\frac{2\left(\sqrt{3}\right)^2}{2(3)}=\frac{6}{6}=1$$

Thus, the numbers are indeed reciprocals.

Chapter 5
of
Fractions, Decimals, & Percents

Percents

In This Chapter…

Chapter 5

Percents

The third component of the FDP trifecta is percents. Percent literally means "per one hundred." You can think of a percent as simply a special type of fraction or decimal that involves the number 100:

> 75% of the students like chocolate ice cream.

This means that, out of every 100 students, 75 like chocolate ice cream.

In fraction form, you write this as 75/100, which simplifies to 3/4.

In decimal form, you write this as 0.75.

One common mistake is the belief that 100% equals 100. This is not correct. In fact, 100% means 100/100. Therefore, 100% = 1.

Percents as Decimals: Multiplication Shortcut

You can convert percents into decimals by moving the decimal point two spaces to the left:

525% = 5.25	52.5% = 0.525	5.25% = 0.0525	0.525% = 0.00525

A decimal can be converted into a percent by moving the decimal point two spaces to the right:

0.6 = 60%	0.28 = 28%	0.459 = 45.9%	1.3 = 130%

Strategy Tip: Remember, the percent is always "bigger" than the decimal!

Percent, Of, Is, What

These four words are by far the most important when translating percent questions. In fact, many percent word problems can be rephrased in terms of these four words:

Percent	=	divide by 100	(/100)
Of	=	multiply	($\times$)
Is	=	equals	(=)
What	=	unknown value	(x, y, or any variable)

What is 70 percent of 120?

As you read left to right, translate the question into an equation:

x	=	70	/100	$\times$	120
What	is	70	percent	of	120?

Now solve the equation:

$$x = \frac{70}{100} \times 120$$
$$x = \frac{7}{10} \times 120$$
$$x = 7 \times 12$$
$$x = 84$$

This translation works no matter what order the words appear in.

30 is what percent of 50?

This statement can be translated directly into an equation:

30	=	x	/100	×	50
30	is	what	percent	of	50?

Every time you create one of these equations, your goal is the same: Solve for the unknown.

In the above examples, *x* represents the unknown value that you have been asked to find. By isolating *x*, you will answer the question:

$$30 = \frac{x}{100} \times 50$$

$$30 = \frac{x}{2}$$

$$60 = x$$

Look for Percent, Of, Is, and What as you translate percent problems into equations; those four words should provide the necessary structure for each equation. As you get better with translation, you may feel comfortable using a shortcut when the problem asks *30 is what percent of 50*? You can always translate this form as:

$$\frac{30}{50} = x\%$$

Quick Calculations: Building Percents

You can calculate most percentages quickly using a combination of 50%, 10%, 5%, and 1% of the original number.

For example, the previous section asked you to find 70% of 120. Note that 70% is the equivalent of 50% + 10% + 10%. It is much easier to calculate 50% and 10% of a number:

100% (original number)	**50%**	**10%**	**50% + 10% + 10% = 70%**
120	60	12	60 + 12 + 12 = 84

Likewise, to calculate 15% of a number, add 10% and 5%:

What is 15% of 90?

$$
\begin{aligned}
100\% &= 90 \\
10\% &= 9 \\
5\% &= 4.5 \\
15\% &= 9 + 4.5 = 13.5
\end{aligned}
$$

Test your skills on these drills:

1. What is 7% of 50?
2. What is 40% of 30?
3. What is 75% of 20?

Here are the answers:

1. 100% = 50

   5% = 2.5 (10% = 5, so 5% = half of that)

   1% = 0.5

   7% = 5% + 1% + 1% = 2.5 + 0.5 + 0.5 = 3.5

2. 100% = 30

   10% = 3

   40% = (4)(10%) = (4)(3) = 12

3. Don't forget about your fraction-conversion skills! Sometimes, it's easier to convert to fractions and cancel: $75\% = \frac{3}{4}$.

$$\frac{3}{{}_1\cancel{4}}\left(\cancel{20}^{5}\right) = (3)(5) = 15$$

Why is it (arguably) easier to use the **building percents** method on the first two problems, but easier to use fractions on the third problem?

Most people don't memorize the fraction conversion for 7%, so the first problem is definitely easier to build via percents.

The second problem could go either way, but because 40% is a multiple of 10%, and 10% is very easy to find, building the answer is quick.

In the third problem, 75% would take multiple steps to build via the percent method. It turns out that 75% also converts to a very nice fraction: $\frac{3}{4}$. In this case, it will probably be easier to use the fraction here (especially because the starting number, 20, is a multiple of 4, so the denominator will cancel entirely!).

Percent Increase and Decrease

Consider this example:

> The price of a cup of coffee increased from 80 cents to 84 cents. By what percent did the price change?

If you want to find a change, whether in terms of percent or of actual value, use the following equation:

$$\textbf{Percent Change} = \frac{\textbf{Change in Value}}{\textbf{Original Value}}$$

In the coffee example, you want to find the *change* in terms of percent. Write $\frac{x}{100}$ to represent an unknown percent:

$$\begin{aligned} \%\ \text{Change} &= \frac{\text{Change}}{\text{Original}} \\ \frac{x}{100} &= \frac{4}{80} = \frac{1}{20} \\ 20x &= 100 \\ x &= 5 \end{aligned}$$

Therefore, the price has been increased 5%.

Alternatively, a question might ask:

> If the price of a $30 shirt is decreased by 20%, what is the final price of the shirt?

In this case, the question didn't tell you the new percent; rather, it gave the percent decrease. If the price decreases by 20%, then the new price is 100% − 20% = 80% of the original. Use the new percent, not the decrease in percent, to solve for the new price directly. You can use this equation:

$$\text{New Percent} = \frac{\text{New Value}}{\text{Original Value}}$$

Once again, use *x* to represent the value you want, the new price:

$$\frac{80}{100} = \frac{x}{30}$$

$$\frac{4}{5} = \frac{x}{30}$$

$$\frac{4}{1} = \frac{x}{6}$$

$$24 = x$$

The new price of the shirt is $24.

Alternatively, you can solve directly without setting up a proportion. The starting price is $30 and this price is decreased by 20%. Find 20% of $30 and subtract:

$30 − (20%)($30)   20% of 30 is 6
$30 − $6 = $24

Increasing or Decreasing from the Original

The language on the GMAT can sometimes be confusing. The original number is the starting point for a comparison. For example, if a problem asks how much smaller the population was in 1980 than in 1990, then the 1990 population is the starting point, or original number. When talking about a percent change made to a number, always think of the original number as 100%.

For example, if you increase a number by 10%, then the new number will be 110% of the original number. Here are some common language cues for this concept:

> 10% increase = 110% of the original
> 10% greater than = 110% of the original

If you decrease a number, then you subtract from 100%:

> 45% decrease = 55% of the original
> 45% less than = 55% of the original

Use this conversion to save steps on percent problems. For example:

> What number is 50% greater than 60?

50% greater than is the same as *150% of*. Rewrite the question:

> What number is 150% of 60?

Translate into an equation, using 1.5 to represent 150%:

> $x = 1.5 \times 60$
> $x = 90$

Successive Percent Change

Some problems will ask you to calculate successive percents. For example:

> If a ticket increased in price by 20%, and then increased again by 5%, by what percent did the ticket price increase in total?

Although it may seem counterintuitive, the answer is *not* 25%.

Walk through this with real numbers. If the ticket originally cost \$100, then the first increase would bring the ticket price up to 100 plus 20% of 100 (or $20) for a total of $120.

The second increase of 5% is now based on this *new* ticket price, $120:

120 + (0.05)(120) = $126

The price increased from $100 to $126, so the percent increase is the change divided by the original, or $\frac{26}{100}$ = 26%.

Successive percents *cannot* simply be added together; instead, you have to calculate each piece separately. This holds for successive increases, successive decreases, and for combinations of increases and decreases.

Try this problem:

> The cost of a plane ticket is increased by 25%. Later, the ticket goes on sale and the price is reduced 20%. What is the overall percent change in the price of the ticket?

You can *multiply* these changes together; you can't just add or subtract them. A 25% increase followed by a 20% decrease is the same as 125% of 80% of the original number:

$$\left(\frac{125}{100}\right)\left(\frac{80}{100}\right)x =$$

$$\left(\frac{5}{4}\right)\left(\frac{4}{5}\right)x = x$$

The 20% decrease entirely offsets the 25% increase. The new price is exactly the same as the original price. You can also work through the math using a real number, as shown in the previous problem:

$100 + (25% of $100) = $125

$125 − (20% of $125) = $100

Interest Formulas

Certain GMAT problems require a working knowledge of compound interest. If you struggle to memorize formulas and are not looking for an 85th percentile or higher Quant score, you may be able to skip memorizing the formula. Instead, learn how to do some of the easier problems and guess if you hit a harder one.

$$\text{Total Amount} = P\left(1+\frac{r}{n}\right)^{nt}$$

In this equation, P = principal, r = rate (in decimal form), n = number of times per year, and t = number of years.

Some of these can be solved as successive percents problems:

> A bank account with $200 earns 5% annual interest, compounded annually. If there are no deposits or withdrawals, how much money will the account have after 2 years?

First, calculate how many times the interest will compound. It compounds once a year for 2 years, so it will compound twice.

If the account earns 5% interest, that is an increase of 5% each year. In other words, the new value of the account is 105% of 105% of $200.

At the end of the first year, the account will have $200 + 5% of $200. Calculate 5% by taking 10% of $200 and then dividing by 2 to get $10:

200	20	10
starting number	10% of starting number	5% of starting number

At the end of the first, year, then, the account will have $210.

For the second year, your starting point is $210. At the end of the second year, the account will have: 210 + (0.05)(210) = 210 + 10.5 = $220.50. (You can use the same procedure to find 5% here. Also, notice that the total increase is just a little bit more than it would have been without compounding. It would take a lot of compounding periods to make a big difference. This can be important if you're trying to narrow down the answer choices.)

If the interest compounds quarterly, the process is the same, although there is one extra step. For example:

> A bank account with $100 earns 8% annual interest, compounded quarterly. If there are no deposits or withdrawals. how much money will be in the account after 6 months?

First, figure out how many times the interest compounds. If the interest compounds quarterly, it compounds every 3 months. In 6 months, the interest will compound twice.

Next, the 8% interest is spread evenly throughout the year. The interest compounds 4 times each year, so each time it compounds at one-fourth of the total interest.

In other words, each time it compounds, there is a 2% increase. After 6 months, the new value of the account will be 102% of 102% of $100:

At the end of the first year, the account will have $100 plus 2% of $100. Calculate 2% by finding 1% and multiplying by 2:

100	1	2
starting number	1% of starting number	2% of starting number

At the end of the first, year, then, the account will have $102.

At the end of the second year, the account will have: 102 + (0.02)(102) = 102 + 2.04 = $104.04. (You can use the same procedure to find 2% here.)

For shorter calculations, you can avoid the compound interest formula by treating the problem as a successive percent change problem.

Problem Set

1. A stereo was marked down by 30% and sold for $84. What was the presale

price of the stereo?

2. A car loan is offered at 8% annual interest, compounded annually. After the first year, the interest due is $240. What is the principal on the loan?

3. *x* is 40% of *y*. 50% of *y* is 40. 16 is what percent of *x*?

4. A bowl is half full of water. Four cups of water are then added to the bowl, filling the bowl to 70% of its capacity. How many cups of water are now in the bowl?

5. Company X has exactly two product lines and no other sources of revenue. If the consumer products line experiences a *k*% increase in revenue (where *k* is a positive integer) in 2015 from 2014 levels and the machine parts line experiences a *k*% decrease in revenue in 2015 from 2004 levels, did Company X's overall revenue increase or decrease in 2010?

   (1) In 2014, the consumer products line generated more revenue than the machine parts line.

   (2) $k = 8$

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

11. 800, increased by 50% and then decreased by 30%, yields what number?

12. If 1,500 is increased by 20% and then reduced by *y*%, yielding 1,080, what is *y*?

13. A bottle is 80% full. The liquid in the bottle consists of 60% guava juice and 40% pineapple juice. The remainder of the bottle is then filled with 200 mL of rum. How much guava juice is in the bottle?

14. Company Z only sells chairs and tables. What percent of its revenue in 2008 did Company Z derive from its sales of chairs?

    (1) In 2008, the price of tables sold by Company Z was 10% higher than the price of chairs sold by Company Z.

    (2) In 2008, Company Z sold 20% fewer tables than chairs.

Solutions

1. **$120:** You know the new price of the stereo, so you can use the following

formula:

$$\text{New Percent} = \frac{\text{New Value}}{\text{Original Value}}$$

The new value of the stereo is $84. If the price of the stereo was marked down 30%, then the new percent is 100 − 30 = 70%. Finally, the original value is the value you're asked for, so you can replace it with *x*:

$$\frac{70}{100} = \frac{84}{x}$$
$$\frac{7}{10} = \frac{84}{x}$$
$$7x = 840$$
$$x = 120$$

Alternatively, you could rephrase the given information to say the following:

$84 is 70% of the original price of the stereo.

You could then translate this statement into an equation and solve:

$$84 = \left(\frac{70}{100}\right)x$$
$$84 = \left(\frac{7}{10}\right)x$$
$$840 = 7x$$
$$120 = x$$

2. **$3,000:** Although this looks like an interest problem, you can think of it as a percent change problem. The percent change is 8%, and the change in value is $240:

$$\text{Percent Change} = \frac{\text{Change in Value}}{\text{Original Value}}$$

$$\frac{8}{100} = \frac{240}{x}$$

$$8x = 24{,}000$$

$$x = 3{,}000$$

The principal amount of the loan is $3,000.

Alternatively, you could rephrase the given information to say the following:

$240 is 8% of the total loan.

You could then translate this statement into an equation and solve:

$$240 = \left(\frac{8}{100}\right)x$$

$$24{,}000 = 8x$$

$$3{,}000 = x$$

3. **50%:** You can translate the first two sentences directly into equations:

x is 40% of y → $x = \left(\frac{40}{100}\right)y$

50% of y is 40 → $\left(\frac{50}{100}\right)y = 40$

You can solve the second equation for *y*:

$$\left(\frac{50}{100}\right)y = 40$$

$$\left(\frac{1}{2}\right)y = 40$$

$$y = 80$$

Now you can replace *y* with 80 in the first equation to solve for *x*:

$$x = \left(\frac{40}{100}\right)(80)$$
$$x = \frac{4}{10} \times 80$$
$$x = 4 \times 8 = 32$$

Be careful now. The question asks, “16 is what percent of *x*?”

You know that *x* = 32, so this question is really asking, “16 is what percent of 32?”

Create a new variable (*z*) to represent the unknown value in the question and solve:

$$16 = \frac{z}{100} \times 32$$
$$16 \times \frac{100}{32} = z$$
$$1 \times \frac{100}{2} = z$$
$$50 = z$$

Thus, 16 is 50% of *x*.

4. **14 cups of water:** If the bowl was already half full of water, then it was originally 50% full. Adding 4 cups of water increased the percentage by 20% of the total capacity of the bowl.

You can use the percent change formula to solve for the total capacity of the bowl:

$$\text{Percent Change} = \frac{\text{Change in Value}}{\text{Original Value}}$$

$$\frac{20}{100} = \frac{4}{x}$$

$$\frac{1}{5} = \frac{4}{x}$$

$$x = 20$$

The total capacity of the bowl is 20 cups, but the question asks for the total number of cups currently in the bowl. You know the bowl is 70% full. You can ask the question, "What is 70% of 20?" and solve:

$$x = \frac{70}{100} \times 20$$

$$x = \frac{7}{10} \times 20$$

$$10x = 140$$

$$x = 14$$

There are 14 cups of water in the bowl.

Alternatively, you can save time by solving directly for 70% rather than by first solving for the full capacity. You know 4 represents 20% of the capacity. Let *x* represent 70% of the capacity. Set up a proportion and solve for *x*:

$$\frac{4}{x} = \frac{20}{70}$$

$$\frac{4}{x} = \frac{2}{7}$$

$$28 = 2x$$

$$14 = x$$

5. (**A**)**:** This question requires you to employ logic about percents. No calculation is required, or even possible.

Here's what you know so far (use new variables *c* and *m* to keep track of your information):

2014:

consumer products makes c dollars
machine parts makes m dollars
total revenue = $c + m$

2015:

consumer products makes “c dollars increased by k%”
machine parts makes “m dollars decreased by k%”
total revenue = ?

What would you need to answer the question, “Did Company X's overall revenue increase or decrease in 2015?” Certainly, if you knew the values of c, m, and k, you could achieve sufficiency, but the GMAT would never write such an easy problem. What is the *minimum* you would need to know to answer definitively?

Since both changes involve the same percent (k), you know that c increases *by the same percent* by which m decreases. So, you don't actually need to know k. You already know that k percent of whichever number is greater (c or m) will constitute a bigger change to the overall revenue.

All this question is asking is whether the overall revenue went up or down. If c started off greater, then a k% increase in c means more new dollars coming in than you would lose due to a k% decrease in the smaller number, m. If c is smaller, then the k% increase would be smaller than what you would lose due to a k% decrease in the larger number, m.

The question can be rephrased, “Which is greater, c or m?”

(1) SUFFICIENT: This statement tells you that c is greater than m. Thus, a k% increase in c is greater than a k% decrease in m, and the overall revenue went up.

(2) INSUFFICIENT: Knowing the percent change doesn't help, since you don't know whether c or m is bigger.

Note that you could plug in real numbers if you wanted to, although the problem is faster with logic. Using statement (1) only:

2014:

consumer products makes \$200
machine parts makes \$100
total revenue = \$300

2015: *if k = 50*

consumer products makes \$300
machine parts makes \$50
total revenue = \$350

This yields an answer of yes—the overall revenue did increase. However, you might have to test several sets of numbers to establish that this will always be true. (That's the main reason that logic is faster here!) You can experiment with different values for *c* and *m*, and you can change *k* to any positive integer (you don't need to know what *k* is). As long as *c* is greater than *m*, you will get the same result. The increase to the larger *c* will be greater than the decrease to the smaller *m*.

The correct answer is **(A)**.

11. **840:** This is a successive percent question. 800 increased by 50% and decreased by 30% is the same as 150% of 70% of 800:

$$\frac{150}{100} \times \frac{70}{100} \times 800 =$$
$$\frac{3}{2} \times \frac{7}{10} \times 800 =$$
$$\frac{21}{20} \times 800 =$$
$$21 \times 40 = 840$$

12. **40:** Break the question into two parts.

First, 1,500 is increased by 20%. Thus, 120% percent of 1,500 is:

$$\left(\frac{120}{100}\right)1{,}500 =$$

$$\left(\frac{6}{5}\right)1{,}500 =$$

$$(6)300 = 1{,}800$$

You are solving for *y*, which represents the percent change of 1,800:

$$\text{Percent Change} = \frac{\text{Change in Value}}{\text{Original Value}}$$

The change in value from 1,800 to 1,080 is 1,800 − 1,080 = 720:

$$\frac{y}{100} = \frac{720}{1{,}800}$$

You can save time by noticing that 720 and 1,800 are both divisible by 360:

$$\frac{y}{100} = \frac{2}{5}$$

$$y = \frac{200}{5} = 40$$

13. **480 mL**: You can begin by figuring out what the total amount of liquid is.

If the bottle was 80% full, and adding 200 mL of rum made the bottle full, then 200 mL is equal to 20% of the total capacity of the bottle. Let *b* be the total capacity of the bottle:

$$200 = \frac{20}{100}b$$

$$200 = \frac{1}{5}b$$

$$1{,}000 = b$$

The bottle has a total capacity of 1,000 mL.

Now you can use a successive percent to figure out the total amount of guava juice. You know 80% of the bottle is filled with juice, and 60% of the juice is guava juice. In other words, guava juice is 60% of 80% of 1,000 mL:

$$g = \left(\frac{60}{100}\right)\left(\frac{80}{100}\right)1{,}000$$
$$g = \left(\frac{3}{5}\right)\left(\frac{4}{5}\right)1{,}000$$
$$g = \left(\frac{3}{1}\right)\left(\frac{4}{1}\right)40$$
$$g = 3 \times 160$$
$$g = 480$$

There is 480 mL of guava juice.

Note that the last calculation, 3 × 4 × 40, can be done in any order that is easiest for you. Most of the time, multiplying the largest numbers together first is easier in the long run

14. **(C):** First of all, notice that the question is only asking for the *percent* of its revenue the company derived from chairs. The question is asking for a relative value.

The revenue for the company can be expressed by the following equation:

$$\text{Revenue}_{\text{Company Z}} = R_T + R_C$$

If you can find the relative value of *any* two of these revenues, you will have enough information to answer the question.

Also, note that the GMAT will expect you to know that Revenue = Price × Quantity Sold. This relationship is discussed in more detail in Chapter 1 of the *Word Problems GMAT Strategy Guide*.

The revenue derived from tables is the price per table multiplied by the number of tables sold. The revenue derived from chairs is the price per chair multiplied by the number of chairs sold. You can create some variables to represent these unknown values:

$$R_T = P_T \times Q_T$$

$$R_C = P_C \times Q_C$$

(1) INSUFFICIENT: This statement gives you the relative value of the price of tables to the price of chairs. If the price of tables was 10% higher than the price of chairs, then the price of tables was 110% of the price of chairs:

$$P_T = 1.1P_C$$

However, without any information on quantity, this information by itself does not give you the relative value of their revenues.

(2) INSUFFICIENT: If the company sold 20% fewer tables than chairs, then the number of tables sold is 80% of the number of chairs sold:

$$Q_T = 0.8Q_C$$

This information by itself, without any information about price, does not give you the relative value of their revenues.

(1) AND (2) SUFFICIENT: Look again at the equation for revenue derived from the sale of tables:

$$R_T = P_T \times Q_T$$

Replace P_T with $1.1P_C$ and replace Q_T with $0.8Q_C$:

$$P_T \times Q_T = (1.1P_C) \times (0.8Q_C)$$

$$P_T \times Q_T = (0.88)(P_C \times Q_C)$$

$$R_T = (0.88)R_C$$

Taken together, the two statements provide the relative value of the revenues. No further calculation is required to know that you *can* find the percent of revenue generated from the sale of chairs. In fact, if you know that you can relate tables to chairs in terms of both quantity and price, it isn't even necessary to calculate revenue at all. The statements together are sufficient.

If you do want to perform the calculation, it would look something like this:

$$\frac{R_C}{\text{Total Revenue}} = \frac{R_C}{R_T + R_C} = \frac{R_C}{(0.88)R_C + R_C} = \frac{\cancel{R_C}}{(1.88)\,\cancel{R_C}} = \frac{1}{1.88} \approx 53\%$$

Save time on Data Sufficiency questions by avoiding unnecessary computation. Once you know you can find the percent, stop and move on to the next problem.

The correct answer is **(C)**.

Chapter 6
of

Fractions, Decimals, & Percents

Strategy: Choose Smart Numbers

In This Chapter…

Chapter 6

Strategy: Choose Smart Numbers

Some algebra problems—problems that involve unknowns, or variables—can be turned into arithmetic problems instead. Such problems are commonly tested with fractions, percents, and even ratios. You're better at arithmetic than algebra (everybody is!), so turning an annoying variable-based problem into one that uses real numbers can save time and aggravation on the GMAT.
Which of the below two problems is easier for you to solve?

50% of 10% of a number is what percent of that number?	50% of 10% of 100 is what percent of 100?
(A) 1% (B) 5% (C) 10%	(A) 1% (B) 5% (C) 10%

In the first problem, you would assign a variable to the unknown *number* mentioned, and then you would use algebra to solve. You may think that this version is not particularly difficult, but no matter how easy you think it is, it's still easier to work with real numbers.

The set-up of the two problems is identical—and this feature is at the heart of how you can turn algebra into arithmetic.

How Do Smart Numbers Work?

Here's how to solve the algebra version of the above problem using smart numbers:

Step 1: *Choose smart numbers* to replace the unknowns.

How do you know you can choose a random number in the first place? The problem talks about a number but never supplies a real value for that number anywhere in the problem or in the answers. If you were to do this problem algebraically, you would have to assign a variable.

Instead, choose a real number. In general, 100 is a great number to choose on percent problems.

Step 2: Solve the problem using your chosen smart numbers.
Wherever the problem talks about the *number*, assume it now says 100:

> 50% of 10% of 100 is what percent of 100?

Do the math! 50% of 100 is 50; 10% of 50 is 5. Therefore, the question is asking: 5 is what percent of 100?

Now you can see the beauty of starting with 100 on a percent problem: $\frac{5}{100}$ = 5%. You don't actually have to convert to a percent in the end!

Step 3: Find a match in the answers.
The correct answer is **(B)**.

FDP smart numbers problems can appear with percents, fractions, or ratios in the answers, in which case you'll use the three-step process shown above. Problems can also appear with variables in the answers, in which that third step (find a match) will take a little more work. You'll try one of these later in this chapter.

Smart Numbers with Percents

It's crucial to know when you're allowed to use this technique. It's also crucial to know how *you* are going to decide whether to use textbook math or to choose smart numbers; you will typically have time to try just one of the two techniques during your 2 minutes on the problem.

The *choose smart numbers* technique can be used any time a problem contains only *unspecified* values. The easiest example of such a problem is one that contains variables, percents, fractions, or ratios throughout. It does not provide real numbers for those variables, even in the answer choices. Whenever a problem has this characteristic, you can choose your own smart numbers to turn the problem into arithmetic.

There is some cost to doing so: it can take extra time compared to the "pure" textbook solution. As a result, the technique is most useful when the problem is a hard one for you. If you find the abstract math involved to be very easy, then you may not want to take the time to transform the problem into arithmetic. As the math gets more complicated, however, the arithmetic form becomes comparatively easier and faster to use. For instance, in the initial example, maybe you quickly saw that 50% of 10% is equal to 5%, and you didn't see the need for smart numbers. Don't let this put you off of using the strategy when the going gets tough. Test-takers at every level are likely to encounter at least a few problems that are much easier to solve with real numbers than with unknowns.

Try this problem using smart numbers:

> The price of a certain computer is increased by 10%, and then the new price is increased by an additional 5%. The new price is what percent of the original price?
>
> (A) 120%
>
> (B) 115.5%
>
> (C) 115%
>
> (D) 112.5%
>
> (E) 110%

First, how do you know that you can choose smart numbers on this problem? The problem talks about the price of a computer but never mentions a real number for that price anywhere along the way.

Step 1: Choose smart numbers.

This percent problem doesn't already use 100, so choose 100 for the starting price of the computer.

Step 2: Solve.

First, the computer's price is increased by 10%, so the new price is: \$100 + \$10 = \$110. Next, the *new* price is increased by a further 5%, so the price becomes: \$110 + (0.05)(\$110) = \$110 + \$5.50 = \$115.50.

The new price is $115.50 and the original price is $100, so $\frac{115.50}{100}$ equals 115.5%. In other words, you can ignore the denominator; your final number already represents the desired percent.

Step 3: Find a match.

(A) 120%

(B) 115.5% → match

(C) 115%

(D) 112.5%

(E) 110%

The correct answer is **(B)**. Notice that the answer is just slightly larger than you'd get if you simply increased the price by 15%.

Let's say that you weren't able to choose $100 for some reason. Instead, you chose an initial price of $20. How would the problem work? Try it out before continuing to read.

If the initial price is $20, then the first increase of 10% would bring the price to: $20 + $2 = $22. The next increase of 5% would bring the price to: $22 + (0.05)($22) = $22 + $1.1 = $23.10.

Tip: to calculate 5% quickly, first find 10% of the desired number, then halve the number. For example, to find 5% of 22, first find 10%: 2.2. Then halve that number: $\frac{2.2}{2} = 1.1$.

Now what? $23.10 isn't in the answers. Remember that you didn't start with 100! The new price as a percent of the original is $\frac{23.10}{20}$. How do you turn that into a percent?

First, remember that percent means "per 100." Manipulate the fraction until you get 100 on the bottom.

$$\frac{23.10}{20} \times \frac{5}{5} = \frac{115.5}{100}$$

The answer is 115.5%.

That last calculation is annoying—you don't want to do it unless you have to. Therefore, if you *can* pick 100 on a percent problem, do so.

Smart Numbers with Fractions

Try this problem:

> Two libraries are planning to combine a portion of their collections in one new space. $\frac{1}{3}$ of the books from Library A will be housed in the new space, along with $\frac{1}{4}$ of the books from Library B. If there are twice as many books in Library B as in Library A, what proportion of the books in the new space will have come from Library A?
>
> (A) $\frac{1}{3}$
>
> (B) $\frac{2}{5}$
>
> (C) $\frac{1}{2}$
>
> (D) $\frac{3}{5}$
>
> (E) $\frac{7}{12}$

Step 1: Choose smart numbers.
When working with fraction problems, choose a common denominator of the fractions given in the problem. In this case, the problem contains the fractions $\frac{1}{3}$

and $\frac{1}{4}$, so use the common denominator of 12.

Note two things. First, technically, any multiple of 12 will work nicely, but keep things simple: use the smallest common denominator possible.

Second, what should you call 12? All of the books from Library A? Or from Library B? The number of books that will be moved from Library A to the new space? Or from Library B to the new space?

Assign the value 12 to the total for the smaller library—in this case, Library A—because the other library has twice as many books. That is, Library B's capacity, 24, is a multiple of Library A's capacity; Library B's capacity is also a multiple of the denominators 3 and 4.

Step 2: Solve.

> If Library A has 12 books total, then Library B must have 24 books total.
> $\frac{1}{3}$ of Library A's books, or $(12)\left(\frac{1}{3}\right)=4$, will move to the new space.
> $\frac{1}{4}$ of Library B's books, or $(24)\left(\frac{1}{4}\right)=6$, will move to the new space.
> The new space will therefore contain 10 books total. Because 4 out of 10 of those books came from Library A, 40%, or $\frac{2}{5}$ of the books in the new space will have come from Library A.

Step 3: Find a match.

The correct answer is **(B)**.

Bonus Exercise: Take a look at the wrong answers. Can you figure out how someone would have gotten to any of them?

Answer (A), $\frac{1}{3}$, is the original figure given for the portion of Library A's books moved to the new space. This represents the moved books as a proportion of the original library's capacity, not the new space's capacity.

Answer (D), $\frac{3}{5}$, represents the proportion of Library B's books in the new space. If you calculated this answer, then you solved for the wrong thing.

Answer (E), $\frac{7}{12}$, represents the sum of $\frac{1}{3}$ and $\frac{1}{4}$. You can't just add up the two starting fractions, however, because the two libraries have a different number of books to start.

Smart Numbers with Variables in the Answers

This same strategy works when there are variables, rather than percents or fractions, in the answers. However, you'll have to add one more piece to the third step.

Try this problem:

> The Crandall's hot tub has a capacity of *x* liters and is half full. Their swimming pool, which has a capacity of *y* liters, is filled to four-fifths of its capacity. If enough water is drained from the swimming pool to fill the hot tub to capacity, the pool is now how many liters short of full capacity, in terms of *x* and *y*?
>
> (A) $0.8y - 0.5x$
>
> (B) $0.8y + 0.5x$
>
> (C) $0.2y + 0.5x$
>
> (D) $0.3(y - x)$
>
> (E) $0.3(y + x)$

Step 1: Choose smart numbers.
The problem keeps talking about the capacity of the hot tub and pool but never offers a real number for either. Instead, it introduces the variables *x* and *y*, which also appear in the answers. You can use smart numbers on this problem.

The two variables are not connected (that is, once you pick one, the other one is not automatically determined), so you'll have to pick two numbers. Pick something divisible by 2 for x and divisible by 5 for y:

$x = 4$ and $y = 20$

Step 2: Solve.

The hot tub, with a capacity of 4, is half full, so there are 2 liters of water in the hot tub. The pool, with a capacity of 20, is four-fifths full, so there are 16 liters in the pool.

The hot tub needs another 2 liters to be full, so the pool will have to lose 2 liters; the pool is now down to 14 liters. Since its total capacity is 20, the pool is 6 liters short of capacity.

Step 3: Find a match.

When the answers contain variables, you have to add an intermediate step: plug your starting values, $x = 4$ and $y = 20$, into the answers to find the match:

(A) $0.8y - 0.5x = (0.8)(20) - (0.5)(4) = 16 - 2 = 14$

(B) $0.8y + 0.5x = (0.8)(20) + (0.5)(4) =$ too big, since (A) was too big

(C) $0.2y + 0.5x = (0.2)(20) + (0.5)(4) = 4 + 2 = 6$ Match!

(D) $0.3(y - x) = 0.3(20 - 4) = 0.3(16) =$ not an integer

(E) $0.3(y + x) = 0.3(20+4) = 0.3(24) =$ not an integer

The correct answer is **(C)**.

Note that some of the work shown above seems to be incomplete; that is, each answer was not calculated fully. Your goal is to find a match; in this case, you want to find the answer that matches 6. You do not actually need to figure out the value of each answer. As soon as you can tell that the value will *not* be 6, stop and cross off that answer.

When the answers contain variables, there is one potential problem to watch for when you choose smart numbers. In certain circumstances, the number(s) you choose will work for more than one answer choice. In that case, you can guess and move on, but it's generally not too hard to adjust by changing one of your numbers and seeing which answer choice still works.

If you follow the guidelines for choosing numbers, you will greatly reduce the chances that your smart number(s) will work for more than one answer:

- Do not pick 0 or 1.
- Do not pick numbers that appear elsewhere in the problem.
- If you have to choose multiple numbers, choose different numbers, ideally with different properties (e.g., odd and even). The above case used even numbers for both due to the fractions mentioned in the problem, but note that the numbers chosen were fairly far apart.

To summarize the choose smart numbers strategy:

Step 0: Recognize that you can choose smart numbers.

The problem talks about some values but doesn't provide real numbers for those values. Rather, it uses variables or only refers to fractions or percents. The answer choices consist of variable expressions, fractions, or percents.

Step 1: Choose smart numbers.

Follow all constraints given in the problem. If the problem says that x is odd, pick an odd number for x. If the problem says that $x + y = z$, then note that once you pick for x and y, you have to calculate z. Don't pick your own random number for z! Pay attention to the following:

- If you have to pick for more than one variable, pick different numbers for each one. If possible, pick numbers with different characteristics (e.g., one even and one odd).
- Follow any constraints given in the problem. You may be restricted to positive numbers or to integers, for example, depending upon the way the problem is worded.
- Avoid choosing 0, 1, or numbers that already appear in the problem.
- Choose numbers that work easily in the problem. Typically, 100 is often the best number to use for percent problems. On fraction problems, try the common denominator of any fractions that appear in the problem.

Step 2: Solve the problem using your chosen smart numbers.

Wherever the problem used to have variables or unknowns, read the problem as though it now contains the real numbers that you've chosen. Solve the problem arithmetically and find your target answer.

Step 3: Find a match in the answers.

1. Pick the matching fraction or percent in the answers, or
2. plug your smart numbers into the variables in the answer choices and look for the choice that matches your target. If, at any point, you can tell that a particular answer will *not* match your target, stop calculating that answer. Cross it off and move to the next answer.

How to Get Better at Smart Numbers

First, practice the problems at the end of this chapter. Try each problem two times: once using smart numbers and once using the “textbook” method. (Time yourself separately for each attempt.)

When you're done, ask yourself which way you prefer to solve *this* problem and why. On the real test, you won't have time to try both methods; you'll have to make a decision and go with it. Learn *how* to make that decision while studying; then, the next time a new problem pops up in front of you that could be solved by choosing smart numbers, you'll be able to make a quick (and good!) decision.

One important note: at first, you may find yourself always choosing the textbook approach. You've practiced algebra for years, after all, and you've only been using the smart numbers technique for a short period of time. Keep practicing; you'll get better! Every high-scorer on the Quant section will tell you that choosing smart numbers is invaluable for getting through Quant on time and with a high enough performance to reach a top score.

When NOT to Use Smart Numbers

There are certain scenarios in which a problem contains some of the smart numbers characteristics but not all.

For example, why can't you use smart numbers on this problem?

Four brothers split a sum of money between them. The first brother received 50% of the total, the second received 25% of the total, the third received 20% of the total, and the fourth received the remaining $4. How many dollars did the four brothers split?

(A) $50

(B) $60

(C) $75

(D) $80

(E) $100

The problem starts by listing percents of an unknown sum. So far, so good. Towards the end, though, it does give you one real value: $4. Because the "remaining" percent has to equal $4 exactly, this problem has just one numerical answer. You can't pick any starting point that you want. One way to tell this right away is that the answer choices are actual values rather than variable expressions or fractions or percents of an unknown whole. (The answer to the problem is (D), by the way.)

Problem Set

It's time to test out your smart numbers skills. Because recognition is a key part of using a strategy effectively, **not every question in this set can be answered using smart numbers**.

First, decide whether the problem can be answered using smart numbers. If it cannot be answered using smart numbers, answer the question algebraically. If it can, try the problem twice, once algebraically and once using smart numbers. Decide which method you prefer for each problem.

1. Bradley owns *b* video game cartridges. If Bradley's total is one-third the total owned by Andrew and four times the total owned by Charlie, how many video game cartridges do the three of them own altogether, in terms of *b*?

(A) $\frac{16}{3}b$ (B) $\frac{17}{4}b$ (C) $\frac{13}{4}b$ (D) $\frac{19}{12}b$ (E) $\frac{7}{12}b$

2. Rob spends $\frac{1}{2}$ of his monthly paycheck, after taxes, on rent. He spends $\frac{1}{3}$ on food and $\frac{1}{8}$ on entertainment. If he donates the entire remainder, $500, to charity, what is Rob's monthly income, after taxes?

3. Lisa spends $\frac{3}{8}$ of her monthly paycheck on rent and $\frac{5}{12}$ on food. Her roommate, Carrie, who earns twice as much as Lisa, spends $\frac{1}{4}$ of her monthly paycheck on rent and $\frac{1}{2}$ on food. If the two women decide to donate the remainder of their money to charity each month, what fraction of their combined monthly income will they donate?

Solutions

1. **(B):** This problem can be answered very effectively by picking numbers to represent how many video game cartridges everyone owns. Bradley owns 4 times as many cartridges as Charlie, so you should pick a value for *b* that is a multiple of 4.

If $b = 4$, then Charlie owns 1 cartridge and Andrew owns 12 cartridges. Together they own 4 + 1 + 12 = 17 cartridges. Plug $b = 4$ into the answer choices and look for the one that yields 17:

(A) $\frac{16}{3}(4) = \frac{64}{3} = 21\frac{1}{3}$

(B) $\frac{17}{4}(4) = 17$ Match!

(C) $\frac{13}{4}(4) = 13$

(D) $\frac{19}{12}(4) = \frac{19}{3} =$ too small to be 17

(E) $\frac{7}{12}(4) = \frac{7}{3} =$ too small to be 17

The correct answer is **(B)**.

2. **$12,000:** You cannot use smart numbers in this problem, because it asks you for an actual dollar amount, not a variable expression, fraction, or percent. A portion of the total is specified, so there can be only one correct amount for that total. Clearly, if you assign a number to represent the total, you will not be able to accurately find the total.

First, use addition to find the fraction of Rob's money that he spends on rent, food, and entertainment: $\frac{1}{2}+\frac{1}{3}+\frac{1}{8}=\frac{12}{24}+\frac{8}{24}+\frac{3}{24}=\frac{23}{24}$. Therefore, the $500 that he donates to charity represents $\frac{1}{24}$ of his total monthly paycheck. We can set up a proportion: $\frac{500}{x}=\frac{1}{24}$ Thus, Rob's monthly income is $500 × 24, or $12,000.

3. $\frac{\mathbf{17}}{\mathbf{72}}$**:** Use smart numbers to solve this problem. Since the denominators in the problem are 8, 12, 4, and 2, assign Lisa a monthly paycheck of $24. Assign her roommate, who earns twice as much, a monthly paycheck of $48. The two women's monthly expenses break down as follows:

	Rent	Food	Left over
Lisa	$\frac{3}{8}$ of $24 = 9$	$\frac{5}{12}$ of $24 = 10$	$24 - (9 + 10) = 5$
Carrie	$\frac{1}{4}$ of $48 = 12$	$\frac{1}{2}$ of $48 = 24$	$48 - (12 + 24) = 12$

The women will donate a total of $17 out of their combined monthly income of $72.

Chapter 7

of

Fractions, Decimals, & Percents

Ratios

In This Chapter…

Chapter 7

Ratios

A ratio expresses a particular relationship between two or more quantities. Here are some examples of ratios:

> The ratio of men to women in the room is 3 to 4. For every 3 men, there are 4 women.
>
> Three sisters invest in a certain stock in the ratio of 2 to 3 to 8. For every $2 the first sister invests, the second sister invests $3, and the third sister invests $8.
>
> Two partners spend time working in the ratio of 1 to 3. For every hour the first partner works, the second partner works 3 hours.

Ratios can be expressed in three different ways:

1. Using the word *to*, as in 3 to 4
2. Using a colon, as in 3 : 4 or 2 : 4 : 7
3. By writing a fraction, as in $\frac{3}{4}$ (only for ratios of two quantities)

Ratios can express a part–part relationship or a part–whole relationship:

> A part–part relationship: The ratio of men to women in the office is 3 : 4.
> A part–whole relationship: There are 3 men for every 7 employees.

Notice that if there are only two parts in the whole, you can derive a part–whole ratio from a part–part ratio, and vice versa. For example, if the ratio of men to women in the office is 3 : 4, then the "total" is 7, so there are 4 women for every 7 employees.

The relationship that ratios express is division:

If the ratio of men to women in the office is 3 : 4, then the number of men *divided by* the number of women equals $\frac{3}{4}$, or 0.75. In the office, there are 0.75 men for every woman.

Ratios express a *relationship* between two or more items; they do not tell you the exact quantity for each item. For example, knowing that the ratio of men to women in an office is 3 to 4 does NOT indicate the actual number of men and women in the office. There could be 3 men and 4 women, or 6 men and 8 women, or any other combination that works out to 3 men for every 4 women. Despite this, ratios are surprisingly powerful on Data Sufficiency. They often provide enough information to answer the question.

Label Each Part of the Ratio with Units

The order in which a ratio is given is vital. For example, “the ratio of dogs to cats is 2 : 3” is very different from “the ratio of dogs to cats is 3 : 2.” Match the first item (dogs) to the first number (2, in the first example). Match the second item (cats) to the second number (3, in the first example).

It is very easy to accidentally reverse the order of a ratio—especially on a timed test like the GMAT. In order to avoid these reversals, write units on either the ratio itself or on the variables you create, or on both.

Thus, if the ratio of dogs to cats is 2 : 3, you can write $\frac{D}{C} = \frac{2\text{ dogs}}{3\text{ cats}}$, where *D* and *C* are variables standing for the *number* of dogs and cats (as opposed to the ratio).

The Unknown Multiplier

All ratios include something called the **unknown multiplier**. If there are 4 dogs for every 7 cats, for example, then the actual number of dogs will be a multiple of 4 and the actual number of cats will be a multiple of 7.

You can use the unknown multiplier to solve for various parts of the ratio. If the ratio of dogs to cats is 4 to 7 and there are 8 dogs total, what else can you figure out?

	Dogs	Cats	Total
Ratio	4	7	
Multiplier			
Actual	8		

The multiplier for dogs is 8/4 = 2. Here's the key: the multiplier is always the same for all parts of a ratio. Therefore, the multiplier for this entire ratio is 2:

	Dogs		Cats		Total
Ratio	4	+	7	=	
	×		×		
Multiplier	2	=	2	=	2
	=		=		
Actual	8	+		=	

Now, you can determine that there are 14 cats. You can also calculate the total number of animals, either by adding dogs and cats (8 + 14 = 22) or by multiplying the ratio total (4 + 7 = 11) by the multiplier, 2.

Try the below problem:

> The ratio of men to women in a room is 3 : 4. If there are 56 people in the room, how many are men?

Draw a table and begin to fill it in:

	Men	Women	Total
Ratio	3	4	
Multiplier			
Actual	◯		56

You can add the top row to obtain a total of 7. The ratio of men to women to total is 3 : 4 : 7. Now you can calculate the multiplier:

	Men	Women	Total
Ratio	3	4	7
Multiplier	8	8	8
Actual	24		56

There are 3 × 8 = 24 men in the room.

If you prefer, you can also solve algebraically. Call the unknown multiplier x. The ratio is 3 : 4 and the actual numbers of men and women are $3x$ and $4x$, respectively.

The problem indicates that the total number of people equals 56:

$$\begin{aligned} \text{Men} + \text{Women} &= \text{Total} \\ 3x + 4x &= 56 \\ 7x &= 56 \\ x &= 8 \end{aligned}$$

Plug the multiplier into the figure for men ($3x$) to determine how many men are in the room: (3)(8) = 24. There are 24 men in the room.

The unknown multiplier is particularly useful with three-part ratios. For example:

> A recipe calls for amounts of lemon juice, wine, and water in the ratio of 2 : 5 : 7. If all three combined yield 35 milliliters of liquid, how much wine was included?

Here's how to set it up algebraically:

$$\begin{array}{ccccccccc} \text{Lemon Juice} & + & \text{Wine} & + & \text{Water} & = & \text{Total} \\ 2x & + & 5x & + & 7x & = & 14x \end{array}$$

Now solve: $14x = 35$, or $x = 2.5$. Thus, the amount of wine is: $5x = 5(2.5) = 12.5$ milliliters.

In this problem, the unknown multiplier turns out not to be an integer. This result is fine, because the problem deals with continuous quantities (milliliters of liquids). In problems like the first one, which deals with integer quantities (men and women), make sure that your multiplier produces integer amounts. In that specific problem, the multiplier is literally the number of “complete sets” of 3 men and 4 women each.

Relative Values and Data Sufficiency

Some problems will give you concrete values while others will provide only relative values.

> **Concrete** values are actual amounts (# of tickets sold, liters of water, etc.).
>
> **Relative** values relate two quantities using fractions, decimals, percents, or ratios (twice as many, 60% less, ratio of 2 : 3, etc.).

Try this problem:

> A company sells only two kinds of pie: apple pie and cherry pie. What fraction of the total pies sold last month were apple pies?
>
> (1) The company sold 460 pies last month.
>
> (2) The company sold 30% more cherry pies than apple pies last month.

When a question asks for a relative value, not a concrete one, you don't need as much information in order to solve.

The question asks what fraction of the total pies sold were apple pies:

$$\frac{\text{apple pies}}{\text{total pies}} = ? \qquad \text{or} \qquad \frac{a}{a+c} = ?$$

Statement (1) indicates that the total number of pies sold was 460, so $a + c = 460$:

$$\frac{a}{460} = ?$$

The value of a is still unknown, so this statement is not sufficient. Eliminate answer choices (A) and (D).

Statement (2) indicates that the company sold 30% more cherry pies than apple pies; in other words, the number of cherry pies sold was 130% of the number of apple pies sold:

$$1.3a = c$$

On the surface this may not seem like enough information. But watch what happens when you replace *c* with 1.3*a* in the rephrased question.

$$\frac{a}{a+c} = ?$$

$$\frac{a}{a+1.3a} =$$

$$\frac{a}{2.3a} = \frac{1}{2.3}$$

Statement (2) does provide enough information to find the value of the fraction. The correct answer is **(B)**.

How could you recognize more easily that Statement (2) is sufficient?

Remember that this Data Sufficiency question was asking for a relative value (What *fraction* of the total pies…). Relative values are really just ratios in disguise. The ratio in this question is:

apple pies sold : cherry pies sold : total pies sold

The question asks for the ratio of apple pies sold to total pies sold. Statement (2) provides the ratio of apple pies sold to cherry pies sold.

If you know any two pieces of this ratio, you can determine the third piece, so statement (2) is sufficient.

While you can determine the relative value, statement (2) does not provide enough information to calculate the actual number of pies. If the question had asked for a concrete number, such as the number of apple pies, you would have needed both statements to solve.

Multiple Ratios: Make a Common Term

You may encounter two ratios containing a common element. To combine the ratios, you can use a process remarkably similar to creating a common denominator for fractions.

Consider the following problem:

> In a box containing action figures of the three Fates from Greek mythology, there are three figures of Clotho for every two figures of Atropos, and five figures of Clotho for every four figures of Lachesis.
>
> (a) What is the least number of action figures that could be in the box?
>
> (b) What is the ratio of Lachesis figures to Atropos figures?

Because ratios act like fractions, you can multiply both pieces of a ratio (or all pieces, if there are more than two) by the same number, just as you can multiply the numerator and denominator of a fraction by the same number. You can change *fractions* to have common *denominators*. Likewise, you can change *ratios* to have common *terms* corresponding to the same quantity.

(a) In symbols, this problem tells you that $C : A = 3 : 2$ and $C : L = 5 : 4$. The terms for C are different (3 and 5), so you cannot immediately combine these two ratios into one. However, you can fix that problem by multiplying each ratio by the right number, turning both C's into the same number:

$C : A : L$				$C : A : L$
3 : 2	→	Multiply by 5	→	15 : 10
5 : : 4	→	Multiply by 3	→	15 : : 12
		This is the combined ratio:		$\boxed{15 : 10 : 12}$

Once the C's are the same (15), combine the two ratios. Note: do not add the two C's together. Just use the base number, 15.

The *actual* numbers of action figures are these three numbers multiplied by an unknown multiplier, which must be a positive integer. Using the smallest possible multiplier, 1, there are $15 + 12 + 10 = 37$ action figures.

(b) Once you have combined the ratios, you can extract the numbers corresponding to the quantities in question and disregard the others: $L : A = 12 : 10$, which reduces to $6 : 5$.

Problem Set

Solve the following problems using the strategies you have learned in this section. Use proportions and the unknown multiplier to organize ratios.

For problems 1–4, assume that neither x nor y is equal to 0, to permit division by x and by y.

1. $48 : 2x$ is equivalent to $144 : 600$. What is x?

2. $2x : y$ is equivalent to $4x : 8{,}500$. What is y?

3. Initially, the men and women in a room were in the ratio of $5 : 7$. Six women leave the room. If there are 35 men in the room, how many women are left in the room?

4. What is the ratio $x : y : z$?

   (1) $x + y = 2z$

   (2) $2x + 3y = z$

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

5. The amount of time that three people worked on a special project was in the ratio of 2 to 3 to 5. If the project took 110 hours, what is the difference between the number of hours worked by the person who worked for the longest time and the person who worked for the shortest time?

6. Alexandra needs to mix cleaning solution in the ratio of 1 part bleach for every 4 parts water. When mixing the solution, Alexandra makes a mistake and mixes in half as much bleach as she ought to have. The total solution consists of 27 mL. How much bleach did Alexandra put into the solution?

Solutions

1. **100:**

$\frac{48}{2x} = \frac{144}{600}$ Simplify the ratios and cancel factors horizontally across the equals sign.

$$\frac{\cancel{48}\ 24}{\cancel{2x}\ x} = \frac{\cancel{144}\ 6}{\cancel{600}\ 25}$$

$$\frac{\cancel{24}\ 4}{x} = \frac{\cancel{6}\ 1}{25}$$

$x = 100$ Then, cross-multiply: $x = 100$.

2. **4,250:** First, simplify the ratio on the right-hand side of the equation to match the one on the left.

$$\frac{2x}{y} = \frac{\cancel{4x}\ 2x}{\cancel{8{,}500}\ 4{,}250}$$

Since the numerators are already identical, y must equal 4,250.

3. **43:** You can use the unknown multiplier very effectively here:

Men = $5x$ Women = $7x$
$5x = 35$
$x = 7$
Women = $7x = 49$

If 6 women leave the room, there are 49 − 6, or 43, women left.

4. **(C):** For this problem, you do not necessarily need to know the value of x, y, or z. You simply need to know the ratio $x : y : z$ (in other words, the value of $\frac{x}{y}$ AND the value of $\frac{y}{z}$). You need to manipulate the information given to see

whether you can determine this ratio.

(1) INSUFFICIENT: There is no way to manipulate this equation to solve for a ratio. If you simply solve for $\frac{x}{y}$, for example, you get a variable expression on the other side of the equation:

$$x + y = 2z$$
$$x = 2z - y$$
$$\frac{x}{y} = \frac{2z - y}{y} = \frac{2z}{y} - 1$$

(2) INSUFFICIENT: As in the previous example, there is no way to manipulate this equation to solve for a ratio. If you simply solve for $\frac{x}{y}$, for example, you get a variable expression on the other side of the equation:

$$2x + 3y = z$$
$$2x = z - 3y$$
$$\frac{x}{y} = \frac{z - 3y}{2y} = \frac{z}{2y} - \frac{3}{2}$$

(1) AND (2) SUFFICIENT: Use substitution to combine the equations:

$$x + y = 2z$$
$$2x + 3y = z$$

Since $z = 2x + 3y$, you can substitute:

$$x + y = 2(2x + 3y)$$
$$x + y = 4x + 6y$$

Therefore, you can arrive at a value for the ratio $x : y$:

$$-3x = 5y$$

$$\frac{-3x}{y} = \frac{5\cancel{y}}{\cancel{y}} \qquad \text{Divide by } y.$$

$$\frac{\cancel{-3}x}{\cancel{-3}y} = \frac{5}{-3} \qquad \text{Divide by } -3.$$

$$\frac{x}{y} = \frac{5}{-3}$$

You can also substitute for *x* to get a value for the ratio *y* : *z*:

$$x + y = 2z$$

$$x = 2z - y$$

$$2x + 3y = z$$

$$2(2z - y) + 3y = z$$

$$4z - 2y + 3y = z$$

$$y = -3z$$

$$\frac{y}{z} = -3$$

This tells you that *x* : *y* = −5/3, and *y* : *z* = −3/1. Both ratios contain a 3 for the *y* variable and both also contain a negative sign, so assign the value −3 to *y*. This means that *x* must be 5 and *z* must be 1. Therefore, the ratio *x* : *y* : *z* = 5 : −3 : 1.

You can test the result by choosing *x* = 5, *y* = −3, and *z* = 1, or *x* = 10, *y* = −6, and *z* = 2. In either case, the original equations hold up.

The correct answer is **(C)**.

5. **33 hours:** Use an equation with the unknown multiplier to represent the total hours put in by the three people:

$$\begin{aligned} 2x + 3x + 5x &= 110 \\ 10x &= 110 \\ x &= 11 \end{aligned}$$

Therefore, the person who worked for the longest time put in 5(11) = 55 hours, and the person who worked for the shortest time put in 2(11) = 22 hours. This represents a difference of 55 − 22 = 33 hours.

6. **3 mL:** The correct ratio is 1 : 4, which means that there should be *x* parts bleach and 4*x* parts water. However, Alexandra put in half as much bleach as she should have, so she put in $\frac{x}{2}$ parts bleach. You can represent this with an equation: $\frac{x}{2} + 4x = 27$. Now solve for *x*:

$$\begin{aligned} x + 8x &= 54 \\ 9x &= 54 \\ x &= 6 \end{aligned}$$

You were asked to find how much bleach Alexandra used. This equaled $\frac{x}{2}$, so Alexandra used $\frac{6}{2} = 3$mL of bleach.

Chapter 8

of

Fractions, Decimals, & Percents

Strategy: Estimation

In This Chapter…

Chapter 8

Strategy: Estimation

You can estimate your way to an answer on problems with certain characteristics. Try these two problems:

1. $\frac{7}{13}+\frac{5}{11}$ is approximately equal to

   (A) 0 (B) 1 (C) 2

2. Of 450 employees at a company, 20% are managers and the rest are not managers. If 60% of the managers work in the engineering department, how many managers do not work in the engineering department?

   (A) 36 (B) 54 (C) 90

Before you look at the solutions in the next section, try to figure out how you would know that you can estimate on these two problems.

How to Estimate

The first problem actually tells you that you can estimate. Whenever a problem contains the word approximately (or an equivalent word), do not even try to do exact calculations. Take the problem at its word and estimate!

1. $\frac{7}{13}+\frac{5}{11}$ is approximately equal to

   (A) 0 (B) 1 (C) 2

Converting to common denominators here would be pretty annoying, as would converting the fractions to decimals or percents. Instead, round those fractions to easier ones and find an approximate answer. $\frac{7}{13}$ is very close to $\frac{7}{14}$, or $\frac{1}{2}$, so call that first fraction 0.5.

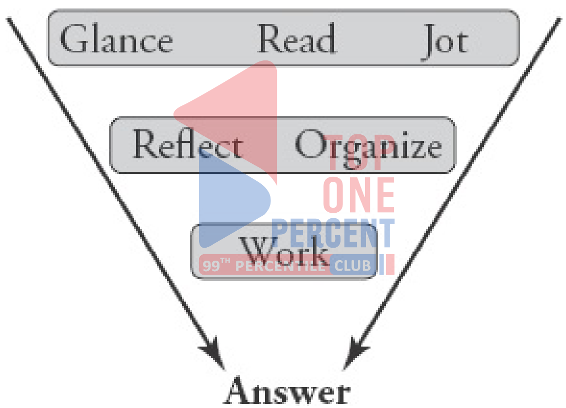
Note that $\frac{7}{13}$ is a little larger than $\frac{7}{14}$; in other words, you rounded down. Try to offset the error by rounding in the other direction (up) next time.

Since $\frac{5}{11}$ is a little bit less than $\frac{1}{2}$, you can round up this time. Call the second fraction 0.5 as well.

In this case, then, 0.5 + 0.5 = 1, so the answer is **(B)**.

The second problem doesn't tell you that you can estimate; nevertheless, it contains an important clue that points towards estimation.

Remember this graphic from Chapter 1?



On all Problem Solving problems, get in the habit of glancing at the answers during your first step (Glance, Read, Jot). Whenever the answers are far apart, you can estimate. (Note that the numbers only need to be far apart in relative terms—in the last problem, the numbers are only 1 apart, but 2 is twice as big as 1, and there's a big difference between 1 and 0.)

2. Of 450 employees at a company, 20% are managers and the rest are not managers. If 60% of the managers work in the engineering department, how many managers do not work in the engineering department?

   (A) 36 (B) 54 (C) 90

The three answers are pretty far apart, so plan to estimate wherever it makes sense in the process.

The question asks for the number of managers who do not work in engineering. First, find the number of managers, which is 20% of 450: 10% of 450 is 45, so 20% is twice as much, or 90.

There are 90 managers total, so (C) cannot be the answer; cross it off. (If there were larger answers, you could of course cross those off, too.) Further, if 60% of those managers work in engineering, then 40% (or less than half) do not. Half of 90 is 45, so the number of managers not in engineering must be less than 45. The only possible answer is **(A)**.

When to Estimate

Estimate whenever the problem explicitly asks for an approximate answer. In addition, consider estimating when the answers are far apart or when they cover certain "divided" characteristics.

In some cases, this estimation will get you all the way to the correct answer. In others, you may be able to eliminate some answers before guessing on a hard problem.

Consider these possible answers:

(A) −6 (B) −3 (C) −2 (D) 1 (E) 2

If you are running out of time or are not sure how to answer the question in the normal way, you may at least be able to tell whether the answer should be positive or negative. If so, you'll be able to eliminate two or three answers before making a guess.

A fraction problem might have some answers greater than 1 and others less than 1. You may also be able to estimate here in order to eliminate some answers.

Benchmark Values

Benchmark values are common percents or fractions that make estimation easier. The easiest percent benchmarks are 50%, 10%, and 1%. The easiest fraction benchmarks are $\frac{1}{2}$, the quarters $\left(\frac{1}{4}, \frac{3}{4}\right)$, and the thirds $\left(\frac{1}{3}, \frac{2}{3}\right)$.

You may be able to use the building percents calculation method, first introduced in chapter 5, to estimate.

Try this problem:

> When Karen bought a new television, originally priced at $700, she used a coupon for a 12% discount. What price did she pay for the television?
>
> (A) $630 (B) $616 (C) $560

Because 10% of 700 is 70, you know Karen received something more than a $70 discount. She must have paid a bit less than $630. Answer (A) can't be correct.

A 20% discount would have resulted in another $70 off, for a total $140 discount, or $560. Answer (C) is too small. The correct answer must be **(B)**.

(Indeed, if you check the math, 10% + 1% + 1% = 70 + 7 + 7 = $84, and 700 − 84 = $616. But don't do this math on the test unless it's necessary!)

You can also use benchmark values to compare fractions:

> Which is greater: $\frac{127}{255}$ or $\frac{162}{320}$?

How does each fraction compare to $\frac{1}{2}$? It turns out that 127 is less than half of 255 and 162 is more than half of 320, so $\frac{162}{320}$ is the greater fraction.

You can also use benchmark values to estimate computations involving fractions:

> What is $\frac{10}{22}$ of $\frac{5}{18}$ of 2,000?

Once you determine that $\frac{10}{22}$ is a little bit less than $\frac{1}{2}$ and $\frac{5}{18}$ is a little bit more than $\frac{1}{4}$, you can use these benchmarks to estimate:

$$\frac{1}{2} \text{ of } \frac{1}{4} \text{ of } 2{,}000 = 250$$

Therefore, $\frac{10}{22}$ of $\frac{5}{18}$ of $2{,}000 \approx 250$.

Notice that the rounding errors compensated for each other:

$\frac{10}{22} \approx \frac{10}{20} = \frac{1}{2}$ You decreased the denominator, so you rounded up: $\frac{10}{22} < \frac{1}{2}$.

$\frac{5}{18} \approx \frac{5}{20} = \frac{1}{4}$ You increased the denominator, so you rounded down: $\frac{5}{18} > \frac{1}{4}$.

If you had rounded $\frac{5}{18}$ to $\frac{6}{18} = \frac{1}{3}$ instead, then you would have rounded both fractions up. This would lead to a slight but systematic overestimation:

$$\frac{1}{2} \times \frac{1}{3} \times 2{,}000 \approx 333$$

Try to make your rounding errors cancel by rounding some numbers up and others down.

How to Get Better at Estimation

First, try the problems associated with this chapter in your online *Official Guide* problem sets. Try doing the official math *and* estimating to see how much time and effort estimation can save you.

Chapter 9

of

Fractions, Decimals, & Percents

Extra FDPs

In This Chapter…

Chapter 9

Extra FDPs

This chapter outlines miscellaneous extra topics within the areas of *fractions, decimals, percents, and ratios.* If you have mastered the material in earlier chapters and are aiming for an especially high Quant score, then learn this material. If not, then you may be able to skip this section.

Exponents and Roots

To square or cube a decimal, you can always multiply it by itself once or twice. However, to raise a decimal to a higher power, you can rewrite the decimal as the product of an integer and a power of 10, and then apply the exponent:

$(0.5)^4 = ?$	
$0.5 = 5 \times 10^{-1}$	Rewrite the decimal.
$(5 \times 10^{-1})^4 = 5^4 \times 10^{-4}$	Apply the exponent to each part.
$5^4 = 25^2 = 625$	Compute the first part and combine.
$625 \times 10^{-4} = 0.0625$	

Solve for roots of decimals the same way. Recall that a root is a number raised to a fractional power: a square root is a number raised to the 1/2 power, a cube root is a number raised to the 1/3 power, etc.:

$$\sqrt[3]{0.000027} = ?$$

Rewrite the decimal. Make the first number something you can take the cube root of easily:

$0.000027 = 27 \times 10^{-6}$	
$(0.000027)^{1/3} = (27 \times 10^{-6})^{1/3}$	Write the root as a fractional

	exponent.
$(27)^{1/3} \times (10^{-6})^{1/3} = (27)^{1/3} \times 10^{-2}$	Apply the exponent to each part.
$(27)^{1/3} = 3$ (since $3^3 = 27$)	Compute the first part and combine.
$3 \times 10^{-2} = 0.03$	

Strategy Tip: Powers and roots: Rewrite the decimal using powers of 10!

Once you understand the principles, you can take a shortcut by counting decimal places. For instance, the number of decimal places in a cubed decimal is 3 times the number of decimal places in the original decimal:

$(0.04)^3 = 0.000064$	$(0.04)^3$	$= 0.000064$
	2 places	*2 × 3 = 6 places*

Likewise, the number of decimal places in a cube root is 1/3 the number of decimal places in the original decimal:

$\sqrt[3]{0.000000008} = 0.002$	$\sqrt[3]{0.000000008}$	$= 0.002$
	9 places	$9 \div 3 = 3$ places

Repeating Decimals

Dividing an integer by another integer yields a decimal that either terminates or never ends and repeats itself:

$2 \div 9 = ?$

$$\begin{array}{r} 0.222\ldots \\ 9\overline{)2.000} \\ \underline{1.8} \\ 20 \\ \underline{18} \\ 20 \end{array}$$

$2 \div 9 = 0.2222\ldots = 0.\overline{2}$

Generally, do long division to determine the repeating cycle. However, you can use any patterns you've memorized to reduce your workload. For instance, if you've memorized that 1/9 is $0.\overline{1}$, you can determine that 2/9 is $0.\overline{2}$.

It can also be helpful to know that if the denominator is 9, 99, 999 or another number equal to a power of 10 minus 1, then the numerator tells you the repeating digits. Here are two examples:

$$23 \div 99 = 0.2323\ldots = 0.\overline{23} \qquad \frac{3}{11} = \frac{27}{99} = 0.2727\ldots = 0.\overline{27}$$

Terminating Decimals

Some numbers, like $\sqrt{2}$ and π, have decimals that never end and *never* repeat themselves. On certain problems, it can be useful to use approximations for these decimals (e.g., $\sqrt{2} \approx 1.4$). Occasionally, though, the GMAT asks you about properties of "terminating" decimals; that is, decimals that end. You can tack on zeroes, of course, but they don't matter. Some examples of terminating decimals are 0.2, 0.47, and 0.375.

Terminating decimals can all be written as a ratio of integers (which might be reducible):

$$\frac{\text{Some integer}}{\text{Some power of 10}}$$

$$0.2 = \frac{2}{10} = \frac{1}{5} \qquad 0.47 = \frac{47}{100} \qquad 0.375 = \frac{375}{1{,}000} = \frac{3}{8}$$

Positive powers of 10 are composed of only 2's and 5's as prime factors. When you reduce this fraction, you only have prime factors of 2's and/or 5's in the denominator. Every terminating decimal shares this characteristic. If, after being fully reduced, the denominator has any prime factors besides 2 or 5, then its decimal will not terminate. If the denominator only has factors of 2 and/or 5, then the decimal will terminate.

Using Place Value on the GMAT

Some difficult GMAT problems require the use of place value with unknown digits. For example:

> A and B are both two-digit numbers, and $A > B$. If A and B contain the same digits, but in reverse order, what integer must be a factor of $(A - B)$?
>
> (A) 4 (B) 5 (C) 6 (D) 8 (E) 9

To solve this problem, assign two variables to be the digits in A and B: x and y. Let $A = \boxed{x}\boxed{y}$ (***not*** the product of x and y: x is in the tens place, and y is in the units place). The boxes remind you that x and y stand for digits. A is therefore the sum of x tens and y ones. Using algebra, write $A = 10x + y$.

Since B's digits are reversed, $B = \boxed{y}\boxed{x}$. Algebraically, B can be expressed as $10y + x$. The difference of A and B can be expressed as follows:

$$A - B = 10x + y - (10y + x) = 9x - 9y = 9(x - y)$$

Therefore, 9 must be a factor of $A - B$. The correct answer is **(E)**.

You can also make up digits for x and y and plug them in to create A and B. This will not necessarily yield the unique right answer, but it will help you eliminate wrong choices.

In general, for unknown digits problems, be ready to create variables (such as x, y, and z) to represent the unknown digits. Recognize that each unknown is restricted to at most 10 possible values (0 through 9). Then apply any given constraints, which may involve number properties such as divisibility or odds and evens.

The Last Digit Shortcut

Sometimes the GMAT asks you to find a units digit, or a remainder after division by 10:

> What is the units digit of $(7)^2(9)^2(3)^3$?

In this problem, you can use the **last digit shortcut:**

> To find the units digit of a product or a sum of integers, only pay attention to the units digits of the numbers you are working with. Drop any other digits.

This shortcut works because only units digits contribute to the units digit of the product:

Step 1:	$7 \times 7 = 49$	Drop the tens digit and keep only the last digit: 9.
Step 2:	$9 \times 9 = 81$	Drop the tens digit and keep only the last digit: 1.
Step 3:	$3 \times 3 \times 3 = 27$	Drop the tens digit and keep only the last digit: 7.
Step 4:	$9 \times 1 \times 7 = 63$	Multiply the last digits of each of the products.

The units digit of the final product is 3.

Unknown Digits Problems

Occasionally, the GMAT asks tough problems involving unknown digits. These problems look like "brainteasers"; it seems it could take all day to test the possible digits.

However, like all other GMAT problems, these digit "brainteasers" must be solvable under time constraints. As a result, there are always ways of reducing the number of possibilities:

Principles:

- Look at the answer choices first, to limit your search.
- Use other given constraints to rule out additional possibilities.
- Focus on the units digit in the product or sum. This units digit is affected by the fewest other digits.
- Test the remaining answer choices.

Example:

$$\begin{array}{r} AB \\ \times\ CA \\ \hline DEBC \end{array}$$

In the multiplication above, each letter stands for a different non-zero digit, with $A \times B < 10$. What is the two-digit number AB?

(A) 23 (B) 24 (C) 25 (D) 32 (E) 42

It is often helpful to look at the answer choices. Here, you see that the possible digits for A and B are 2, 3, 4, and 5.

Next, apply the given constraint that $A \times B < 10$. This rules out answer choice (C), 25, since $2 \times 5 = 10$.

Now, test the remaining answer choices. Notice that $A \times B = C$, the units digit of the product. Therefore, you can find all the needed digits and complete each multiplication.

Compare each result to the template. The two positions of the B digit must match (note that it's not really necessary to calculate past the tens digit):

$$\begin{array}{r} 23 \\ \times\ 62 \\ \hline 1{,}426 \end{array}$$

The B's do not match

$$\begin{array}{r} 24 \\ \times\ 82 \\ \hline 1{,}968 \end{array}$$

The B's do not match

$$\begin{array}{r} 32 \\ \times\ 63 \\ \hline 2{,}016 \end{array}$$

The B's do not match

$$\begin{array}{r} 42 \\ \times\ 84 \\ \hline 3{,}528 \end{array}$$

The B's match

The correct answer is **(E)**.

Note that you could have used the constraints to derive the possible digits (2, 3, and 4) without using the answer choices. However, for these problems, take advantage of the answer choices to restrict your search quickly.

Problem Set

1. What is the units digit of $\left(\dfrac{6^6}{6^5}\right)^6$?

2. What is the units digit of $(2)^5(3)^3(4)^2$?

3. Order from least to greatest:

$$\dfrac{\frac{3}{5}}{\frac{8}{10}} \qquad \dfrac{0.00751}{0.01} \qquad \dfrac{200}{3}\times 10^{-2}$$

4. What is the length of the sequence of different digits in the decimal equivalent of $\dfrac{3}{7}$?

5. Which of the following fractions will terminate when expressed as a decimal? (Choose all that apply.)

(A) $\dfrac{1}{256}$ (B) $\dfrac{27}{100}$ (C) $\dfrac{100}{27}$ (D) $\dfrac{231}{660}$ (E) $\dfrac{7}{105}$

6. The number A is a two-digit positive integer; the number B is the two-digit positive integer formed by reversing the digits of A. If $Q = 10B - A$, what is the value of Q?

(1) The tens digit of A is 7.

(2) The tens digit of B is 6.

7.
```
   ●◆
×  ■◆
 ▲■◆
```

In the multiplication above, each symbol represents a different unknown digit, and ● × ■ × ◆ = 36. What is the three-digit integer ●■◆?

(A) 263 (B) 236 (C) 194 (D) 491 (E) 452

Solutions

1. **6:** First, use the rules for combining exponents to simplify the expression. Subtract the exponents to get $\frac{6^6}{6^5} = 6^1$. Then, raise this to the sixth power: $(6^1)^6 = 6^6$. Ignore any digits other than the units digit. No matter how many times you multiply 6 × 6, the result will still end in 6. The units digit is 6.

2. **4:** Use the last digit shortcut, ignoring all digits but the last in any intermediate products:

Step 1:	$2^5 = 32$	Drop the tens digit and keep only the last digit: 2.
Step 2:	$3^3 = 27$	Drop the tens digit and keep only the last digit: 7.
Step 3:	$4^2 = 16$	Drop the tens digit and keep only the last digit: 6.
Step 4:	$2 \times 7 \times 6 = 84$	Multiply the last digits of each of the products and keep only the last digit: 4.

3. $$\frac{200}{3} \times 10^{-2} < \frac{3}{5} \div \frac{8}{10} < \frac{0.00751}{0.01}$$

First, simplify all terms and express them in decimal form:

$$\frac{3}{5} \div \frac{8}{10} = \frac{3}{5} \times \frac{10}{8} = \frac{3}{4} = 0.75$$

$$\frac{0.00751}{0.01} = \frac{0.751}{1} = 0.751$$

$$\frac{200}{3} \times 10^{-2} = \frac{2}{3} = 0.\overline{6}$$

$$0.\overline{6} < 0.75 < 0.751$$

4. **6:** Generally, the easiest way to find the pattern of digits in a non-terminating decimal is to simply do the long division and wait for the pattern to repeat (see long division at right). This results in a repeating pattern of $0.\overline{428571}$.

$$
\begin{array}{r}
0.4285714 \\
7\overline{)3.0000000} \\
0 \\
\hline
3.0 \\
2.8 \\
\hline
20 \\
-14 \\
\hline
60 \\
-56 \\
\hline
40 \\
-35 \\
\hline
50 \\
-49 \\
\hline
10 \\
-7 \\
\hline
30 \\
-28 \\
\hline
2
\end{array}
$$

5. **(A), (B), and (D):** Recall that in order for the decimal version of a fraction to terminate, the fraction's denominator in fully reduced form must have a prime factorization that consists of only 2's and/or 5's. The denominator in (A) is composed of only 2's ($256 = 2^8$). The denominator in (B) is composed of only 2's and 5's ($100 = 2^2 \times 5^2$). In fully reduced form, the fraction in (D) is equal to $\frac{7}{20}$, and 20 is composed of only 2's and 5's ($20 = 2^2 \times 5$). By contrast, the denominator in (C) has prime factors other than 2's and 5's ($27 = 3^3$), and in fully reduced form, the fraction in (E) is equal to $\frac{1}{15}$, and 15 has a prime factor other than 2's and 5's ($15 = 3 \times 5$).

6. **(B):** Write A as XY, where X and Y are digits (X is the tens digit of A and Y is the units digit of A). Then B can be written as YX, with reversed digits. Writing

these numbers in algebraic rather than digital form, you have $A = 10X + Y$ and $B = 10Y + X$.

Therefore, $Q = 10B - A = 10(10Y + X) - (10X + Y) = 100Y + 10X - 10X - Y = 99Y$. The value of Q only depends on the value of Y, which is the tens digit of B. The value of X is irrelevant to Q. Therefore, statement (2) alone is SUFFICIENT.

You can also test cases to get the same result, although algebra is probably faster if you are comfortable with the setup. To test cases here, you'd need to try different two-digit numbers that fit the constraints for each statement:

(1) INSUFFICIENT:

$A = 72$, $B = 27$, $10B - A = 198$
$A = 73$, $B = 37$, $10B - A = 297$

(2) SUFFICIENT:

$A = 26$, $B = 62$, $10B - A = 596$
$A = 76$, $B = 67$, $10B - A = 596$

Without the benefit of algebra, it may not be clear why statement (2) always produces the same answer, but if you get the exact same number from two different values on a problem with such involved math, chances are good that a consistent pattern is emerging.

7. **(B):** The three symbols ●, ■, and ◆ multiply to 36 and each must represent a different digit. Break 36 into its primes: $2 \times 2 \times 3 \times 3$. What three different digits can you create using two 2's and two 3's? 2, 3, and 6. So these three symbols (●, ■, and ◆) must equal 2, 3, and 6, but which is which? (Note that at this point, only (A) or (B) can be the answer.) Notice in the multiplication problem given that the units digit indicates that ◆ × ◆ = ◆. Which of the three numbers will give this result? Only the number 6 ($6 \times 6 = 36$; the units digit is the same). So ◆ must equal 6 and the units digit of ●■◆ must be 6. Only answer choice (B) fits this requirement.

Appendix A

of

Fractions, Decimals, & Percents

Data Sufficiency

In This Chapter…

Appendix A

Data Sufficiency

Data Sufficiency (DS) problems are a cross between math and logic. Imagine that your boss just walked into your office and dumped a bunch of papers on your desk, saying, “Hey, our client wants to know whether they should raise the price on this product. Can you answer that question from this data? If so, which pieces do we need to prove the case?” What would you do?

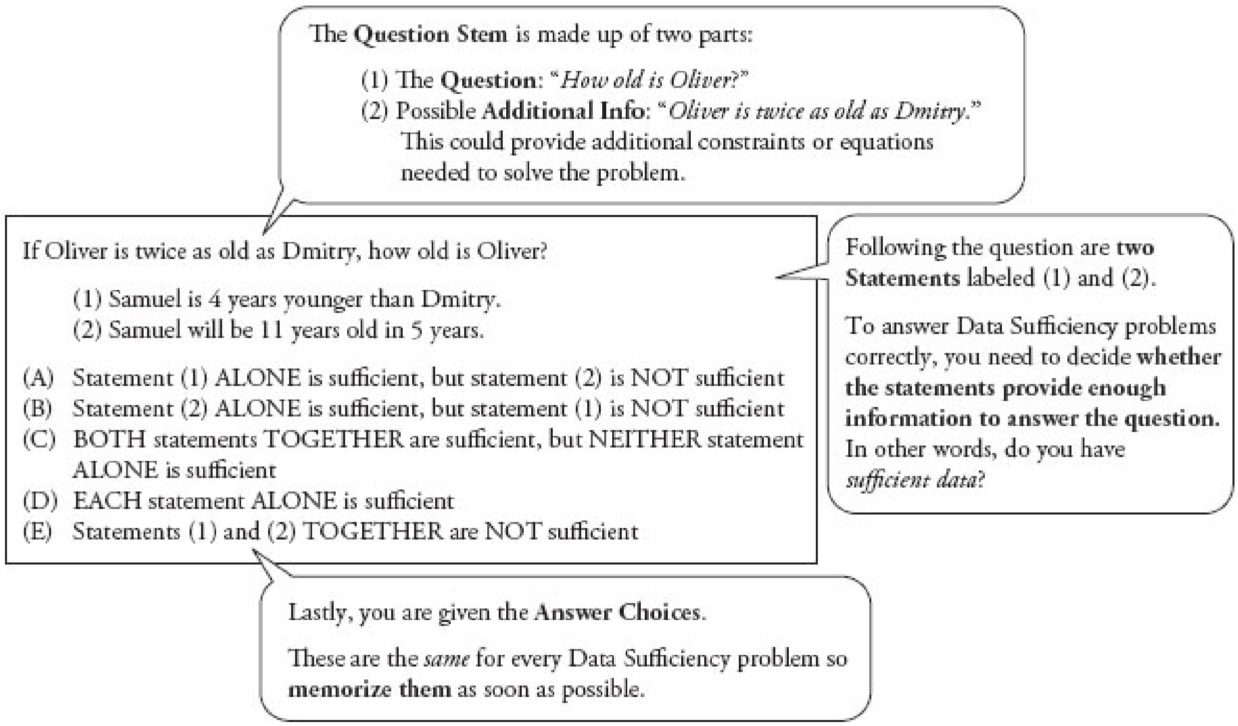
The client has asked a specific question: should the company raise the price? You have to decide which pieces of information will allow you to answer that question—or, possibly, that you don't have enough information to answer the question at all.

This kind of logical reasoning is exactly what you use when you answer DS questions.

How Data Sufficiency Works

If you already feel comfortable with the basics of Data Sufficiency, you may want to move quickly through this particular section of the chapter—but you are encouraged to read it. There are a few insights that you may find useful.

Every DS problem has the same basic form:



The question stem contains the question you need to answer. The two statements provide *given* information, information that is true. DS questions look strange but you can think of them as deconstructed Problem Solving (PS) questions. Compare the DS-format problem above to the PS-format problem below:

> Samuel is 4 years younger than Dmitry, and Samuel will be 11 years old in 5 years. If Oliver is twice as old as Dmitry, how old is Oliver?"

The two questions contain exactly the same information; that information is just presented in a different order. The PS question stem contains all of the givens as well as the question. The DS problem moves some of the givens down to statement (1) and statement (2).

As with regular PS problems, the given information in the DS statements is always true. In addition, the two statements won't contradict each other. In the same way that a PS question wouldn't tell you that $x > 0$ *and* $x < 0$, the two DS statements won't do that either.

In the PS format, you would go ahead and calculate Oliver's age. The DS format works a bit differently. Here is the full problem, including the answer choices:

If Oliver is twice as old as Dmitry, how old is Oliver?

(1) Samuel is 4 years younger than Dmitry.

(2) Samuel will be 11 years old in 5 years.

(A) Statement (1) ALONE is sufficient, but statement (2) is NOT sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) is NOT sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

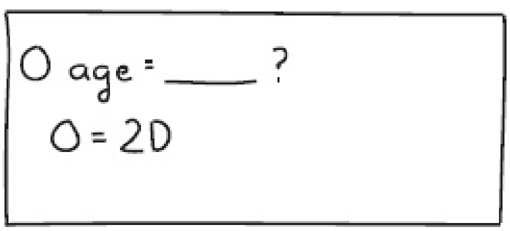
(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.

Despite all appearances, the question is not actually asking you to calculate Oliver's age. Rather, it's asking *which pieces of information* would allow you to calculate Oliver's age.

You may already be able solve this one on your own, but you'll see much harder problems on the test, so your first task is to learn how to work through DS questions in a systematic, consistent way.

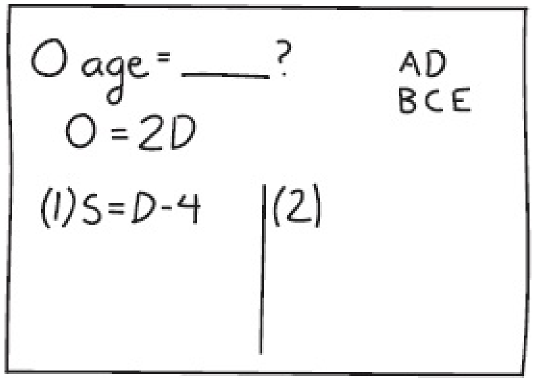
As you think the problem through, jot down information from the question stem:



Hmm. If they tell you Dmitry's age, then you can find Oliver's age. Remember that!

Take a look at the first statement. Also, write down the $\frac{AD}{BCE}$ answer grid (you'll learn why as you work through the problem):

(1) Samuel is 4 years younger than Dmitry.



The first statement doesn't allow you to figure out anyone's real age. Statement (1), then, is *not sufficient*. Now you can cross off the top row of answers, (A) and (D).

Why? Here's the text for answers (A) and (D):

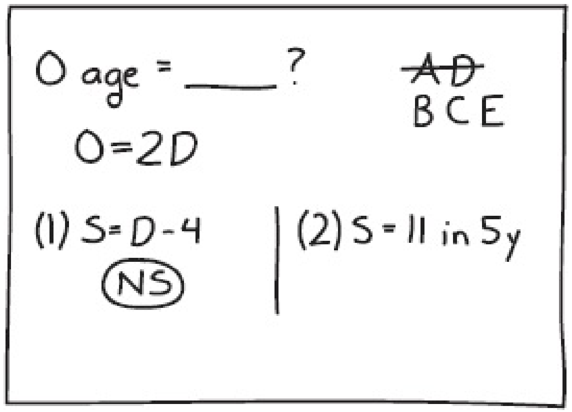
> (A) Statement (1) ALONE is sufficient, but statement (2) is NOT sufficient.
>
> (D) EACH statement ALONE is sufficient.

Both answers indicate that statement (1) is sufficient to answer the question. Because statement (1) is *not* sufficient to find Oliver's age, both (A) and (D) are wrong.

The answer choices will always appear in the order shown for the above problem, so any time you decide that statement (1) is not sufficient, you will always cross off answers (A) and (D). That's why your answer grid groups these two answers together.

Next, consider statement (2), but remember one tricky thing: forget what statement (1) told you. Because of the way DS is constructed, you must evaluate the two statements separately before you look at them together:

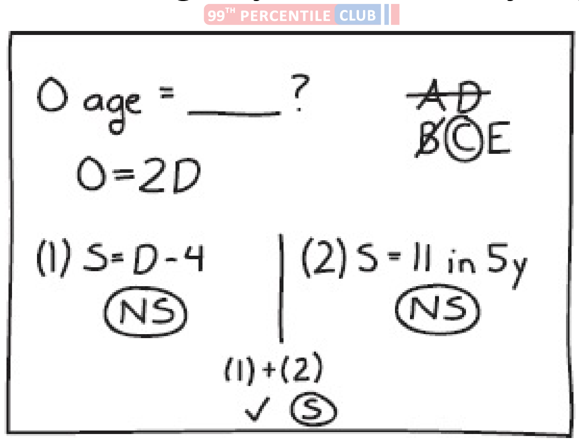
> (2) Samuel will be 11 years old in 5 years.



It's useful to write the two statements side-by-side, as shown above, to help remember that statement (2) is separate from statement (1) and has to be considered by itself first.

Statement (2) does indicate how old Sam is now, but says nothing about Oliver or Dmitry. (Remember, you're looking *only* at statement (2) now.) By itself, statement (2) is not sufficient, so cross off answer (B).

Now that you've evaluated each statement by itself, take a look at the two statements together. Statement (2) provides Sam's age, and statement (1) allows you to calculate Dmitry's age if you know Sam's age. Finally, the question stem allows you to calculate Oliver's age if you know Dmitry's age:



As soon as you can tell that you *can* solve, put down a check mark or write an S with a circle around it (or both!). Don't actually calculate Oliver's age; the GMAT doesn't give you any extra time to calculate a number that you don't need.

The correct answer is **(C)**.

The Answer Choices

The five Data Sufficiency answer choices will always be exactly the same (and presented in the same order), so memorize them before you go into the test.

Here are the five answers written in an easier way to understand:

(A) Statement (1) does allow you to answer the question, but statement (2) does not.

(B) Statement (2) does allow you to answer the question, but statement (1) does not.

(C) Neither statement works on its own, but you can use them *together* to answer the question.

(D) Statement (1) works by itself *and* statement (2) works by itself.

(E) Nothing works. Even if you use both statements together, you still can't answer the question.

Answer (C) specifically says that neither statement works on its own. For this reason, you are required to look at each statement by itself *and decide that neither one works* before you are allowed to evaluate the two statements together.

Here's an easier way to remember the five answer choices; we call this the “twelve-ten” mnemonic (memory aid):

1	only statement 1
2	only statement 2
T	together
E	either one
N	neither/nothing

Within the next week, memorize the DS answers. If you do a certain number of practice DS problems in that time frame, you'll likely memorize the answers without conscious effort—and you'll solidify the DS lessons you're learning right now.

Starting with Statement (2)

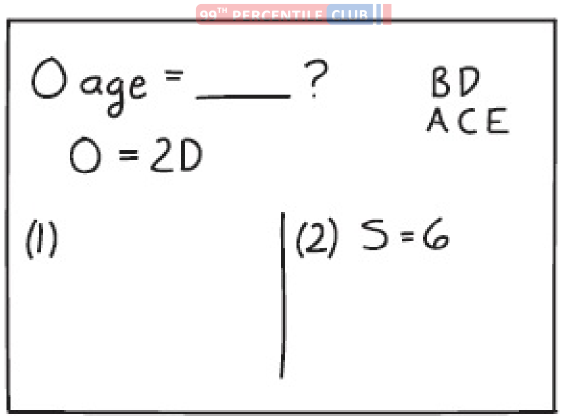
If statement (1) looks hard, start with statement (2) instead. Your process will be the same, except you'll make one change in your answer grid.

Try this problem:

> If Oliver is twice as old as Dmitry, how old is Oliver?
>
> (1) Two years ago, Dmitry was twice as old as Samuel.
>
> (2) Samuel is 6 years old.

(From now on, the answer choices won't be shown. Start memorizing!) Statement (1) is definitely more complicated than statement (2), so start with statement (2) instead. Change your answer grid to BD ACE. (You'll learn why in a minute.)

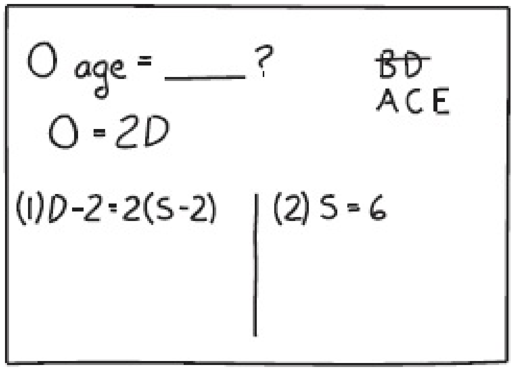
> (2) Samuel is 6 years old.



Statement (2) is not sufficient to determine Oliver's age, so cross off the answers that say statement (2) is sufficient: (B) and (D). Once again, you can cross off the entire top row; when starting with statement (2), you always will keep or eliminate these two choices at the same time.

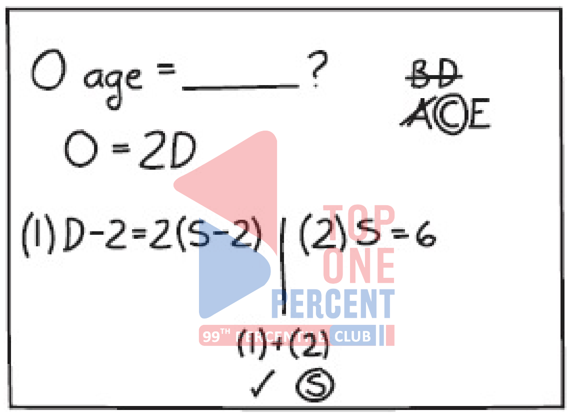
Now assess statement (1):

> (1) Two years ago, Dmitry was twice as old as Samuel.



Forget all about statement (2); only statement (1) exists. By itself, is the statement sufficient?

Nope! Too many variables. Cross off (A), the first of the remaining answers in the bottom row, and assess the two statements together:



You can plug Samuel's age (from the second statement) into the formula from statement (1) to find Dmitry's age, and then use Dmitry's age to find Oliver's age. Together, the statements are sufficient.

The correct answer is **(C)**.

The two answer grids work in the same way, regardless of which one you use. As long as you use the AD/BCE grid when starting with statement (1), or the BD/ACE grid when starting with statement (2), you will always:

- cross off the *top* row if the first statement you try is *not* sufficient;
- cross off the *bottom* row if the first statement you try *is* sufficient; and
- assess the remaining row of answers one answer at a time.

Finally, remember that you must assess the statements separately before you can try them together—and you'll only try them together if neither one is sufficient on its own. You will only consider the two together if you have already crossed off answers (A), (B), and (D).

Value vs. Yes/No Questions

Data Sufficiency questions come in two “flavors”: Value or Yes/No.

On Value questions, it is necessary to find a single value in order to answer the question. If you can't find any value or you can find two or more values, then the information is not sufficient.

Consider this statement:

(1) Oliver's age is a multiple of 4.

Oliver could be 4 or 8 or 12 or any age that is a multiple of 4. Because it's impossible to determine one particular value for Oliver's age, the statement is not sufficient

What if the question changed?

Is Oliver's age an even number?

(1) Oliver's age is a multiple of 4.

(2) Oliver is between 19 and 23 years old.

This question is a Yes/No question. There are three possible answers to a Yes/No question:

1. Always Yes: Sufficient!
2. Always No: Sufficient!
3. Maybe (or Sometimes Yes, Sometimes No): Not Sufficient

It may surprise you that Always No is sufficient to answer the question. Imagine that you ask a friend to go to the movies with you. If she says, “No, I'm sorry, I can't,” then you did receive an answer to your question (even though the answer is negative). You know she can't go to the movies with you.

Apply this reasoning to the Oliver question. Is statement 1 sufficient to answer the question *Is Oliver's age an even number*?

(1) Oliver's age is a multiple of 4.

If Oliver's age is a multiple of 4, then Yes, he must be an even number of years old. The information isn't enough to tell how old Oliver actually is—he could be 4, 8, 12, or any multiple of 4 years old. Still, the information is sufficient to answer the specific question asked.

Because the statement tried first is sufficient, cross off the bottom row of answers, (B), (C), and (E).

Next, check statement (2):

(2) Oliver is between 19 and 23 years old.

Oliver could be 20, in which case his age is even. He could also be 21, in which case his age is odd. The answer here is Sometimes Yes, Sometimes No, so the information is not sufficient to answer the question.

The correct answer is **(A)**: the first statement is sufficient but the second is not.

The DS Process

This section summarizes everything you've learned in one consistent DS process. You can use this on every DS problem on the test.

<u>Step 1: Determine whether the question is Value or Yes/No.</u>

Value: The question asks for the value of an unknown (e.g., What is *x*?).

A statement is **Sufficient** when it provides **1 possible value**.

A statement is **Not Sufficient** when it provides **more than 1 possible value** (or none at all).

Yes/No: The question asks whether a given piece of information is true (e.g., Is *x* even?).

Most of the time, these will be in the form of Yes/No questions.

A statement is **Sufficient** when the answer is **Always Yes** or **Always No**.

A statement is **Not Sufficient** when the answer is **Maybe** or **Sometimes Yes, Sometimes No**.

Step 2: Separate given information from the question itself.

If the question stem contains given information—that is, any information other than the question itself—then write down that information separately from the question itself. This is true information that you must consider or use when answering the question.

Step 3: Rephrase the question.

Most of the time, you will not write down the entire question stem exactly as it appears on screen. At the least, you'll simplify what is written on screen. For example, if the question stem asks, "What is the value of *x*?" then you might write down something like *x* = ________?

For more complicated question stems, you may have more work to do. Ideally, before you go to the statements, you will be able to articulate a fairly clear and straightforward question. In the earlier example, *x* = ________? is clear and straightforward.

What if this is the question?

If $xyz \neq 0$, is $\frac{3x}{2} + y + 2z = \frac{7x}{2} + y$?

(1) $y = 3$ and $x = 2$

(2) $z = -x$

Do you need to know the individual values of *x*, *y*, and *z* in order to answer the question? Would knowing the value of a combination of the variables, such as $x + y + z$, work? It's tough to tell.

In order to figure this out, ***rephrase*** the question stem, which is a fancy way of saying: simplify the information a lot. Take the time to do this before you address the statements; you'll make your job much easier!

If you're given an equation, the first task is to put the “like” variables together. Also, when working with the question stem, make sure to carry the question mark through your work:

$$y - y + 2z = \frac{7x}{2} - \frac{3x}{2}?$$

That's interesting: the two *y* variables cancel out. Keep simplifying:

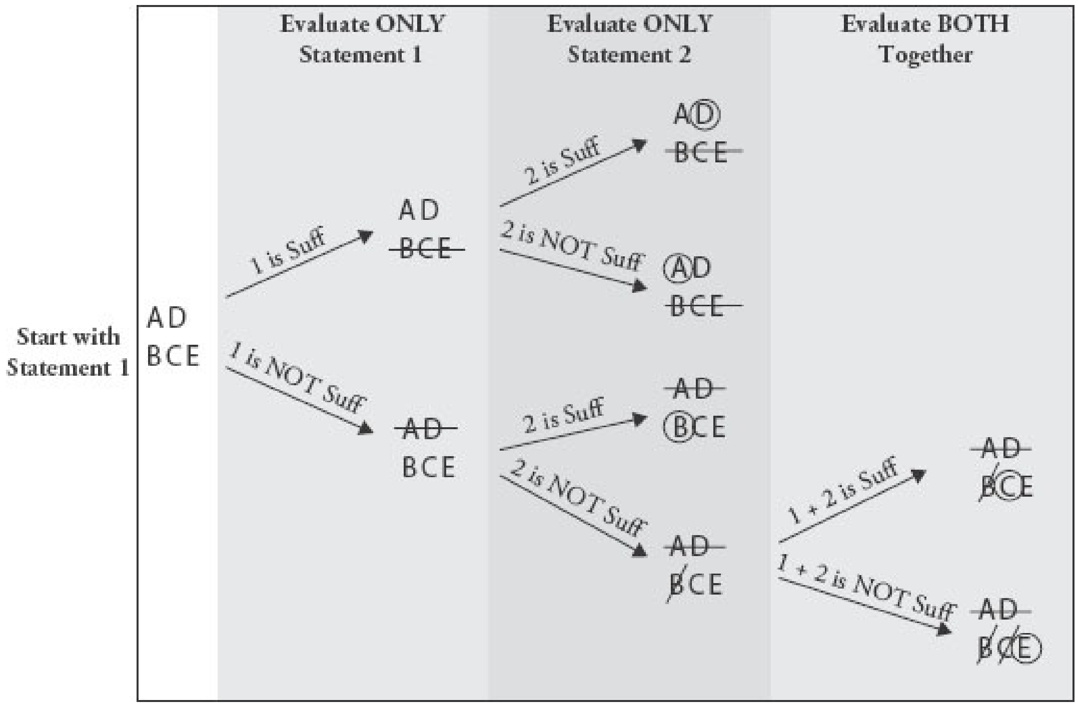
$$2z = \frac{4x}{2}?$$
$$2z = 2x?$$
$$z = x?$$

That whole crazy equation is really asking a much simpler question: is $z = x$?

It might seem silly to keep writing that question mark at the end of each line, but don't skip that step or you'll be opening yourself up to a careless error. By the time you get to the end, you don't want to forget that this is still a *question*, not a statement or given.

<u>Step 4: Use the Answer Grid to Evaluate the Statements</u>

If you start with statement 1, then write the AD/BCE grid on your scrap paper.



Here is the rephrased problem:

If $xyz \neq 0$, is $z = x$?

(1) $y = 3$ and $x = 2$

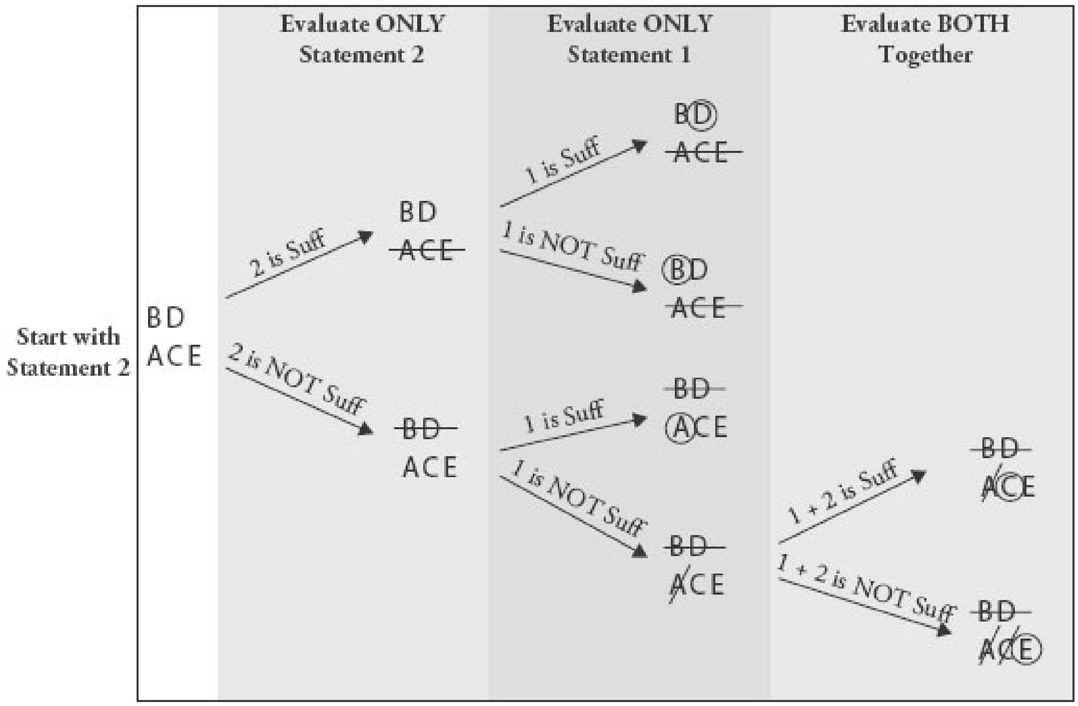
(2) $z = -x$

Statement (1) is useless by itself because it says nothing about z. Cross off the top row of answers: $\frac{\cancel{AD}}{BCE}$

Statement (2) turns out to be very useful. None of the variables is 0, so if $z = -x$, then those two numbers cannot be equal to each other. This statement is sufficient to answer the question: no, z does not equal x. You can circle B on your grid: $\frac{\cancel{AD}}{BCE}$

The correct answer is **(B)**.

If you decide to start with statement (2), your process is almost identical, but you'll use the BD/ACE grid instead. For example:



First, evaluate statement (1) by itself and, if you've crossed off answers (A), (B), and (D), then evaluate the two statements together.
Whether you use AD/BCE or BD/ACE, remember to

- cross off the *top* row if the first statement you try is *not* sufficient, and
- cross off the *bottom* row if the first statement you try *is* sufficient.

Pop Quiz! Test Your Skills

Have you learned the DS process? If not, go back through the chapter and work through the sample problems again. Try writing out each step yourself.

If so, prove it! Give yourself up to four minutes total to try the following two problems:

1. Are there more engineers than salespeople working at SoHo Corp?

   (1) SoHo Corp employs $\frac{2}{3}$ as many clerical staff as engineers and salespeople combined.

   (2) If 3 more engineers were employed by SoHo Corp and the number of salespeople remained the same, then the number of engineers would be

double the number of salespeople employed by the company.

2. At SoHo Corp, what is the ratio of managers to non-managers?

(1) If there were 3 more managers and the number of salespeople remained the same, then the ratio of managers to non-managers would double.

(2) There are 4 times as many non-managers as managers at SoHo Corp.

How did it go? Are you very confident in your answers? Somewhat confident? Not at all confident?

Before you check your answers, go back over your work, using the DS process discussed in this chapter as your guide. Where can you improve? Did you write down (and use!) your answer grid? Did you look at each statement separately before looking at them together (if necessary)? Did you mix up any of the steps of the process? How neat is the work on your scrap paper? You may want to rewrite your work before you review the answers.

Pop Quiz Answer Key

1. Engineers vs. Salespeople

Step 1: Is this a Value or Yes/No question?

1. Are there more engineers than salespeople working at SoHo Corp?

This is a Yes/No question.

Steps 2 and 3: What is given and what is the question? Rephrase the question.

The question stem doesn't contain any given information. In this case, the question is already about as simplified as it can get: are there more engineers than salespeople?

Step 4: Evaluate the statements.

If you start with the first statement, use the AD/BCE answer grid.

(1) SoHo Corp employs $\frac{2}{3}$ as many clerical staff as engineers and

salespeople combined.

If you add up the engineers and salespeople, then there are fewer people on the clerical staff…but this indicates nothing about the relative number of engineers and salespeople. This statement is not sufficient. Cross off (A) and (D), the top row, of your answer grid.

(2) If 3 more engineers were employed by SoHo Corp and the number of salespeople remained the same, then the number of engineers would be double the number of salespeople employed by the company.

This one sounds promising. If you add only 3 engineers, then you'll have twice as many engineers as salespeople. Surely, that means there are more engineers than salespeople?

Don't jump to any conclusions. Test some possible numbers; think about fairly extreme scenarios. What if you start with just 1 engineer? When you add 3, you'll have 4 engineers. If there are 4 engineers, then there are half as many, or 2, salespeople. In other words, you start with 1 engineer and 2 salespeople, so there are more salespeople. Interesting.

According to this one case, the answer to the Yes/No question *Are there more engineers than salespeople?* is no.

Can you find a yes answer? Try a larger set of numbers. If you start with 11 engineers and add 3, then you would have 14 total. The number of salespeople would have to be 7. In this case, then, there are more engineers to start than salespeople, so the answer to the question *Are there more engineers than salespeople?* is yes.

Because you can find both yes and no answers, statement (2) is not sufficient. Cross off answer (B).

Now, try the two statements together. How does the information about the clerical staff combine with statement (2)?

Whenever you're trying some numbers and you have to examine the two statements together, see whether you can reuse the numbers that you tried earlier.

If you start with 1 engineer, you'll have 2 salespeople, for a total of 3. In this case, you'd have 2 clerical staff, and the answer to the original question is no.

If you start with 11 engineers, you'll have 7 salespeople, for a total of 18. In this case, you'd have 12 clerical staff, and the answer to the original question is yes.

The correct answer is **(E)**. The information is not sufficient even when both statements are used together.

2. Managers vs. Non-Managers

Step 1: Is this a Value or a Yes/No question?

2. At SoHo Corp, what is the ratio of managers to non-managers?

This is a Value question. You need to find one specific ratio—or know that you can find one specific ratio—in order to answer the question.

Steps 2 and 3: What is given and what is the question? Rephrase the question.

Find the ratio of managers to non-managers, or M : N.

Step 4: Evaluate the statements.

If you start with the second statement, use the BD/ACE answer grid. (Note: this is always your choice; the solution with the BD/ACE grid shown is just for practice.)

(2) There are 4 times as many non-managers as managers at SoHo Corp.

If there is 1 manager, there are 4 non-managers. If there are 2 managers, there are 8 non-managers. If there are 3 managers, there are 12 non-managers.

What does that mean? In each case, the ratio of managers to non-managers is the same, 1 : 4. Even though you don't know how many managers and non-managers there are, you do know the ratio. (For more on ratios, see the Ratios chapter of this book.

This statement is sufficient; cross (A), (C), and (E), the bottom row, off of the grid.

(1) If there were 3 more managers and the number of salespeople remained the same, then the ratio of managers to non-managers would double.

First, what does it mean to *double* a ratio? If the starting ratio were 2 : 3, then doubling that ratio would give you 4 : 3. The first number in the ratio doubles relative to the second number.

Test some cases. If you start with 1 manager, then 3 more would bring the total number of managers to 4. The *manager* part of the ratio just quadrupled (1 to 4), not doubled, so this number is not a valid starting point. Discard this case.

If you have to add 3 and want that number to double, then you need to start with 3 managers. When you add 3 more, that portion of the ratio doubles from 3 to 6. The other portion, the non-managers, remains the same.

Notice anything? The statement says nothing about the relative number of non-managers. The starting ratio could be 3 : 2 or 3 : 4 or 3 : 14, for all you know. In each case, doubling the number of managers would double the ratio (to 6 : 2, or 6 : 4, or 6 : 14). You can't figure out the specific ratio from this statement.

The correct answer is **(B)**: statement (2) is sufficient, but statement (1) is not.

Proving Insufficiency

The Pop Quiz solutions used the Testing Cases strategy: testing real numbers to help determine whether a statement is sufficient. You can do this whenever the problem allows for the possibility of multiple numbers or cases.

When you're doing this, your goal is to try to prove the statement insufficient. For example:

If x and y are positive integers, is the sum of x and y between 20 and 26, inclusive?

(1) $x - y = 6$

Test your first case. You're allowed to pick any numbers for x and y that make statement 1 true *and* that follow any constraints given in the question stem. In this case, that means the two numbers have to be positive integers and that $x - y$ has to equal 6.

Case #1: $20 - 14 = 6$. These numbers make statement 1 true and follow the constraint in the question stem, so these are legal numbers to pick. Now, try to answer the Yes/No question: $20 + 14 = 34$, so no, the sum is not between 20 and 26, inclusive.

You now have a *no* answer. Can you think of another set of numbers that will give you the opposite, a *yes* answer?

Case #2: $15 - 9 = 6$. In this case, the sum is 24, so the answer to the Yes/No question is yes, the sum is between 20 and 26, inclusive.

Because you have found both a yes and a no answer, the statement is not sufficient.

Here's a summary of the process:

1. Notice that you can test cases. You can do this when the problem allows for multiple possible values.
2. Pick numbers that make the statement true and that follow any givens in the question stem. If you realize that you picked numbers that make the statement false or contradict givens in the question stem, *discard* those numbers and start over.
3. Your first case will give you one answer: a yes or a no on a Yes/No problem, or a numerical value on a value problem.
4. Try to find a second case that gives you a *different* answer. On a Yes/No problem, you'll be looking for the opposite of what you found for the first case. For a Value problem, you'll be looking for a different numerical answer. (Don't forget that whatever you pick still has to make the statement true and follow the givens in the question stem!)

The usefulness of trying to prove insufficiency is revealed as soon as you find two different answers. You're done! That statement is not sufficient, so you can cross off an answer or answers and move to the next step.

What if you keep finding the same answer? Try this:

> If *x* and *y* are positive integers, is the product of *x* and *y* between 20 and 26, inclusive?
>
> (1) *x* is a multiple of 17.

Case #1: Test *x* = 17. Since *y* must be a positive integer, try the smallest possible value first: *y* = 1. In this case, the product is 17, which is not between 20 and 26 inclusive. The answer to the question is *no*; can you find the opposite answer?

Case #2: If you make *x* = 34, then *xy* will be too big, so keep *x* = 17. The next smallest possible value for *y* is 2. In this case, the product is 34, which is also not between 20 and 26 inclusive. The answer is again no.

Can you think of a case where you will get a *yes* answer? No! The smallest possible product is 17, and the next smallest possible product is 34. Any additional values of *x* and *y* you try will be equal to or larger than 34.

You've just proved the statement sufficient because it is impossible to find a yes answer. Testing Cases can help you to figure out the "theory" answer, or the mathematical reasoning that proves the statement is sufficient.

This won't always work so cleanly. Sometimes, you'll keep getting all no answers or all yes answers but you won't be able to figure out the theory behind it all. If you test three or four different cases, and you're actively seeking out the opposite answer but never find it, then go ahead and assume that the statement is sufficient, even if you're not completely sure why.

Do make sure that you're trying numbers with different characteristics. Try both even and odd. Try a prime number. Try zero or a negative or a fraction. (You can only try numbers that are allowed by the problem, of course. In the case of the above problems, you were only allowed to try positive integers.)

Here's how Testing Cases would work on a Value problem:

> If *x* and *y* are prime numbers, what is the product of *x* and *y*?
>
> (1) The product is even.

Case #1: *x* = 2 and *y* = 3. Both numbers are prime numbers and their product is even, so these are legal numbers to try. In this case, the product is 6. Can you

Case #1: x = 2 and y = 3. Both numbers are prime numbers and their product is even, so these are legal numbers to try. In this case, the product is 6. Can you choose numbers that will give a different product?
Case #2: x = 2 and y = 5. Both numbers are prime numbers and their product is even, so these are legal numbers to try. In this case, the product is 10.
The statement is not sufficient because there are at least two different values for the product of x and y.

Chapter 1
of
Algebra

PEMDAS

In This Chapter…

Chapter 1

PEMDAS

When simplifying an algebraic expression, you have to follow a specific order of operations. The correct order of operations is: Parentheses-Exponents-(Multiplication/Division)-(Addition/Subtraction), or PEMDAS in the United States. If you learned math in other English-speaking countries, you may have memorized slightly different acronyms; the rules are still the same. Multiplication and division are in parentheses because they are on the *same* level of priority. The same is true of addition and subtraction. When two or more operations are at the same level of priority, always work from left to right.

Simplify $5 + (2 \times 4 + 2)^2 - |7(-4)| + 18 \div 3 \times 5 - 8$.

P = PARENTHESES. First, perform all of the operations that are *inside* parentheses. Note that in terms of order of operations, absolute value signs are equivalent to parentheses. In this expression, there are two groups of parentheses:	$(2 \times 4 + 2)$ and $\lvert 7(-4) \rvert$
In the first group, there are two operations to perform, multiplication and addition. According to PEMDAS, multiplication must come before addition:	$(2 \times 4 + 2) = (8 + 2) = 10$
In the second group, perform the operation inside first (multiplication), then take the absolute value of that number:	$\lvert 7(-4) \rvert = \lvert -28 \rvert = 28$
Now the original expression looks like this:	$\mathbf{5 + 10^2 - 28 + 18 \div 3 \times 5 - 8}$

E = EXPONENTS. Second, take care of any exponents in the expression:	$10^2 = 100$
Now the expression looks like this:	$\mathbf{5 + 100 - 28 + 18 \div 3 \times 5 - 8}$
M&D = MULTIPLICATION & DIVISION. Next, perform all the multiplication and division. Work from left to right:	$\underbrace{18 \div 3}_{6} \times 5 = 30$
Now the expression reads:	$\mathbf{5 + 100 - 28 + 30 - 8}$
A&S = ADDITION & SUBTRACTION. Lastly, perform all the addition and subtraction. Work from left to right.	$5 + 100 - 28 + 30 - 8$ $105 - 28 + 30 - 8$ $77 + 30 - 8$ $107 - 8$
The answer:	**99**

Subtraction of Expressions

One of the most common errors involving the order of operations occurs when an expression with multiple terms is subtracted. The subtraction must occur across *every* term within the expression. Each term in the subtracted part must have its sign reversed. For example:

$x - (y - z) = x - y + z$	(Note that the signs of both y and $-z$ have been reversed.)
$x - (y + z) = x - y - z$	(Note that the signs of both y and z have been reversed.)
$x - 2(y - 3z) = x - 2y + 6z$	(Note that the signs of both y and $-3z$ have been reversed.)

Now try another example:

What is $5x - [y - (3x - 4y)]$?

Both expressions in parentheses must be subtracted, so the signs of each term must be reversed for *each* subtraction. Note that the square brackets are just fancy parentheses, used so that you avoid having parentheses right next to each other.

$$5x – [y – (3x – 4y)] =$$
$$5\ x – (y – 3x + 4y) =$$
$$5x – (5y – 3x) =$$
$$5x – 5y + 3x = \mathbf{8x – 5y}$$

Fraction Bars as Grouping Symbols

Even though fraction bars do not fit into the PEMDAS hierarchy, they do take precedence. In any expression with a fraction bar, pretend that there are parentheses around the numerator and denominator of the fraction. This may be obvious as long as the fraction bar remains in the expression, but it is easy to forget if you eliminate the fraction bar or add or subtract fractions. For example:

$$\text{Simplify: } \frac{x-1}{2} - \frac{2x-1}{3}$$

The common denominator for the two fractions is 6, so multiply the numerator and denominator of the first fraction by 3, and those of the second fraction by 2:

$$\frac{x-1}{2}\left(\frac{3}{3}\right) - \frac{2x-1}{3}\left(\frac{2}{2}\right) = \frac{3x-3}{6} - \frac{4x-2}{6}$$

Treat the expressions $3x – 3$ and $4x – 2$ as though they were enclosed in parentheses! Accordingly, once you make the common denominator, actually put in parentheses for these numerators. Then reverse the signs of both terms in the second numerator:

$$\frac{(3x-3)-(4x-2)}{6} = \frac{3x-3-4x+2}{6} = \frac{-x-1}{6} = -\frac{x+1}{6}$$

Problem Set

1. Evaluate: $(4 + 12 \div 3 - 18) - [-11 - (-4)]$

2. Evaluate: $-|-13 - (-17)|$

3. Evaluate: $\left(\frac{4+8}{2-(-6)}\right) - (4 + 8 \div 2 - (-6))$

4. Simplify: $x - (3 - x)$

5. Simplify: $(4 - y) - 2(2y - 3)$

Solutions

1. **–3:** $(4 + 12 \div 3 - 18) - (-11 - (-4)) =$

$(4 + 4 - 18) - (-11 + 4) =$ Division before addition/subtraction

$(-10) - (-7) =$ Subtraction of negative = addition

$-10 + 7 = -3$ Arithmetic—watch the signs!

2. **–4:** $-|-13 - (-17)| =$

$-|-13 + 17| =$ Subtraction of negative = addition

$-|4| = -4$

Note that the absolute value *cannot* be made into 13 + 17. You must perform the arithmetic inside grouping symbols *first*, whether inside parentheses or inside absolute value bars. *Then* you can remove the grouping symbols.

3. $-\frac{25}{2}$ **or** $-12\frac{1}{2}$: $\left[\frac{4+8}{\underbrace{2-(-6)}}\right] - [4 + 8 \div 2 - (-6)] =$

$$\underbrace{\left(\frac{4+8}{2+6}\right)} - (4 + 8 \div 2 + 6) =$$

$$\left(\frac{12}{8}\right) - \left(\underbrace{4 + 4 + 6}\right) =$$

$$\frac{3}{2} - 14 =$$

$$\frac{3}{2} - \frac{28}{2} = -\frac{25}{2} \text{ or } -12\frac{1}{2}$$

4. **$2x$ – 3:** Do not forget to reverse the signs of every term in a subtracted expression:

$$x - (3 - x) = x - 3 + x = 2x - 3$$

5. **– $5y$ + 10 (or 10 – $5y$):** Do not forget to reverse the signs of every term in a subtracted expression:

$$(4 - y) - 2(2y - 3) = 4 - y - 4y + 6 = -5y + 10 \text{ (or } 10 - 5y)$$

Chapter 2
of
Algebra

Linear Equations

In This Chapter…

Chapter 2

Linear Equations

Linear equations are equations in which all variables have an exponent of 1. For example, the equation $x - 13 = 24$ is linear because the variable x is raised to the first power.

Expressions vs. Equations

The most basic difference between expressions and equations is that equations contain an equals sign, and expressions do not.

An expression, even one that contains variables, represents a value. When manipulating or simplifying expressions, you have to follow certain rules to ensure that you don't change the value of the expression.

There are several methods for simplifying expressions. You can:

1. Combine like terms	$6z + 5z \rightarrow 11z$
2. Find a common denominator	$\frac{1}{12} + \frac{3x^3}{4} \times \left(\frac{3}{3}\right) \rightarrow \frac{1}{12} + \frac{9x^3}{12} = \frac{9x^3 + 1}{12}$
3. Pull out a common factor	$2ab + 4b \rightarrow 2b(a + 2)$
4. Cancel common factors	$\frac{5y^3}{25y} \rightarrow \frac{y^2}{5}$

What all of these moves have in common is that the value of the expression stays the same. If you plug numbers into the original and simplified forms, the value is the same. For example, replace *z* in the first expression with 3:

6z + 5z	11z
6(3) + 5(3)	11(3)
18 + 15	33
33	

Thus, 6z + 5z is equivalent to 11*z*.

Equations behave differently. An equation contains an equals sign. In order to keep the two sides of the equation equal, any change you make to one side must also be made to the other side. Also, while the two sides are still equal, the change may alter the values on both sides of the equation.

3 = 3	An equivalence
2 × (3) = (3) × 2	Multiply both sides by 2
6 = 6	The two sides are still equal, but have different values

In general, there are six operations you can perform to both sides of an equation. Remember to perform the action on the *entire* side of the equation. For example, if you were to square both sides of the equation $\sqrt{x}+1=x$, you would have to square the entire expression ($\sqrt{x}+1$), as opposed to squaring each term individually.

You can:

1. Add the same thing to both sides

$$\begin{array}{l} z-13=-14 \\ \quad +13 \quad +13 \\ \hline z \qquad =-1 \end{array}$$

2. Subtract the same thing from both sides

$$\begin{array}{l} x+8=34 \\ \quad -8 \quad -8 \\ \hline x \qquad =26 \end{array}$$

3. Multiply both sides by the same thing

$$\frac{4}{a} = a + b$$

$$a \times \left(\frac{4}{a}\right) = (a + b) \times a$$

$$4 = a^2 + ab$$

4. Divide both sides by the same thing

$$3x = 6y + 12$$

$$\frac{3x}{3} = \frac{6y + 12}{3}$$

$$x = 2y + 4$$

5. Raise both sides to the same power

$$\sqrt{y} = y + 2$$

$$\left(\sqrt{y}\right)^2 = (y + 2)^2$$

$$y = (y + 2)^2$$

6. Take the same root of both sides

$$x^3 = 125$$

$$\sqrt[3]{x^3} = \sqrt[3]{125}$$

$$x = 5$$

Solving One-Variable Equations

In order to solve one-variable equations, isolate the variable on one side of the equation. In doing so, make sure you perform identical operations to both sides of the equation. Here are three examples:

$3x + 5 = 26$	Subtract 5 from both sides.
$3x = 21$	Divide both sides by 3.
$x = 7$	

$w = 17w - 1$	Subtract *w* from both sides.
$0 = 16w - 1$	Add 1 to both sides.
$1 = 16w$	Divide both sides by 16.
$\frac{1}{16} = w$	

$$\frac{p}{9} + 3 = 5$$ Subtract 3 from both sides.

$$\frac{p}{9} = 2$$ Multiply both sides by 9.

$$p = 18$$

Simultaneous Equations: Solving by Substitution

Sometimes the GMAT asks you to solve a system of equations with more than one variable. You might be given two equations with two variables, or perhaps three equations with three variables. In either case, there are two primary ways of solving simultaneous equations: by substitution or by combination.

Use substitution to solve the following for *x* and *y*.

$x + y = 9$
$2x = 5y + 4$

First, solve the first equation for *x*.

$$x + y = 9$$
$$x = 9 - y$$

Next, substitute the expression $9 - y$ into the second equation wherever x appears.

$$2x = 5y + 4$$
$$2(9 - y) = 5y + 4$$

Then, solve the second equation for *y*. You will now get a number for *y*.

$$2(9 - y) = 5y + 4$$
$$18 - 2y = 5y + 4$$
$$14 = 7y$$
$$2 = y$$

Finally, substitute your solution for *y* into either of the original equations in order to solve for *x*.

$$\begin{aligned} x + y &= 9 \\ x + 2 &= 9 \\ x &= 7 \end{aligned}$$

You could also have started this process by solving the first equation for *y* and then substituting the expression $9 - x$ in place of *y* in the second equation.

Simultaneous Equations: Solving by Combination

Alternatively, you can solve simultaneous equations by combination. In this method, add or subtract the two equations to eliminate one of the variables.

Solve the following for *x* and *y*.

$$\begin{aligned} x + y &= 9 \\ 2x &= 5y + 4 \end{aligned}$$

To start, line up the terms of the equations.

$$\begin{aligned} x + y &= 9 \\ 2x - 5y &= 4 \end{aligned}$$

The goal is to make one of two things happen: either the coefficient in front of one of the variables (say, *x*) is the same in both equations, in which case you subtract one equation from the other, or the coefficient in front of one of the variables is the same but with opposite signs, in which case you add the two equations. You do this by multiplying one of the equations by some number. For example, multiply the first equation by –2:

$$\begin{aligned} -2(x + y = 9) \quad &\rightarrow \quad -2x - 2y = -18 \\ 2x - 5y = 4 \quad &\rightarrow \quad 2x - 5y = 4 \end{aligned}$$

Next, add the equations to eliminate one of the variables.

$$\begin{array}{rr} & -2x-2y=-18 \\ + & 2x-5y=\quad 4 \\ \hline & -7y=-14 \end{array}$$

Now, solve the resulting equation for the unknown variable.

$$\begin{aligned} -7y &= -14 \\ y &= 2 \end{aligned}$$

Finally, substitute into one of the original equations to solve for the second variable.

$$\begin{aligned} x+y &= 9 \\ x+2 &= 9 \\ x &= 7 \end{aligned}$$

Absolute Value Equations

Absolute value refers to the *positive* value of the expression within the absolute value brackets. Equations that involve absolute value generally have two solutions. In other words, there are *two* numbers that the variable could equal in order to make the equation true, because the value of the expression inside the absolute value brackets could be *positive or negative*. For instance, if you know $|x| = 5$, then x could be either 5 or –5 and the equation would still be true.

Use the following two-step method when solving for a variable expression inside absolute value brackets.

Solve for w, given that $12 + |w - 4| = 30$.

Step 1: Isolate the expression within the absolute value brackets.

$$\begin{aligned} 12 + |w-4| &= 30 \\ |w-4| &= 18 \end{aligned}$$

Step 2: Once you have an equation of the form $|x| = a$ with $a > 0$, you know that $x = \pm a$. Remove the absolute value brackets and solve the equation for two different cases:

CASE 1: $x = a$ (x is positive)

$$w - 4 = 18$$
$$w = 22$$

CASE 2: $x = -a$ (x is negative)

$$w - 4 = -18$$
$$w = -14$$

Problem Set

Now that you've finished the chapter, do the following problems.

1. Solve for x: $2(2 - 3x) - (4 + x) = 7$

2. Solve for z: $\frac{4z-7}{3-2z} = -5$

3. Solve for y: $22 - |y + 14| = 20$

4. Solve for x and y: $y = 2x + 9$ and $7x + 3y = -51$

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

5. Every attendee at a monster truck rally paid the same admission fee. How many people attended the rally?

   (1) If the admission fee had been raised to \$20 and twice as many people had attended, the total admission fees collected would have been three times greater.

   (2) If the admission fee had been raised to \$30 and two-thirds as many people had attended, the total admission fees collected would have been 150% of the actual admission fees collected.

6. Solve for x: $x\left(x - \frac{5x+6}{x}\right) = 0$

7. Solve for x: $\frac{3x-6}{5} = x - 6$

8. Solve for x: $\frac{x+2}{4+x} = \frac{5}{9}$

Solutions

1. **–1:**

$$\begin{aligned} 2(2-3x)-(4+x)&=7 \\ 4-6x-4-x&=7 \\ -7x&=7 \\ x&=-1 \end{aligned}$$

2. $\mathbf{\frac{4}{3}}$**:**

$$\frac{4z-7}{3-2z}=-5$$

$$\begin{aligned} 4z-7&=-5(3-2z) \\ 4z-7&=-15+10z \\ 8&=6z \\ z&=\frac{8}{6}=\frac{4}{3} \end{aligned}$$

3. ***y*** = {**–16, –12}:** First, isolate the expression within the absolute value brackets. Then, solve for two cases, one in which the expression is positive and one in which it is negative:

$$\begin{aligned} 22-|y+14|&=20 \\ 2&=|y+14| \end{aligned}$$

Case 1: $y+14=2$
$y=-12$

Case 2: $y+14=-2$
$y=-16$

4. ***x*** **= –6;** ***y*** = –**3:** Solve this system by substitution. Substitute the value given for *y* in the first equation into the second equation. Then, distribute, combine like terms, and solve. Once you get a value for *x*, substitute it back into the first equation to obtain the value of *y*:

$$y = 2x + 9 \qquad 7x + 3y = -51$$

$$\begin{aligned} 7x + 3(2x + 9) &= -51 \\ 7x + 6x + 27 &= -51 \\ 13x + 27 &= -51 \\ 13x &= -78 \\ x &= -6 \end{aligned}$$

$$y = 2x + 9 = 2(-6) + 9 = -3$$

5. **(E):** This question asks how many people attended a monster truck rally. The number of attendees times the admission fee equals the total amount collected, as such:

Total = Attendees × Price
$T = A \times P$

You want to know A.
(1) INSUFFICIENT: If the price had been $20 and twice as many people had attended, the total would be three times greater. Therefore:

$3T = 2A \times 20$
$3T = 40A$

This is not sufficient to solve for A.

(2) INSUFFICIENT: If the price had been $30 and two-thirds as many people had attended, the total would be 150% of the actual total. Therefore:

$$1.5T = \frac{2}{3}A \times 30$$
$$1.5T = 20A$$

This is not sufficient to solve for A.

(1) AND (2) INSUFFICIENT: Don't fall for the trap that two equations for two variables is enough to solve. Notice that $3T = 40A$ and $1.5T = 20A$ are identical. Combining the two statements is therefore no more sufficient than either statement alone.

The correct answer is (E).

6. **{6, –1}:** Distribute the multiplication by *x*. Note that, when you cancel the *x* in the denominator, the quantity 5*x* + 6 is implicitly enclosed in parentheses!

$$x\left(x-\frac{5x+6}{x}\right)=0$$
$$x^2-(5x+6)=0$$
$$x^2-5x-6=0$$
$$(x-6)(x+1)=0$$
$$x=6 \text{ or } -1$$

Note also that the value 0 is impossible for *x*, because *x* is in a denominator by itself in the original equation. You are not allowed to divide by 0. Do not look at the product in the original equation and deduce that *x* = 0 is a solution.

7. **12:** Solve by multiplying both sides by 5 to eliminate the denominator. Then, distribute and isolate the variable:

$$\frac{3x-6}{5}=x-6$$
$$3x-6=5(x-6)$$
$$3x-6=5x-30$$
$$24=2x$$
$$12=x$$

8. $\boldsymbol{x=\frac{1}{2}}$: Cross-multiply to eliminate the denominators. Then, distribute and solve:

$$\frac{x+2}{4+x}=\frac{5}{9}$$

$$9(x+2)=5(4+x)$$

$$9x+18=20+5x$$

$$4x=2$$

$$x=\frac{1}{2}$$

Chapter 3
of
Algebra

Strategy: Choose Smart Numbers

In This Chapter…

Chapter 3

Strategy: Choose Smart Numbers

Some algebra problems—problems that involve unknowns, or variables—can be turned into arithmetic problems instead. You're better at arithmetic than algebra (everybody is!), so turning an annoying variable-based problem into one that uses real numbers can save time and aggravation on the GMAT.

Which of the below two problems is easier for you to solve?

> If *n* employees are fulfilling orders at the rate of 3 orders per employee per hour, how many orders are filled in 4 hours?
>
> (A) 3*n* (B) 4*n* (C) 12*n*

> If 2 employees are fulfilling orders at the rate of 3 orders per employee per hour, how many orders are filled in 4 hours?
>
> (A) 6 (B) 12 (C) 24

In the first problem, you would write an expression using the variable *n* and then you would use algebra to solve. You may think that this version is not particularly difficult, but no matter how easy you think it is, it's still easier to work with real numbers.

The set-up of the two problems is identical—and this feature is at the heart of how you can turn algebra into arithmetic.

How Do Smart Numbers Work?

Here's how to solve the algebra version of the above problem using smart numbers.

Step 1: ***Choose smart numbers*** to replace the unknowns.

How do you know you can choose a random number in the first place? The problem talks about a number but only supplies a variable for that number. It never supplies a real value for that number anywhere in the problem or in the answers.

Instead, choose a real number. In general, 2 is a often a good number to choose on algebraic smart number problems.

Step 2: ***Solve*** the problem using your chosen smart numbers.

Wherever the problem talks about the *number*, it now says 2:

> If 2 employees are fulfilling orders at the rate of 3 orders per employee per hour, how many orders are filled in 4 hours?

Do the math! If 2 employees fulfill 3 orders per hour, then together they fulfill 6 orders per hour. In 4 hours, they'll fulfill 24 orders.

Step 3: ***Find a match*** in the answers. Plug $n = 2$ into the answers.

> (A) $3n = (3)(2) = 6$
> (B) $4n = (4)(2) = 8$
> (C) $12n = (12)(2) = 24$

The correct answer is **(C).**

Keep an eye out for problems that contain variable expressions (no equations or inequality signs) in the answers; many of these problems can be solved by choosing smart numbers.

When to Choose Smart Numbers

It's crucial to know when you're allowed to use this technique. It's also crucial to know how *you* are going to decide whether to use textbook math or choose smart numbers; you will typically have time to try just one of the two techniques during your two minutes on the problem.

The *Choose Smart Numbers* technique can be used any time a problem contains only *unspecified* values. The easiest example of such a problem is one that contains variables, percents, fractions, or ratios throughout. It does not provide real numbers for those variables, even in the answer choices. Whenever a problem has this characteristic, you can choose your own smart numbers to turn the problem into arithmetic.

There is some cost to doing so: it can take extra time compared to the "pure" textbook solution. As a result, the technique is most useful when the problem is a hard one for you. If you find the algebra involved to be very easy, then you may not want to take the time to transform the problem into arithmetic. As the math gets more complicated, however, the arithmetic form becomes comparatively easier and faster to use.

Try this problem. Solve it twice; once using textbook math and once using smart numbers:

> A store bought a box of 50 t-shirts for a total of x dollars. The store then sold each t-shirt for a premium of 25% over the original cost per shirt. In terms of x, how much did the store charge for each shirt?
>
> (A) $\frac{x}{4}$
>
> (B) $\frac{5x}{4}$
>
> (C) $\frac{x}{40}$
>
> (D) $125x$
>
> (E) $\frac{125}{x}$

First, how do you know that you can choose smart numbers on this problem? The problem talks about the price of a box of t-shirts but never mentions a real number for that price anywhere along the way.

Step 1: Choose smart numbers.

Make your life easy and choose a number that will work nicely in the problem. What is the first math operation you'll need to do?

You need the cost per t-shirt, so you'll have to divide *x* by 50. Pick a number, then, that is not already in the problem (so don't use 50 itself) but that will divide evenly by 50. The number 100 is the smallest number that fits the bill.

Step 2: Solve.

The store paid \$100 for the box of 50 t-shirts, or \$2 per t-shirt.

The store then charged a 25% premium. Take 25% of \$2 and add it to the total:

$$\$2 + (0.25)(\$2) = \$2 + \$0.50 = \$2.50$$

The store sold the t-shirts for \$2.50 each.

Step 3: Find a match.

Plug $x = 100$ into the answers. At any point that you can tell that a particular answer will not equal \$2.50, stop and cross off that answer.

(A) $\frac{x}{4} = \frac{100}{4} = 25$

(B) $\frac{5x}{4} = \frac{5(100)}{4}$ = to big

(C) $\frac{x}{40} = \frac{100}{40} = \frac{10}{4} = 2.5$ Match!

(D) $125x = 125(100)$ = too big

(E) $\frac{125}{x} = \frac{125}{100} = 1.25$

The correct answer is **(C)**.

Here's the algebraic solution:

The store bought 50 t-shirts for a total of *x* dollars, or $\frac{x}{50}$ dollars per shirt. The store then sold the shirts for a 25% premium over that original cost. Set up an equation to solve:

Sale price per shirt = (Cost per shirt) + (Premium applied to cost)

$$S = \frac{x}{50} + (0.25)\left(\frac{x}{50}\right)$$

$$(1.25)\ S = \frac{x}{50} = \left(\frac{x}{50}\right)\left(\frac{5}{4}\right) = \left(\frac{x}{10}\right)\left(\frac{1}{4}\right) = \frac{x}{40}$$

The correct answer is **(C).** That may seem like less work, but take a look at some of the wrong answers:

(B) $\frac{5x}{4}$	Mistake: assume x is cost per shirt, instead of $\frac{x}{50}$.
(D) $125x$	Mistake: assume x is cost per shirt *and* multiply by 125 instead of 125%.

Answers (A) and (E) also involve mixing up legitimate calculations.

As a general rule, if you find the algebra easy, go ahead and solve that way. When the algebra becomes hard for you, though, then switch to smart numbers. If you realize you made a careless mistake with the algebra, that may be a signal that you should have used smart numbers instead.

How to Pick Good Numbers

Half of the battle lies in learning how to choose numbers that work well with the given problem. Try this one:

> A truck can carry x shipping containers and each container can hold y gallons of milk. If one truck is filled to capacity and a second one is half full, how many gallons of milk are they carrying, in terms of x and y?
>
> (A) $x + 0.5y$
> (B) $x + y$
> (C) $0.5xy$
> (D) $1.5xy$
> (E) $2xy$

Step 1: Choose smart numbers.

Try $x = 1$ and $y = 2$.

Step 2: Solve.

Each truck has one shipping container. The first truck is filled to capacity, so it carries 2 gallons of milk. The second is half full, so it carries 1 gallon. Together, the trucks carry 3 gallons of milk.

Step 3: Find a match.

(A) $x + 0.5y = 1 + (0.5)(2) = 2$
(B) $x+y = 1 + 2 = 3$ Match!
(C) $0.5xy = (0.5)(1)(2) = 1$
(D) $1.5xy = (1.5)(1)(2) = 3$ Wait a second—this one matches too!
(E) $2xy = 2(1)(2) = 4$

In rare circumstances, the number you choose could work for more than one answer choice. Now what? While the clock is still ticking, either guess between (B) and (D) or, if you have time, try a different set of numbers in the problem.

Afterwards, learn a valuable lesson about how to choose the best smart numbers: if you choose 0 or 1, you increase the chances that more than one answer will work, because those two numbers both have strange properties.

Try $x = 2$ and $y = 3$ instead. The first truck is carrying $(2)(3) = 6$ gallons of milk. The second carries half that, or 3 gallons. Together, they carry 9 gallons of milk.

You've already eliminated answers (A), (C), and (E), so you only need to try (B) and (D):

(B) $x+y = 2 + 3 = 5$ Not a match
(D) $1.5xy = (1.5)(2)(3) = 9$ Match!

The correct answer is **(D)**.

If you follow the guidelines below for choosing numbers, then the above situation is much less likely to occur:

- Do not pick 0 or 1.
- Do not pick numbers that appear elsewhere in the problem.
- If you have to choose multiple numbers, choose different numbers, ideally with different properties (e.g., odd and even). The second case above used one odd and one even number, just in case.

If you do accidentally find yourself in this situation and you have the time, then go back, change one of the numbers in your problem, and do the math again. If you don't have time, just pick one of the two answers that did work.

Here's a summary of the Choose Smart Numbers strategy:

Step 0: Recognize that you can choose smart numbers.

The problem talks about some values but doesn't provide real numbers for those values. Rather, it uses variables or only refers to fractions or percents. The answer choices consist of variable expressions, fractions, or percents. (See the *Fractions, Decimals, & Percents GMAT Strategy Guide* for more on using smart numbers.)

Step 1: Choose smart numbers.

1. If you have to pick for more than one variable, pick different numbers for each one. If possible, pick numbers with different characteristics (e.g., one even and one odd).
2. Follow any constraints given in the problem. You may be restricted to positive numbers or to integers, for example, depending upon the way the problem is worded.
3. Avoid choosing 0, 1, or numbers that already appear in the problem.
4. Choose numbers that work easily in the problem. The numbers 2, 3, and 5 are often good numbers to use for algebra problems. If you have to divide, try to pick a number that will yield an integer after the division.

Step 2: Solve the problem using your chosen smart numbers.

Wherever the problem used to have variables or unknowns, it now contains the real numbers that you've chosen. Solve the problem arithmetically and find your target answer.

Step 3: Find a match in the answers.

> Plug your smart numbers into the variables in the answer choices and look for the choice that matches your target. If, at any point, you can tell that a particular answer will *not* match your target, stop calculating that answer. Cross it off and move on to the next answer.

How to Get Better at Smart Numbers

Practice makes perfect! First, try the problem sets associated with this book. When you think smart numbers can be used, try each problem two times: once using smart numbers and once using the “textbook” method. (Time yourself separately for each attempt.)

When you're done, ask yourself which way you prefer to solve *this* problem and why. On the real test, you won't have time to try both methods; you'll have to make a decision and go with it. Learn *how* to make that decision while studying; then, the next time a new problem pops up in front of you that could be solved by choosing smart numbers, you'll be able to make a quick (and good!) decision.

Keep an eye out for other opportunities to choose smart numbers throughout the rest of this guide, as well as other guides. This strategy is very useful!

Finally, one important note: at first, you may find yourself always choosing the textbook approach. You've practiced algebra for years, after all, and you've only been using the choose smart numbers technique for a short period of time. Keep practicing; you'll get better! Every high scorer on the Quant section will tell you that choosing smart numbers is invaluable to getting through Quant on time and with a consistent enough performance to reach a top score.

When NOT to Use Smart Numbers

There are certain scenarios in which a problem contains some of the smart numbers characteristics but not all. For example, why can't you use smart numbers on this problem?

> Four brothers split a sum of money between them. The first brother received 50% of the total, the second received 25% of the total, the third received 20% of the total, and the fourth received the remaining $4. How

many dollars did the four brothers split?

(A) 50
(B) 60
(C) 75
(D) 80
(E) 100

The problem talks about a sum of money but, at first, tells you nothing concrete about this sum of money. Towards the end, though, it does give you one real value: $4. Because the "remaining" percent has to equal $4 exactly, this problem has just one numerical answer. You can't pick any starting point that you want. (The answer to the above problem is (D), by the way!)

Problem Set

1. Seamus has 3 times as many marbles as Ronit, and Taj has 7 times as many marbles as Ronit. If Seamus has s marbles then, in terms of s, how many marbles do Seamus, Ronit, and Taj have together?

   (A) $\frac{3}{7}s$
   (B) $\frac{7}{3}s$
   (C) $\frac{11}{3}s$
   (D) $7s$
   (E) $11s$

2. If $x = a + b$ and $y = a + 2b$, then what is $a - b$, in terms of x and y?

   (A) $2y - 3x$
   (B) $3y - 2x$
   (C) $2x - 3y$
   (D) $2x + 3y$
   (E) $3x - 2y$

3. Cost is expressed by the formula tb^4. If b is doubled and t remains the same, the new cost is how many times greater than the original cost?

   (A) 1.2
   (B) 2
   (C) 6
   (D) 8
   (E) 16

Solutions

1. **(C):** The problem will be easier to solve if you can choose numbers that will give you all integers as you solve. Both Seamus and Taj have a multiple of the number of marbles that Ronit has, so begin by picking for Ronit, not for Seamus.

If Ronit has 2 marbles, then Seamus has (3)(2) = 6 marbles and Taj has (7)(2) = 14 marbles. Together, the three have 22 marbles.

Plug $s = 6$ into the answers (remember that the problem asks about Seamus's starting number, not Ronit's!) and look for a match of 22:

(A) $\frac{3}{7}s$ = not an integer

(B) $\frac{7}{3}s = \frac{7}{3}(6) = 14$. Not a match.

(C) $\frac{11}{3}s = \frac{11}{3}(6) = 22$. Match!

(D) $7s = 42$. Not a match.

(E) $11s$ = Too large.

Alternatively, you can use an algebraic approach. Begin by translating the first sentence into equations:

$$s = 3r$$
$$t = 7r$$

The question asks for the sum of the three:

$$s + r + t = ?$$

The answers use only s, so figure out how to substitute to leave only s in the equation:

$r = \frac{s}{3}$

$t = 7r = 7\left(\frac{s}{3}\right)$

Substitute those into the question:

$s + r + t$

$s + \frac{s}{3} + 7\left(\frac{s}{3}\right)$

$\frac{3s}{3} + \frac{s}{3} + \frac{7s}{3}$

$\frac{11s}{3}$

The correct answer is **(C)**.

2. **(E):** With so many variables, choosing smart numbers will probably be more efficient. Because *x* and *y* can be found by certain sums of *a* and *b*, pick for *a* and *b*, then calculate *x* and *y*.

If $a = 5$ and $b = 2$, then $x = 5 + 2 = 7$ and $y = 5 + 2(2) = 9$. The difference $a – b = 5 – 2 = 3$.

Plug $x = 7$ and $y = 9$ into the answers and look for a match of 3:

(A) $2y – 3x = 2(9) – 3(7) = 18 – 21$ = negative
(B) $3y – 2x = 3(9) – 2(7) = 27 – 14 = 13$
(C) $2x – 3y = 2(7) – 3(9) = 14 – 27$ = negative
(D) $2x + 3y = 2(7) + 3(9)$ = too big
(E) $3x – 2y = 3(7) – 2(9) = 21 – 18 = 3$. Match!

You can also use an algebraic approach.

Given: $x = a + b$

Given: $y = a + 2b$
What is $a - b$?

The answers use only x and y, so figure out how to rewrite the given equations to plug into the question, using only x and y.

If you subtract the two equations, you'll get x and y in terms of b alone:

$$\begin{array}{r} y = a + 2b \\ \underline{-(x = a + b)} \\ y - x = b \end{array}$$

Multiply the $x = a + b$ equation by 2 and perform the same operation to get x and y in terms of a alone:

$$\begin{array}{r} 2x = 2a + 2b \\ \underline{-(y = a + 2b)} \\ 2x - y = a \end{array}$$

Then, find $a - b$:

$(2x - y) - (y - x)$
$2x - y - y + x$
$3x - 2y$

The correct answer is **(E)**.

3. **(E):** The problem gives the formula $C = tb^4$ but never offers a real number for the cost or for any of the other variables in the problem. Choose your own smart numbers!

First, use your numbers to find the original cost. Then, double the value of b and find the new cost. Finally divide the new cost by the original cost to determine how many times greater it is.

If, for the *original* cost, $b = 2$ and $t = 3$, then the original cost was $(3)(2^4) = (3)(16)$. Don't multiply that out—remember that you're going to divide later and t doesn't change, so you'll be able to cross 3 off later.

For the new cost, $b = 4$ and $t = 3$, so the cost is $(3)(4^4) = (3)(4^4)$:

$$\frac{\text{new cost}}{\text{original cost}} = \frac{(3)(4^4)}{(3)(2^4)}$$

Simplify before you multiply. The 3*s* cancel out. Write out the remaining terms:

$$\frac{(4 \times 4 \times 4 \times 4)}{(2 \times 2 \times 2 \times 2)}$$

All of the 2's on the bottom cancel out two of the 4's on top, leaving you with (4)(4) = 16.

Alternatively, use an algebraic approach.

Original cost = tb^4

New cost = $t(2b)^4 = 16tb^4$

$$\frac{\text{new cost}}{\text{original cost}} = \frac{16tb^4}{tb^4} = 16$$

The correct answer is **(E)**.

Chapter 4
of
Algebra

Exponents

In This Chapter…

Chapter 4

Exponents

The mathematical expression 4^3 consists of a base (4) and an exponent (3).

The base (4) is multiplied by itself as many times as the power requires (3):

$4^3 = 4 \times 4 \times 4 = 64$

In other words, exponents are actually shorthand for repeated multiplication.

Two exponents have special names: the exponent 2 is called the square, and the exponent 3 is called the cube:

5^2 can be read as five squared ($5^2 = 5 \times 5 = 25$).
5^3 can be read as five cubed ($5^3 = 5 \times 5 \times 5 = 125$).

All About the Base

A Variable Base

Variables can also be raised to an exponent, and they behave the same as numbers:

$y^4 = y \times y \times y \times y$

Base of 0 or 1

0 raised to *any* power equals 0.

1 raised to *any* power equals 1.

For example, $0^3 = 0 \times 0 \times 0 = 0$ and $0^4 = 0 \times 0 \times 0 \times 0 = 0$.

Similarly, $1^3 = 1 \times 1 \times 1 = 1$ and $1^4 = 1 \times 1 \times 1 \times 1 = 1$.

If you are told that $x = x^2$, then x must be either 0 or 1.

A Fractional Base

When the base of an exponential expression is a positive proper fraction (in other words, a fraction between 0 and 1), an interesting thing occurs: as the exponent increases, the value of the expression decreases! For example:

$$\left(\frac{3}{4}\right)^1 = \frac{3}{4} \qquad \left(\frac{3}{4}\right)^2 = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16} \qquad \left(\frac{3}{4}\right)^3 = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$$

Notice that $\frac{3}{4} > \frac{9}{16} > \frac{27}{64}$. Positive fractions get smaller, not larger, when raised to higher powers.

You could also distribute the exponent before multiplying. For example:

$$\left(\frac{3}{4}\right)^1 = \frac{3^1}{4^1} = \frac{3}{4} \qquad \left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16} \qquad \left(\frac{3}{4}\right)^3 = \frac{3^3}{4^3} = \frac{27}{64}$$

Note that, just like proper fractions, decimals between 0 and 1 decrease as their exponent increases:

$$(0.6)^2 = 0.36 \qquad (0.5)^4 = 0.0625 \qquad (0.1)^5 = 0.00001$$

A Compound Base

Just as an exponent can be distributed to a fraction, it can also be distributed to a product:

$$10^3 = (2 \times 5)^3 = (2)^3 \times (5)^3 = 8 \times 125 = 1{,}000$$

This also works if the base includes variables:

$(3x)^4 = 3^4 \times x^4 = 81x^4$

A Base of –1

$(-1)^1 = -1$ $(-1)^2 = -1 \times -1 = 1$ $(-1)^3 = -1 \times -1 \times -1 = -1$

This pattern repeats indefinitely. In general:

$(-1)^{\text{ODD}} = -1$ $(-1)^{\text{EVEN}} = 1$

A Negative Base

When dealing with negative bases, pay particular attention to PEMDAS. Unless the negative sign is inside parentheses, the exponent does not distribute. For example:

$(-2)^4$	$\neq$	-2^4
$-2^4 = -1 \times 2^4 = -16$		$(-2)^4 = (-1)^4 \times (2)^4 = 1 \times 16 = 16$

As with a base of −1, any negative bases raised to an odd exponent will be negative, and any negative bases raised to an even exponent will be positive.

Combining Exponential Terms with Common Bases

The rules in this section *only* apply when the terms have the *same* base.

As you will see, all of these rules are related to the fact that exponents are shorthand for repeated multiplication.

Multiply Terms: Add Exponents

When *multiplying* two exponential terms with the same base, *add the exponents*. This rule is true no matter what the base is.

$z^2 \times z^3 = (z \times z) \times (z \times z \times z) = z \times z \times z \times z \times z = z^5$
$4 \times 4^2 = (4) \times (4 \times 4) = 4 \times 4 \times 4 = 4^3$

Fortunately, once you know the rule, you can simplify the computation greatly:

$$\left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^{2+4} = \left(\frac{1}{2}\right)^6$$

Divide Terms: Subtract Exponents

When *dividing* two exponential terms with the same base, *subtract the exponents*. This rule is true no matter what the base is.

$$\frac{5^6}{5^2} = \frac{5\times5\times5\times5\times\cancel{5}\times\cancel{5}}{\cancel{5}\times\cancel{5}} = 5\times5\times5\times5 = 5^4$$

Fortunately, once you know the rule, you can simplify the computation greatly:

$$\frac{x^{15}}{x^8} = x^{15-8} = x^7$$

Anything Raised to the Zero Equals 1

This rule is an extension of the previous rule. If you divide something by itself, the quotient is 1:

$$\frac{a^3}{a^3} = \frac{\cancel{a}\times\cancel{a}\times\cancel{a}}{\cancel{a}\times\cancel{a}\times\cancel{a}} = 1$$

Look at this division by subtracting exponents:

$$\frac{a^3}{a^3} = a^{3-3} = a^0$$

Therefore, $a^0 = 1$.

Any base raised to the 0 power equals 1. The one exception is a base of 0.

Note that 0^0 is *undefined*. That's because $\frac{0}{0}$ is undefined (but the GMAT does not test undefined numbers, so you don't need to memorize this).

Negative Exponents

The behavior of negative exponents is also an extension of the rules for dividing exponential terms.

$$\frac{y^2}{y^5} = \frac{y \times y}{y \times y \times y \times y \times y} = \frac{1}{y^3}$$

Look at this division by subtracting exponents:

$$\frac{y^2}{y^5} = y^{2-5} = y^{-3}$$

Therefore, $y^{-3} = \dfrac{1}{y^3}$.

This is the general rule: *something with a negative exponent is just "one over" that same thing with a positive exponent*. You can rewrite the above expression by taking the reciprocal of y^3 and dropping the negative sign.

Here are some additional examples of how to take the reciprocal and drop the negative sign:

$$\frac{1}{3^{-3}} = 3^3 \qquad\qquad \left(\frac{x}{4}\right)^{-2} = \frac{4^2}{x^2}$$

Nested Exponents: Multiply Exponents

How can you simplify $(z^2)^3$? Expand this term to show the repeated multiplication.

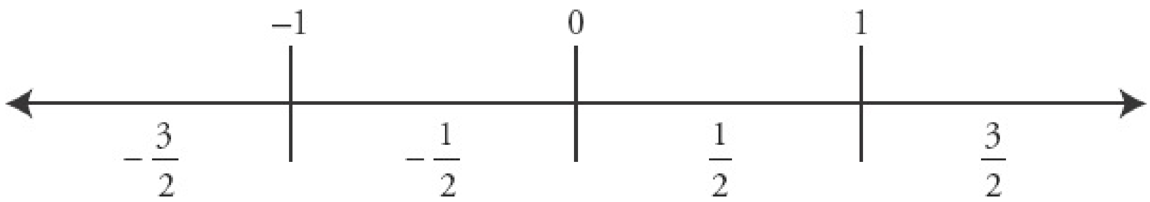
$$(z^2)^3 = (z^2) \times (z^2) \times (z^2) = z^{2+2+2} = z^6$$

When you raise an exponential term to an exponent, multiply the exponents.

$$(a^5)^4 = a^{5 \times 4} = a^{20}$$

Fractions and Exponents

There are four broad categories of fractions that all behave differently when raised to a power. The result depends on the size and the sign of the fraction, as well as on the power. While it is not necessary to memorize all of the cases below, it is important that you be able to re-create them when necessary.



EVEN EXPONENTS (such as 2):

Less than –1	Between –1 and 0	Between 0 and 1	Greater than 1
$\left(-\frac{3}{2}\right)^2=\frac{9}{4}$	$\left(-\frac{1}{2}\right)^2=\frac{1}{4}$	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$	$\left(\frac{3}{2}\right)^2=\frac{9}{4}$
$-\frac{3}{2}<\frac{9}{4}$	$-\frac{1}{2}<\frac{1}{4}$	$\frac{1}{2}>\frac{1}{4}$	$\frac{3}{2}<\frac{9}{4}$
Result is bigger.	Result is bigger.	Result is *smaller*.	Result is bigger.

ODD EXPONENTS (such as 3):

Less than –1	Between –1 and 0	Between 0 and 1	Greater than 1
$\left(-\frac{3}{2}\right)^3=-\frac{27}{8}$	$\left(-\frac{1}{2}\right)^3=-\frac{1}{8}$	$\left(\frac{1}{2}\right)^3=\frac{1}{8}$	$\left(\frac{3}{2}\right)^3=\frac{27}{8}$
$-\frac{3}{2}>-\frac{27}{8}$	$-\frac{1}{2}<-\frac{1}{8}$	$\frac{1}{2}>\frac{1}{8}$	$\frac{3}{2}<\frac{27}{8}$
Result is *smaller*.	Result is bigger.	Result is *smaller*.	Result is bigger.

As you can see, the effect of raising a fraction to a power varies depending upon the fraction's value, the sign, and the exponent.

To raise a fraction to a negative power, raise the reciprocal to the equivalent positive power.

$$\left(\frac{3}{7}\right)^{-2}=\left(\frac{7}{3}\right)^{2}=\frac{7^2}{3^2}=\frac{49}{9} \qquad \left(\frac{x}{y}\right)^{-w}=\left(\frac{y}{x}\right)^{w}=\frac{y^w}{x^w}$$

Factoring Out a Common Term

Normally, exponential terms that are added or subtracted cannot be combined. However, if two terms with the same base are added or subtracted, you can factor out a common term. For example:

$$11^3 + 11^4 \rightarrow 11^3(11^0 + 11^1) \rightarrow 11^3(1 + 11) \rightarrow 11^3(12)$$

On the GMAT, it generally pays to factor exponential terms that have bases in common.

If $x = 4^{20} + 4^{21} + 4^{22}$, what is the largest prime factor of x?

If you want to know the prime factors of x, you need to express x as a product. Factor 4^{20} out of the expression on the right side of the equation.

$$x = 4^{20} + 4^{21} + 4^{22}$$
$$x = 4^{20}(4^0 + 4^1 + 4^2)$$
$$x = 4^{20}(1 + 4 + 16)$$
$$x = 4^{20}(21)$$
$$x = 4^{20}(3 \times 7)$$

Now that you have expressed x as a product, you can see that 7 is the largest prime factor of x.

Equations with Exponents

Exponents can also appear in equations. In fact, the GMAT often complicates equations by including exponents or roots with unknown variables.

Here are a few situations to look out for when equations contain exponents.

Even Exponents Hide the Sign of the Base

Any number raised to an even exponent becomes positive. For example:

$3^2 = 9$ AND $(-3)^2 = 9$

Another way of saying this is that an even exponent hides the sign of its base. Compare the following two equations:

$x^2 = 25$ $|x| = 5$

Do you see what they have in common? In both cases, $x = \pm 5$. The equations share the same two solutions. In fact, there is an important relationship: **for any *x***, $\sqrt{x^2} = |x|$.

Here is another example:

$a^2 - 5 = 12$	By adding 5 to both sides, you can rewrite this equation as $a^2 = 17$. This equation has two solutions: $\sqrt{17}$ and $-\sqrt{17}$.

You can also say that the equation $a^2 = 17$ has two roots (the word *root* is a synonym for the word *solution*).

Also note that not all equations with even exponents have two solutions. For example:

$x^2 + 3 = 3$	By subtracting 3 from both sides, you can rewrite this equation as $x^2 = 0$, which has only one solution: 0.

Odd Exponents Keep the Sign of the Base

Equations that involve only odd exponents or cube roots have only one solution:

$x^3 = -125$	Here, x has only one solution, −5, because $(-5)(-5)(-5) = -125$. This will not work with positive 5.
$243 = y^5$	Here, y has only one solution, 3, because $(3)(3)(3)(3)(3) = 243$. This will not work with −3.

If an equation includes some variables with odd exponents and some variables with even exponents, treat it as dangerous, as it is likely to have two solutions. Any even exponents in an equation signal two potential solutions.

Same Base or Same Exponent

In problems that involve exponential expressions on *both* sides of the equation, it is imperative to rewrite the bases so that either the same base or the same exponent appears on both sides of the exponential equation. Once you do this, you can usually eliminate the bases or the exponents and rewrite the rest as an equation. Consider this example"

> Solve the following equation for w: $(4^w)^3 = 32^{w-1}$

To start, rewrite the bases so that the same base appears on both sides of the equation. Right now, the left side has a base of 4 and the right side has a base of 32. Both 4 and 32 can be expressed as powers of 2, so you can rewrite 4 as 2^2 and you can rewrite 32 as 2^5.

Next, plug the rewritten bases into the original equation:

$$(4^w)^3 = 32^{w-1}$$
$$((2^2)^w)^3 = (2^5)^{w-1}$$

Now, simplify the equation using the rules of exponents:

$$((2^2)^w)^3 = (2^5)^{w-1}$$
$$2^{6w} = 2^{5w-5}$$

When the bases are identical (and no other bases exist), you can drop the bases, rewrite the exponents as an equation, and solve:

$$6w = 5w - 5$$
$$w = -5$$

Be very careful if 0, 1, or −1 is the base (or could be the base), since the outcome of raising those bases to powers is not unique. For instance, $0^2 = 0^3 = 0^{29} = 0$. So if $0^x = 0^y$, you cannot claim that $x = y$. Likewise, $1^2 = 1^3 = 1^{29} = 1$, and $(-1)^2 = (-1)^4 = (-1)^{\text{even}} = 1$, while $(-1)^3 = (-1)^5 = (-1)^{\text{odd}} = -1$. Fortunately, the GMAT rarely tries to trick you this way.

Problem Set

Now that you've finished the chapter, do the following problems.

For problems 1 and 2, determine whether the inequality is TRUE or FALSE:

1. $\left(-\frac{3}{4}\right)^3 > -\frac{3}{4}$

2. $\left(\frac{x+1}{x}\right)^{-2} > \frac{x+1}{x}$, if $x > 0$.

3. $x^3 < x^2$. Describe the possible values of x.

4. Simplify: $\frac{m^8p^7r^{12}}{m^3r^9p} \times p^2r^3m^4$

5. If $p = \frac{x^{a+b}}{x^b}$, what is the value of positive integer p?

   (1) $x = 5$
   (2) $a = 0$

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

6. Which of the following expressions has the largest value?

   (A) $(3^4)^{13}$

   (B) $\left[(3^{30})^{12}\right]^{\frac{1}{10}}$

   (C) $3^{30} + 3^{30} + 3^{30}$

   (D) $4(3^{51})$

(E) $4(3^{100})^{\frac{1}{2}}$

7. Simplify: $(4^y + 4^y + 4^y + 4^y)(3^y + 3^y + 3^y)$

(A) $44^y \times 3^{3y}$

(B) 12^{y+1}

(C) $16^y \times 9^y$

(D) 12^y

(E) $4^y \times 12^y$

8. If $4^a + 4^{a+1} = 4^{a+2} - 176$, what is the value of a?

9. If m and n are positive integers and $(2^{18})(5^m) = (20^n)$, what is the value of m?

10. Which of the following is equivalent to $\left(\frac{1}{3}\right)^{-4}\left(\frac{1}{9}\right)^{-3}\left(\frac{1}{27}\right)^{-2}$?

(A) $\left(\frac{1}{3}\right)^{-8}$

(B) $\left(\frac{1}{3}\right)^{-9}$

(C) $\left(\frac{1}{3}\right)^{-16}$

(D) $\left(\frac{1}{3}\right)^{-18}$

(E) $\left(\frac{1}{3}\right)^{-144}$

11. If $B^3A < 0$ and $A > 0$, which of the following must be negative?

(A) AB

(B) B^2A

(C) B^4

(D) $\frac{A}{B^2}$

(E) $-\frac{B}{A}$

Solutions

1. **TRUE:** Raising a proper fraction to a power causes that fraction to move closer to 0 on a number line. Raising any negative number to an odd power will result in a negative number. The number $\left(-\frac{3}{4}\right)^3$, therefore, will be to the right of $-\frac{3}{4}$ on the number line.

2. **FALSE:** Any number $\frac{x+1}{x}$, where x is positive, will be greater than 1. Therefore, raising that number to a negative exponent will result in a number smaller than 1 whenever x is a positive number:

$$\left(\frac{x+1}{x}\right)^{-2} = \left(\frac{x}{x+1}\right)^2 < \frac{x+1}{x}$$

3. **Any non-zero number less than 1:** As positive proper fractions are multiplied, their value decreases. For example, $\left(\frac{1}{2}\right)^3 < \left(\frac{1}{2}\right)^2$. Also, any negative number will make this inequality true. A negative number cubed is negative. Any negative number squared is positive. For example, $(-3)^3 < (-3)^2$. The number 0 itself, however, does not work, since $0^3 = 0^2$.

4. $\boldsymbol{m^9p^8r^6}$**:** $\frac{m^8p^7r^{12}}{m^3r^9p} \times p^2r^3m^4 = \frac{m^{12}p^9r^{15}}{m^3r^9p} = m^{(12-3)}p^{(9-1)}r^{(15-9)} = m^9p^8r^6$

5. **(B):** This question isn't really about p. It's about the expression $\frac{x^{a+b}}{x^b}$, which can be simplified by subtracting the exponent in the denominator from the exponent in the numerator:

$$\frac{x^{a+b}}{x^b} = x^{a+b-(b)} = x^a$$

So this question may be rephrased as simply, What is x^a?

Be careful, though—sufficiency in this case does *not* necessarily mean that you need to know x and a individually. (As just one example, if x is 1, then a is not needed, because 1 to any power is still 1.)

(1) INSUFFICIENT: Knowing that x is 5 is not sufficient without knowing a.

(2) SUFFICIENT: Anything to the 0 power is 1. The only exception to the rule is 0, because 0^0 is undefined. However, you have been told that p is a positive integer, so you know that x cannot equal 0.

The correct answer is (**B**).

6. **(D):** Use the rules of exponents to simplify each expression:

(A) $(3^4)^{13} = 3^{52}$

(B) $\left[(3^{30})^{12}\right]^{\frac{1}{10}} = (3^{360})^{\frac{1}{10}} = 3^{\frac{360}{10}} = 3^{36}$

(C) $3^{30} + 3^{30} + 3^{30} = 3\,(3^{30}) = 3^{31}$

(D) $4(3^{51})$ cannot be simplified further.

(E) $(3^{100})^{\frac{1}{2}} = 3^{\frac{100}{2}} = 3^{50}$

Answer choice (A) is clearly larger than (B), (C), and (E). You must now compare $4(3^{51})$ to 3^{52}. To make them most easily comparable, factor one 3 out of 3^{52}: $3^{52} = 3(3^{51})$. Thus, $4(3^{51})$ is greater than $3(3^{51})$, so (D) is the correct answer.

7. **(B):** $(4^y + 4^y + 4^y + 4^y)(3^y + 3^y + 3^y) = (4 \times 4^y)(3 \times 3^y) = (4^{y+1})(3^{y+1}) = (4 \times 3)^{y+1} = (12)^{y+1}$

8. **2:** The key to this problem is to express all of the exponential terms in terms of the greatest common factor of the terms: 4^a. Using the addition rule (or the corresponding numerical examples), you get:

$$
\begin{aligned}
4^a + 4^{a+1} &= 4^{a+2} - 176 \\
176 &= 4^{a+2} - 4^a - 4^{a+1} \\
176 &= 4^a \times (4^2) - 4^a - 4^a \cdot (4^1) \\
176 &= 4^a \times (4^2 - 4^0 - 4^1) \\
176 &= 4^a \times (16 - 1 - 4) \\
176 &= 4^a \times (11) \\
4^a &= 176 \div 11 = 16 \\
a &= 2
\end{aligned}
$$

9. **9:** With exponential equations such as this one, the key is to recognize that as long as the exponents are all integers, each side of the equation must have the same number of each type of prime factor. Break down each base into prime factors and set the exponents equal to each other:

$$
\begin{aligned}
&(2^{18})(5^m) = (20^n) \\
&2^{18} \cdot 5^m = (2 \cdot 2 \cdot 5)^n \\
&2^{18} \times 5^m = 2^{2n} \times 5^n \\
&18 = 2n;\ m = n \\
&n = 9;\ m = n = 9
\end{aligned}
$$

Because m and n have to be integers, there must be the **same number of 2's** on either side of the equation and there must be the **same number of 5's** on either side of the equation. Thus, $18 = 2n$ and $m = n$.

10. **(C):** Once again, you should break each base down into its prime factors first. Then, apply the negative exponent by taking the reciprocal of each term, and making the exponent positive:

$$\left(\frac{1}{3}\right)^{-4}\left(\frac{1}{9}\right)^{-3}\left(\frac{1}{27}\right)^{-2} = \left(\frac{1}{3}\right)^{-4}\left(\frac{1}{3^2}\right)^{-3}\left(\frac{1}{3^3}\right)^{-2} = 3^4 \times (3^2)^3 \times (3^3)^2 = 3^4 \times 3^6 \times 3^6 = 3^{4+6+6} = 3^{16}$$

Because all of the answer choices have negative exponents, you can perform the same transformation on them—simply take the reciprocal of each and change the exponent to a positive:

(A) $\left(\dfrac{1}{3}\right)^{-8} = 3^8$

(B) $\left(\dfrac{1}{3}\right)^{-9} = 3^9$

(C) $\left(\frac{1}{3}\right)^{-16} = 3^{16}$

(D) $\left(\frac{1}{3}\right)^{-18} = 3^{18}$

(E) $\left(\frac{1}{3}\right)^{-144} = 3^{144}$

11. **(A):** If *A* is positive, B^3 must be negative. Therefore, *B* must be negative. If *A* is positive and *B* is negative, the product *AB* must be negative.

Chapter 5
of
Algebra

Roots

In This Chapter…

Chapter 5

Roots

A Square Root Has Only One Value

Compare the following two equations:

$x^2 = 16$ $\qquad\qquad$ $x = \sqrt{16}$

Although they may seem very similar, there is an important difference. There are two solutions to the equation on the left: $x = 4$ *or* $x = –4$. There is *one* solution to the equation on the right: $x = 4$.

If a given equation contains a square root symbol on the GMAT, *only* use the positive root.

If an equation contains a squared variable, and *you* take the square root, use both the positive and the negative solutions.

This rule applies for any even root (square root, 4th root, 6th root, etc.). For instance:

$$\sqrt[4]{81} = 3$$

Odd roots (cube root, 5th root, 7th root, etc.) also have only one solution.

Odd roots, like odd exponents, keep the sign of the base.

If $\sqrt[3]{-27} = x$, what is x?

The correct answer is –3, because (–3)(–3)(–3) = –27.

Roots and Fractional Exponents

Fractional exponents are the link between roots and exponents. Within the exponent fraction, the numerator tells you what power to raise the base to, and the denominator tells you which root to take. You can raise the base to the power and take the root in *either* order. For example:

What is $216^{\frac{1}{3}}$?

The numerator of the fraction is 1, so raise the base to the power of 1. The denominator is 3, so you need to take the 3rd (cube) root of 216^1. In order to determine that root, break 216 into prime factors:

$216 = 3 \times 3 \times 3 \times 2 \times 2 \times 2 = 6^3$.

216 is equal to 6^3, so $216^{\frac{1}{3}} = \sqrt[3]{216} = \sqrt[3]{6^3} = 6$.

What is $\left(\frac{1}{8}\right)^{-\frac{4}{3}}$?

Because the exponent is negative, first take the reciprocal of the base of $\left(\frac{1}{8}\right)$ and change the exponent to its positive equivalent. Then take the 3rd (cube) root to the 4th power:

$$\left(\frac{1}{8}\right)^{-\frac{4}{3}} = 8^{\frac{4}{3}} = \sqrt[3]{8^4} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16$$

You need to know how to take fractional exponents and rewrite them as roots and powers, as above, and you also need to know how to express roots as fractional exponents. The resulting expression may be much easier to simplify. Just remember that a root becomes the denominator of a fractional exponent.

Express $\sqrt[4]{\sqrt{x}}$ as a fractional exponent.

Transform the individual roots into exponents. The square root is equivalent to an exponent of $\frac{1}{2}$, and the fourth root is equivalent to an exponent of $\frac{1}{4}$.

Therefore, this expression becomes $\left(x^{\frac{1}{2}}\right)^{\frac{1}{4}}$, or $x^{\frac{1}{8}}$.

Note that this is equivalent to $\sqrt[8]{x}$.

Simplifying a Root

Sometimes there are two numbers inside the radical sign. In order to simplify this type of root, it is often helpful to split up the numbers into two roots and then solve. At other times, the opposite is true: you have two roots that you would like to simplify by combining them under one radical sign.

When Can You Simplify Roots?

You can only simplify roots in the ways described below when the roots are connected via multiplication or division. If two roots are added or subtracted, you cannot use this method.

How Can You Simplify Roots?

When multiplying roots, you can split up a larger product into its separate factors. Creating two separate radicals and simplifying each one individually before multiplying can save you from having to compute large numbers. For example:

$$\sqrt{25 \times 16} = \sqrt{25} \times \sqrt{16} = 5 \times 4 = 20$$
$$\sqrt{50} \times \sqrt{18} = \sqrt{50 \times 18} = \sqrt{2 \times 25 \times 2 \times 9} = \sqrt{4 \times 25 \times 9} = 2 \times 5 \times 3 = 30$$

In the first example, 25 and 16 are perfect squares. Because they are multiplied together, you can take the square root of each separately. In the second example, 50 and 18 are not perfect squares. In this case, break down the numbers into factors in order to find any perfect squares, then take the square root.

Division of roots works the same way. You can split a larger quotient into the dividend and divisor. You can also combine two roots that are being divided into a single root of the quotient. For example:

$$\sqrt{144 \div 16} = \sqrt{144} \div \sqrt{16} = 12 \div 4 = 3$$
$$\sqrt{72} \div \sqrt{8} = \sqrt{72 \div 8} = \sqrt{9} = 3$$

The GMAT may try to trick you into splitting the sum or difference of two numbers inside a radical into two individual roots. Also, the GMAT may try to trick you into combining the sum or difference of two roots inside one radical sign. Remember that you may only separate or combine the *product* or *quotient* of two roots. You cannot separate or combine the *sum* or *difference* of two roots.

The 16 and 9 cannot be split into separate roots:

$\sqrt{16+9} \neq \sqrt{16} + \sqrt{9}$ This move is illegal.

Instead, first add the numbers together, then take the square root:

$$\sqrt{16+9} = \sqrt{25} = 5$$

Imperfect vs. Perfect Squares

Not all square roots yield an integer. For example, $\sqrt{52}$ is the root of an imperfect square. It will not yield an integer answer because no integer multiplied by itself will yield 52.

Simplifying Roots of Imperfect Squares

Some imperfect squares can be simplified into multiples of smaller square roots. For an imperfect square such as $\sqrt{52}$, you can rewrite $\sqrt{52}$ as a product of primes under the radical:

$$\sqrt{52} = \sqrt{2 \times 2 \times 13}$$

You can simplify any pairs inside the radical. In this case, there is a pair of 2's.

Since $\sqrt{2 \times 2} = \sqrt{4} = 2$, you can rewrite $\sqrt{52}$ as follows:

$$\sqrt{52} = \sqrt{2 \times 2 \times 13} = 2 \times \sqrt{13}$$

This is more typically written $2\sqrt{13}$. Look at another example:

Simplify: $\sqrt{72}$

You can rewrite $\sqrt{72}$ as a product of primes: $\sqrt{72} = \sqrt{2\times2\times2\times3\times3}$. Since there are a pair of 2's and a pair of 3's inside the radical, you can simplify them:

$$\sqrt{72} = 2\times3\times\sqrt{2} = 6\sqrt{2}$$

Memorize: Squares and Square Roots

Memorize the following squares and square roots, as they often appear on the GMAT.

$1^2 = 1$	$\sqrt{1} = 1$
$1.4^2 \approx 2$	$\sqrt{2} \approx 1.4$
$1.7^2 \approx 3$	$\sqrt{3} \approx 1.7$
$2.25^2 \approx 5$	$\sqrt{5} \approx 2.25$
$2^2 = 4$	$\sqrt{4} = 2$
$3^2 = 9$	$\sqrt{9} = 3$
$4^2 = 16$	$\sqrt{16} = 4$
$5^2 = 25$	$\sqrt{25} = 5$
$6^2 = 36$	$\sqrt{36} = 6$
$7^2 = 49$	$\sqrt{49} = 7$
$8^2 = 64$	$\sqrt{64} = 8$
$9^2 = 81$	$\sqrt{81} = 9$
$10^2 = 100$	$\sqrt{100} = 10$
$11^2 = 121$	$\sqrt{121} = 11$
$12^2 = 144$	$\sqrt{144} = 12$
$13^2 = 169$	$\sqrt{169} = 13$
$14^2 = 196$	$\sqrt{196} = 14$
$15^2 = 225$	$\sqrt{225} = 15$
$16^2 = 256$	$\sqrt{256} = 16$
$20^2 = 400$	$\sqrt{400} = 20$
$25^2 = 625$	$\sqrt{625} = 25$
$30^2 = 900$	$\sqrt{900} = 30$

Memorize: Cubes and Cube Roots

Memorize the following cubes and cube roots, as they often appear on the GMAT.

$1^3 = 1$	$\sqrt[3]{1} = 1$
$2^3 = 8$	$\sqrt[3]{8} = 2$
$3^3 = 27$	$\sqrt[3]{27} = 3$
$4^3 = 64$	$\sqrt[3]{64} = 4$
$5^3 = 125$	$\sqrt[3]{125} = 5$
$10^3 = 1{,}000$	$\sqrt[3]{1{,}000} = 10$

Problem Set

Now that you've finished the chapter, do the following problems.

1. For each of these statements, indicate whether the statement is TRUE or FALSE:

   (a) If $x^2 = 11$, then $x = \sqrt{11}$.

   (b) If $x^3 = 11$, then $x = \sqrt[3]{11}$.

   (c) If $x^4 = 16$, then $x = 2$.

   (d) If $x^5 = 32$, then $x = 2$.

Solve or simplify the following problems, using the properties of roots:

2. $\sqrt{18} \div \sqrt{2}$

3. Evaluate the following expression: $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$

4. $\left(\frac{1}{81}\right)^{-\frac{1}{4}}$

5. $\sqrt{63} + \sqrt{28}$

6. $\sqrt[3]{100 - 36}$

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

7. $\sqrt{150} - \sqrt{96}$

8. Estimate: $\sqrt{60}$

9. $\sqrt{20a} \times \sqrt{5a}$, assuming a is positive

10. $10\sqrt{12} \div 2\sqrt{3}$

11. $\sqrt{x^2y^3 + 3x^2y^3}$, assuming *x* and *y* are positive

12. $\sqrt{0.0081}$

13. $\dfrac{\sqrt[4]{64}}{\sqrt[4]{4}}$

Solutions

1. (a) **FALSE:** Even exponents hide the sign of the original number, because they always result in a positive value. If $x^2 = 11$, then $|x| = \sqrt{11}$. Thus, x could be either $\sqrt{11}$ or $-\sqrt{11}$.

(b) **TRUE:** Odd exponents preserve the sign of the original expression. Therefore, if x^3 is positive, then x must itself be positive. If $x^3 = 11$, then x must be $\sqrt[3]{11}$.

(c) **FALSE:** Even exponents hide the sign of the original number, because they always result in a positive value. If $x^4 = 16$, then x could be either 2 or –2.

(d) **TRUE:** Odd exponents preserve the sign of the original expression. Therefore, if x^5 is positive, then x must itself be positive. If $x^5 = 32$, then x must be 2.

2. **3:** $\sqrt{18} \div \sqrt{2} = \sqrt{9} = 3$

3. $\mathbf{\frac{27}{8}}$**:** $\left(\frac{4}{9}\right)^{-\frac{3}{2}} = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \sqrt{\left(\frac{9}{4}\right)^3} = \left(\sqrt{\left(\frac{9}{4}\right)}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{3^3}{2^3} = \frac{27}{8}$

4. **3:** $\left(\frac{1}{81}\right)^{-\frac{1}{4}} = 81^{\frac{1}{4}} = \sqrt[4]{81} = 3$

Note: On the GMAT, when given a square root symbol with a number beneath, you are supposed to take only the positive root. This restriction does not apply when given exponents (e.g., $x^2 = 16$ does give you both 4 and –4 as possible solutions).

5. $\mathbf{5\sqrt{7}}$**:** $\sqrt{63} + \sqrt{28} = \left(\sqrt{9} \times \sqrt{7}\right) + \left(\sqrt{4} \times \sqrt{7}\right) = 3\sqrt{7} + 2\sqrt{7} = 5\sqrt{7}$

6. **4:** $\sqrt[3]{100 - 36} = \sqrt[3]{64} = 4$

7. $\mathbf{\sqrt{6}}$**:** $\sqrt{150} - \sqrt{96} = \left(\sqrt{25} \times \sqrt{6}\right) - \left(\sqrt{16} \times \sqrt{6}\right) = 5\sqrt{6} - 4\sqrt{6} = \sqrt{6}$.

8. **7.7:** 60 is in between two perfect squares: 49, which is 7^2, and 64, which is 8^2. The difference between 64 and 49 is 15, so 60 is a little more than $\frac{2}{3}$ of the way toward 64 from 49. A reasonable estimate for $\sqrt{60}$, then, would be about 7.7, which is a little more than $\frac{2}{3}$ towards 8 from 7.

9. **10*a*:** $\sqrt{20a} \times \sqrt{5a} = \sqrt{100a^2} = 10a$

10. **10:** $10\sqrt{12} \div 2\sqrt{3} = \dfrac{10\left(\sqrt{4} \times \sqrt{3}\right)}{2\sqrt{3}} = \dfrac{20\sqrt{3}}{2\sqrt{3}} = 10$

11. $\mathbf{2xy\sqrt{y}}$**:** Notice that you have two terms under the radical that both contain x^2y^3. You can add like terms together if they are under the same radical: $\sqrt{x^2y^3 + 3x^2y^3} = \sqrt{(1+3)x^2y^3} = \sqrt{4x^2y^3}$. Now, factor out all squares and isolate them under their own radical sign:

$$\sqrt{4x^2y^3} = \sqrt{4} \times \sqrt{x^2} \times \sqrt{y^2} \times \sqrt{y} = 2xy\sqrt{y}$$

(Note that since *x* and *y* are positive, $\sqrt{x^2} = x$ and $\sqrt{y^2} = y$.)

12. **0.09:** Since (0.09)(0.09) = 0.0081, $\sqrt{0.0081}$ = 0.09. You can also rewrite 0.0081 as 81×10^{-4}:

$$\sqrt{81 \times 10^{-4}} = \sqrt{81} \times \sqrt{10^{-4}} = 9 \times \left(10^{-4}\right)^{\frac{1}{2}} = 9 \times 10^{-2} = 0.09$$

13. **2:** $\dfrac{\sqrt[4]{64}}{\sqrt[4]{4}} = \sqrt[4]{\dfrac{64}{4}} = \sqrt[4]{16} = 2$

Chapter 6
of
Algebra

Quadratic Equations

In This Chapter…

Chapter 6

Quadratic Equations

One special type of exponent equation is called the quadratic equation. Here are some examples of quadratic equations:

$x^2 + 3x + 8 = 12$ $\qquad$ $w^2 - 16w + 1 = 0$ $\qquad$ $2y^2 - y + 5 = 8$

The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b, and c are constants and a does not equal 0.

Here are other ways of writing quadratics (in non-standard form):

$x^2 = 3x + 4$ $\qquad$ $a = 5a^2$ $\qquad$ $6 - b = 7b^2$

Like other even exponent equations, quadratic equations generally have two solutions. That is, there are usually two possible values of x (or whatever the variable is) that make the equation *true*.

Factoring Quadratic Equations

The following example illustrates the process for solving quadratic equations:

Given that $x^2 + 3x + 8 = 12$, what is x?

To start, move all the terms to the left side of the equation, combine them, and put them in the form $ax^2 + bx + c$ (where a, b, and c are integers). The right side of the equation should be set to 0:

$x^2 + 3x + 8 =$ $\qquad$ Subtracting 12 from both sides of the equation puts

12	all the
$x^2 + 3x - 4 = 0$	terms on the left side and sets the right side to 0.

Next, factor the equation. In order to factor, you generally need to think about two terms in the equation. Assuming that $a = 1$ (which is usually the case on GMAT quadratic equation problems), focus on the two terms b and c. (If a is not equal to 1, simply divide the equation through by a.)

In the equation $x^2 + 3x - 4 = 0$, $b = 3$ and $c = -4$. In order to factor this equation, you need to find two integers whose product is –4 and whose sum is 3. The only two integers that work are 4 and –1, since $4(-1) = -4$ and $4 + (-1) = 3$.

Now, rewrite the equation in the form $(x + ?)(x + ?)$, where the question marks represent the two integers you solved for in the previous step:

$$x^2 + 3x - 4 = 0$$
$$(x + 4)(x - 1) = 0$$

The left side of the equation is now a product of two factors in parentheses: $(x + 4)$ and $(x - 1)$. Since this product equals 0, one or both of the factors must be 0.

For instance, if you know that $M \times N = 0$, then you know that either $M = 0$ or $N = 0$ (or both M and N are 0).

In this problem, set each factor in parentheses independently to 0 and solve for x:

$$x + 4 = 0 \qquad \text{OR} \qquad x - 1 = 0$$
$$x = -4 \qquad\qquad\qquad x = 1$$

Therefore, the two solutions or roots of the quadratic equation $x^2 + 3x + 8 = 12$ are –4 and 1.

Disguised Quadratics

The GMAT will often attempt to disguise quadratic equations by putting them in forms that do not quite look like the traditional form of $ax^2 + bx + c = 0$.

Here is a very common "disguised" form for a quadratic:

$$3w^2 = 6w$$

This is certainly a quadratic equation. However, it is very tempting to try to solve this equation without thinking of it as a quadratic. This classic mistake looks like this:

$$\begin{aligned} 3w^2 &= 6w && \text{Divide both sides by } w. \\ 3w &= 6 && \text{Divide both sides by 3.} \\ w &= 2 \end{aligned}$$

If you solve this equation without factoring it like a quadratic, you will miss one of the solutions! Here is how it should be solved:

$$\begin{aligned} 3w^2 &= 6w \\ 3w^2 - 6w &= 0 \\ w(3w - 6) &= 0 \end{aligned}$$

Setting both factors equal to 0 yields the following solutions:

$$w = 0 \qquad \text{OR} \qquad \begin{aligned} 3w - 6 &= 0 \\ 3w &= 6 \\ w &= 2 \end{aligned}$$

If you recognize that $3w^2 = 6w$ is a disguised quadratic, you will find both solutions instead of accidentally missing one (in this case, the solution $w = 0$).

Here is another example of a disguised quadratic:

Solve for b, given that $\frac{36}{b} = b - 5$.

At first glance, this does not look like a quadratic equation. The first simplification step is to get rid of that fraction. Watch what happens:

Multiply both sides of the equation by b.

$$\frac{36}{b} = b - 5$$

$36 = b^2 - 5b$

Now this looks like a quadratic! Solve it by factoring:

$36 = b^2 - 5b$	Subtract 36 from both sides to set the equation equal to 0.
$b^2 - 5b - 36 = 0$	
$(b - 9)(b + 4) = 0$	Thus, $b = 9$ or $b = -4$.

Some quadratics are hidden within more difficult equations, such as higher order equations (in which a variable is raised to the power of 3 or more). On the GMAT, these equations can almost always be factored to find the hidden quadratic expression. For example:

Solve for x, given that $x^3 + 2x^2 - 3x = 0$.

$x^3 + 2x^2 - 3x = 0$	Factor out an x from each term.
$x(x^2 + 2x - 3) = 0$	

Now factor the quadratic:

$x(x^2 + 2x - 3) = 0$	
$x(x + 3)(x - 1) = 0$	$x = 0$ OR $x + 3 = 0$ OR $x - 1 = 0$. This equation has *three* solutions: 0, −3, and 1.

This example illustrates a general rule:

If you have a quadratic expression equal to 0, *and* you can factor an x out of the expression, then $x = 0$ is a solution of the equation.

Do not just divide both sides by x. If you do so, you will eliminate the solution $x = 0$. You are only allowed to divide by a variable if you are absolutely sure that the variable does not equal 0.

Taking the Square Root

So far you have seen how to solve quadratic equations by setting one side of the equation equal to 0 and factoring. However, some quadratic problems can be solved without setting one side equal to 0. If the other side of the equation is a perfect square, the problem can be solved by taking the square root of both sides of the equation. For example:

If $(z + 3)^2 = 25$, what is z?

You could solve this problem by distributing the left-hand side of the equation, setting the right-hand side equal to 0, and factoring. However, it would be much easier to take the square root of both sides of the equation to solve for z. You just have to consider both the positive and the negative square root:

$$\begin{aligned} \sqrt{(z+3)^2} &= \sqrt{25} \\ z+3 &= \pm 5 \\ z &= -3 \pm 5 \\ z &= \{2, -8\} \end{aligned}$$

Going in Reverse: Use FOIL

Instead of starting with a quadratic equation and factoring it, you may need to start with factors and rewrite them as a quadratic equation. To do this, use a multiplication process called FOIL: First, Outer, Inner, Last.

To change the expression $(x + 7)(x – 3)$ into a quadratic equation, use FOIL as follows:

First: Multiply the first term of each factor together: $x \times x = x^2$

Outer: Multiply the outer terms of the expression together: $x(–3) = –3x$

Inner: Multiply the inner terms of the expression together: $7(x) = 7x$

Last: Multiply the last term of each factor together: $7(–3) = –21$

Now, there are four terms: $x^2 – 3x + 7x – 21$. By combining the two middle terms, you have your quadratic expression: $x^2 + 4x – 21$.

If you encounter a quadratic equation or expression, try factoring it. On the other hand, if you encounter the product of factors such as $(x + 7)(x - 3)$, you may need to use FOIL. Note that if the product of factors equals 0, you should be ready to *interpret* the meaning. For instance, if you are given $(x + k)(x - m) = 0$, then you know that $x = -k$ or $x = m$.

One-Solution Quadratics

Not all quadratic equations have two solutions. Some have only one solution. One-solution quadratics are also called perfect square quadratics, because both roots are the same. Consider the following examples:

$$x^2 + 8x + 16 = 0$$
$$(x + 4)(x + 4) = 0$$
$$(x + 4)^2 = 0$$

Here, the only solution for x is -4.

$$x^2 - 6x + 9 = 0$$
$$(x - 3)(x - 3) = 0$$
$$(x - 3)^2 = 0$$

Here, the only solution for x is 3.

When you see a quadratic equation, look for two solutions, but be aware that some circumstances will lead to just one solution. As long as you understand how the math works, you'll know when you should have two solutions and when you should have just one.

Zero in the Denominator: Undefined

When 0 appears in the denominator of an expression, then that expression is called *undefined*. How does this convention affect quadratic equations? Consider the following:

What are the solutions to the following equation?

$$\frac{x^2 + x - 12}{x - 2} = 0$$

The numerator contains a quadratic equation. Since it is a good idea to start solving quadratic equations by factoring, factor this numerator as follows:

$$\frac{x^2 + x - 12}{x - 2} = 0 \quad \rightarrow \quad \frac{(x-3)(x+4)}{x-2} = 0$$

If either of the factors in the numerator is 0, then the entire expression equals 0. Thus, the solutions to this equation are $x = 3$ or $x = -4$.

Note that making the denominator of the fraction equal to 0 would *not* make the entire expression equal to 0. Recall that if 0 appears in the denominator, the expression becomes undefined. Thus, $x = 2$ (which would make the denominator equal to 0) is *not* a solution to this equation. In fact, since setting x equal to 2 would make the denominator 0, the value 2 is illegal: *x cannot* equal 2.

The Three Special Products

Three quadratic expressions called *special products* come up so frequently on the GMAT that it pays to memorize them. They are GMAT favorites! Make yourself some flash cards and drill them until you immediately recognize these three expressions and know how to factor (or distribute) each one automatically. This will usually put you on the path toward the solution to the problem.

Memorize these!

Special Product #1:	$x^2 - y^2 = (x + y)(x - y)$
Special Product #2:	$x^2 + 2xy + y^2 = (x + y)(x + y) = (x + y)^2$
Special Product #3:	$x^2 - 2xy + y^2 = (x - y)(x - y) = (x - y)^2$

You may also need to identify these products when they are presented in other forms. For example, $a^2 - 1$ can be factored as $(a + 1)(a - 1)$. Similarly, $(a + b)^2$ can be distributed as $a^2 + 2ab + b^2$.

Within an equation, you may need to recognize these special products in pieces. For instance, if you see $a^2 + b^2 = 9 + 2ab$, move the $2ab$ term to the left, yielding $a^2 - 2ab + b^2 = 9$. This quadratic can then be factored to $(a - b)^2 = 9$, or $a - b = \pm 3$.

Simplify: $\frac{x^2+4x+4}{x^2-4}$, given that x does not equal 2 or –2.

Both the numerator and denominator of this fraction can be factored:

$$\frac{(x+2)(x+2)}{(x+2)(x-2)}$$

The expression $x + 2$ can be cancelled out from the numerator and denominator:

$$\frac{x^2+4x+4}{x^2-4}=\frac{x+2}{x-2}$$

Problem Set

Now that you've finished the chapter, do the following problems.

1. $\left(3-\sqrt{7}\right)\left(3+\sqrt{7}\right)=$

2. If –4 is a solution for x in the equation $x^2 + kx + 8 = 0$, what is k?

3. If 8 and –4 are the solutions for x, which of the following could be the equation?

   (A) $x^2 - 4x - 32 = 0$
   (B) $x^2 - 4x + 32 = 0$
   (C) $x^2 + 4x - 12 = 0$
   (D) $x^2 + 4x + 32 = 0$
   (E) $x^2 + 4x + 12 = 0$

4. If $x^2 + k = G$ and x is an integer, which of the following could be the value of $G - k$?

   (A) 7
   (B) 8
   (C) 9
   (D) 10
   (E) 11

5. What is x?

   (1) $x = 4y - 4$
   (2) $xy = 8$

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

6. Given that $\frac{d}{4}+\frac{8}{d}+3=0$, what is d?

7. Given that $\frac{x^2+6x+9}{x+3}=7$, what is x?

8. Given that $(p-3)^2-5=0$, what is p?

9. Given that $z^2-10z+25=9$, what is z?

10. Hugo lies on top of a building, throwing pennies straight down to the street below. The formula for the height, H, that a penny falls is $H = Vt + 5t^2$, where V is the original velocity of the penny (how fast Hugo throws it when it leaves his hand) and t is equal to the time it takes to hit the ground. The building is 60 meters high, and Hugo throws the penny down at an initial speed of 20 meters per second. How long does it take for the penny to hit the ground?

11. If $a \neq 2$, which of the following is equal to $\frac{b(a^2-4)}{ab-2b}$?

(A) ab
(B) a
(C) $a + 2$
(D) a^2
(E) $2b$

Solutions

1. **2:** Use FOIL to simplify this product:

F: $3 \times 3 = 9$
O: $3 \times \sqrt{7} = 3\sqrt{7}$
I: $-\sqrt{7} \times 3 = -3\sqrt{7}$
L: $-\sqrt{7} \times \sqrt{7} = -7$

$$9 + 3\sqrt{7} - 3\sqrt{7} - 7 = 2$$

Alternatively, recognize that the original expression is in the form $(x - y)(x + y)$, which is one of the three special products and which equals $x^2 - y^2$ (the difference of two squares). Thus, the expression simplifies to $3^2 - (\sqrt{7})^2 = 9 - 7 = 2$.

2. **6:** If –4 is a solution, then you know that (x + 4) must be one of the factors of the quadratic equation. The other factor is (x + ?). You know that the product of 4 and ? must be equal to 8; thus, the other factor is (x + 2). You know that the sum of 4 and 2 must be equal to k. Therefore, k = 6.

Alternatively, if –4 is a solution, then it is a possible value for x. Plug it into the equation for x and solve for k:

$$\begin{aligned} x^2 + kx + 8 &= 0 \\ 16 - 4k + 8 &= 0 \\ 24 &= 4k \\ k &= 6 \end{aligned}$$

3. **(A):** If the solutions to the equation are 8 and –4, the factored form of the equation is: (x – 8)(x + 4) = 0

Use FOIL to find the quadratic form: x^2 – 4x – 32 = 0. Therefore, the correct equation is (A).

4. **(C):** $$\begin{aligned} x^2 + k &= G \\ x^2 &= G - k \end{aligned}$$

Because you know that x is an integer, x^2 is a perfect square (the square of an integer). Therefore, G – k is also a perfect square. The only perfect square among the answer choices is (C), 9.

5. **(E):** Each statement alone is not enough information to solve for *x*. Using statements (1) and (2) combined, if you substitute the expression for *x* in the first equation, into the second, you get two different answers:

$$x = 4y - 4$$
$$xy = (4y - 4)y = 8$$
$$4y^2 - 4y = 8$$
$$y^2 - y - 2 = 0$$
$$(y + 1)(y - 2) = 0$$

$$y = \{-1, 2\}$$
$$x = \{-8, 4\}$$

6. {**–8, –4**}: Multiply the entire equation by 4*d* (to eliminate the denominators) and factor:

$$d^2 + 32 + 12d = 0$$
$$d^2 + 12d + 32 = 0$$
$$(d + 8)(d + 4) = 0$$

$d + 8 = 0$ OR $d + 4 = 0$

$d = -8$ $\quad$ $d = -4$

7. **4:** Cross-multiply, simplify, and factor to solve:

$$\frac{x^2 + 6x + 9}{x + 3} = 7$$

$$x^2 + 6x + 9 = 7x + 21$$
$$x^2 - x - 12 = 0$$
$$(x + 3)(x - 4) = 0$$

$x + 3 = 0$ OR $x - 4 = 0$

$x = -3$ $\quad$ $x = 4$

Discard –3 as a value for *x*, since this value would make the denominator 0; thus, the fraction would be undefined.

8. **$\{3 + \sqrt{5}, 3 - \sqrt{5}\}$:**

$$(p - 3)^2 - 5 = 0$$
$$(p - 3)^2 = 5$$
$$\sqrt{(p-3)^2} = \sqrt{5}$$
$$p - 3 = \pm\sqrt{5}$$
$$p = 3 \pm \sqrt{5}$$

Note that if you try to distribute out $(p - 3)^2$ and solve as a quadratic, you will realize there is a non-integer solution and you can't easily solve that way. You would get:

$$p^2 - 6p + 9 - 5 = 0$$
$$p^2 - 6p + 4 = 0$$

There aren't any integers that multiply to 4 and add to 6; at this point, you could choose to use the quadratic equation to solve or you could solve using the method shown at the beginning of this explanation.

9. **{2, 8}:** Since you recognize that the left-hand side of the equation is a perfect square quadratic, you will factor the left side of the equation first, instead of trying to set everything equal to 0.

$$z^2 - 10z + 25 = 9$$
$$(z - 5)^2 = 9$$
$$\sqrt{(z-5)^2} = \sqrt{9}$$
$$z - 5 = \pm 3$$
$$z = 5 \pm 3$$

10. **2:**

$$H = Vt + 5t^2$$
$$60 = 20t + 5t^2$$
$$5t^2 + 20t - 60 = 0$$
$$5(t^2 + 4t - 12) = 0$$
$$5(t + 6)(t - 2) = 0$$

$t + 6 = 0$ OR $t - 2 = 0$ Since a time must be positive, discard the
$t = -6$ $t = 2$ negative value for t.

11. **(C):** Choose numbers. The number 2 is not allowed and the number 4 appears in the expression, so try plugging a = 3 and b = 5 into the expression:

$$\frac{b(a^2 - 4)}{ab - 2b} =$$

$$\frac{(5)((3)^2 - 4)}{(3)(5) - 2(5)} =$$

$$\frac{5(9 - 4)}{15 - 10} =$$

$$\frac{5(5)}{5} = 5$$

Now, plug a = 3 and b = 5 into the answer choices and look for a matching answer of 5:

(A) $ab = (3)(5) = 15$
(B) $a = (3) = 3$
(C) $a + 2 = (3) + 2 = 5$ Match!
(D) $a^2 = (3)^2 = 9$
(E) $2b = 2(5) = 10$

The correct answer is **(C)**.

Alternatively, solve algebraically. Begin by factoring the given expression, then simplify:

$$\frac{b(a+2)(a-2)}{b(a-2)} = a+2$$

Everything divides out except for the *a* + 2 term. If you spot that quickly, then the algebraic solution is faster. If not, then the smart numbers solution may be faster.

Chapter 7
of
Algebra

Strategy: Combos

In This Chapter…

Chapter 7

Strategy: Combos

Combo problems look, at first glance, much like certain algebra questions you learned in school.

What is the value of $\frac{x}{y}$?

(1) $\frac{x+y}{y}=3$

(2) $y = 4$

It wasn't unusual to be asked, in school, to solve for $\frac{x}{y}$, or $x + y$, or any similar combination of variables.

Find x, find y, and voilà! You can calculate any combination of the variables, too.

GMAT combo problems however, have one key difference: your goal is to solve directly for the *combination* of variables, not for each individual variable.

How to Solve for Combos

Here's how to solve for combos in the above problem:

Step 1: Notice that the question asks for a combo.

When a question asks directly for a combination of variables, you have a combo problem. (There are ways to disguise a combo—you'll see an example later in this chapter.)

Step 2: Manipulate any given information to try to match the combo.

The question stem doesn't contain any given information. The question itself is already simplified: $\frac{x}{y} = ?$

Jump into the statements:

(1) $\frac{x+y}{y} = 3$

If you weren't looking for the combo, you might do something like this: $x + y = 3y$. The combo contains a fraction $\left(\frac{x}{y}\right)$, so you actually do not want to get rid of that denominator. How else can you manipulate the equation while preserving the fraction? Try splitting the numerator:

$$\frac{x}{y} + \frac{y}{y} = 3$$

Bingo! The fraction on the left matches the combo and the one on the right goes away:

$$\frac{x}{y} + 1 = 3$$

$$\frac{x}{y} = 2$$

Statement (1) is actually sufficient!

AD
~~BCE~~

Note that you cannot find the individual values for x and y from this statement. If you follow your “school” instincts and try to solve for each variable, you'll answer this question incorrectly. Always try to solve for the combo.

Here's statement (2):

(2) $y = 4$

The statement provides no information about x, so it is not sufficient. The correct answer is **(A)**.

A ~~D~~

~~BCE~~

When you solve for a combo in Data Sufficiency (DS), your ultimate goal is to try to find a single match for the desired combo. If you can, then the statement is sufficient.

The above problem also contains a common DS trap called the C-trap. Problems with this trap appear to work when both statements are used together—that is, the answer appears to be (C). In actuality, one of the two statements works by itself and (C) is incorrect. Take a look at the two statements again:

(1) $\frac{x+y}{y} = 3$

(2) $y = 4$

If you are trying to solve for x and y individually, you will realize pretty quickly that neither statement alone will get you there. Put the two statements together, however, and it is possible to find the values of both x and y.

There's just one hitch: the problem didn't ask for the values of x and y. It asked for the value of $\frac{x}{y}$, and statement (1) is sufficient all by itself to find that combo.

Keep an eye out for the C-trap on Data Sufficiency. If it is obvious that the two statements do work together, reexamine each one individually; one might work all by itself. Combo problems are a very common place for C-traps to occur.

Try another combo problem:

If $x \neq y$, what is the value of $x + y$?

(1) $x - y = 1$

(2) $x^2 - y^2 = x - y$

Step 1: Notice that the question asks for a combo.

The question stem asks directly for a combo: $x + y$.

Step 2: Manipulate any given information to try to match the combo

Write down $x \neq y$ on your scrap paper. You may or may not have to use this piece of information, but either way, you don't want to forget it. Now look at statement (1):

(1) $x - y = 1$

Hmm. How can you turn this into $x + y$? How about $x = y + 1$. Good, now there's an addition sign—but it's not between the x and y. What next?

It turns out that, no matter how you manipulate the equation, you can't change the original subtraction relationship between x and y. As a general rule, if you are given just one linear equation with only basic math operations (addition, subtraction, multiplication, or division), you cannot alter the initial relationship between the two variables. If it starts out as subtraction, it will remain subtraction, no matter what you do to the equation.

This statement is not sufficient; cross off answers (A) and (D).

~~AD~~
BCE

Now look at statement (2):

(2) $x^2 - y^2 = x - y$

This one must also not work, since it also has $x - y$, right? Hang on. This one also contains some other terms. What can you do with them?

If you have already memorized the three special products, you'll recognize $x^2 - y^2$. If you haven't yet done so, make a flash card for yourself right now and start drilling.

As a general rule, whenever you see one of the special products, write down the given form *and* the other form:

$$x^2 - y^2 \Leftrightarrow (x + y)(x - y)$$

Substitute the other form into the given equation:

$$(x + y)(x - y) = x - y$$

What next?

The expression $(x - y)$ appears on both sides of the equation. You can divide it out as long as you know that it does not equal 0. That's why the question stem says that $x \neq y$! If so, then $x - y$ cannot equal 0. Divide both sides of the equation by $(x - y)$:

$$(x + y) = 1$$

This statement is sufficient to answer the question. The answer is **(B)**.

~~AD~~
BCE

In sum:

Step 1: Notice that the question asks for a combo.

A question stem may ask for the combo directly or it may try to disguise the question (see the next section for more).

Step 2: Manipulate any given information to try to match the combo.

Your goal is to try to match the combination of variables. Most of the time, if you try to solve for each variable individually, you will get the question wrong. Go for the combo!

When to Solve for Combos

Whenever the problem asks you for a combination, try to solve directly for it. Sometimes, a question stem will obviously ask for a combo; other times, the question will come in disguise.

Try this problem:

If $a = 3bc$ and $abc \neq 0$, what is the value of c?

(1) $a = 10 - b$
(2) $3a = 4b$

This question definitely does *not* look like a combo question. This time, though, the question stem also contains a given equation: $a = 3bc$. Solve this for c:

$$c = \frac{a}{3b}$$

What would you need to know in order to calculate c? Take a look at the equation this way:

$$c = \left(\frac{1}{3}\right)\left(\frac{a}{b}\right)$$

In other words, if you can find a value for the combination $\frac{a}{b}$, then you can calculate c. This problem is a combo problem in disguise!

Step 1: Notice that the question asks for a combo.

If the question stem contains given information in addition to a question that asks for a single variable, see how the information can be combined. It's possible that you have a combo question in disguise.

Step 2: Manipulate any given information to try to match the combo. Look at statement (1):

(1) $a = 10 - b$

Is there any way to find a value for $\frac{a}{b}$? First, put the variables on the same side of the equals sign: $a + b = 10$. There's no way to turn an addition relationship into a division relationship without further information.

This statement is not sufficient to answer the question. Now look at statement (2): ~~AD~~

BCE

(2) $3a = 4b$

Get the variables on the same side:

$$\frac{3a}{4b} = 1$$

Perfect: a division relationship! If you aren't sure, take one more step to solve, but if you can see that you will be able to find a value for $\frac{a}{b}$, then you're done:

$$\frac{a}{b} = \frac{4}{3}$$

This statement is sufficient; the correct answer is **(B)**.

~~AD~~
BCE

How to Get Better at Combos

To get better at combos, practice the problems at the end of this chapter. When the question doesn't obviously ask for a combination of variables, ask yourself what to do to strip off the disguise and find the combo.

Afterwards, review the problem. In particular, see whether you can articulate both how to reveal the combo (where necessary) and how to find a match or prove that no match exists. Could you explain to a fellow student who is confused?

If so, then you are starting to learn both the process by which you use combos and the underlying principles that these kinds of problems test. You're ready to try *Official Guide* problems or move on to other topics.

If not, then review the solution, search online, or ask an instructor or fellow student for help. When doing *OG* problems, review the solutions in our GMAT Navigator™ program.

Problem Set

1. What is the sum of x, y, and z?

   $x + y = 8$
   $x + z = 11$
   $y + z = 7$

2. If x and y are integers, what is $x + y$?

   (1) $3^x = 81$
   (2) $5^x = \frac{25}{5^y}$

3. If x and y are integers, what is the value of $x^2 + 2xy + y^2$?

   (1) $x + y = 7$
   (2) $2x = \frac{28 - 4y}{2}$

4. If $xy \neq 0$ and $\sqrt{\frac{xy}{3}} = x$, what is y?

   (1) $\frac{x}{y} = \frac{1}{3}$
   (2) $x = 3$

5. If $A = \frac{\frac{x}{3}}{\frac{2}{y}}$, what is A?

   (1) $xy = 8$
   (2) $\frac{x}{y} = 2$

6. If $x = \dfrac{9b - 3ab}{\dfrac{3}{a} - \dfrac{a}{3}}$, what is x?

(1) $\dfrac{9ab}{3+a} = \dfrac{18}{5}$

(2) $b = 1$

Solutions

1. **13:** It is possible to solve for *x*, *y*, and *z* individually, but you will save a significant amount of time by solving for the combo. What is $x + y + z$?

The equations collectively contain exactly two “copies” of each variable and these variables are always added. Add the three equations together:

$$\begin{array}{r} x + y = 8 \\ x + z = 11 \\ + \quad y + z = 7 \\ \hline 2x + 2y + 2z = 26 \end{array}$$

Divide the equation by 2: $x + y + z = 13$.

2. **(B):** The question asks for the combo $x + y$ and specifies that *x* and *y* are integers.

(1) INSUFFICIENT: $3^x = 81$

You could solve for the value of *x*, but the statement does not provide any information about the value of *y*, so this statement is not sufficient. Don't solve for *x* now; check statement (2) first.

(2) SUFFICIENT:

$$5^x = \frac{25}{5^y}$$

$$\begin{aligned} (5^x)(5^y) &= 25 \\ 5^{x+y} &= 5^2 \\ x + y &= 2 \end{aligned}$$

Note that, if you do not do the math (or you do it incorrectly), you may think that this statement is not enough to answer the question. In that case, you may have fallen into a C-trap: the two statements together are definitely enough, but the answer cannot be (C) because one of the statements works by itself.

The correct answer is **(B)**.

3. **(D):** The question stem specifies that x and y are integers and asks for the value of $x^2 + 2xy + y^2$. Since the expression is one of the common quadratic identities, write down its other form as well: $(x + y)^2$.

(1) SUFFICIENT: The work is made much easier if you recognized the quadratic identity and wrote down both forms. Knowing the value of $x + y$ is enough to find the value of $(x + y)^2$.

(2) SUFFICIENT:

$$2x = \frac{28-4y}{2}$$

$$4x = 28 - 4y$$
$$4x + 4y = 28$$
$$x + y = 7$$

The correct answer is **(D)**.

4. **(B):** The question asks for the value of y, so isolate y in the given equation:

$$\sqrt{\frac{xy}{3}} = x$$

$$\frac{xy}{3} = x^2$$

$xy = 3x^2$ It's okay to divide by x since you know that x is not 0.

$$y = 3x$$

In other words, if you can find the value of x, then you can also find the value of y.

(1) INSUFFICIENT: Statement (1) provides the value of $\frac{x}{y}$, but does not provide the

value of x and y individually. For example, x could be 1 and y could be 3, or x could be 2 and y could be 6. Ironically, the combo is not helpful here since this same combo is given in the stem.

(2) SUFFICIENT: With x, you can find y using the rephrased equation from the question stem.

This problem is a reminder that sometimes what appears to be a combo question might just involve “school” algebra of solving for a single variable.

The correct answer is **(B)**.

5. **(A):** This question is a combo problem in disguise. The question asks for A, but the value of A is dependent upon x and y. Before diving into the statements, simplify the given equation:

$$A = \frac{\frac{x}{3}}{\frac{2}{y}}$$

$$A = \frac{x}{3} \times \frac{y}{2}$$

$$A = \frac{xy}{6}$$

If you can find the value of xy, you will have enough information to answer the question.

(1) SUFFICIENT: Statement (1) matches the rephrased question, so it is sufficient to answer the question.

(2) SUFFICIENT: It is not possible to find the value for xy from the value for $\frac{x}{y}$. For example, x could be 2 and y could be 1, in which case xy is 2. Alternatively, x could be 4 and y could be 2, in which case xy is 8. Put differently, the quotient combo and the product combo are not one in the same.

The correct answer is **(A)**.

6. **(A):** This question is really a combo problem in disguise. Notice that the question asks for *x*, and the question stem contains an equation with *x*. You need to simplify the expression on the right side of the equation to solve for the simplest combo possible. As a general rule, simplify fractions as much as possible (eliminate them entirely if possible). Also as a general rule, if the same variable exists in more than one place in the question, attempt to combine like terms.

Begin by getting a common denominator on the bottom:

$$x = \frac{9b - 3ab}{\frac{3}{a} - \frac{a}{3}} = \frac{9b - 3ab}{\frac{9}{3a} - \frac{a^2}{3a}} = \frac{9b - 3ab}{\frac{9 - a^2}{3a}}$$

Now that you have only a single fraction on the bottom, you can flip it over (thus multiplying the numerator by the reciprocal of the denominator):

$$9b - 3ab \times \frac{3a}{9 - a^2} = \frac{3a(9b - 3ab)}{9 - a^2}$$

A lot of factoring can be done here! Factor the common term 3*b* out of the parentheses in the numerator, and factor the denominator as the difference of squares:

$$\frac{3a(9b - 3ab)}{9 - a^2} = \frac{3a(3b)(3 - a)}{(3 - a)(3 + a)}$$

Cancel (3 – *a*) from both the top and the bottom:

$$\frac{3a(3b)}{(3 + a)} = \frac{9ab}{3 + a}$$

Thus, the question is, “What is the value of $\frac{9ab}{3 + a}$?” You can solve the statements directly for this combo.

(1) SUFFICIENT: This statement directly gives you the value of the combo.

(2) INSUFFICIENT: Knowing the value of *b* does not give you the value of the

combo $\frac{9ab}{3+a}$.

The correct answer is **(A)**.

Chapter 8
of
Algebra

Formulas

In This Chapter…

Chapter 8
Formulas

Formulas are another means by which the GMAT tests your ability to work with unknowns. Formulas are specific equations that can involve multiple variables. There are four major types of formula problems on the GMAT:

1. Plug-in formulas
2. Functions
3. Strange symbol formulas
4. Sequence formulas

The GMAT uses formulas both in abstract problems and in real-life word problems.

Plug-in Formulas

The most basic GMAT formula problems provide you with a formula and ask you to solve for one of the variables in the formula by plugging in given values for the other variables. For example:

> The formula for determining an individual's comedic aptitude, C, on a given day is defined as $\frac{QL}{J}$, where J represents the number of jokes told, Q represents the overall joke quality on a scale of 1 to 10, and L represents the number of individual laughs generated. If Nicole told 12 jokes, generated 18 laughs, and earned a comedic aptitude of 10.5, what was the overall quality of her jokes?

Plug the given values into the formula in order to solve for the unknown variable *Q*:

$$C=\frac{QL}{J} \rightarrow 10.5=\frac{18Q}{12} \rightarrow Q=\frac{10.5(12)}{18} \rightarrow Q=\frac{10.5(2)}{3} \rightarrow \frac{21}{3} \rightarrow Q=7$$

The quality of Nicole's jokes was rated a 7.

Notice that you will typically have to do some rearrangement after plugging in the numbers in order to isolate the desired unknown. The actual computations are typically not very complex (though do remember to simplify before you multiply!). Formula problems are tricky because the given formula is unfamiliar. Do not be intimidated. Write the equation down, plug in the numbers carefully, and solve for the required unknown.

Be sure to write the formula as a part of an equation. For instance, do not just write $\frac{QL}{J}$ on your paper.

Rather, write $C=\frac{QL}{J}$. Look for language such as *is defined as* to identify what equals what.

Functions

Functions are very much like the "magic boxes" you may have learned about in elementary school.

> You put a 2 into the magic box, and a 7 comes out. You put a 3 into the magic box, and a 9 comes out. You put a 4 into the magic box, and an 11 comes out. What is the magic box doing to your number?

There are many possible ways to describe what the magic box is doing to your number. One possibility is as follows: The magic box is doubling your number and adding 3:

2(2) + 3 = 7 　　　 2(3) + 3 = 9 　　　 2(4) + 3 = 11

Assuming that this is the case (it is possible that the magic box is actually doing something different to your number), this description would yield the following rule for this magic box: $2x + 3$. This rule can be written in function form as:

$$f(x) = 2x + 3$$

The function f represents the rule that the magic box is using to transform your number.

The magic box analogy is a helpful way to conceptualize a function as a *rule* built on an independent variable. The value of a function changes as the value of the independent variable changes. In other words, the value of a function is dependent on the value of the independent variable. Examples of functions include:

$f(x) = 4x^2 - 11$	The value of the function, f, is dependent on the independent variable, x.
$g(t) = t^3 + \sqrt{t} - \frac{2t}{5}$	The value of the function, g, is dependent on the independent variable, t.

You can think of functions as consisting of an *input* variable (the number you put into the magic box) and a corresponding *output* value (the number that comes out of the box). The function is the rule that turns the input variable into some output value.

By the way, the expression $f(x)$ is pronounced "f of x," not "fx." It does not mean "*f times x*"! The letter f does *not* stand for a variable; rather, it stands for the rule that dictates how the input x changes into the output $f(x)$.

The *domain* of a function indicates the possible inputs. The *range* of a function indicates the possible outputs. For instance, the function $f(x) = x^2$ can take any input but never produces a negative number. So the domain is all numbers, but the range is $f(x) \geq 0$.

The most basic type of function problem asks you to input the numerical value (say, 5) in place of the independent variable (x) to determine the value of the function.

If $f(x) = x^2 - 2$, what is the value of $f(5)$?

In this problem, you are given a rule for $f(x)$: square x and subtract 2. Then, you are asked to apply this rule to the number 5. Square 5 and subtract 2 from the result:

$$f(5) = (5)^2 - 2 = 25 - 2 = 23$$

Variable Substitution in Functions

This type of function problem is slightly more complicated. Instead of finding the output value for a numerical input, you must find the output when the input is an algebraic expression.

If $f(z) = z^2 - \frac{z}{3}$, what is the value of $f(3w + 6)$?

Input the variable expression ($3w + 6$) in place of the independent variable (z) to determine the value of the function:

$$f(3w+6) = (3w+6)^2 - \frac{3w+6}{3}$$

Compare this equation to the equation for $f(z)$. The expression ($3w + 6$) has taken the place of every z in the original equation. In a sense, you are treating the expression ($3w + 6$) as one thing, as if it were a single letter or variable.

The rest is algebraic simplification:

$$\begin{aligned} f(3w+6) &= (3w+6)(3w+6) - (w+2) \\ &= 9w^2 + 36w + 36 - w - 2 \\ &= 9w^2 + 35w + 34 \end{aligned}$$

Strange Symbol Formulas

Another type of GMAT formula problem involves the use of strange symbols. In these problems, the GMAT introduces an arbitrary symbol and uses it to define a certain procedure. People sometimes panic, thinking they forgot to study the weird symbol. Don't worry! The question will tell you what the symbol means.

It is helpful to break the operations down one by one and say them aloud (or in your head)—to "hear" them explicitly. Here are some examples:

Formula Definition	**Step-by-Step Breakdown**
$x \heartsuit y = x^2 + y^2 - xy$	"The first number squared, plus the second number squared, minus the product of the two…"
$s \circ t = (s - 2)(t + 2)$	"Subtract two from the first number, add two to the second number, then multiply them together…"
$\boxed{x}$ is defined as the product of all integers smaller than x but greater than 0…	"…x minus 1, times x minus 2, times x minus 3…Aha! So this is $(x - 1)$ factorial!"

Notice that it can be helpful to refer to the variables as "the first number," "the second number," and so on. In this way, you use the physical position of the numbers to keep them straight in relation to the strange symbol.

Now that you understand what the formula means, you can calculate a solution for the formula with actual numbers. Consider the following example:

$W \psi F = \left(\sqrt{W}\right)^F$ for all integers W and F. What is $4 \psi 5$?

The symbol ψ between two numbers signals the following procedure: take the square root of the first number and then raise that value to the power of the second number.

$$4 \psi 5 = \left(\sqrt{4}\right)^5 = 2^5 = 32$$

Watch out for symbols that invert the order of an operation. It is easy to automatically translate the function in a left-to-right manner even when that is *not* what the function specifies.

$W \Phi F = \left(\sqrt{F}\right)^W$ for all integers W and F. What is $4 \Phi 9$?

It would be easy to perform the first operation using W. However, notice that the order of the operation is *reversed*—you need to take the square root of the *second* number, raised to the power of the *first* number:

$$4 \ \Phi \ 9 = \left(\sqrt{9}\right)^4 = 3^4 = 81$$

More challenging strange-symbol problems require you to use the given procedure more than once. For example:

$W \ \Phi \ F = \left(\sqrt{F}\right)^W$ for all integers *W* and *F*. What is 2 Φ (3 Φ 16)?

Always perform the procedure inside the parentheses first:

$$3 \ \Phi \ 16 = \left(\sqrt{16}\right)^3 = 4^3 = 64$$

Now rewrite the original formula as follows: 2 Φ (3 Φ 16) = 2 Φ 64.

Performing the procedure a second time yields the answer:

$$2 \ \Phi \ 64 = \left(\sqrt{64}\right)^2 = 8^2 = 64$$

Squaring a square root will take you back to your starting point; if you notice this, you can cancel the two operations and you're left with 64.

Formulas That Act on Decimals

Occasionally, you might encounter a formula or special symbol that acts on decimals. Follow the formula's instructions *precisely*.

Define symbol [*x*] to represent the largest integer less than or equal to *x*:

What is [5.1]?

According to the definition you are given, [5.1] is the largest integer less than or equal to 5.1. That integer is 5, so [5.1] = 5. Try another example:

What is [0.8]?

According to the definition again, [0.8] is the largest integer less than or equal to 0.8. That integer is 0. So [0.8] = 0. Notice that the result is *not* 1. This particular definition does not round the number. Rather, the operation *seems* to be truncation—simply cutting off the decimal. However, you must be careful with negatives. For example:

What is [–2.3]?

Once again, [–2.3] is the largest integer less than or equal to –2.3. Remember that “less than” on a number line means “to the left of.” A “smaller” negative number is further away from 0 than a “bigger” negative number. So the largest integer less than –2.3 is –3, and [–2.3] = –3. Notice that the result is *not* –2; this bracket operation is *not* truncation.

Be sure to follow the instructions exactly whenever you are given a special symbol or formula involving decimals. It is easy to jump to conclusions about how an operation works; for instance, finding the largest integer less than x is *not* the same as rounding x or truncating x in all cases. Also, do not confuse this particular set of brackets $[x]$ with parentheses (x) or absolute value signs $|x|$.

Sequence Formulas

A sequence is a collection of numbers in a set order. Every sequence is defined by a rule, which you can use to find the values of terms:

$$A_n = 9n + 3$$

You can find the first term (A_1) by plugging $n = 1$ into the equation. $A_1 = 12$

You can find the second term (A_2) by plugging $n = 2$ into the equation. $A_2 = 21$

You can find the nth term (A_n) by plugging n into the equation.

If $S_n = 15n – 7$, what is the value of $S_7 – S_5$?

This question is asking for the difference between the seventh term and the fifth term of the sequence.

$$S_7 = 15(7) – 7 = 105 – 7 = 98$$
$$S_5 = 15(5) – 7 = 75 – 7 = 68$$
$$S_7 – S_5 = (98) – (68) = 30$$

Recursive Sequences

Occasionally, a sequence will be defined *recursively*. A recursive sequence defines each term relative to other terms. For example:

> If $a_n = 2a_{n-1} - 4$, and $a_6 = -4$, what is the value of a_4?

If a_n represents the nth term, then a_{n-1} is the term right before a_n. You are given the value of the 6th term, and need to figure out the value of the 4th term. Keep track of this on your scrap paper.

____	____	-4
a_4	a_5	a_6

Use the value of the sixth term (a_6) to find the value of the fifth term (a_5):

$$\begin{aligned} a_6 &= 2a_5 - 4 \\ (-4) &= 2a_5 - 4 \\ 0 &= 2a_5 \\ 0 &= a_5 \end{aligned}$$

The value of the fifth term is 0:

____	0	-4
a_4	a_5	a_6

Now use the fifth term to find the fourth term:

$$\begin{aligned} a_5 &= 2a_4 - 4 \\ (0) &= 2a_4 - 4 \\ 4 &= 2a_4 \\ 2 &= a_4 \end{aligned}$$

The value of the fourth term is 2.

When a sequence is defined recursively, the question will have to give you the value of at least one of the terms. Use that value to find the value of the desired term.

Linear Sequence Problems: Alternative Method

For linear sequences, in which the same number is added to any term to yield the next term, you can use the following alternative method:

> If each number in a sequence is 3 more than the previous number, and the 6th number is 32, what is the 100th number?

Instead of finding the rule for this sequence, consider the following reasoning: From the 6th to the 100th term, there are 94 “jumps” of 3. Since 94 × 3 = 282, there is an increase of 282 from the 6th term to the 100th term:

$32 + 282 = 314$

Problem Set

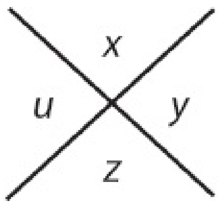
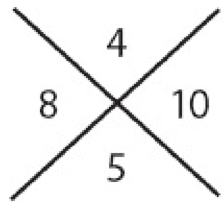
Now that you've finished the chapter, do the following problems.

For problems 1 and 2, use the following sequence: $A_n = 3 - 8n$.

1. What is A_1?

2. What is $A_{11} - A_9$?

3. Given that $A \lozenge B = 4A - B$, what is the value of $(3 \lozenge 2) \lozenge 3$?

4. Given that  $= \dfrac{u+y}{x+z}$, what is  ?

5. If $f(x) = 2x^4 - x^2$, what is the value of $f\left(2\sqrt{3}\right)$?

6. If $a_n = \dfrac{a_{n-1} \times a_{n-2}}{2}$, $a_5 = -6$, and $a_6 = -18$, what is the value of a_3?

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

7. Life expectancy is defined by the formula $\dfrac{2SB}{G}$, where S = shoe size, B = average monthly electric bill in dollars, and G = GMAT score. If Melvin's GMAT score is twice his monthly electric bill, and his life expectancy is 50, what is his shoe size?

8. The “competitive edge” of a baseball team is defined by the formula $\sqrt{\dfrac{W}{L}}$, where W represents the number of the team's wins and L represents the number of the team's losses. This year, the GMAT All-Stars had 3 times as many wins and one-half as many losses as they had last year. By what factor did their “competitive edge” increase?

9. If $k(x) = 4x^3a$, and $k(3) = 27$, what is $k(2)$?

10. The first term in an arithmetic sequence is –5 and the second term is –3. What is the 50th term? (Recall that in an arithmetic sequence, the difference between successive terms is constant.)

11. Given that $\boxed{\begin{matrix} A \\ B \end{matrix}} = A^2 + B^2 + 2AB$, what is $A + B$, if $\boxed{\begin{matrix} A \\ B \end{matrix}} = 9$?

12. If $f(x) = 2x^2 - 4$ and $g(x) = 2x$, for what values of x will $f(x) = g(x)$?

Solutions

1. **–5:** $A_n = 3 - 8n$

$$A_1 = 3 - 8(1) = 3 - 8 = -5$$

2. **–16:** $A_n = 3 - 8n$

$$A_{11} = 3 - 8(11) = 3 - 88 = -85$$
$$A_9 = 3 - 8(9) = 3 - 72 = -69$$
$$A_{11} - A_9 = -85 - (-69) = -16$$

3. **37:** First, simplify 3 ◊ 2: 4(3) – 2 = 12 – 2 = 10. Then, solve 10 ◊ 3: 4(10) – 3 = 40 – 3 = 37.

4. **2:** Plug the numbers in the grid into the formula, matching up the number in each section with the corresponding variable in the formula: $\frac{u+y}{x+z} = \frac{8+10}{4+5} = \frac{18}{9} = 2$.

5. **276:**

$$\begin{aligned} f(x) &= 2\left(2\sqrt{3}\right)^4 - \left(2\sqrt{3}\right)^2 \\ &= 2(2)^4\left(\sqrt{3}\right)^4 - (2)^2\left(\sqrt{3}\right)^2 \\ &= (2\times16\times9) - (4\times3) \\ &= 288 - 12 = 276 \end{aligned}$$

6. –**2:** According to the formula, $a_3 = \frac{a_2 \times a_1}{2}$. But you aren't given a_1 or a_2. Instead, you're given a_5 and a_6. You have to work backwards from the fifth and sixth terms of the sequence to find the third term. Notice what happens if you plug n = 6 into the formula:

$$a_6 = \frac{a_5 \times a_4}{2}$$

If you plug in the values of a_5 and a_6, you can solve for the value of a_4:

$$\begin{aligned}-18 &= \frac{-6 \times a_4}{2} \\ -36 &= -6 \times a_4 \\ 6 &= a_4\end{aligned}$$

Now you can use the fourth and fifth terms of the sequence to solve for a_3:

$$\begin{aligned}a_5 &= \frac{a_4 \times a_3}{2} \\ -6 &= \frac{6 \times a_3}{2} \\ -12 &= 6 \times a_3 \\ -2 &= a_3\end{aligned}$$

7. **Size 50:** Substitute 2*B* for *G* in the formula. Note that the term 2*B* appears in both the numerator and denominator, so they cancel out.

$$\begin{aligned}\frac{2SB}{2B} &= 50 \\ S &= 50\end{aligned}$$

8. $\sqrt{6}$: Let *c* = competitive edge:

$$c = \sqrt{\frac{W}{L}}$$

Pick numbers to see what happens to the competitive edge when *W* is tripled and *L* is halved. If the original value of *W* is 4 and the original value of *L* is 2, the original value of *c* is $\sqrt{\frac{4}{2}} = \sqrt{2}$. If *W* triples to 12 and *L* is halved to 1, the new value of *c* is $\sqrt{\frac{12}{1}} = \sqrt{12}$. The competitive edge has increased from $\sqrt{2}$ to $\sqrt{12}$. Therefore:

$$\frac{\sqrt{12}}{\sqrt{2}} = \sqrt{\frac{12}{2}} = \sqrt{6}$$

The competitive edge has increased by a factor of $\sqrt{6}$.

9. **8:** If $k(3) = 27$, then you know the following:

$$\begin{aligned} 4(3)^3a &= 27 \\ 4(27)a &= 27 \\ 4a &= 1 \\ a &= \frac{1}{4} \end{aligned} \qquad k(x) = 4x^3\left(\frac{1}{4}\right) = x^3 \quad \rightarrow \quad k(2) = (2)^3 = 8$$

10. **93:** The first term is –5 and the second term is –3, so you are adding +2 to each successive term. How many times do you have to add 2? There are 50 – 1 = 49 additional "steps" after the 1st term, so you have to add +2 a total of 49 times, beginning with your starting point of –5: –5 + 2(49) = 93.

11. **{3, –3}:** First, set the formula equal to 9. Then, factor the expression $A^2 + B^2 + 2AB$. Unsquare both sides, taking both the positive and negative roots into account.

$$\begin{aligned} A^2 + B^2 + 2AB &= 9 \\ (A + B)^2 &= 9 \end{aligned}$$

$$A + B = 3 \qquad \text{OR} \qquad A + B = -3$$

12. **{–1, 2}:** To find the values for which $f(x) = g(x)$, set the functions equal to each other:

$$\begin{aligned} 2x^2 - 4 &= 2x \\ 2x^2 - 2x - 4 &= 0 \\ 2(x^2 - x - 2) &= 0 \\ 2(x - 2)(x + 1) &= 0 \end{aligned}$$

$$\begin{aligned} x - 2 &= 0 \qquad \text{OR} \qquad & x + 1 &= 0 \\ x &= 2 & x &= -1 \end{aligned}$$

Chapter 9
of
Algebra

Inequalities

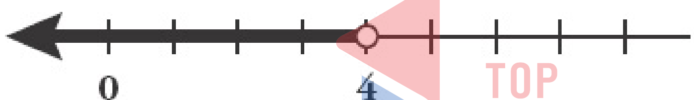
In This Chapter…

Chapter 9

Inequalities

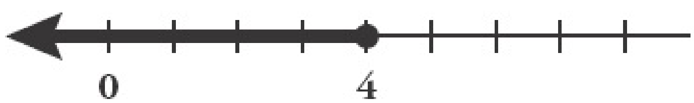
Unlike equations, which relate two equivalent quantities, inequalities compare quantities that have different values. Inequalities are used to express four kinds of relationships, illustrated by the following examples.

1. x is less than 4



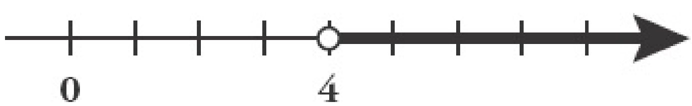
$x < 4$

2. x is less than or equal to 4



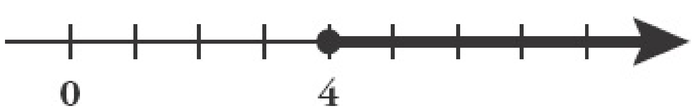
$x \leq 4$

3. x is greater than 4



$x > 4$

4. x is greater than or equal to 4



$x \geq 4$

Number lines, such as those shown above, are an excellent way to visualize exactly what a given inequality means.

When you see inequalities with 0 on one side of the inequality, consider using positive/negative analysis to help solve the problem!

Here are some common inequality statements on the GMAT, as well as what they imply.

STATEMENT	IMPLICATION
$xy > 0$	x and y are *both positive* OR *both negative*
$xy < 0$	x and y have *different signs* (one positive, one negative)
$x^2 - x < 0$	$x^2 < x$, so $0 < x < 1$

Flip the Sign

Most operations that can be performed on equations can be performed on inequalities. For example, in order to simplify an inequality (e.g., $2 + x < 5$), you can add or subtract a constant on both sides:

$$\begin{array}{rl} 2+x < & 5 \\ \underline{-2 \qquad} & \underline{-2} \\ x < & 3 \end{array} \qquad\qquad \begin{array}{rl} x-5 < & 9 \\ \underline{\;+5 \qquad} & \underline{+5} \\ x \qquad < & 14 \end{array}$$

You can also add or subtract a variable expression on both sides:

$$\begin{array}{rl} y+x & <5 \\ \underline{-y \qquad} & \underline{\qquad -y} \\ x & <5-y \end{array} \qquad\qquad \begin{array}{rl} x-ab & <9 \\ \underline{\;+ab \qquad} & \underline{\quad +ab} \\ x \qquad & <9+ab \end{array}$$

You can multiply or divide by a *positive* number on both sides:

$$\begin{array}{rl} 2x < & 6 \\ \underline{\div 2 \quad} & \underline{\div 2} \\ x < & 3 \end{array} \qquad\qquad \begin{array}{rl} 0.2x < & 1 \\ \underline{\times 5 \qquad} & \underline{\times 5} \\ x < & 5 \end{array}$$

One procedure, however, is very different for inequalities: When you multiply or divide an inequality by a negative number, the inequality sign flips! For example:

Given that $4 - 3x < 10$, what is the range of possible values for x?

$$\begin{array}{rr} 4 - 3x < & 10 \\ \underline{-4 \qquad} & \underline{-4} \\ -3x < & 6 \\ \underline{\div(-3)} & \underline{\div(-3)} \\ x > & -2 \end{array}$$

First, subtract 4 from both sides.

Next, divide by −3. Because you're dividing by a negative, flip the inequality sign.

Do not multiply or divide an inequality by a variable unless you know the sign of the number that the variable stands for. If you don't know whether that number is positive or negative, then you don't know whether to flip the inequality sign.

Combining Inequalities: Line 'Em Up!

Many GMAT inequality problems involve more than one inequality. To solve such problems, you may need to convert several inequalities to a compound inequality, which is a series of inequalities strung together, such as $2 < 3 < 4$. To convert multiple inequalities to a compound inequality, first line up the variables, then combine.

If $x > 8$, $x < 17$, and $x + 5 < 19$, what is the range of possible values for x?

First, solve any inequalities that need to be solved. In this example, only the last inequality needs to be solved:

$$\begin{aligned} x + 5 &< 19 \\ x &< 14 \end{aligned}$$

Second, simplify the inequalities so that all the inequality symbols point in the same direction, and then line up the common variables in the inequalities:

$8 < x$

$\quad x < 17$

$\quad x < 14$

Finally, put the information together. Notice that $x < 14$ is more limiting than $x < 17$ (in other words, whenever $x < 14$, x will always be less than 17, but not vice versa.) The range, then, is $8 < x < 14$ rather than $8 < x < 17$. Discard the less limiting inequality, $x < 17$. Try another example:

Given that $u < t$, $b > r$, $f < t$, and $r > t$, is $b > u$?

Combine the four given inequalities by simplifying and lining up the common variables.

Align all inequalities in the same direction: $u < t$, $r < b$, $f < t$, and $t < r$.

Then, line up the variables…

$$\begin{array}{ccccccc} u & < & t & & & & \\ & & & & r & < & b \\ f & < & t & & & & \\ & & t & < & r & & \end{array}$$

…and combine.

$$\begin{array}{ccccccc} u & < & t & < & r & < & b \\ f & < & t & < & r & < & b \end{array}$$

It is not always possible to combine all the information into a single compound inequality, as you see in this example. You know that both u and f are less than t, but you do not know the relationship between u and f.

The answer to the question is yes, b is greater than u.

Manipulating Compound Inequalities

Sometimes a problem with compound inequalities will require you to manipulate the inequalities in order to solve the problem. You can perform operations on a compound inequality as long as you remember to perform those operations on every term in the inequality, not just the outside terms. For example:

$x + 3 < y < x + 5 \not\rightarrow x < y < x + 2$ **WRONG:** You must subtract 3

	from *every* term in the inequality
$x + 3 < y < x+5 \rightarrow x < y - 3 < x + 2$	CORRECT
$\frac{c}{2} \leq b - 3 \leq \frac{d}{2} \quad \rightarrow \quad c \leq b - 3 \leq d$	**WRONG:** You must multiply by 2 in *every* term in the inequality
$\frac{c}{2} \leq b - 3 \leq \frac{d}{2} \quad \rightarrow \quad c \leq 2b - 6 \leq d$	CORRECT

If $1 > 1 - ab > 0$, which of the following must be true?

I. $\frac{a}{b} > 0$

II. $\frac{a}{b} < 1$

III. $ab < 1$

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I and III only

You can manipulate the original compound inequality as follows, making sure to perform each manipulation on every term:

$1 > 1 - ab > 0$	Subtract 1 from all three terms.
$0 > \quad - ab > -1$	Multiply all three terms by −1 and flip the inequality signs.
$0 < \quad ab < 1$	

Therefore, you know that $0 < ab < 1$. This tells you that ab is positive, so $\frac{a}{b}$ must be positive (a and b have the same sign). Therefore, statement I must be true. However, you do not know whether $\frac{a}{b} < 1$, so statement II is not necessarily true. But you do know that ab must be less than 1, so statement III must be true. Therefore, the correct answer is **(E).**

Combining Inequalities: Add 'Em Up!

You can combine inequalities by adding the inequalities together. In order to add inequalities, the inequality signs must face in the same direction.

Is $a + 2b < c + 2d$?

(1) $a < c$
(2) $d > b$

Assume that you've already tried the two statements individually and neither was sufficient by itself. In order to test the statements together, add the inequalities together to see whether they match the question. First, line up the inequalities so that they are all facing the same direction:

$a < c$
$b < d$

Then, take the sum of the two inequalities to prove the result. You will need to add the second inequality *twice:*

$$\begin{array}{rl} & a \qquad < c \\ + & \underline{\qquad b < \qquad d} \\ & a + \; b < c + \; d \\ + & \underline{\qquad b < \qquad d} \\ & a + 2b < c + 2d \end{array}$$

If you use both statements, you can answer the question. Therefore, the answer is **(C).**

Notice that you also could have multiplied the second inequality by 2 before summing, so that the result matched the original question:

$$\begin{array}{rl} & a \qquad < c \\ + & \underline{\quad 2(b < \qquad d)} \\ & a + 2b < c + 2d \end{array}$$

Adding inequalities together is a powerful technique on the GMAT. However, never subtract or divide two inequalities. You can multiply inequalities together as long as all possible values of the inequalities are positive, but the GMAT rarely tests this skill.

Square-Rooting Inequalities

Just like equations involving even exponents, inequality problems involving even exponents require you to consider *two* scenarios. Consider this example:

> If $x^2 < 4$, what are the possible values for x?

To solve this problem, recall that $\sqrt{x^2} = |x|$. For example, $\sqrt{3^2} = 3$ and $\sqrt{(-5)^2} = 5$. Therefore, when you take the square root of both sides of the inequality, you get:

$$\sqrt{x^2} < \sqrt{4}$$
$$|x| < 2$$

If x is positive, then $x < 2$. On the other hand, if x is negative, then $x > -2$.

Here is another example:

> If $10 + x^2 \geq 19$, what is the range of possible values for x?

$$10 + x^2 \geq 19$$
$$x^2 \geq 9$$
$$|x| \geq 3$$

If x is positive, then $x \geq 3$. If x is negative, then $x \leq -3$.

Note that you can *only* take the square root of an inequality for which both sides are definitely *not* negative, since you cannot take the square root of a negative number. Restrict this technique to situations in which the square of a variable or expression must be positive.

Problem Set

Now that you've finished the chapter, do the following problems.

1. Which of the following is equivalent to $-3x + 7 \leq 2x + 32$?

   (A) $x \geq -5$
   (B) $x \geq 5$
   (C) $x \leq 5$
   (D) $x \leq -5$

2. If $G^2 < G$, which of the following could be G?

   (A) 1
   (B) $\frac{23}{7}$
   (C) $\frac{7}{23}$
   (D) -4
   (E) -2

3. If $5B > 4B + 1$, is $B^2 > 1$?

4. If $|A| > 19$, which of the following could not be equal to A?

   (A) 26
   (B) 22
   (C) 18
   (D) -20
   (E) -24

5. If $|10y - 4| > 7$ and $y < 1$, which of the following could be y?

   (A) -0.8
   (B) -0.1

(C) 0.1

(D) 0

(E) 1

6. A retailer sells only radios and clocks. If she currently has 44 total items in inventory, how many of them are radios?

(1) The retailer has more than 28 radios in inventory.

(2) The retailer has less than twice as many radios as clocks in inventory.

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

7. If $a > 7$, $a + 4 > 13$, and $2a < 30$, which of the following must be true?

(A) $9 < a < 15$

(B) $11 < a < 15$

(C) $15 < a < 20$

(D) $13 < a < 15$

8. If $d > a$ and $L < a$, which of the following cannot be true?

(A) $d + L = 14$

(B) $d - L = 7$

(C) $d - L = 1$

(D) $a - d = 9$

(E) $a + d = 9$

9. If $\frac{AB}{7} > \frac{1}{14}$ and $A = B$, which of the following must be greater than 1?

(A) $A + B$

(B) $1 - A$

(C) $2A^2$

(D) $A^2 - \frac{1}{2}$

(E) A

10. If $4x - 12 \geq x + 9$, which of the following must be true?

(A) $x > 6$
(B) $x < 7$
(C) $x > 7$
(D) $x > 8$
(E) $x < 8$

11. If $0 < ab < ac$, is a negative?

(1) $c < 0$
(2) $b > c$

12. Eco Wildlife Preserve contains $5x$ zebras and $2x$ lions, where x is a positive integer. If the lions succeed in killing z of the zebras, is the new ratio of zebras to lions less than 2 to 1?

(1) $z > x$
(2) $z = 4$

Solutions

1. **(A):** $-3x + 7 \leq 2x + 32$

$$-5x \leq 25$$
$$x \geq -5$$

2. **(C):** If $G^2 < G$, then G must be positive (since G^2 will never be negative), and G must be less than 1, because otherwise $G^2 > G$. Thus, $0 < G < 1$. You can eliminate (D) and (E), since they violate the condition that G be positive. Then test (A): 1 is not less than 1, so you can eliminate (A). (B) is larger than 1, so only (C) satisfies the inequality.

3. **YES:** $5B > 4B + 1$

$$B > 1$$

The squares of all numbers greater than 1 are also greater than 1, so $B^2 > 1$.

4. **(C):** If $|A| > 19$, then $A > 19$ OR $A < -19$. The only answer choice that does not satisfy either of these inequalities is (C), 18.

5. **(A):** First, eliminate any answer choices that do not satisfy the simpler of the two inequalities, $y < 1$. Based on this inequality alone, you can eliminate (E). Then, simplify the first inequality:

$$\begin{array}{ccc} 10y - 4 > 7 & \text{OR} & -10y + 4 > 7 \\ 10y > 11 & & 10y < -3 \\ y > 1.1 & & y < -\dfrac{3}{10} \end{array}$$

The only answer choice that satisfies this inequality is (A), −0.8.

6. **(C):** First assign r equal to the number of radios the retailer has in inventory and c equal to the number of clocks the retailer has in inventory. You can translate the information in the question stem:

$$r + c = 44$$

The question now becomes: What is r?

(1) INSUFFICIENT: This only tells you that $r \geq 29$. r could equal 29, 30, 40, etc.

(2) INSUFFICIENT: This only tells you that $r < 2c$. Combining this information with the original equation from the problem gives you:

$$r < 2c$$
$$r + c = 44$$

If you isolate c in the second equation, you can then substitute into the inequality:

$$c = 44 - r$$

$$\begin{aligned} r &< 2c \\ r &< 2(44 - r) \\ r &< 88 - 2r \\ 3r &< 88 \\ r &< \frac{88}{3} \end{aligned}$$

Thus, $\frac{88}{3}$ is equal to $29\frac{1}{3}$. Therefore, you know that r must be less than or equal to 29 (because r must be an integer). This information on its own, however, is insufficient.

(1) AND (2) SUFFICIENT: Statement (1) tells you $r \geq 29$ and statement (2) tells you $r \leq 29$. Therefore, r must equal 29.

The correct answer is **(C).**

7. **(A):** First, solve the second and third inequalities. Simplify the inequalities, so that all the inequality symbols point in the same direction. Then, line up the inequalities as shown. Finally, combine the inequalities:

$$9 < a$$
$$a < 15 \longrightarrow 9 < a < 15$$
$$7 < a$$

Notice that all the wrong answers are more constrained: the low end is too high. The right answer will both keep out all the impossible values of *a and* let in all the possible values of a.

8. **(D):** Simplify the inequalities, so that all the inequality symbols point in the same direction. Then, line up the inequalities as shown. Finally, combine the inequalities:

$$L < a$$
$$a < d \longrightarrow L < a < d$$

Since *d* is a larger number than *a*, *a* – *d* cannot be positive. Therefore, (D) cannot be true.

9. **(C):** $\frac{AB}{7} > \frac{1}{14}$

$14AB > 7$	Cross-multiply across the inequality.
$2AB > 1$	Divide both sides by 7.
$2A^2 > 1$	Since you know that $A = B$, then $2AB = 2A^2$.

Note that you need to get the expression > 1 on the right because the question asked what must be greater than 1.

10. **(A):** $4x - 12 \geq x + 9$

$$3x \geq 21$$
$$x \geq 7$$

You were asked to pick the answer that *must be* true. If *x* is greater than or equal to 7, then *x* could be 7, 7.3, 8, 9.2, and so on. Which of the five answers contains an expression that covers all possible values of *x*? Most people will immediately look at answer (C) *x* > 7, but be careful! Does *x* have to be greater than 7? No; *x* could be 7 itself, in which case answer (C) is inaccurate. Similarly, answers (D) and (E) cover some of the possible values for *x*, but not *all* of them. Answer (B) doesn't share anything in common with *x* > 7, so it's wrong. You're left with answer (A). Why must it be true that *x* is greater than 6? Because *x* could be 7, 7.3, 8, 9.2, and so on. All of those possible values for x are greater than 6. The logic here is very similar to that of Data Sufficiency: if $x \geq 7$ were a statement, it would be sufficient to establish that $x > 6$.

11. **(D):** By the transitive property of inequalities, if $0 < ab < ac$, then $0 < ac$. Therefore, a and c must have the same sign.

(1) SUFFICIENT: Statement (1) tells you that c is negative. Therefore, a is negative.

(2) SUFFICIENT: Statement (2) is trickier. The statement indicates that $b > c$, but the question stem also told you that $ab < ac$. When you multiply both sides of $b > c$ by a, the sign gets flipped. For inequalities, what circumstance needs to be true in order to flip the sign when you multiply by something? You multiply by a negative. Therefore, a must be negative, because multiplying the two sides of the equation by a results in a flipped inequality sign.

The correct answer is (D).

12. **(A):** The ratio of zebras to lions can be written as $\frac{5x}{2x}$.

If z zebras then meet a sad ending, the new ratio can be written as $\frac{5x-z}{2x}$.

(Note that it's fine to “mix” the ratio with the variable z, since the ratio itself already contains the variable x, which is the multiplier—that is, x is the number you would multiply 5 and 2 by to get the real, original numbers of zebras and lions.

Thus, you can rephrase the question as:

$$\frac{5x-z}{2x} < \frac{2}{1}?$$

But you can keep simplifying! (If the DS question contains fractions, or the same variable in more than one place, try to simplify a bit more.) Since you know that x is positive, you can cross-multiply:

$$5x - z < 4x?$$
$$-z < -x?$$
$$z > x?$$

The question is asking, “Is $z > x$?”

(1) SUFFICIENT: This statement answers the rephrased question directly.

Alternatively, plugging in values for z and x would also show the statement to be sufficient if you didn't take the algebraic route. For instance, if $z = 3$ and $x = 2$, then you would start with 10 zebras and 4 lions, and then losing three zebras would give you 7 zebras to 4 lions, which is less than a 2 to 1 ratio. Additional examples will yield the same results.

(2) INSUFFICIENT: Knowing that $z = 4$ is not sufficient without knowing something about x. For instance, if $x = 1$ and you began with 5 zebras and 2 lions, then losing 4 zebras would certainly shift the ratio below 2 to 1. But what if x were 100? If you began with 500 zebras and 200 lions, then the loss of 4 zebras would not shift the ratio below 2 to 1.

The correct answer is **(A).**

Chapter 10 *of* Algebra

Strategy: Test Cases

In This Chapter…

Chapter 10

Strategy: Test Cases

Certain problems allow for multiple possible scenarios, or cases. When you **Test Cases**, you try different numbers in a problem to see whether you have the same outcome or different outcomes.

The strategy plays out a bit differently for Data Sufficiency (DS) versus Problem Solving. This chapter will focus on DS problems; if you have not yet studied DS, please see Appendix A of this guide. For a full treatment of Problem Solving, see the Test Cases strategy chapter in the *Number Properties GMAT Strategy Guide*.

Try this problem, using any solution method you like:

Is $m < n$?

(1) $m < n^2$
(2) $|m| < n$

How to Test Cases

Here's how to test cases to solve the above problem.

Step 1: Set up the base scenario: what possible cases are allowed?

The problem asks about the variables m and n but does not give any constraints. (A constraint, for example, might be, “If m and n are integers…,” in which case you are only allowed to try integers for the two variables.)

Step 2: Remind yourself: choose numbers that make the statement true.

Before you dive into the work, remember this crucial rule:

Tip: When choosing numbers to Test Cases, ONLY choose numbers that make the statement true.

If you inadvertently choose numbers that make the statement false, discard that case and try again.

Step 3: Try to prove the statement *Insufficient*.

Here's how:

(1) $m < n^2$

What numbers would make this statement true?

Case 1: $m = 1$, $n = 2$

Statement True? ($m < n^2$)	Is $m < n$?
$1 < 4$ ✓	YES

First, ensure that the value you've chosen to test does make the statement true. In this case, m is indeed smaller than n^2.

Second, answer the question asked. If $m = 1$ and $n = 2$, then Yes, $m < n$.

Next, ask yourself: Is there another possible case that would give you a *different* outcome?

Case 2: $m = 1$, $n = –2$

Statement True? ($m < n^2$)	Is $m < n$?
$1 < 4$ ✓	NO

Because the answer is Sometimes Yes, Sometimes No, this statement is not sufficient; ~~AD~~ BCE cross off answers (A) and (D). Try statement (2) next.

(2) $|m| < n$

Case 1: $m = 1$, $n = 2$

Statement True? ($\lvert m\rvert < n$)	Is $m < n$?
$1 < 2$ ✓	YES

Is there another possible case that would give you a different outcome?

Case 2: $m = 1$, $n = –2$

Statement True? ($\lvert m\rvert < n$)	Is $m < n$?
$1 < –2$ ✗	

Careful! This time, the numbers don't work. You are required to pick values that make statement (2) true. Discard this case. (Literally cross it off on your scrap paper.)

What else might give a No outcome?

Case 3: $m = –2$, $n = –2$

Statement True? ($\lvert m\rvert < n$)	Is $m < n$?
$2 < –2$ ✗	

Nope, they can't both be negative either, nor can they equal each other.

It turns out that, no matter how many cases you try for statement (2), you are always going to get a Yes outcome. In other words, the only cases that will make the statement true are those that return a Yes outcome to the question. Why?

Taking the absolute value of *m* turns that number positive or keeps it 0 (if $m = 0$). If $|m| < n$, then *n* has to be positive. Further, *n* has to be greater than the positive version of *m*, in order for the statement to be true. This statement is sufficient.

The correct answer is **(B).** ~~AD~~
BCE

When you test cases in Data Sufficiency, your ultimate goal is to try to prove the statement insufficient, if you can. The first case you try will give you one outcome. For the next case, think about numbers that would be likely to give a *different* outcome.

As soon as you do find two different outcomes, as in statement (1) above, you know the statement is not sufficient, and you can cross off some answer choices and move on.

If you cannot find two different outcomes, then you may be able to prove to yourself why you will always get the same outcome, as in statement (2) above. If you have tried several times to prove the statement insufficient but you keep getting the same outcome, then that statement probably is sufficient.

Try another one:

Is $d > 0$?

(1) $bc < 0$
(2) $cd > 0$

Step 1: Set up the base scenario: what possible cases are allowed?

The question stem does not provide any constraints, so think about trying negatives or fractions, where appropriate.

Step 2: Remind yourself: choose numbers that make the statement true.

Separate your evaluation into two parts: first, have you chosen numbers that make this statement true? Second, is the outcome to the question Yes or No based on this one case?

Step 3: Try to prove the statement *insufficient.*

(1) $bc < 0$

At the most basic level, this statement cannot be sufficient because it mentions nothing about d. It does, though, tell you one piece of info: if *b* and *c* multiply to a negative number, then *b* and *c* must have opposite signs. File that piece of information away for later, just in case.

Also, note that this approach used number theory to analyze the problem. Trying real numbers to figure out this theory would also work.

This statement is not sufficient to answer the question. Cross off answers (A) and (D).

~~AD~~
BCE

(2) $cd > 0$

If *c* and *d* multiply to a positive number, then *c* and *d* must have the same sign: both positive or both negative.

In this case, *d* could be positive or negative, so this statement is not sufficient to answer

~~AD~~
~~B~~CE

the question. Cross off (B) and try the two statements together:

(1) $bc < 0$
(2) $cd > 0$

Map out the scenarios to make sure that you correctly keep track of it all. If *b* is positive, then *c* has to be negative. In that case, *d* also has to be negative:

b	c	d	$bc < 0$?	$cd > 0$?	$d > 0$?
+	−	−	✓	✓	No
−	+	+	✓	✓	Yes

If *b* is negative, however, then *c* has to be positive. In that case, *d* also has to be positive.

A Sometimes Yes, Sometimes No answer is not sufficient. The correct answer is

(E). ~~AD~~
~~B~~ ~~C~~ E

You can test cases by using real numbers or by thinking through the math theoretically, whatever you find best for a particular problem. In general, when you really understand the theory, that path will be fastest. If you are at all unsure about the theory, then it's best to test real numbers.

In sum, when you have decided to test cases, follow three main steps:

Step 1: What possible cases are allowed?

Before you start solving, make sure you know what restrictions have been placed on the basic problem in the question stem. You may be told that the number is an integer, or positive, or odd, and so on. Follow these restrictions when choosing numbers to try later in your work.

Step 2: Choose numbers that make the statement true.

Pause for a moment to remind yourself that you are only allowed to choose numbers for each statement that make that particular statement true. With enough practice, this will begin to become second nature. If you answer a testing cases problem incorrectly but aren't sure why, see whether you accidentally tested cases that weren't allowed because they didn't make the statement true.

Step 3: Try to prove the statement *insufficient*.

<u>Value</u>

Sufficient: single numerical answer
Not Sufficient: two or more possible answers

<u>Yes/No</u>

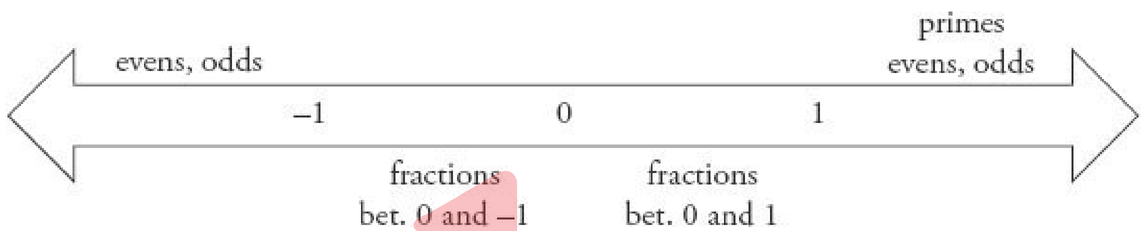
Sufficient: Always Yes *or* Always No
Not Sufficient: Maybe or Sometimes Yes, Sometimes No

When to Test Cases

You can test cases whenever the statements of a DS problem allow multiple possible starting points or scenarios that fulfill the conditions. In that case, try some of the different possibilities allowed in order to see whether these different scenarios, or cases, result in different answers or in the same answer to the question.

When testing cases, your initial starting point is every possible number on the number line. However, many problems give you restrictions that narrow the possible values, such as specifying that a number has to be an integer, or less than 0, or even. Write down your restrictions before you begin testing cases.

Think about different classes of numbers that are commonly tested on the GMAT. For example:



Try the following problem:

Is $b < 0$?

(1) $b^3 < b$
(2) $b^2 > b$

Step 1: What possible cases are allowed?

The question stem does not restrict the possible values for *b*.

Step 2: Choose numbers that make the statement true.

Step 3: Try to prove the statement *insufficient*.

(1) $b^3 < b$

Case 1: If $b = \frac{1}{2}$, then $\frac{1}{8} < \frac{1}{2}$. Therefore, $\frac{1}{2}$ is a possible value for *b*. In this case, no, *b* is not less than 0.

Case 2: If b = 1, then it is not true that 1 < 1. Discard this case. Any number greater than 1 will not make this statement true.

Case 3: If b = –2, then –8 < –2. Therefore, –2 is a possible value for b. In this case, yes, b is less than 0.

This statement is insufficient; eliminate answers (A) and (D). Now look at statement (2): ~~AD~~
BCE

(2) $b^2 > b$

What numbers make this statement true?

Case 1: If b = 2, then 4 > 2. Therefore, 2 is a possible value and, in this case, no, b is not less than 0.

Case 2: If $b = \frac{1}{2}$, then it is not true that $\frac{1}{4} > \frac{1}{2}$. Discard this case.

Case 3: If b = –2, then 4 > –2. Therefore, –2 is a possible value and, in this case, yes, b is less than 0.

Statement (2) is also insufficient; eliminate answer (B). ~~AD~~
~~B~~CE

Try the two statements together:

(1) $b^3 < b$
(2) $b^2 > b$

Statement (1) allows b to be a fraction between 0 and 1 or a negative number smaller than –1.

Statement (2) allows b to be a positive number greater than 1 or any negative number.

The two statements only overlap for negative numbers smaller than –1. As a result, b must be a negative number. Together, the statements are sufficient.

The correct answer is (C). ~~AD~~
~~B~~CE

How to Get Better at Testing Cases

Practice makes perfect. First, try the problems at the end of this chapter using the three-step process detailed above. If you mess up any part of the process, try the problem again, making sure to write out all of your work.

Afterwards, review the problem. In particular, see whether you can articulate the reason why certain statements are sufficient (as the solutions to the earlier problems did). Could you explain to a fellow student who is confused? If so, then you are starting to learn both the process by which you test cases and the underlying principles that these kinds of problems test.

Next, try some problems from the online *Official Guide* problem sets. Again, review your work afterward. If you have any difficulties, look up the solution in the GMAT Navigator™ program, search online, or ask an instructor or fellow student for help.

Problem Set

It's time to practice your testing cases skills.

1. If $y > 0$, what is the value of y?

   (1) $y^2 < y$
   (2) y is an integer.

2. If n is a one-digit positive integer, what is n?

   (1) The units digit of 4^n is 4.
   (2) The units digit of n^4 is n.

3. Is $z > 0$?

   (1) $(z + 1)(z)(z - 1) < 0$
   (2) $|z| < 1$

Solutions

1. **(C):** The question stem allows any positive values for *y*, including fractions. The second statement is considerably easier than the first, so you might choose to start there.

(2) INSUFFICIENT: The statement indicates that *y* is an integer. The value of *y* could be 1, 2, 14, 192, or any other positive integer.

(1) INSUFFICIENT: What numbers make $y^2 \leq y$ true?

Case 1: If $y = 1$, then $1 \leq 1$. Therefore, 1 is a possible value for *y*.

Case 2: If $y = \frac{1}{2}$, then $\frac{1}{4} \leq \frac{1}{2}$. Therefore, $\frac{1}{2}$ is a possible value for *y*.

There are at least two possible values for *y*.

(1) AND (2) SUFFICIENT: Together, the two statements eliminate the fraction case $y = \frac{1}{2}$, but $y = 1$ is still a valid case. In order for $y^2 \leq y$ to be true, *y* must equal 0, 1, or a fraction between 0 and 1. If *y* is a positive integer, then it cannot be 0 or a fraction. The two statements together, then, are sufficient to answer the question: the value of *y* is 1.

The correct answer is (C).

2. **(E):** If *n* is a one-digit positive integer, it has to be 1, 2, 3, 4, 5, 6, 7, 8, or 9.

(1) INSUFFICIENT: The units digit of 4^n is 4.

Case	n	The units digit of 4^n is 4.	What is n?
#1	1	$4^1 = 4$ ✓	1
#2	2	$4^2 = 16$ ✗	
#3	3	$4^3 = 64$ ✓	3

Since *n* could be 1 or 3, statement (1) is not sufficient. (You might notice a pattern. It turns out that every *n* = odd will return a units digit of 4. Every *n* = even will return a units digit of 6.)

(2) INSUFFICIENT: The units digit of n^4 is n.

Case	n	The units digit of n^4 is n.	What is n?
#1	1	$1^4 = 1$ ✓	1
#2	2	$2^4 = 16$ ✗	
#3	3	$3^4 = 81$ ✗	

You can continue to test each possible value for *n* in order, or you can think about any patterns you know for raising a number to a power.

For example, raising 5 to any power will always return a number that ends in 5. Therefore 5^4 will end in 5, so 5 is a valid number for *n*.

For example, raising 5 to any power will always return a number that ends in 5. Therefore 5^4 will end in 5, so 5 is a valid number for *n*.

n	**The units digit of n^4 is n.**	**What is n?**
5	$5^4 = 625$ ✓	5

Because there are at least two possible values for *n*, statement (2) is not sufficient.

(1) AND (2) INSUFFICIENT: Both statements allow *n* = 1. Statement (2) does not allow 3, but does allow 5. Does *n* = 5 work for the first statement?

n	**The units digit of 4^n is 4.**	**What is n?**
5	4^5 = (ends in 4) ✓	4

Note that you do *not* actually multiply out 4^5. Instead, note the pattern:

4^n	**Units digit**

4^1	4
4^2	6
4^3	4
4^4	6

This pattern repeats to infinity.

Because both 1 and 5 work for each statement, even the two statements together are not sufficient to answer the question.

The correct answer is (E).

3. **(C):** This is a tough one. The question stem asks whether *z* is positive. Both statements look fairly complicated, so start with whichever one looks better to you.

(1) INSUFFICIENT: Many people will try *z* = 0, 1, or 2 first. All of these cases are invalid (0 and 1 return a product of 0, which is not *less* than 0, and 2 returns a positive product). Even –1 doesn't work! Think outside of the box: what weirder numbers can you try?

Case	z	$(z+1)(z)(z-1) < 0$	Is $z > 0$?
#1	-2	$(-1)(-2)(-3) = -6$ ✓	No
#2	$\frac{1}{2}$	$\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{3}{8}$ ✓	Yes

Careful! While the three terms *z* + 1, *z*, and *z* –1 appear to represent consecutive integers, the problem never specifies that *z* is an integer. When you pick a fraction, $z = \frac{1}{2}$, you find a case that makes the statement true and also answers the question with a Yes.

(2) INSUFFICIENT: If you understand absolute value, then you might recognize that the statement |*z*| < 1 establishes that *z* is between –1 and 1. If not, test some cases.

In general, start by trying the numbers that worked in the last statement. Are they valid for this statement as well? If so, this could save you time in evaluating these cases and also make it easier if and when you get to the stage of combining the two statements:

Case	z	$\lvert z \rvert < 1$	Is $z > 0$?
#1	–2	$2 < 1$ ✗	
#2	$\frac{1}{2}$	$\frac{1}{2} < 1$ ✓	Yes
#3	$-\frac{1}{2}$	$-\frac{1}{2} < 1$ ✓	No

For this statement, –2 is an invalid case, but $\frac{1}{2}$ is valid. That valid case returns a Yes answer, so try to find a case that will return a No answer instead. Case 3 does just that.

(1) AND (2) SUFFICIENT: Statement (2) allows any values between –1 and 1. From among these possible values, any numbers between 0 and 1 will also make statement (1) true. Therefore, *z* can be greater than 0.

The answer could still be (E), though, if any numbers between –1 and 0 make statement (1) true as well. Try plugging $z = -\frac{1}{2}$ into statement (1):

$$\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right) < 0?$$

$$\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) < 0?$$

Don't multiply out the left side of the equation! The two negative terms will multiply to a positive number, leaving:

positive < 0?

This, of course, is never true. It may take a little more work or reasoning to realize that this result will be repeated for any *z* you pick between –1 and 0. Therefore, only numbers between 0 and 1 work for both statements. The value of *z* has to be positive.

The correct answer is **(C).**

Chapter 11
of
Algebra

Extra Equations Strategies

In This Chapter…

Chapter 11

Extra Equations Strategies

Simultaneous Equations: Three Equations

The procedure for solving a system of three equations with three variables is exactly the same as for a system with two equations and two variables. You can use substitution or combination. This example uses both:

Solve the following for *w*, *x*, and *y*.

$$x + w = y$$
$$2y + w = 3x - 2$$
$$13 - 2w = x + y$$

The first equation is already solved for *y:*

$$y = x + w$$

Substitute for *y* in the second and third equations.

Substitute for *y* in the second equation:

$$2(x + w) + w = 3x - 2$$
$$2x + 2w + w = 3x - 2$$
$$-x + 3w = -2$$

Substitute for *y* in the third equation:

$$13 - 2w = x + (x + w)$$
$$13 - 2w = 2x + w$$
$$13 = 2x + 3w$$

Next, subtract one equation from the other to drop the *w* term.

$$\begin{array}{r} 2x + 3w = 13 \\ -\ (-x + 3w = -2) \\ \hline 2x - (-x) = 13 - (-2) \\ 3x = 15 \\ x = 5 \end{array}$$

Therefore, $x = 5$

Use your solution for x to determine solutions for the other two variables:

$2x + 3w = 13$	$y = x + w$
$10 + 3w = 13$	$y = 5 + 1$
$3w = 3$	$y = 6$
$w = 1$	

The preceding example requires a lot of steps to solve. The GMAT is unlikely to ask you to solve for all three variables, but could ask you to solve for just one or for some other combination of variables. For more on this topic, see Chapter 7, (Strategy: Combos).

Complex Absolute Value Equations

Earlier, you learned about absolute value equations that have one unknown inside one absolute value expression. These equations can also become more complicated by including more than one absolute value expression. There are two primary types of these complex absolute value equations:

1. The equation contains *two* or more variables in more than one absolute value expression. These equations, which usually lack constants, are generally testing the concept of positives and negatives. You can learn about a more conceptual approach to positives and negatives in the *Test Cases* strategy chapter of our *Number Properties GMAT Strategy Guide*.

2. The equation contains *one* variable and at least one *constant* in more than one absolute value expression. These equations are usually easier to solve with an algebraic approach than with a conceptual

approach. For example:

If $|x - 2| = |2x - 3|$, what are the possible values for x?

Because there are two absolute value expressions, each of which yields two algebraic cases, it seems that you need to test *four* cases overall: positive/positive, positive/negative, negative/positive, and negative/negative.

1. The positive/positive case: $(x - 2) = (2x - 3)$
2. The positive/negative case: $(x - 2) = -(2x - 3)$
3. The negative/positive case: $-(x - 2) = (2x - 3)$
4. The negative/negative case: $-(x - 2) = -(2x - 3)$

However, note that case (1) and case (4) yield the same equation. Likewise, case (2) and case (3) yield the same equation. Thus, you only need to consider two real cases: one in which neither expression changes sign, and another in which one expression changes sign:

CASE A: Same sign CASE	B: Different signs
$(x - 2) = (2x - 3)$	$(x - 2) = -(2x - 3)$
$1 = x$	$3x = 5$
	$x = 5/3$

Complex absolute value equations have one other catch: After you finish solving the different cases, you have to check the validity of the solutions by plugging them back into the original equation. Ironically, complex absolute value equations can sometimes yield results that are not actually valid solutions.

Both solutions are valid here, because $|1 - 2| = |2(1) - 3| = 1$ and $\left|\frac{5}{3} - 2\right| = \left|2\left(\frac{5}{3}\right) - 3\right| = \frac{1}{3}$.

Integer Constraints

Occasionally, a GMAT algebra problem contains integer constraints. In such a case, there might be many possible solutions among all numbers but only one *integer* solution.

> $2y - x = 2xy$ and $x \neq 0$. If x and y are integers, which of the following could equal y?
>
> (A) 2
> (B) 1
> (C) 0
> (D) −1
> (E) −2

Test the possibilities for y, using the answer choices, and find the answer that also makes x an integer. The case $y = 0$ produces $x = 0$, but this outcome is disallowed by the condition that $x \neq 0$. The only other case that produces an integer value for x is $y = -1$, yielding $x = 2$. Thus, the answer is **(D)**.

Integer constraints together with *inequalities* can also lead to just one solution.

> If x and y are nonnegative integers and $x + y = 25$, what is x?
>
> (1) $20x + 10y < 300$
> (2) $20x + 10y > 280$

First, simplify the inequality in statement (1): $2x + y < 30$. Since x and y have to be positive integers, the smallest possible value for $2x + y$ is when $x = 0$ and $y = 25$: $2x + y = 25$.

If $x = 1$ and $y = 24$, then $2x + y$ is 26. Statement (1), then, yields multiple possible values for x; thus, it is not sufficient.

Simplify statement (2): $2x + y > 28$. This statement also allows multiple possible values of x. (If you're not sure, test out some numbers. If $x = 10$ and $y = 15$, then $2x + y = 35$. If $x = 15$ and $y = 10$, then $2x + y = 40$.)

Here's what happens when you combine the two statements:

Substituting $(25 - x)$ for y:

$$28 < 2x + y < 30$$
$$28 < 2x + (25 - x) < 30$$
$$28 < x + 25 < 30$$

$$3 < x < 5$$

Since x must be an integer, x must equal 4. Therefore, the answer is **(C)**.

Using FOIL with Square Roots

Some GMAT problems ask you to solve factored expressions that involve roots. For example, the GMAT might ask you to solve the following:

What is the value of $\left(\sqrt{8}-\sqrt{3}\right)\left(\sqrt{8}+\sqrt{3}\right)$?

Even though these problems do not involve any variables, you can solve them just like you would solve a pair of quadratic factors: use FOIL:

FIRST:	$\sqrt{8}\times\sqrt{8}=8$	OUTER:	$\sqrt{8}\times\sqrt{3}=\sqrt{24}$
INNER:	$\sqrt{8}\times\left(-\sqrt{3}\right)=-\sqrt{24}$	LAST:	$\left(\sqrt{3}\right)\left(-\sqrt{3}\right)=-3$

The four terms are: $8+\sqrt{24}-\sqrt{24}-3$.

You can simplify this expression by removing the two middle terms (they cancel each other out) and subtracting: $8+\sqrt{24}-\sqrt{24}-3=8-3=5$. Although the problem looks complex, using FOIL reduces the entire expression to 5.

Quadratic Formula

The vast majority of quadratic equations on the GMAT can be solved by the factoring or square-rooting techniques described earlier in this guide. However, very occasionally you might encounter a problem difficult to solve with these techniques. Such a problem requires an understanding of the quadratic formula, which can solve any quadratic equation but is cumbersome to memorize and use. Unless you are aiming to score a 51 (the top score) on the Quant section of the GMAT, you can skip this lesson.

Quadratic Formula: For any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c are constants, the solutions for x are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Consider the following: If $x^2 + 8x + 13 = 0$, what is x?

This problem cannot be factored because there are no two integers for which the sum is 8 and the product is 13. However, you can find the solutions by plugging the coefficients from the equation into the quadratic formula:

$$x = \frac{-8 \pm \sqrt{8^2 - 4(1)(13)}}{2(1)} = \frac{-8 \pm \sqrt{64 - 52}}{2(1)} = -4 \pm \frac{\sqrt{12}}{2} = \{-4 + \sqrt{3}, -4 - \sqrt{3}\}$$

It is not imperative that you memorize the quadratic formula, but the expression underneath the radical in the formula ($b^2 - 4ac$, called the discriminant) can convey important information: it can tell you how many solutions the equation has. If the discriminant is greater than 0, there will be two solutions. If the discriminant is equal to 0, there will be one solution. If the discriminant is less than 0, there will be no solutions. For example:

Which of the following equations has no solution for x?

(A) $x^2 - 8x - 11 = 0$
(B) $x^2 + 8x + 11 = 0$
(C) $x^2 + 7x + 11 = 0$
(D) $x^2 - 6x + 11 = 0$
(E) $x^2 - 6x - 11 = 0$

None of these equations can be solved by factoring. However, you can determine which of the equations has no solution by determining which equation has a negative discriminant (and note that you can stop at any point that you realize the answer will not be negative):

(A) $b^2 - 4ac = (-8)^2 - 4(1)(-11) = 64 + 44 = 108$ ⟵ For example, a positive plus a positive will be positive, so you could stop this calculation early.

(B) $b^2 - 4ac = (8)^2 - 4(1)(11) = 64 - 44 = 20$

(C) $b^2 - 4ac = (7)^2 - 4(1)(11) = 49 -$

44 = 5

(D) $b^2 - 4ac = (-6)^2 - 4(1)(11) = 36 - 44 = -8$

(E) $b^2 - 4ac = (-6)^2 - 4(1)(-11) = 36 + 44 = 80$

The correct answer is **(D).** Again, it is very rare for a GMAT problem to require familiarity with the quadratic formula. The vast majority of quadratic equations can be factored through conventional methods.

Using Conjugates to Rationalize Denominators

Occasionally, GMAT problems involve fractions that contain square roots in the denominator. When the denominator is a square root alone, you can simplify the fraction by multiplying the numerator and denominator by the square root:

Simplify $\dfrac{4}{\sqrt{2}}$

By multiplying the numerator and denominator by the square root, you remove the root from the denominator entirely:

$$\frac{4}{\sqrt{2}} \times \left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

However, simplifying a denominator that contains the sum or difference of a square root *and* another term is more difficult:

Simplify $\dfrac{4}{3-\sqrt{2}}$

To simplify this type of problem, you need to use the conjugate of the denominator. The conjugate for any square root expression involving addition or subtraction is defined as follows:

For $a+\sqrt{b}$, the conjugate is given by $a-\sqrt{b}$.

For $a-\sqrt{b}$, the conjugate is given by $a+\sqrt{b}$.

In other words, change the sign of the second term to find the conjugate. By multiplying the numerator and denominator by the conjugate, you eliminate the square root from the denominator:

$$\frac{4}{3-\sqrt{2}}\left(\frac{3+\sqrt{2}}{3+\sqrt{2}}\right)=\frac{4\left(3+\sqrt{2}\right)}{\left(3-\sqrt{2}\right)\left(3+\sqrt{2}\right)}=\frac{12+4\sqrt{2}}{9+3\sqrt{2}-3\sqrt{2}-2}=\frac{12+4\sqrt{2}}{7}$$

Problem Set

Solve each problem, applying the concepts and rules you learned in this section.

1. Given that $ab = 12$ and $\frac{c}{a} + 10 = 15$, what is bc?

2. If $|x + 1| = |3x - 2|$, what are the possible values for x?

   (A) $\frac{1}{4}$ and $\frac{3}{4}$
   (B) $\frac{1}{4}$ and $\frac{3}{2}$
   (C) $\frac{2}{3}$ and $\frac{3}{2}$
   (D) $\frac{2}{3}$ and $\frac{4}{3}$
   (E) $\frac{3}{4}$ and $\frac{4}{3}$

3. If $xy = 2$, $xz = 8$, and $yz = 5$, then the value of xyz is closest to:

   (A) 5
   (B) 9
   (C) 15
   (D) 25
   (E) 75

4. If $c + d = 11$ and c and d are positive integers, which of the following is a possible value for $5c + 8d$?

   (A) 55
   (B) 61
   (C) 69
   (D) 83

(E) 88

5. If $mn = 3(m + 1) + n$ and m and n are integers, m could be any of the following values EXCEPT:

(A) 2
(B) 3
(C) 4
(D) 5
(E) 7

6. Which of the following equations has no solution for a?

(A) $a^2 - 6a + 7 = 0$
(B) $a^2 + 6a - 7 = 0$
(C) $a^2 + 4a + 3 = 0$
(D) $a^2 - 4a + 3 = 0$
(E) $a^2 - 4a + 5 = 0$

7. Which of the following is equal to $\frac{6+\sqrt{5}}{2-\sqrt{5}}$?

(A) 17
(B) −17
(C) $17 + 8\sqrt{5}$
(D) $-17 - 8\sqrt{5}$
(E) $12 + 12\sqrt{5}$

8. Solve for a, b, and c: $a + b = 10$, $b + c = 12$, and $a + c = 16$.

Solutions

1. **60:** You can first solve for $\frac{c}{a}$; then multiply the two equations together to quickly solve for bc:

$$\frac{c}{a} = 15 - 10 = 5$$

$$(\cancel{a}b)\left(\frac{c}{\cancel{a}}\right) = 12(5) \qquad bc = 12(5) = 60$$

2. **(B):** This is a complex absolute value problem, so you first must decide on an approach. The equation $|x + 1| = |3x - 2|$ has one variable (x) and several constants (1, 3, and −2). Thus, you should take an algebraic approach.

In theory, with two absolute value expressions, you would set up four cases. However, those four cases collapse to just two cases: 1) the two expressions inside the absolute value symbols are given the same sign, and 2) the two expressions are given the opposite sign.

Case (1): Same Sign	Case (2): Opposite Sign
$x + 1 = 3x - 2$	$x + 1 = -(3x - 2) = -3x + 2$
$3 = 2x$	$4x = 1$
$x = \frac{3}{2}$	$x = \frac{1}{4}$

Testing each solution in the original equation, you verify that both solutions are valid:

$$\left|\frac{3}{2} + 1\right| = \left|3\left(\frac{3}{2}\right) - 2\right| \qquad \left|\frac{1}{4} + 1\right| = \left|3\left(\frac{1}{4}\right) - 2\right|$$

$$\left|\frac{5}{2}\right| = \left|\frac{9}{2} - 2\right| \qquad \left|\frac{5}{4}\right| = \left|\frac{3}{4} - 2\right| = \left|\frac{-5}{4}\right|$$

$$\frac{5}{2} = \frac{5}{2} \qquad \frac{5}{4} = \frac{5}{4}$$

3. **(B):** Multiplying together all three equations gives $x^2y^2z^2 = 80$. As a result, $xyz = \sqrt{80}$, which is very close to $xyz = 9$.

4. **(B):** Because *c* and *d* must be positive integers and $c + d = 11$, there are only 10 possible values for $5c + 8d$ (starting with $c = 1$ and $d = 10$, then $c = 2$ and $d = 9$, and so on). In other words, if your starting point is $5c + 8d = 58$, where $c = 10$ and $d = 1$, if you reduce *c* by 1 and increase *d* by 1, the resulting sum will increase by 3; this pattern will continue to occur all the way to your largest possible value, 85. Starting with 58, then, keep adding 3 until you reach a number found in the answers. 58 + 3 = 61, and 61 is one of the answer choices.

Alternatively, you can notice that consecutive values of $5c + 8d$ differ by 3. In other words, every possible value of $5c + 8d$ equals a multiple of 3 plus some constant. By inspection, you see that the values of $5c + 8d$ are all one more than a multiple of 3: for instance, the value 82 = 81 + 1. The only answer choice that equals a multiple of 3 plus 1 is 61: 60 + 1.

5. **(D):** First, you need to solve for *n*. The reason you solve for *n* is that the answer choices list possible values for *m*, the other variable. If you solve for *n*, then you can plug the possible values for *m* into the formula and see when you get a non-integer for *n*, since *n* must be an integer:

$$mn = 3(m+1) + n$$
$$mn - n = 3(m+1)$$
$$n(m-1) = 3(m+1)$$
$$n = \frac{3(m+1)}{(m-1)} \longrightarrow$$

Only a value of 5 for *m* does not produce an integer for *n*.

m	$n=\frac{3(m+1)}{(m-1)}$
2	$n=\frac{3(2+1)}{(2-1)}=9$
3	$n=\frac{3(3+1)}{(3-1)}=6$
4	$n=\frac{3(4+1)}{(4-1)}=5$
5	$n=\frac{3(5+1)}{(5-1)}=\frac{18}{4}=\frac{9}{2}$
7	$n=\frac{3(7+1)}{(7-1)}=4$

6. **(E):** You can determine which of the equations has no solution by determining which equation has a negative discriminant:

(A) $b^2 - 4ac = (-6)^2 - 4(1)(7) = 36 - 28 = 8$
(B) $b^2 - 4ac = (6)^2 - 4(1)(-7) = 36 + 28 = 64$
(C) $b^2 - 4ac = (4)^2 - 4(1)(3) = 16 - 12 = 4$
(D) $b^2 - 4ac = (-4)^2 - 4(1)(3) = 16 - 12 = 4$
(E) $b^2 - 4ac = (-4)^2 - 4(1)(5) = 16 - 20 = -4$

7. **(D):** In order to simplify a fraction that has a difference involving a square root in the denominator, you need to multiply the numerator and denominator by the sum of the same terms (this is also known as the "conjugate"):

$$\frac{6+\sqrt{5}}{2-\sqrt{5}}=\frac{6+\sqrt{5}}{2-\sqrt{5}}\times\frac{2+\sqrt{5}}{2+\sqrt{5}}=\frac{(6+\sqrt{5})(2+\sqrt{5})}{2^2-(\sqrt{5})^2}=\frac{12+2\sqrt{5}+6\sqrt{5}+5}{4-5}=\frac{17+8\sqrt{5}}{-1}=-17-8\sqrt{5}$$

8. ***a* = 7; *b* = 3; *c* = 9:** This problem could be solved by an elaborate series of substitutions. However, because the coefficients on each variable in each equation are equal to 1, combination proves easier. Here is one way, though certainly not the only way, to solve the problem:

First, combine all three equations by adding them together. Then divide by 2 to get the sum of all three equations. Subtracting any of the original equations from this new equation will solve for one of the variables, and the rest can be solved by plugging back into the original equations.

$$\begin{array}{r} a + b = 10 \\ b + c = 12 \\ \underline{a + c = 16} \\ 2a + 2b + 2c = 38 \end{array}$$

$$\begin{array}{r} a + b + c = 19 \\ \underline{-(a + b = 10)} \\ c = 9 \end{array}$$

$$\begin{array}{l} b + 9 = 12 \\ b = 3 \end{array} \qquad \begin{array}{l} a + 9 = 16 \\ a = 7 \end{array}$$

Chapter 12
of
Algebra

Extra Formulas Strategies

In This Chapter…

Chapter 12

Extra Formulas Strategies

Sequences and Patterns

Some sequences are easier to look at in terms of patterns, rather than rules. For example, consider the following:

> If $S_n = 3^n$, what is the units digit of S_{65}?

Clearly, you cannot be expected to multiply out 3^{65} on the GMAT. Therefore, there must be a pattern in the units digits of the powers of three:

$3^1 = \mathbf{3}$

$3^2 = \mathbf{9}$

$3^3 = 2\mathbf{7}$

$3^4 = 8\mathbf{1}$

$3^5 = 24\mathbf{3}$

$3^6 = 72\mathbf{9}$

$3^7 = 2{,}18\mathbf{7}$

$3^8 = 6{,}56\mathbf{1}$

Note the pattern of the units digits in the powers of 3: 3, 9, 7, 1, [repeating]…. Also note that the units digit of S_n, when n is a multiple of 4, is always equal to 1. You can use the multiples of 4 as "anchor points" in the pattern. Since 65 is 1 more than 64 (the closest multiple of 4), the units digit of S_{65} will be 3, which always follows 1 in the pattern.

As a side note, most sequences on the GMAT are defined for integer $n \geq 1$. That is, the sequence S_n almost always starts at S_1. Occasionally, a sequence might start at S_0, but in that case, you are told that $n \geq 0$. Notice that the *first* term in the sequence would then be S_0, the *second* term would be S_1, the *third* term would be S_2, and so on.

Compound Functions

Compound functions give you two different rules to use.

If $f(x) = x^3 + \sqrt{x}$ and $g(x) = 4x - 3$, what is $f(g(3))$?

The expression $f(g(3))$, pronounced "*f* of *g* of 3", looks ugly, but the key to solving compound function problems is to work from the *inside out*. In this case, start with $g(3)$. Start by putting the number 3 into the function $g(x)$:

$g(3) = 4(3) - 3 = 12 - 3 = 9$

Use the result from the *inner* function g as the new input variable for the *outer* function f:

$f(g(3)) = f(9) = (9)^3 + \sqrt{9} = 729 + 3 = 732$	The final result is 732.

Note that changing the order of the compound functions changes the answer:

If $f(x) = x^3 + \sqrt{x}$ and $g(x) = 4x - 3$, what is $g(f(3))$?

Again, work from the inside out. This time, start with $f(3)$ (which is now the inner function):

$$f(3) = (3)^3 + \sqrt{3} = 27 + \sqrt{3}$$

Use the result from the *inner* function, f, as the new input variable for the *outer* function g:

$$g(f(3)) = g(27 + \sqrt{3}) = 4(27 + \sqrt{3}) - 3 = 108 + 4\sqrt{3} - 3 = 105 + 4\sqrt{3}$$

Thus, $g(f(3)) = 105 + 4\sqrt{3}$.

In general, $f(g(x))$ and $g(f(x))$ are not the same rule overall and will often lead to different outcomes. As an analogy, think of “putting on socks” and “putting on shoes” as two functions: the order in which you perform these steps obviously matters!

You may be asked to *find* a value of x for which $f(g(x)) = g(f(x))$. In that case, use *variable substitution*, working as always from the inside out:

> If $f(x) = x^2 + 1$, and $g(x) = 2x$, for what positive value of x does $f(g(x)) = g(f(x))$?

Evaluate as shown in the problems above, using x instead of an actual value:

$$f(g(x)) = g(f(x))$$
$$f(2x) = g(x^2 + 1)$$
$$(2x)^2 + 1 = 2(x^2 + 1)$$

$$4x^2 + 1 = 2x^2 + 2$$
$$2x^2 = 1$$
$$x = \sqrt{\frac{1}{2}}$$

Take only the positive root since x must be positive.

Functions with Unknown Constants

On the GMAT, you may be given a function with an unknown constant. You will also be given the value of the function for a specific number. You can combine these pieces of information to find the complete function rule:

> If $f(x) = ax^2 – x$, and $f(4) = 28$, what is $f(−2)$?

Solve these problems in three steps. First, use the value of the input variable and the corresponding output value of the function to solve for the unknown constant:

$$f(4) = a(4)2 – 4 = 28$$
$$16a – 4 = 28$$

So, $16a = 32$, $a = 2$. $f(−2)$ 10

Common Function Types

Though the GMAT could pose function questions in many different forms, several different themes occur through many of them. This section explores some of these common types of functions.

Population Problems

In these problems, some population typically increases by a common factor every time period. These can be solved with a population chart. Consider the following example:

> The population of a certain type of bacterium triples every 10 minutes. If the population of a colony 20 minutes ago was 100, in approximately how many minutes from now will the bacteria population reach 24,000?

Make a table with a few rows, labeling one of the middle rows as NOW. Work forward, backward, or both (as necessary in the problem), obeying any conditions given in the problem statement about the rate of growth or decay. In this case, triple each population number as you move down a row. Notice that while the population increases by a constant *factor*, it does *not* increase by a constant *amount* each time period.

For this problem, the population chart at right shows that the bacterial population will reach 24,000 about 30 minutes from now.

Time Elapsed	Population
20 minutes ago	100
10 minutes ago	300
NOW	900
in 10 minutes	2,700
in 20 minutes	8,100
in 30 minutes	24,300

In some cases, you might pick a *smart number* for a starting point in your population chart. If you do so, pick a number that makes the computations as easy as possible.

Proportionality

Many GMAT problems, especially those concerned with real-life situations, will use direct or inverse proportionality between the input and the output values.

Direct proportionality means that the two quantities always change by the same factor and in the same direction. For instance, tripling the input will cause the output to triple as well. Cutting the input in half will also cut the output in half. Direct proportionality relationships are of the form $y = kx$, where x is the input value, y is the output value, and k is called the proportionality constant. This equation can also be written as $\frac{y}{x} = k$, which means that the ratio of the output and input values is always constant.

> The maximum height reached by an object thrown directly upward is directly proportional to the square of the velocity with which the object is thrown. If an object thrown upward at 16 feet per second reaches a maximum height of 4 feet, with what speed must the object be thrown upward to reach a maximum height of 9 feet?

Typically, with direct proportion problems, you will be given *before* and *after* values. Set up ratios to solve the problem. For example, $\frac{y_1}{x_1}$ can be used for the *before* values and $\frac{y_2}{x_2}$ can be used for the *after* values. Write $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, since both ratios are equal to the same constant k. Finally, solve for the unknowns. In the problem given above, be sure to note that the direct proportion is between the height and the square of the velocity, not the velocity itself. Therefore, write the proportion as $\frac{h_1}{{v_1}^2} = \frac{h_2}{{v_2}^2}$. Substitute the known values $h_1 = 4$, $v_1 = 16$, and $h_2 = 9$:

$$\frac{4}{16^2} = \frac{9}{{v_2}^2} \qquad {v_2}^2 = 9\left(\frac{16^2}{4}\right) \qquad {v_2}^2 = 9(64) = 576 \qquad v_2 = 24$$

Thus, the object must be thrown upward at 24 feet per second.

Inverse proportionality means that the two quantities change by *reciprocal* factors. Cutting the input in half will actually double the output. Tripling the input will cut the output to one-third of its original value.

Inverse proportionality relationships are of the form $y = \frac{k}{x}$, where x is the input value, y is the output value, and k is the proportionality constant. This equation can also be written as $yx = k$, which means that the product of the output and input values is always constant.

As with other proportion problems, you will typically be given *before* and *after* values. However, this time you set up products, not ratios, to solve the problem—for example, y_1x_1 can be used for the *before* values and y_2x_2 can be used for the *after* values. Next, write $y_1x_1 = y_2x_2$, since each product equals the same constant k. Finally, use algebra to solve for the unknowns in the problem. Try this example:

> The amount of electrical current that flows through a wire is inversely proportional to the resistance in that wire. If a wire currently carries 4 amperes of electrical current, but the resistance is then cut to one-third of its original value, how many amperes of electrical current will flow through the wire?

While you are not given precise amounts for the *before* or *after* resistance in the wire, you can pick numbers. Using 3 as the original resistance and 1 as the new resistance, the new electrical current will be 12 amperes:

$C_1R_1 = C_2R_2$ $\quad$ $4(3) = C_2(1)$ $\quad$ $12 = C_2$

Linear Growth

Many GMAT problems, especially word problems, feature quantities with linear growth (or decay), that is, they grow (or decline) at a constant rate. Such quantities are determined by the linear function: $y = mx + b$. In this equation, the slope m is the constant rate at which the quantity grows. The y-intercept b is the value of the quantity at time zero, and the variable (in this case, x) stands for time. You can also use t to represent time.

For instance, if a baby weighs 9 pounds at birth and gains 1.2 pounds per month, then the baby's weight can be written as $W = 1.2t + 9$, where t is the baby's age in months. Note that $t = 0$ represents the birth of the baby.

> Jake was 4½ feet tall on his 12th birthday, when he began to have a growth spurt. Between his 12th and 15th birthdays, he grew at a constant rate. If Jake was 20% taller on his 15th birthday than on his 13th birthday, how many inches per year did Jake grow during his growth spurt? (12 inches = 1 foot)

In this problem, the constant growth does not begin until Jake has reached his 12th birthday, so in order to use the constant growth function $y = mx + b$, let time $x = 0$ (the initial state) stand for Jake's 12th birthday. Therefore, $x = 1$ stands for his 13th birthday, $x = 2$ stands for his 14th birthday, and $x = 3$ stands for his 15th birthday.

The problem asks for an answer in inches but gives you information in feet. Therefore, convert to inches at the beginning of the problem: 4½ feet = 54 inches = b. Since the growth rate m is unknown, the growth function can be written as $y = mx + 54$. Jake's height on his 13th birthday, when $x = 1$, was $54 + m$, and his height on his 15th birthday, when $x = 3$, was $54 + 3m$. Because the problem also states that Jake was 20% taller on his 15th birthday than on his 13th, you can write an equation:

$$54 + 3m = (54 + m) + 0.20(54 + m)$$
$$54 + 3m = 1.2(54 + m)$$
$$54 + 3m = 64.8 + 1.2m$$
$$1.8m = 10.8$$
$$m = 6$$

Therefore, Jake grew at a rate of 6 inches each year.

Symmetry

Some difficult GMAT function questions revolve around symmetry, or the property that two seemingly different inputs to the function always yield the same output.

For which of the following functions does $f(x) = f\left(\frac{1}{x}\right)$, given that $x \neq -2, -1, 0,$ or 1?

(A) $f(x) = \left|\frac{x+1}{x}\right|$

(B) $f(x) = \left|\frac{x+1}{x-1}\right|$

(C) $f(x) = \left|\frac{x-1}{x}\right|$

(D) $f(x) = \left|\frac{x}{x+1}\right|$

(E) $f(x) = \left|\frac{x+1}{x+2}\right|$

There are two primary ways that you can set about solving this problem. First, you could substitute $\frac{1}{x}$ in for x in each of the functions and simplify, to see which of the functions yields the same result. Alternatively, you could pick a number for x and see which of the functions produces an equal output for both x and $\frac{1}{x}$. In most cases, the latter strategy will probably be easier. For example, you could choose $x = 3$:

		$f(3)$	$f\left(\frac{1}{3}\right)$
(A)	$f(x)=\left\lvert\frac{x+1}{x}\right\rvert$	$\left\lvert\frac{3+1}{3}\right\rvert=\frac{4}{3}$	$\left\lvert\frac{\frac{1}{3}+1}{\frac{1}{3}}\right\rvert=\left\lvert\frac{\frac{4}{3}}{\frac{1}{3}}\right\rvert=4$
(B)	$f(x)=\left\lvert\frac{x+1}{x-1}\right\rvert$	$\left\lvert\frac{3+1}{3-1}\right\rvert=\frac{4}{2}=2$	$\left\lvert\frac{\frac{1}{3}+1}{\frac{1}{3}-1}\right\rvert=\left\lvert\frac{\frac{4}{3}}{-\frac{2}{3}}\right\rvert=2$
(C)	$f(x)=\left\lvert\frac{x-1}{x}\right\rvert$	$\left\lvert\frac{3-1}{3}\right\rvert=\frac{2}{3}$	$\left\lvert\frac{\frac{1}{3}-1}{\frac{1}{3}}\right\rvert=\left\lvert\frac{-\frac{2}{3}}{\frac{1}{3}}\right\rvert=2$
(D)	$f(x)=\left\lvert\frac{x}{x+1}\right\rvert$	$\left\lvert\frac{3}{3+1}\right\rvert=\frac{3}{4}$	$\left\lvert\frac{\frac{1}{3}}{\frac{1}{3}+1}\right\rvert=\left\lvert\frac{\frac{1}{3}}{\frac{4}{3}}\right\rvert=\frac{1}{4}$
(E)	$f(x)=\left\lvert\frac{x+1}{x+2}\right\rvert$	$\left\lvert\frac{3+1}{3+2}\right\rvert=\frac{4}{5}$	$\left\lvert\frac{\frac{1}{3}+1}{\frac{1}{3}+2}\right\rvert=\left\lvert\frac{\frac{4}{3}}{\frac{7}{3}}\right\rvert=\frac{4}{7}$

Only answer **(B)** returns the same result for 3 and $\frac{1}{3}$, so it is the correct answer.

Note that, if you are confident with your math, you can stop after testing (B) and finding that it does work. You can also prove that (B) is the correct answer algebraically:

$$f(x)=\left|\frac{x+1}{x-1}\right|$$

$$f\left(\frac{1}{x}\right)=\left|\frac{\frac{1}{x}+1}{\frac{1}{x}-1}\right|=\left|\frac{\frac{x+1}{x}}{\frac{1-x}{x}}\right|=\left|\frac{x+1}{1-x}\right|=\left|\frac{x+1}{-(1-x)}\right|=\left|\frac{x+1}{x-1}\right|$$

In the second to last step, $-(1 - x)$ replaces $1 - x$ from the previous step. Technically, one of these expressions is positive and the other is negative; because they are inside of an absolute value symbol, however, they both become positive.

Problem Set

1. If $g(x) = \frac{x^3 - ax}{4}$, and $g(2) = \frac{1}{2}$, what is the value of $g(4)$?

2. The velocity of a falling object in a vacuum is directly proportional to the amount of time the object has been falling. If after 5 seconds an object is falling at a speed of 90 miles per hour, how fast will it be falling after 12 seconds?

3. If $S_n = (4^n) + 3$, what is the units digit of S_{100}?

4. The "luminous flux," or perceived brightness, of a light source is measured in lumens and is inversely proportional to the square of the distance from the light. If a light source produces 200 lumens at a distance of 3 meters, at what distance will the light source produce a luminous flux of 25 lumens?

5. For which of the following functions does $f(x) = f(2 - x)$?

   (A) $f(x) = x + 2$
   (B) $f(x) = 2x - x^2$
   (C) $f(x) = 2 - x$
   (D) $f(x) = (2 - x)^2$
   (E) $f(x) = x^2$

6. If $f(x) = (x + \sqrt{3})^4$, what is the range of the function $f(x)$?

   (A) $\sqrt{3} < f(x) < 4$
   (B) $f(x) \geq 0$
   (C) $f(x) < 0$
   (D) $f(x) \neq 0$

7. If $g(x) = 3x + \sqrt{x}$, what is the value of $g(d^2 + 6d + 9)$?

Solutions

1. **13:** $g(2)=\frac{(2)^3-a(2)}{4}=\frac{1}{2}$

$$8-2a=2$$
$$2a=6 \quad\rightarrow\quad g(x)=\frac{x^3-3x}{4} \quad\rightarrow\quad g(4)=\frac{(4)^3-3(4)}{4}=\frac{\cancel{4}(4^2-3(1))}{\cancel{4}}=\frac{16-3}{1}=13$$
$$a=3$$

2. **216 miles per hour:** Because the velocity and the time spent falling are directly proportional, you can simply set the ratio of the "before" velocity and time to the "after" velocity and time:

$$\frac{v_1}{w_1}=\frac{v_2}{w_2}$$
$$\frac{90\text{ mph}}{5\text{ sec}}=\frac{v_2}{12\text{ sec}}$$
$$v_2=\frac{90(12)}{5}=216\text{ mph}$$

3. **9:** Begin by listing the first few terms of the sequence in order to find the pattern:

$$S^1 = 4^1 + 3 = 4 + 3 = 7$$
$$S^2 = 4^2 + 3 = 16 + 3 = 19$$
$$S^3 = 4^3 + 3 = 64 + 3 = 67$$
$$S^4 = 4^4 + 3 = 256 + 3 = 259$$

The units digit of all odd-numbered terms is 7. The units digit of all even-numbered terms is 9. Because S_{100} is an even-numbered term, its units digit will be 9.

4. $6\sqrt{2}$ **meters (or** $\sqrt{72}$ **meters):** Because the intensity of the light source and the *square* of the distance are inversely proportional, you can write the product of the "before" intensity and distance squared and the product of the "after" intensity and distance squared. Then set these two products equal to each other:

$$I_1 \times d_1^{\,2} = I_2 \times d_2^{\,2}$$

$$(200 \text{ lumens})(3 \text{ meters})^2 = (25 \text{ lumens}) \times d_2^{\,2}$$

$$d_2^{\,2} = \frac{(200 \text{ lumens})(3 \text{ meters})^2}{(25 \text{ lumens})}$$

$$d_2 = 6\sqrt{2} \text{ meters}$$

5. **(B):** This is a "symmetry function" type of problem. Generally the easiest way to solve these kinds of problems is to pick numbers and plug them into each function to determine which answer gives the desired result. For example, you could pick *x* = 4:

		$f(4)$	$f(2-4)$ or $f(-2)$
(A)	$f(x) = x + 2$	$4 + 2 = 6$	$-2 + 2 = 0$
(B)	$\boldsymbol{f(x) = 2x - x^2}$	$\boldsymbol{2(4) - 4^2 = -8}$	$\boldsymbol{2(-2) - (-2)^2 = -8}$
(C)	$f(x) = 2 - x$	$2 - 4 = -2$	$2 - (-2) = 4$
(D)	$f(x) = (2 - x)^2$	$(2 - 4)^2 = 4$	$[2 - (-2)]^2 = 4^2 = 16$
(E)	$f(x) = x^2$	$4^2 = 16$	$(-2)^2 = 4$

6. **(B):** If $f(x) = (x + \sqrt{3})^4$, the range of outputs, or *y*-values, can never be negative. Regardless of the value of *x*, raising $x + \sqrt{3}$ to an even power will result in a non-negative *y*-value. Therefore, the range of the function is all non-negative numbers, or $f(x) \geq 0$.

7. **$3d^2 + 19d + 30$ OR $3d^2 + 17d + {}^{2}4$:**

$$\begin{aligned}
g(d^2 + 6d + 9) &= 3(d^2 + 6d + 9) + \sqrt{d^2 + 6d + 9} \\
&= 3d^2 + 18d + 27 + \sqrt{(d+3)^2} \\
&= 3d^2 + 18d + 27 + d + 3 && \text{OR} \quad 3d^2 + 18d + 27 - (d + 3) \\
&= 3d^2 + 19d + 30 && \text{OR} \quad 3d^2 + 17d + 24 \\
&\quad (\text{if } d + 3 > 0) && \qquad\quad (\text{if } d + 3 < 0)
\end{aligned}$$

Chapter 13
of
Algebra

Extra Inequalities Strategies

In This Chapter…

Chapter 13

Extra Inequalities Strategies

Optimization Problems

Optimization problems involve minimizing or maximizing values. In these problems, you need to focus on the largest and smallest possible values for each of the variables, as some combination of them will usually lead to the largest or smallest possible result.

If $2y + 3 \leq 11$ and $1 \leq x \leq 5$, what is the maximum possible value for xy?

Test the extreme values for x and for y to determine which combinations of extreme values will maximize xy:

$2y + 3 \leq 11$ $\quad$ $2y \leq 8$ $\quad$ $y \leq 4$

Extreme Values for x	**Extreme Values for y**
The lowest value for x is 1.	There is no lower limit to y.
The highest value for x is 5.	The highest value for y is 4.

Now consider the different extreme value scenarios for x, y, and xy. Since y has no lower limit and x is positive, the product xy has no lower limit. Using *y's* highest value (4), test the extreme values of x (1 and 5). The first extreme value generates $xy = (1)(4) = 4$. The second extreme value generates $xy = (5)(4) = 20$.

In this case, xy is maximized when $x = 5$ and $y = 4$, with a result that $xy = 20$.

If $-7 \leq a \leq 6$ and $-7 \leq b \leq 8$, what is the maximum possible value for ab?

Once again, you are looking for a maximum possible value, this time for *ab*. Test the extreme values for *a* and for *b* to determine which combinations of extreme values will maximize *ab*:

Extreme Values for a	**Extreme Values for b**
The lowest value for *a* is −7.	The lowest value for *b* is −7.
The highest value for *a* is 6.	The highest value for *b* is 8.

Now consider the different extreme value scenarios for *a*, *b*, and *ab*:

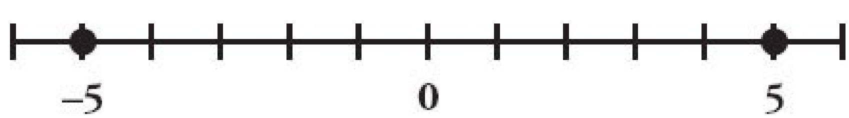
a	***b***	***ab***
Min −7	Min −7	$(-7) \times (-7) =$ **49**
Min −7	Max 8	$(-7) \times 8 = -56$
Max 6	Min −7	$6 \times (-7) = -42$
Max 6	Max 8	$6 \times 8 = 48$

This time, *ab* is maximized when you take the *negative* extreme values for both *a* and *b*, resulting in *ab* = 49. Notice that you could have focused right away on the first and fourth scenarios, because they are the only scenarios that produce positive products.

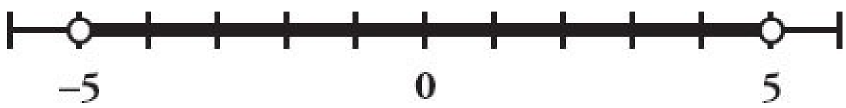
Inequalities and Absolute Value

Absolute value can be a confusing concept—particularly in a problem involving inequalities. For these types of problems, it is often helpful to try to visualize the problem with a number line.

For an equation such as $|x| = 5$, the graph of the solutions looks like this:



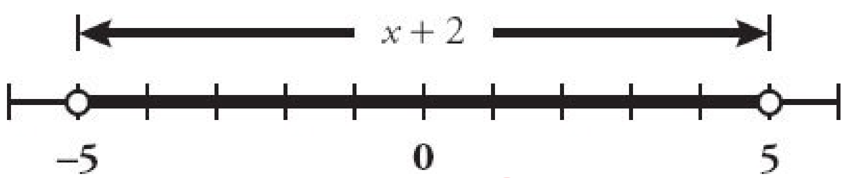
When absolute value is used in an inequality, the unknown generally has more than two possible solutions. Indeed, for an inequality such as $|x| < 5$, the graph of the solutions covers a range:



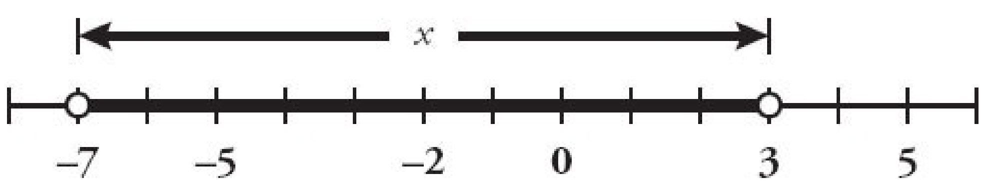
One way to understand this inequality is to say “*x* must be less than 5 units from 0 on the number line.” Indeed, one interpretation of absolute value is simply distance on the number line. For a simple absolute value expression such as $|x|$, you are evaluating distance from 0.

Absolute values can be more difficult to graph than the one above. Consider, for instance, the inequality $| x + 2 | < 5$. The “+ 2” term complicates things.

However, there is a relatively straightforward way to think about this problem. First, create a number line for the term inside the absolute value bars:



In other words, $x + 2$ must be less than 5 units away from 0 on the number line. Next, how does the “+ 2” change the graph? It shifts the entire graph down by 2, because the absolute value expression will be equal to 0 when $x = -2$. Thus, the graph for *x* alone will look like this:

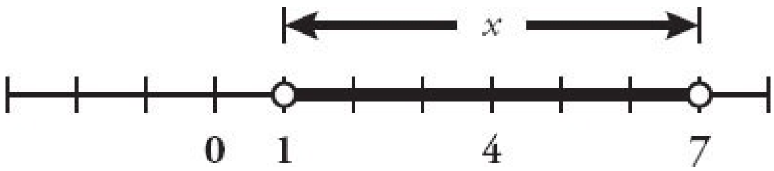


Notice that the center point for the possible values of *x* is now −2, which is the value for *x* that fits $x + 2 = 0$. This is the *center point* for the number line graph. The distance from the center point (−2) to either end point remains the same.

From this example, you can extract a standard formula for interpreting absolute value. When $| x + b | = c$, the center point of the graph is $-b$. The equation indicates that *x* must be *exactly c* units away from $-b$. Similarly, for the inequality $| x + b | < c$, the center point of the graph is $-b$, and *x* must be *less than c* units away from $-b$.

What is the graph of $|x - 4| < 3$?

Based on this formula, the center point of the graph is −(− 4) = 4, and x must be less than 3 units away from that point:



You can also solve these types of problems algebraically. Recall that equations involving absolute value require you to consider *two* scenarios: one where the expression inside the absolute value brackets is positive and one where the expression is negative. The same is true for inequalities. For example:

Given that $|x - 2| < 5$, what is the range of possible values for *x?*

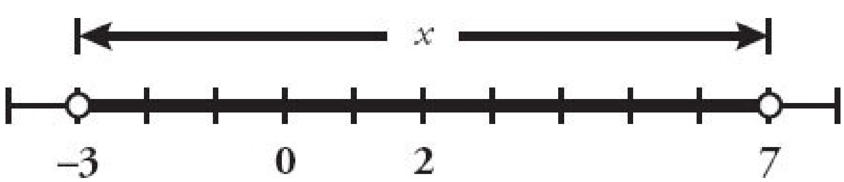
To work out the positive scenario, remove the absolute value brackets and solve:

$|x - 2| < 5$ $\qquad$ $x - 2 < 5$ $\qquad$ $x < 7$

To work out the negative scenario, reverse the signs of the terms inside the absolute value brackets, remove the brackets, and solve again:

$$\begin{aligned} |x-2| &< 5 \\ -(x-2) &< 5 \\ -x+2 &< 5 \\ -x &< 3 \\ x &> -3 \end{aligned}$$

Combine these two scenarios into one range of values for x: $-3 < x < 7$. This range is illustrated by the following number line:



Note that this range fits in perfectly with the number-line interpretation of absolute value: this graph is the set of all points such that x is less than 5 units away from −(−2) = 2.

As an aside, *never* change $|x - 5|$ to $x + 5$. This is a common mistake. Remember, when you drop the absolute value signs, you either leave the expression alone or enclose the *entire* expression in parentheses and put a negative sign in front.

Reciprocals of Inequalities

Taking reciprocals of inequalities is similar to multiplying or dividing by negative numbers. You need to consider the positive and negative cases of the variables involved. The general rule is that **if $x < y$, then:**

- $\frac{1}{x} > \frac{1}{y}$ **when x *and* y are positive.** Flip the inequality. If $3 < 5$, then $\frac{1}{3} > \frac{1}{5}$.
- $\frac{1}{x} > \frac{1}{y}$ **when x and y are negative.** Flip the inequality. If $-4 < -2$, then $\frac{1}{-4} > \frac{1}{-2}$.
- $\frac{1}{x} < \frac{1}{y}$ **when x is negative and y is positive.** Do *not* flip the inequality.

  If $-6 < 7$, then $\frac{1}{-6} < \frac{1}{7}$. The left side is negative, while the right side is positive.

If you do not know the sign of x or y, you cannot take reciprocals.

In summary, if you know the signs of the variables, flip the inequality *unless* x and y have different signs.

Given that $ab < 0$ and $a > b$, which of the following must be true?

I. $a > 0$
II. $b > 0$

III. $\frac{1}{a} > \frac{1}{b}$

(A) I only
(B) II only
(C) I and III only
(D) II and III only
(E) I, II and III

If $ab < 0$, then a and b have different signs. Since $a > b$, a must be positive and b must be negative. Therefore, statement I is true, while statement II is not true.

You also know from the discussion on reciprocals that if $a > b$, then $\frac{1}{a} > \frac{1}{b}$ when a and b have different signs.

Therefore, statement III is also true and the correct answer is (C).

Squaring Inequalities

As with reciprocals, you cannot square both sides of an inequality unless you know the signs of both sides of the inequality. However, the rules for squaring inequalities are somewhat different than those for reciprocating inequalities:

- **If both sides are known to be negative, then flip the inequality sign when you square.** For instance, if $x < -3$, then the left side must be negative. Since both sides are negative, you can square both sides and reverse the inequality sign: $x^2 > 9$. However, if you are given an inequality such as $x > -3$, then you cannot square both sides, because it is unclear whether the left side is positive or negative. If x is negative, then $x^2 < 9$, but if x is positive, then x^2 could be either greater than 9 or less than 9.

- **If both sides are known to be positive, then do not flip the inequality sign when you square.** For instance, if $x > 3$, then the left side must be positive; since both sides are positive, you can square both sides to yield $x^2 > 9$. If you are given an inequality such as $x < 3$, however, then you cannot square both sides, because it is unclear whether the left side is positive or negative. If x is positive, then $x^2 < 9$, but if x is negative, then x^2 could be either greater than 9 or less than 9.

- **If one side is positive and one side is negative, then you cannot square.** If you know that $x < y$, x is negative, and y is positive, you cannot make any determination about x^2 vs. y^2. If, for example, $x = -2$ and $y = 2$, then $x^2 = y^2$. If $x = -2$ and $y = 3$, then $x^2 < y^2$. If $x = -2$ and $y = 1$, then $x^2 > y^2$. If one side of the inequality is negative and the other side is positive, then squaring is probably not warranted—some other technique is likely needed to solve the problem.

- **If the signs are unclear, then you cannot square.** Put simply, you would not know whether to flip the sign of the inequality once you have squared it.

Is $x^2 > y^2$?

(1) $x > y$

(2) $x > 0$

In this problem, statement (1) is insufficient, because you do not know whether x and y are positive or negative numbers. For example, if $x = 5$ and $y = 4$, then $x^2 > y^2$. However, if $x = -4$ and $y = -5$, then $x > y$ but $x^2 < y^2$.

Statement (2) does not tell you anything about y, so it too is insufficient.

Combined, you know that x is positive and larger than y. This is still insufficient, because y could be a negative number of larger magnitude than x. For example, if $x = 3$ and $y = 2$, then $x^2 > y^2$, but if $x = 3$ and $y = -4$, then $x^2 < y^2$. Therefore, the correct answer is **(E).**

Problem Set

1. If a and b are integers and $-4 \le a \le 3$ and $-4 \le b \le 5$, what is the maximum possible value for ab?

2. Is $mn > -12$?

   (1) $m > -3$
   (2) $n > -4$

3. If $\frac{4}{x} < \frac{1}{3}$, what is the possible range of values for x?

4. If $\frac{4}{x} < -\frac{1}{3}$, what is the possible range of values for x?

5. Is $x < y$?

   (1) $\frac{1}{x} < \frac{1}{y}$
   (2) $\frac{x}{y} < 0$

Solution

1. **16:** In order to maximize *ab*, you need to test the endpoints of the ranges for *a* and *b*:

> If $a = -4$ and $b = -4$, $ab = 16$.
> If $a = -4$ and $b = 5$, the product is negative (smaller than 16).
> If $a = 3$ and $b = -4$, the product is negative (smaller than 16).
> If $a = 3$ and $b = 5$, $ab = 15$.

Thus, the maximum value for *ab* is 16. Notice that this maximum occurs when *a* and *b* are both negative in this case.

2. **(E):** Combining the two statements, it is tempting to conclude that *mn* must either be positive or a negative number larger than −12. However, because either variable could be positive or negative, it is possible to end up with a negative number less than −12. For example, *m* could equal −1 and *n* could equal 50. In that case, $mn = -50$, which is less than −12. Therefore, the two statements combined are INSUFFICIENT. The correct answer is (E).

3. $x < 0$ **OR** $x > 12$**:** For this type of problem, you have to consider two possibilities: *x* could be positive or negative. When you multiply the inequality by x, you will need to flip the sign when *x* is negative, but not flip the sign when *x* is positive:

Case 1: $x > 0$	**Case 2: $x < 0$**
$\frac{4}{x} < \frac{1}{3}$	$\frac{4}{x} > \frac{1}{3}$
$12 < x$	$12 > x$

For Case 1, *x* must be positive AND greater than 12. Thus, $x > 12$.

For Case 2, *x* must be negative AND less than 12. Thus, $x < 0$.

Combined, $x < 0$ OR $x > 12$.

4. **-12 < x < 0:** For this type of problem, you have to consider two possibilities: x could be positive or negative. When you multiply the inequality by x, you will need to flip the sign when x is negative, but not flip the sign when x is positive. However, notice that x *cannot* be positive: the left-hand side of the inequality is less than $-\frac{1}{3}$, which means $\frac{4}{x}$ must be negative. Therefore, x must be negative:

Case 1: $x > 0$	**Case 2: $x < 0$**
Not Possible	$\frac{4}{x} < -\frac{1}{3}$ $12 > -x$ $-12 < x$

Case 1 is not possible.

For Case 2, x must be negative AND greater than -12. Thus, $-12 < x < 0$.

5. (**C**): (1) INSUFFICIENT: The meaning of statement (1) depends on the signs of x and y. If x and y are either both positive or both negative, then you can take reciprocals of both sides, yielding $x > y$. However, this statement could also be true if x is negative and y is positive; in that case, $x < y$.

(2) INSUFFICIENT: Statement (2) tells you that the quotient of x and y is negative. In that case, x and y have different signs: one is positive and the other is negative. However, this does not tell you which one is positive and which one is negative.

(1) AND (2) SUFFICIENT: Combining the two statements, if you know that the reciprocal of x is less than that of y, and that x and y have opposite signs, then x must be negative and y must be positive, so $x < y$.

The correct answer is **(C).**

Chapter 1
of
Word Problems

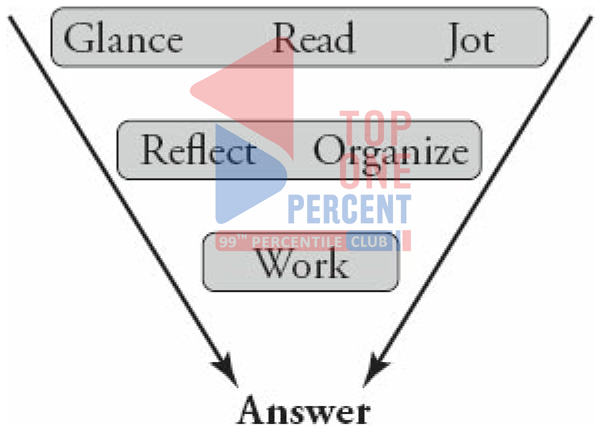
Translations

In This Chapter…

Chapter 1

Translations

Story problems are prevalent on the GMAT and can come in any form: Word Problems, Fractions, Percents, Algebra, and so on. Tackle story problems using your standard three-step approach to solving:



Step 1: Glance, Read, Jot: What's the story?

Glance at the problem: is it Problem Solving or Data Sufficiency? Do the answers or statements give you any quick clues? (Example: variables in the answers might lead you to choose smart numbers.)

Often, on story problems, it's best to finish reading the entire problem before you begin to write.

Step 2: Reflect, Organize: Translate

Your task is to turn the story into math. You can use either the Algebraic method or one of the special strategy methods (work backwards, choose smart numbers, or draw it out, all of which are discussed in this book).

Step 3: Work: Solve

Now that you have the story laid out, you can go ahead and solve.

Try out the three-step process on this problem:

> A candy company sells premium chocolate candies at $5 per pound and regular chocolate candies at $4 per pound in increments of whole pounds only. If Barrett buys a 7-pound box of chocolate candies that costs him $31, how many pounds of premium chocolate candies are in the box?
>
> (A) 1
> (B) 2
> (C) 3
> (D) 4
> (E) 5

Try the algebraic approach first.

Step 1: Glance, Read, Jot

The problem contains a bunch of numbers, but hold off writing them down. Get oriented on the story first so that you can organize the information in a way that makes sense.

Step 2: Reflect, Organize

The problem asks for the number of pounds of premium chocolate candies. Since this is an unknown, assign a variable. Choose variables that tell you what they mean. The variables *x* and *y*, while classic choices, do not indicate whether *x* is premium and *y* is regular or vice versa. The following labels are more useful:

> *p* = pounds of premium chocolate candies
> *r* = pounds of regular chocolate candies

Note that, while the problem asks only for the premium figure, you also want to assign a variable for the regular figure, since this is another unknown in the problem. You would also want to write down something similar to this:

p = ______?

What else can you write down? Barrett bought a 7-pound box of the candies. Both premium and regular make up that 7 pounds, so you can write an equation:

$p + r = 7$

The other given concerns the total cost of the box, \$31. The total cost is equal to the cost of the premium chocolates plus the cost of the regular chocolates.

This relationship is slightly more complicated than it appears, because it involves a relationship the GMAT expects you to know: *Total Cost* = *Unit Price* × *Quantity*. Just as you want to minimize the number of variables you create, you want to minimize the number of equations you have to create. You can express all three terms in the above equation using information you already have:

Total Cost of Box = \$31
Cost of Premiums = (5 \$/pound) × (p pounds) = $5p$
Cost of Regulars = (4 \$/pound) × (r pounds) = $4r$

Note that you can translate "dollars per pound" to "\$/pound." In general, the word "per" is translated as "divided by."

Put that all together, and you have your second equation:

$31 = 5p + 4r$

Step 3: Work

Here's your current scrap paper; how can you solve?

p = # prem choc
r = # reg choc

$p + r = 7$
$31 = 5p + 4r$

$p =$ _____?

When you have two equations with two variables, the most efficient way to find the desired value is to eliminate the unwanted variable in order to solve for the desired variable.

You're looking for *p*. To eliminate *r*, first isolate it in one of the equations. It is easier to isolate *r* in the first equation:

$$r + p = 7 \quad \rightarrow \quad r = 7 - p$$

Now replace *r* with $(7 - p)$ in the second equation and solve for *p*:

$$31 = 5p + 4(7 - p)$$
$$31 = 5p + 28 - 4p$$
$$3 = p$$

The correct answer is **(C)**.

The Work Backwards Method

What if you didn't want to write a bunch of formulas? How else could you solve?

Step 1: Glance, Read, Jot
Glance: you have a story problem. Read the whole thing—including the answer choices—before you start to solve.

Step 2: Reflect, Organize
Notice anything? The answer choices are very "nice" numbers! You don't need to do algebra; instead, you can work backwards from the answers.

Step 3: Work
Start laying out the information you were given and try answer choice (B) first:

#s of p	Cost of p (\$5/#)	Cost of r ($p + r =$ \$31)	Is r an integer? (cost/4)
(A)			
(B) 2	(2)(5) = \$10	\$31 − \$10 = \$21	\$21/4 = No

That didn’t work, so try (D):

(C)			
(D) 4	(4)(5) = \$20	\$31 – \$20 = 11	\$11/4 = No
(E)			

Answer (D) also doesn’t work. Are you noticing any patterns?

In order for *R* to be an integer, what has to happen? In this case, \$31 minus the cost of *P* must be a multiple of 4. Run through the beginning of the calculation, looking for something that will produce a multiple of 4 at the right stage:

#s of *p*	Cost of *p* (\$5/#)	Cost of *r* (*p* + *r* = \$31)	Is *r* an integer? (cost/4)
(A) 1	(1)(5) = \$5	Mult of 4? No.	
(B) 2	(2)(5) = \$10	\$31 – \$10 = \$21	\$21/4 = No
(C) 3	(3)(5) = \$15	\$31 – \$15 = \$16	\$16/4 = 4 Yes!
(D) 4	(4)(5) = \$20	\$31 – \$20 = \$11	\$11/4 = No
(E) 5			

The correct answer is **(C)**.

The GMAT has many ways of making various stages of a Word Problem more difficult, which is why it is so important to have a good process. Train yourself to use these three steps to help assess what you have, figure out an approach, and only then perform the necessary work to get to the solution.

Pay Attention to Units

Unlike problems that test pure algebra, Word Problems have a context. The values, both unknown and known, have a meaning. Practically, this means that every value in a Word Problem has units.

Every equation that correctly represents a relationship has units that make sense. Most relationships are either additive or multiplicative.

Additive Relationships

In the chocolates problem, there were two additive relationships:

$$r + p = 7$$
$$31 = 5p + 4r$$

For each equation, the units of every term are the same; for example, pounds plus pounds equals pounds. Adding terms with the same units does not change the units. Here are the same equations with the units added in parentheses:

r	$+$	p	$=$	7
(pounds)		(pounds)		(pounds)
31	$=$	$5p$	$+$	$4r$
(dollars)		(dollars)		(dollars)

You may be wondering how you can know the units for 5*p* and 4*r* are dollars. That brings us to the second type of relationship.

Rate Relationships

Remember the relationship you used to find those two terms?

Total Cost = *Unit Price* × *Quantity*

Look at them again with units in parentheses:

$$5\left(\frac{\text{dollars}}{\text{pound}}\right) \times p \text{ (pounds)} = 5p \text{ (dollars)}$$

$$4\left(\frac{\text{dollars}}{\text{pound}}\right) \times r \text{ (pounds)} = 4r \text{ (dollars)}$$

For multiplicative relationships, treat units like numerators and denominators. Units that are multiplied together *do* change.

In the equations above, pounds in the denominator of the first term cancel out pounds in the numerator of the second term, leaving dollars as the final units:

$$5\left(\frac{\text{dollars}}{\cancel{\text{pounds}}}\right) \times p\ (\cancel{\text{pounds}}) = 5p\ (\text{dollars})$$

Look at the formula for area to see what happens to the same units when they appear on the same side of the fraction:

l (feet) × w (feet) = lw (feet^2)

Keep track of the units to stay on track in the calculation.

Common Relationships

The GMAT will assume that you have mastered the following relationships. Notice that for all of these relationships, the units follow the rules laid out in the previous section:

- Total Cost ($) = Unit Price ($/unit) × Quantity Purchased (units)
- Profit ($) = Revenue ($) − Cost ($)
- Total Earnings ($) = Wage Rate ($/hour) × Hours Worked (hours)
- Miles = Miles per Hour × Hours
- Miles = Miles per Gallon × Gallons

Units Conversion

When values with units are multiplied or divided, the units change. This property is the basis of using **conversion factors** to convert units. A conversion factor is a fraction whose numerator and denominator have different units but the same value.

For instance, how many seconds are in 7 minutes? If you said 420, you are correct. You were able to make this calculation because you know there are

60 seconds in a minute. In this case, $\frac{60 \text{ seconds}}{1 \text{ minute}}$ is a conversion factor.

Because the numerator and denominator are the same, multiplying by a conversion factor is just a sneaky way of multiplying by 1. The multiplication looks like this:

$$7 \text{ minutes} \times \frac{60 \text{ seconds}}{1 \text{ minute}} = 420 \text{ seconds}$$

Because you are multiplying, you can cancel minutes, leaving you with your desired units (seconds).

Questions will occasionally center around your ability to convert units. Try the following example:

> A certain medicine requires 4 doses per day. If each dose is 150 milligrams, how many milligrams of medicine will a person have taken after the end of the third day, if the medicine is used as directed?

For any question that involves unit conversion, there will have to be some concrete value given. In this case, you were told that the time period is three days, that there are 4 doses/day, and that 1 dose equals 150 milligrams.

Now you need to know what the question wants. It's asking for the number of milligrams of medicine that will be taken in that time. How can you <u>combine all of those givens so that the only units that remain are milligrams?</u>

Combine the calculations into one big expression:

$$3 \text{ days} \times \frac{4 \text{ doses}}{1 \text{ day}} \times \frac{150 \text{ milligrams}}{1 \text{ dose}} = 1{,}800 \text{ milligrams}$$

During the GMAT, you may not actually write out the units for each piece of multiplication. If you don't, however, make sure that your conversion factors are set up properly to cancel out the units you don't want and to leave the units you do want.

Finally, keep an eye out for more of these relationships! For instance, rate and work problems are also built on a common relationship that you’re expected to know for the test; you’ll learn about that relationship in chapter 3.

Problem Set

Solve the following problems using the three-step method outlined in this chapter.

1. United Telephone charges a base rate of $10.00 for service, plus an additional charge of $0.25 per minute. Atlantic Call charges a base rate of $12.00 for service, plus an additional charge of $0.20 per minute. For what number of minutes would the bills for each telephone company be the same?

2. Caleb spends $72.50 on 50 hamburgers for the marching band. If single burgers cost $1.00 each and double burgers cost $1.50 each, how many double burgers did he buy?

3. On the planet Flarp, 3 floops equal 5 fleeps, 4 fleeps equal 7 flaaps, and 2 flaaps equal 3 fliips. How many floops are equal to 35 fliips?

4. Carina has 100 ounces of coffee divided into 5- and 10-ounce packages. If she has 2 more 5-ounce packages than 10-ounce packages, how many 10-ounce packages does she have?

5. A circus earned $150,000 in ticket revenue by selling 1,800 V.I.P. and Standard tickets. They sold 25% more Standard tickets than V.I.P. tickets. If the revenue from Standard tickets represents one-third of the total ticket revenue, what is the price of
a V.I.P. ticket?

Solutions

1. **40 minutes:**

Let x = the number of minutes.

A call made by United Telephone costs \$10.00 plus \$0.25 per minute: $10 + 0.25x$.

A call made by Atlantic Call costs \$12.00 plus \$0.20 per minute: $12 + 0.20x$.

Set the expressions equal to each other:

$$\begin{aligned} 10 + 0.25x &= 12 + 0.20x \\ 0.05x &= 2 \\ x &= 40 \end{aligned}$$

2. **45 double burgers:**

Let s = the number of single burgers purchased.

Let d = the number of double burgers purchased.

Caleb bought 50 burgers:	Caleb spent \$72.50 in all:
$s + d = 50$	$s + 1.5d = 72.50$

Combine the two equations by subtracting equation 1 from equation 2.

$$\begin{array}{r} s + 1.5d = 72.50 \\ -\,(s + \quad d = 50) \\ \hline 0.5d = 22.5 \\ d = 45 \end{array}$$

3. **8 floops:** All of the objects in this question are completely made up, so you can't use intuition to help you convert units. Instead, you need to use the

conversion factors given in the question. Start with 35 fliips, and keep converting until you end up with floops as the units:

$$35\ \cancel{\text{fliips}} \times \frac{2\ \cancel{\text{flaaps}}}{3\ \cancel{\text{fliips}}} \times \frac{4\ \cancel{\text{fleeps}}}{7\ \cancel{\text{flaaps}}} \times \frac{3\ \text{floops}}{5\ \cancel{\text{fleeps}}} = 8\ \text{floops}$$

4. **6:**

Let a = the number of 5-ounce packages.

Let b = the number of 10-ounce packages.

Carina has 100 ounces of coffee:	She has two more 5-ounce packages than 10-ounce packages:
$5a + 10b = 100$	$a = b + 2$

Combine the equations by substituting the value of a from equation 2 into equation 1:

$$\begin{aligned} 5(b+2)+10b &= 100 \\ 5b+10+10b &= 100 \\ 15b+10 &= 100 \\ 15b &= 90 \\ b &= 6 \end{aligned}$$

5. **\$125:** To answer this question correctly, you need to make sure to differentiate between the price of tickets and the *quantity* of tickets sold.

Let V = # of V.I.P. tickets sold.

Let S = # of Standard tickets sold.

The question tells you that the circus sold a total of 1,800 tickets, and that the circus sold 25% more Standard tickets than V.I.P. tickets. You can create two equations:

$$V + S = 1{,}800 \qquad\qquad 1.25V = S$$

You can use these equations to figure out how many of each type of ticket was sold:

$$V + S = 1{,}800$$
$$V + (1.25V) = 1{,}800$$
$$2.25V = 1{,}800$$
$$V = 800$$

Thus, 800 V.I.P. tickets were sold. Next, subtract 800 from the total number of tickets (1,800 − 800) to find that 1,000 Standard tickets were sold.

Now you need to find the cost per V.I.P. ticket. The question states that the circus earned $150,000 in ticket revenue, and that Standard tickets represented one-third of the total revenue. Therefore, Standard tickets accounted for 1/3 × $150,000 = $50,000. V.I.P. tickets then accounted for $150,000 − $50,000 = $100,000 in revenue.

Now, you know that the circus sold 800 V.I.P. tickets for a total of $100,000. Thus, $100,000/800 = $125 per V.I.P. ticket.

Chapter 2
of
Word Problems

Strategy: Work Backwards

In This Chapter…

Chapter 2

Strategy: Work Backwards

Work backwards literally means to start with the answers and do the math in the reverse order described in the problem. You're essentially plugging the answers into the problem to see which one makes the math work.

Try this problem, using any solution method you like:

> Four brothers are splitting a sum of money between them. The first brother receives 50% of the total, the second receives 25% of the total, the third receives 20% of the total, and the fourth receives the remaining \$4. How many dollars are the four brothers splitting?
>
> (A) \$60
>
> (B) \$70
>
> (C) \$80
>
> (D) \$90
>
> (E) \$100

How to Work Backwards

Here's how to Work Backwards to solve the above problem.

Step 1: Start with answer (B) or answer (D). (In many cases, you'll do less work if you start with one of these two; if you're curious as to why, you'll learn later in the chapter!)

Plug that answer into the problem to see whether it works. Use a chart to track your work because you may need to try more than one answer. (Remember that the GMAT gives you graph paper, so you won't have to draw the gridlines.)

The first column is labeled Total because the question stem indicates that the answers represent possible Total amounts of money. Let's say that you start with answer (B). Assume it's correct and start calculating according to the problem:

Total	B#1 (50%)	B#2 (25%)	B#3 (20%)	B#4 (4)	Sum	Match?
(A) 60						
(B) 70	35	17.5	14	4	70.5	No
(C) 80						
(D) 90						
(E) 100						

Assuming that $70 is the total amount of money, the first brother would get 50%, or $35. The second brother, at 25%, would get $17.5, and the third brother, at 20%, would get $14. The fourth brother is given a set amount: $4. Finally, add up the individual amounts. Does it match your starting point of $70?

Close! But not good enough. Answer (B) is incorrect.

Which answer should you try next?

Step 2: Narrow your answers. If the first answer you try works, pick it. If not, cross it off and figure out what to try next.

If you can tell that the starting number has to be smaller than (B), then the answer must be (A) because, in this problem, (A) is the only smaller number.

If you can tell that the starting number has to be larger, then cross off both (A) and (B) and try answer (D) next.

If you can't tell, try answer (D) next. Let's say that you can't tell.

Total	B#1 (50%)	B#2 (25%)	B#3 (20%)	B#4 (4)	Sum	Match?
(A) 60						
(B) 70	30	15	12	4	70.5	No
(C) 80						
(D) 90	45	22.5	18	4	89.5	No
(E) 100						

Answer (D) is also not a match, so it's incorrect. What should you try next?

Step 3: Pick! Actually, you don't have to try another answer. You can pick the correct answer right now! Try to figure out how before you keep reading.

Compare the given answers to your calculated sums. In answer (B), the calculated sum, 70.5, was a bit *higher* than your starting point of 70. In answer (D), by contrast, the calculated sum, 89.5, was *lower* than your starting point of 90. The correct answer, then, should be in between—answer (C).

For these problems, the answers will always be in increasing or decreasing order, so if you try (B) and (D) and neither work, then you can almost always figure out which of the remaining answers must be right without actually checking them. (You'll see more examples of this later in the chapter.)

Here's how the math works for correct answer (C):

Total	B#1 (50%)	B#2 (25%)	B#3 (20%)	B#4 (4)	Sum	Match?
(C) 80	40	20	16	4	80	Yes!

Try ~~another:~~

Machine X produces cartons at a uniform rate of 90 every 3 minutes, and Machine Y produces cartons at a uniform rate of 100 every 2 minutes. Working simultaneously, how many minutes would it take for the two machines to produce a total of 560 cartons?

(A) 7

(B) 6

(C) 5
(D) 4
(E) 3

Step 1: Start with answer (B) or answer (D). Set up your chart and solve:

Minutes	X (90c in 3 min)	Y (100c in 2 min)	Total	= 560?
(B) 6	180	300	480	No

Step 2: Narrow your answers. Answer (B) is incorrect. The total is too low, so you need a higher number; therefore, only answer (A) can work.

Step 3: Pick! If you're confident in your reasoning, pick (A). If not, try answer (A) to confirm.

If you didn't notice that you needed a higher number, you'd try answer (D) next:

Minutes	X (90c in 3 min)	Y (100c in 2 min)	Total	= 560?
(B) 6	180	300	480	No
(D) 4	90 + 30 = 120	200	320	No

The answers are moving in the wrong direction—you're getting even further away from the desired 560! The answer definitely has to be greater than 6. Only answer choice (A) is greater.

When to Work Backwards

The two examples shown above possess a couple of characteristics in common that make working backwards a viable method.

First, the answer choices are numerical and they are what are called “nice” numbers. In the second problem, the answers were small integers. In the first one, the numbers were larger, but they were still integers and they all ended in zero. “Nice” numbers make working backwards easier.

Second, the question stems ask for a discrete number—in the first case, the total, and, in the second case, the number of minutes. <u>A problem that asks for something that you could label with a single variable (for example, *T* or *m*) is more likely to work well using this technique</u> than a problem that asks for something more complicated, such as the difference between two numbers.

In sum, look for “nice” numbers and a question that asks for a single variable. <u>When these characteristics exist, it may be easier to work backwards than forwards!</u>

How to Get Better at Working Backwards

First, practice the problems at the end of this chapter. Try each problem two times: once working backwards and once using the “textbook” method. (Time yourself separately for each attempt.)

When you’re done, ask yourself which way you prefer to solve *this* problem and why. The key to mastering strategies such as working backwards, and others, is developing an instinct for when to use them. On the real test, you won’t have time to try both methods; you’ll have to make a decision and go with it.

Learn *how* to make that decision while studying; then, the next time a new problem pops up in front of you that could be solved by working backwards, you'll be able to make a quick (and good!) decision.

One important note: at first, you may find yourself always choosing the textbook approach. You've practiced algebra for years, after all, and you've only been trying the work backwards technique for a short period of time. Keep practicing; you'll get better! Every high-scorer on the Quant section will tell you that this technique is one of the essential techniques for getting through Quant on time and with a high enough performance to reach a top score.

Try this problem:

> Boys and girls in a class are writing letters. There are twice as many girls as boys in the class, and each girl writes 3 more letters than each boy. If boys write 24 of the 90 total letters written by the class, how many letters does each boy write?
>
> (A) 3
> (B) 4
> (C) 5
> (D) 6
> (E) 8

Step 0: How do you know that you can work backwards on this problem?

> The answers are fairly nice. The question asks for a discrete variable (the number of letters written by each boy).

Step 1: Start with answer (B) or answer (D). Set up your chart and solve.

Letters per boy	# boys (24 letters)	# girls (2 × boys)	L per girl (+3)	Total letters	90?
(B) 4	6	12	7	(6)(4) + (12)(7)	No

Step 2: Narrow your answers: 24 + 84 is more than 90, so (B) can't be correct. Try (D) next.

Letters per boy	# boys (24 letters)	# girls (2 × boys)	L per girl (+3)	Total letters	90?
(B) 4	6	12	7	(6)(4) + (12)(7)	No
(D) 6	4	8	9	(6)(4) + (8)(9)	No

So 24 + 72 is still larger than 90, though not by much. Answer (D) is also incorrect.

Step 3: Because both (B) and (D) are too large, answer (E) must be correct.

If you're not confident in that reasoning, check the math.

Letters per boy	# boys (24 letters)	# girls (2 × boys)	L per girl (+3)	Total letters	90?
(E) 8	3	6	11	(8)(3) + (6)(11)	Yes!

By the time you get to the test, though, make sure you have enough practice with this method that you will be confident in your reasoning. Then, most of the time, you won't need to check more than two answers!

Here's an algebraic solution:

Call the number of boys b and the number of girls g. Call the number of letters for one boy L. Start translating the problem:
There are twice as many girls: $g = 2b$
If each boy writes L letters, then each girl writes $L + 3$ letters.
All of the boys, then, write a total of $bL = 24$ letters.
Together, the boys and girls write a total of $bL + g(L + 3)= 90$ letters.

How to put that all together? Start substituting. Try to reduce the number of variables:

$$\begin{aligned} bL + gL + 3g &= 90 && \text{substitute } g = 2b \\ bL + 2bL + 3(2b) &= 90 && \text{substitute } bL = 24 \\ 24 + 2(24) + 6b &= 90 \\ 6b &= 90 - 24 - 48 \\ 6b &= 18 \\ b &= 3 \end{aligned}$$

There are 3 boys, so they write $3L = 24$, or 8 letters each. The correct answer is **(E)**.

Both solution methods are valid. Which do you prefer?

With the second method, you have to be capable of thinking through a pretty tricky situation in order to set up the math correctly. If you find that straightforward, great! If not, then you may want to work backwards when you can on these kinds of problems.

When *Not* to Work Backwards

There are two scenarios in which working backwards can get messy. The first one is obvious: what if the numbers are really large or ugly? In that case, starting with those numbers doesn't sound like the best idea.

The second is a little more subtle. Take a look at this problem (careful—it's a bit different than the first version you saw!):

> Four brothers are splitting a sum of money between them. The first brother receives 50% of the total, the second receives 25% of the total, the third receives 20% of the total, and the fourth receives the remaining \$4. How much more does the first brother receive than the third brother?

(A) 4

(B) 16

(C) 20

(D) 24

(E) 36

The first version of this problem asked for a discrete variable: the total sum of money. This time, though, the problem asks for the difference between the amounts that two of the brothers receive. How could you do that backwards? Try it out.

$B_1 - B_3$	B#1 (50%)	B#2 (25%)	B#3 (20%)	B#4 (4)	Sum	$B_1 - B_3$	Match?
(B) 16	x?	?	$x - 16$?	4	?		

The value 16 represents the first brother's amount minus the third brother's amount. But how do you find the actual values for brother #1 and brother #3? Maybe you can just pick a number for brother #1 and subtract 16 for brother #2?

This isn't actually making your life any easier. You shouldn't need to pick random numbers; the math should work from the numbers that you were given in the first place. When the question stem asks for a combination of variables, such as $B_1 - B_3$, rather than one discrete variable, such as the total, then solving the normal way is likely a better bet than working backwards.

Why can't I start with answer (C)?

You can, actually. In fact, you might want to when answer (C) is a much nicer number than answers (B) or (D).

In general, though, there is one good reason to use (B) or (D) as the default starting point. If you're really curious what that reason is, read on; if not, feel free to skip this section.

Assume that you start with answer (B)—as opposed to answer (D)—and that the answers go in ascending order, from the smallest at (A) to the largest at (E).

You have a 20% chance that choice (B) will be the correct answer. If (B) is incorrect, then what? If you had started with answer (C), then you could only know whether to try one of the two higher answers or one of the two lower answers—so you'd have only a 20% chance of answering the problem after your first try, when (C) is correct.

If, on the other hand, you start with answer (B), and you realize you need a smaller number, then (A) has to be correct. In other words, you have a 40% chance of getting the right answer, even though you've tried only one answer choice so far!

This is really the only difference between starting with choice (B) and starting with choice (C). Once you try the second answer choice, the odds are all equivalent. Still, it's better to have a 40% chance that you can be done with the problem after the first try rather than just a 20% chance. Let's work through another problem:

> Train X is traveling at a constant speed of 30 miles per hour and Train Y is traveling at a constant speed of 40 miles per hour. If the two trains are traveling in the same direction along the same route but Train X is 25 miles ahead of Train Y, how many hours will it be until Train Y is 10 miles ahead of Train X?
>
> (A) 1.5
> (B) 2.0
> (C) 2.5
> (D) 3.0
> (E) 3.5

Step 0: How do you know that you can work backwards on this problem?

> The answers are fairly nice. The question asks for a discrete variable (the number of hours it takes Train *Y* to travel a certain distance).

Step 1: Start with answer (B) or answer (D). Set up your chart and solve:

Hours	Y (40 mph)	X (30 mph) + 25 miles	Where is Y?	Is Y 10 miles ahead?
(B) 2.0	80 miles	60 + 25 = 85 miles	5 miles behind	No

Step 2: Narrow your answers. Answer (B) is incorrect. Train Y is still behind Train X, so you need a higher number. Try (D) next:

Hours	Y (40 mph)	X (30 mph) + 25 miles	Where is Y?	Is Y 10 miles ahead?
(B) 2.0	80 miles	60 + 25 = 85 miles	5 miles behind	No
(D) 3.0	120 miles	90 + 25 = 115 miles	5 miles ahead	No

Thus, answer (D) is incorrect.

Step 3: Pick! Train Y has passed Train X, though, so you're moving in the right direction. Because Train Y is not yet 10 miles ahead, though, the answer must be larger. Only answer (E) is larger, so it must be correct.

Here's the math:

Hours	Y (40 mph)	X (30 mph) + 25 miles	Where is Y?	Is Y 10 miles ahead?
(E) 3.5	140 miles	105 + 25 = 130 miles	10 miles ahead	Yes!

Here's one algebraic solution:

The two trains are currently 25 miles apart, with X ahead of Y. The problem asks you to solve for the time when Y had moved 10 miles ahead of X. Therefore, Y has to catch up to X to erase that initial 25-mile deficit and then move an additional 10 miles beyond X, for a total of 35 extra miles.

For every hour that the two trains travel, Y goes 10 miles per hour faster (since it travels 40 miles per hour to X's 30 miles per hour). Plug these numbers into your *RTD* formula:

$$\frac{35 \text{ miles}}{10 \text{ miles per hour}} = 3.5 \text{ hours}$$

Here's what you might find in a textbook:

Plug the scenario into an RTD chart:

	R	T	D
Train X	30	t	D
Train Y	40	t	$D + 35$

Note that the time is the same for the two trains. In order for Train Y to catch up to Train X, it must cover an additional 25 miles. In order for Train Y to pull 10 miles ahead of Train X, it must cover an additional 25 + 10 = 35 miles.

Write two equations:

$30t = D$
$40t = D + 35$

Substitute and solve:

$$40t = 30t + 35$$
$$10t = 35$$
$$t = \frac{35}{10} = 3.5$$

All three solution methods are valid. Which do you prefer?

With the second and third methods, you have to be capable of thinking through a pretty tricky situation in order to set up the math correctly. If you find that straightforward, great! If not, then you may want to work backwards when you can on these kinds of problems.

Chapter 3
of
Word Problems

Rates & Work

In This Chapter…

Chapter 3

Rates & Work

Rate problems come in a variety of forms on the GMAT, but all are marked by three primary components: *rate*, *time*, and *distance* or *work*.

These three elements are related by the following equations:

Rate × **Time** = **Distance**
Rate × **Time** = **Work**

These equations can be abbreviated as $RT = D$ or as $RT = W$.

This chapter will discuss the ways in which the GMAT makes rate situations more complicated. Often, $RT = D$ problems will involve more than one person or vehicle traveling. Similarly, many $RT = W$ problems will involve more than one worker.

Let's get started with a review of some fundamental properties of rate problems.

Basic Motion: The RTD Chart

All basic motion problems involve three elements: rate, time, and distance.

Rate is expressed as a ratio of distance and time, with two corresponding units. Some examples of rates include: 30 miles per hour, 10 meters/second, 15 kilometers/day.

Time is expressed using a unit of time. Some examples of times include: 6

hours, 23 seconds, 5 months.

Distance is expressed using a unit of distance. Some examples of distances include: 18 miles, 20 meters, 100 kilometers.

You can make an “RTD chart” to solve a basic motion problem. Read the problem and fill in two of the variables.
Then use the *RT* = *D* formula to find the missing variable. For example:

> If a car is traveling at 30 miles per hour, how long does it take to travel 75 miles?

An RTD chart is shown to the right. Fill in your RTD chart with the given information. Then solve for the time:

	Rate (miles/hr)		*Time* (hr)		*Distance* (miles)
Car	30	×		=	75

30*t* = 75, or *t* = 2.5 hours

Matching Units in the RTD Chart

All the units in your RTD chart must match up with one another. The two units in the rate should match up with the unit of time and the unit of distance. For example:

> It takes an elevator 4 seconds to go up one floor. How many floors will the elevator rise in 2 minutes?

The rate is 1 floor every 4 seconds, or 1/4, which simplifies to 0.25 floors/second. Note: the rate is NOT 4 seconds per floor! This is an extremely frequent error. Always express rates as “distance over time,” not as “time over distance.”

The desired time is 2 minutes. The distance is unknown.

Watch out! There is a problem with this RTD chart on the right. The rate is expressed in floors per second, but the time is expressed in minutes. This will yield an incorrect answer.

	R (floors/sec)	×	T (min)	=	D (floors)
Elevator	0.25	×	2	=	?

To correct this table, change the time into seconds. To convert minutes to seconds, multiply 2 minutes by 60 seconds per minute, yielding 120 seconds, as shown in the chart on the right.

	R (floors/sec)	×	T (sec)	=	D (floors)
Elevator	0.25	×	120	=	?

Once the time has been converted from 2 minutes to 120 seconds, the time unit will match the rate unit, and you can solve for the distance using the $RT = D$ equation:

$$0.25(120) = d$$
$$d = 30 \text{ floors}$$

Thus, the elevator will go up 30 floors in 2 minutes.

Try another example:

> A train travels 90 kilometers/hr. How many hours does it take the train to travel 450,000 meters? (1 kilometer = 1,000 meters)

First, divide 450,000 meters by 1,000 to convert this distance to 450 km. By doing so, you match the distance unit (kilometers) with the rate unit (kilometers per hour). Set up an RTD chart like the one to the right.

	R (km/hr)	$\times$	T (hr)	$=$	D (km)
Train	90	$\times$	?	$=$	450

You can now solve for the time: $90t = 450$. Thus, t is equal to 5 hours. Note that this time is the “stopwatch” time: if you started a stopwatch at the start of the trip, what would the stopwatch read at the end of the trip? This is not what a clock on the wall would read, but if you take the *difference* of the start and end clock times (say, 1pm and 6pm), you will get the stopwatch time of 5 hours.

The RTD chart may seem like overkill for a relatively simple problem such as this one. In fact, for such problems, you can simply set up the equation $RT = D$ or $RT = W$ and then substitute. However, the RTD chart comes into its own when you have more complicated scenarios that contain more than one RTD relationship, as you'll see in the next section.

Multiple Rates

Some rate questions on the GMAT will involve *more than one trip or traveler*. To deal with this, you will need to deal with multiple $RT = D$ relationships. For example:

> Harvey runs a 30-mile course at a constant rate of 4 miles per hour. If Clyde runs the same track at a constant rate and completes the course in 90 fewer minutes, how fast did Clyde run?

An RTD chart for this question would have two rows, one for Harvey and one for Clyde, as shown below:

	R (miles/hr)	×	T (hr)	=	D (miles)
Harvey					
Clyde					

Pay attention to the relationships between these two equations. Try to use the minimum necessary number of variables.

For instance, both Harvey and Clyde ran the same course, so the distance they both ran was 30 miles. Additionally, you know Clyde ran for 90 fewer minutes. To make units match, you can convert 90 minutes to 1.5 hours. If Harvey ran t hours, then Clyde ran (t − 1.5) hours. Fill in this information on your chart:

	R (miles/hr)	×	T (hr)	=	D (miles)
Harvey	4		t		30
Clyde	?		$t - 1.5$		30

Now solve for t:

$$4t = 30$$
$$t = 7.5$$

If $t = 7.5$, then Clyde ran for 7.5 − 1.5 = 6 hours. You can now solve for Clyde's rate. Let r equal Clyde's rate:

$$r \times 6 = 30$$
$$r = 5$$

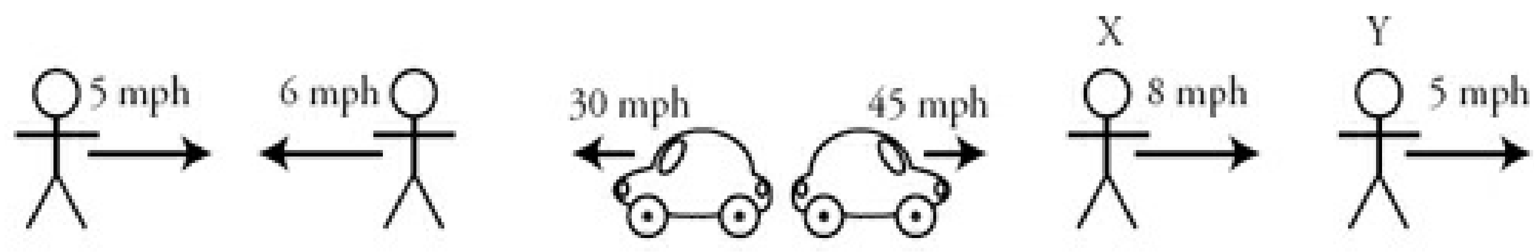
For questions that involve multiple rates, remember to set up multiple $RT = D$ equations and look for relationships between the equations. These relationships will help you reduce the number of variables you need and allow you to solve for the desired value.

Relative Rates

Relative rate problems are a subset of multiple rate problems. The defining aspect of relative rate problems is that two bodies are traveling *at the same time*. There are three possible scenarios:

1. The bodies move towards each other.
2. The bodies move away from each other.
3. The bodies move in the same direction on the same path.

These questions can be dangerous because they can take a long time to solve using the conventional multiple rates strategy (discussed in the last section). You can save valuable time and energy by creating a third $RT = D$ equation for the rate at which the distance between the bodies changes:



Two people decrease the distance between themselves at a rate of $5 + 6 = 11$ mph.

Two cars increase the distance between themselves at a rate of $30 + 45 = 75$ mph.

Persons X and Y decrease the distance between themselves at a rate of $8 - 5 = 3$ mph.

Try an example:

> Two people are 14 miles apart and begin walking towards each other. Person A walks 3 miles per hour, and Person B walks 4 miles per hour. How long will it take them to reach each other?

To answer this question using multiple rates, you would need to make two important inferences: the time that each person walks is exactly the same (t hours) and the total distance they walk is 14 miles. If one person walks d miles, the other walks $(14 - d)$ miles. The chart would look like this:

	R (miles/hr)	×	T (hr)	=	D (miles)
Person A	3		t		d
Person B	4		t		$14 - d$

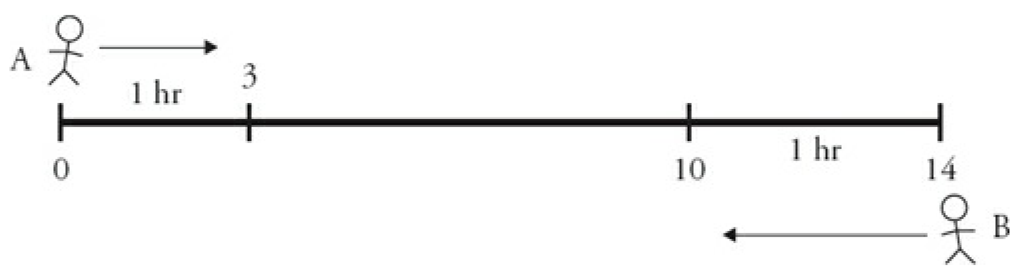
Alternatively, you can create an $RT = D$ equation for the rate at which they're getting closer to each other.

The rate at which they're getting closer to each other is $3 + 4 = 7$ miles per hour. In other words, <u>after every hour they walk, they are 7 miles closer to each other</u>. Now you can create one $RT = D$ equation:

	R (miles/hr)	×	T (hr)	=	D (miles)
$A + B$	7		t		14

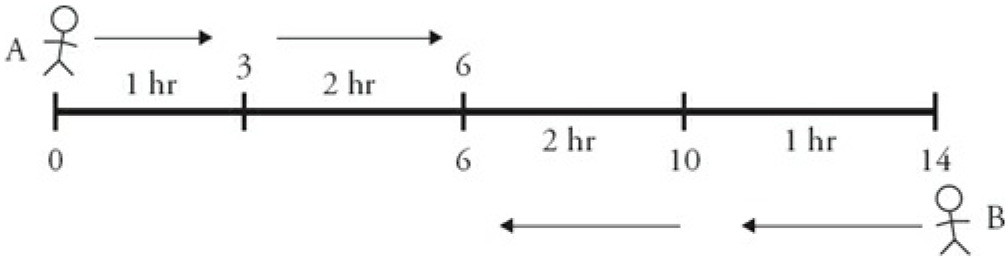
$$7t = 14$$
$$t = 2$$

Alternatively, draw it out!



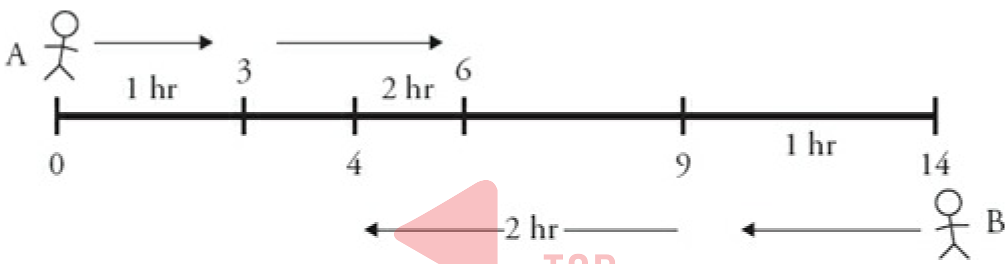
The distance is 14 miles; label one end 0 and the other end 14. After 1 hour, A has traveled 3 miles. B has traveled 4 (so subtract B's distance from 14). Have they met yet?

No. Keep going:

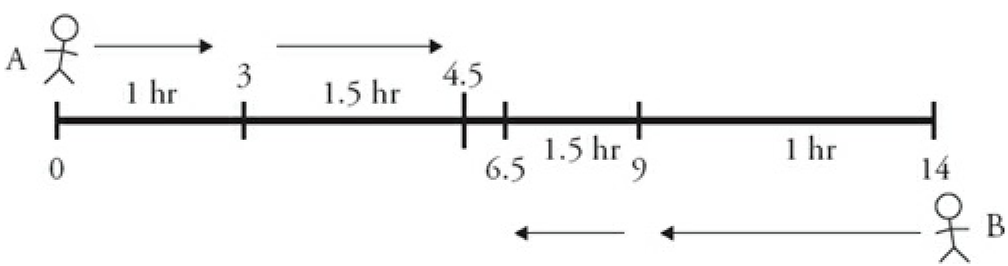


After 2 hours, A has traveled another 3 miles, bringing him up to 6. B has traveled another 4 miles, bringing her all the way to…6! The two people meet at the 2-hour mark.

Note that this technique will still work even when the answer isn't an integer. Let's say that person B is walking at a rate of 5 mph:



They haven't passed at 1 hour, but they have at 2 hours. You would typically be able to eliminate two or three multiple-choice answers at this stage. Next, try 1.5 hours:



They haven't passed yet. This is often enough for you to narrow the choices down to a single answer, though it depends on the exact mix of answer choices.

Average Rate: Find the Total Time

Consider the following problem:

> If Lucy walks to work at a rate of 4 miles per hour, and she walks home by the same route at a rate of 6 miles per hour, what is Lucy's average walking rate for the round trip?

It is very tempting to find an average rate as you would find any other average: add and divide. Thus, you might say that Lucy's average rate is 5 miles per hour (4 + 6 = 10 and 10 ÷ 2 = 5). However, this is incorrect!

If an object moves the same distance twice, but at different rates, then *the average rate will NEVER be the average of the two rates given for the two legs of the journey*. In fact, because the object spends more time traveling at the slower rate, *the average rate will ALWAYS be closer to the slower of the two rates than to the faster*. On DS problems, that knowledge may be enough to answer the question.

In order to find the average rate, first find the *total* combined time for the trips and the *total* combined distance for the trips. Use this formula:

$$\text{Average speed} = \frac{\text{Total Distance}}{\text{Total Time}}$$

The problem never establishes a specific distance. Because she walks the same route to work and back home, the average does not depend upon the specific distance. If Lucy walks 1 mile or 15, the average will be the same.

Pick your own number for the distance. Since 12 is a multiple of the two rates in the problem, 4 and 6, 12 is an ideal choice. (You'll learn more about this technique, choose smart numbers, in the next chapter.)

Set up a multiple RTD chart:

	Rate (miles/hr)	×	*Time* (hr)	=	*Distance* (miles)
Going	4	×	t_g	=	12
Return	6	×	t_r	=	12
Total	?	×		=	24

The times can be found using the RTD equation.

For the GOING trip, $4t_g = 12$, so $t_g = 3$ hours.

For the RETURN trip, $6t_r = 12$, so $t_r = 2$ hours.

Thus, the combined time for the two trips is 5 hours.

	Rate (miles/hr)	×	*Time* (hr)	=	*Distance* (miles)
Going	4	×	**3**	=	12
Return	6	×	**2**	=	12
Total	?	×	5	=	24

Now that you have the total time and the total distance, find the average rate using the RTD formula:

$r(5) = 24$
$r = 4.8$ miles per hour

You can test different numbers for the distance (try 24 or 36) to prove that you will get the same answer, regardless of the number you choose for the distance.

Basic Work Problems

Work problems are just another type of rate problem. Instead of distances, however, these questions are concerned with the amount of "work" done.

Work: Work takes the place of distance. Instead of $RT = D$, use the equation $RT = W$. The amount of work done is often a number of jobs completed or a number of items produced.

Time: This is the time spent working.

Rate: In work problems, the rate expresses the amount of work done in a given amount of time. Rearrange the equation to isolate the rate:

$$R = \frac{W}{T}$$

Be sure to express a rate as work per time (*W*/*T*), NOT time per work (*T*/*W*). For example, if a machine produces pencils at a constant rate of 120 pencils every 30 seconds, the rate at which the machine works is $\frac{120 \text{ pencils}}{30 \text{ seconds}} = 4$ pencils/second.

Many work problems will require you to calculate a rate. Try the following problem:

> Martha can paint $\frac{3}{7}$ of a room in $4\frac{1}{2}$ hours. If Martha finishes painting the room at the same rate, how long will it have taken Martha to paint the entire room?
>
> (A) $8\frac{1}{3}$ hours (B) 9 hours (C) $9\frac{5}{7}$ hours (D) $10\frac{1}{2}$ hours (E) $11\frac{1}{7}$ hours

Your first step in this problem is to calculate the rate at which Martha paints the room. Set up an RTW chart:

	R (rooms/hr)	×	*T* (hr)	=	*W* (rooms)
Martha	r		$\frac{9}{2}$		$\frac{3}{7}$

Now you can solve for the rate:

$$r \times \frac{9}{2} = \frac{3}{7}$$

$$r = \frac{3}{7} \times \frac{2}{9} = \frac{2}{21}$$

The division would be messy, so leave it as a fraction. Martha paints $\frac{2}{21}$ of the room every hour. Now you have what you need to answer the question. You can say that painting the whole room is the same as doing 1 unit of work. Set up another RTW chart:

	R (rooms/hr)	×	T (hr)	=	W (rooms)
Martha	$\frac{2}{21}$		t		1

$$\left(\frac{2}{21}\right)t = 1$$

$$t = \frac{21}{2} = 10\frac{1}{2}$$

The correct answer is **(D)**. Notice that the rate and the time in this case were reciprocals of each other. This will always be true when the amount of work done is 1 unit (because reciprocals are defined as having a product of 1).

Working Together: Add the Rates

More often than not, work problems will involve more than one worker. When two or more workers are performing the same task, their rates can be added together. For instance, if Machine A can make 5 boxes in an hour, and Machine B can make 12 boxes in an hour, then working together the two machines can make 5 + 12 = 17 boxes per hour.

Likewise, if Lucas can complete $\frac{1}{3}$ of a task in an hour and Serena can complete $\frac{1}{2}$ of that task in an hour, then working together they can complete $\frac{1}{3}+\frac{1}{2}=\frac{5}{6}$ of the task every hour.

If, on the other hand, one worker is undoing the work of the other, subtract the rates. For instance, if one hose is filling a pool at a rate of 3 gallons per minute, and another hose is draining the pool at a rate of 1 gallon per minute, the pool is being filled at a rate of 3 − 1 = 2 gallons per minute.

Try the following problem:

> Machine A fills soda bottles at a constant rate of 60 bottles every 12 minutes and Machine B fills soda bottles at a constant rate of 120 bottles every 8 minutes. How many bottles can both machines working together at their respective rates fill in 25 minutes?

To answer these questions quickly and accurately, it is a good idea to begin by expressing rates in equivalent units:

$$\text{Rate}_{\text{Machine A}} = \frac{60 \text{ bottles}}{12 \text{ minutes}} = 5 \text{ bottles/minute}$$

$$\text{Rate}_{\text{Machine B}} = \frac{120 \text{ bottles}}{8 \text{ minutes}} = 15 \text{ bottles/minute}$$

Working together, they fill 5 + 15 = 20 bottles every minute. Now you can fill out an RTW chart. Let *b* be the number of bottles filled:

	R (bottles/min)	×	T (min)	=	W (bottles)
$A + B$	20		25		b

> $b = 20 \times 25 = 500$ bottles

Remember that, even as work problems become more complex, there are still only a few relevant relationships: $RT = W$ and $R_A + R_B = R_{A+B}$.

Try another example:

> Alejandro, working alone, can build a doghouse in 4 hours. Betty can build the same doghouse in 3 hours. If Betty and Carmelo, working together, can build the doghouse twice as fast as Alejandro can alone, how long would it take Carmelo, working alone, to build the doghouse?

Begin by solving for the rate that each person works. Let *c* represent the number of hours it takes Carmelo to build the doghouse.

Alejandro can build $\frac{1}{4}$ of the doghouse every hour, Betty can build $\frac{1}{3}$ of the doghouse every hour, and Carmelo can build $\frac{1}{c}$ of the doghouse every hour.

The problem states that Betty and Carmelo, working together, can build the doghouse twice as fast as Alejandro. In other words, their rate is twice Alejandro's rate:

$$\text{Rate}_B + \text{Rate}_C = 2\left(\text{Rate}_A\right)$$

$$\frac{1}{3} + \frac{1}{c} = 2\left(\frac{1}{4}\right)$$

$$\frac{1}{c} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$c = 6$$

It takes Carmelo 6 hours working by himself to build the doghouse.

When dealing with multiple rates, be sure to express rates in equivalent units. When the the work involves completing a task, remember to treat completing the task as doing one “unit” of work. Once you know the rates of every worker, add the rates of workers who work together to complete the same task.

Problem Set

Solve the following problems using the strategies you have learned in this section. Use RTD or RTW charts as appropriate to organize information.

1. An empty bucket being filled with paint at a constant rate takes 6 minutes to be filled to $\frac{7}{10}$ of its capacity. How much more time will it take to fill the bucket to full capacity?

2. Two hoses are pouring water into an empty pool. Hose 1 alone would fill up the pool in 6 hours. Hose 2 alone would fill up the pool in 4 hours. How long would it take for both hoses to fill up two-thirds of the pool?

3. Did it take a certain ship less than 3 hours to travel 9 kilometers? (1 kilometer = 1,000 meters)

   (1) The ship's average speed over the 9 kilometers was greater than 55 meters per minute.
   (2) The ship's average speed over the 9 kilometers was less than 60 meters per minute.

4. Twelve identical machines, running continuously at the same constant rate, take 8 days to complete a shipment. How many additional machines, each running at the same constant rate, would be needed to reduce the time required to complete a shipment by 2 days?

   (A) 2 (B) 3 (C) 4 (D) 6 (E) 9

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

5. Al and Barb shared the driving on a certain trip. What fraction of the total distance did Al drive?

(1) Al drove for $\frac{3}{4}$ as much time as Barb did.

(2) Al's average driving speed for the entire trip was $\frac{4}{5}$ of Barb's average driving speed for the trip.

6. Nicky and Cristina are running a race. Since Cristina is faster than Nicky, she gives him a 36-meter head start. If Cristina runs at a pace of 5 meters per second and Nicky runs at a pace of only 3 meters per second, how many seconds will Nicky have run before Cristina catches up to him?

(A) 15 seconds (B) 18 seconds (C) 25 seconds (D) 30 seconds (E) 45 seconds

7. Mary and Nancy can each perform a certain task in *m* and *n* hours, respectively. Is $m < n$?

(1) Twice the time it would take both Mary and Nancy to perform the task together, each working at their respective constant rates, is greater than *m*.
(2) Twice the time it would take both Mary and Nancy to perform the task together, each working at their respective constant rates, is less than *n*.

Solutions

1. **$2\frac{4}{7}$ minutes:** Use the *RT* = *W* equation to solve for the rate, with *t* = 6 minutes and *w* = $\frac{7}{10}$:

$$r(6 \text{ minutes}) = \frac{7}{10}$$

$$r = \frac{7}{10} \div 6 = \frac{7}{60} \text{ buckets per minute.}$$

R (bucket/min)	×	*T* (min)	=	*W* (bucket)
r	×	6	=	7/10

Next, substitute this rate into the equation again, using $\frac{3}{10}$ for *w* (the remaining work to be done):

$$\left(\frac{7}{60}\right)t = \frac{3}{10}$$

$$t = \frac{3}{1\cancel{0}} \times \frac{6\cancel{0}}{7} = \frac{18}{7} = 2\tfrac{4}{7} \text{ minutes}$$

R (bucket/min)	×	*T* (min)	=	*W* (bucket)
7/60	×	*t*	=	3/10

2. **$1\frac{3}{5}$ hours:** If Hose 1 can fill the pool in 6 hours, its rate is $\frac{1}{6}$ "pool per hour," or the fraction of the job it can do in one hour. Likewise, if Hose 2 can fill the pool in 4 hours, its rate is $\frac{1}{4}$ pool per hour. Therefore, the combined rate is $\frac{5}{12}$ pool per hour $\left(\frac{1}{4} + \frac{1}{6} = \frac{5}{12}\right)$:

$$RT = W$$

$$(5/12)t = 2/3$$

$$t = \left(\frac{2}{3}\right)\left(\frac{12}{5}\right) = \frac{8}{5} = 1\frac{3}{5} \text{ hours}$$

R (pool/hr)	×	*T* (hr)	=	*W* (pool)
5/12	×	*t*	=	2/3

3. **(A):** Notice that the statements provide rates in meters per minute. A good first step here is to figure out how fast the ship would have to travel to cover

9 kilometers in 3 hours. Create an RTD chart, and convert kilometers to meters and hours to minutes:

	R (meters/min)	×	T (min)	=	D (meters)
	r		180		9,000

$$180r = 9{,}000$$
$$r = 50$$

The question asks whether the ship traveled 9 kilometers in *less than* 3 hours. The ship must travel faster than 50 meters/min to make it in less than 3 hours. Therefore, the question is really asking, is $r > 50$?

(1): SUFFICIENT: If the average speed of the ship was greater than 55 meters per minute, then $r > 55$. Thus, r is definitely greater than 50.

(2): INSUFFICIENT: If the average speed of the ship was less than 60 meters per minute, then $r < 60$. This is not enough information to guarantee that $r > 50$.

4. **(C) 4 additional machines:** Let the work rate of 1 machine be r. Then the work rate of 12 machines is $12r$, and you can set up an RTW chart:

	R	×	T	=	W
Original	$12r$		8		$96r$

The shipment work is then $96r$. To figure out how many machines are needed to complete this work in $8 - 2 = 6$ days, set up another row and solve for the unknown rate:

	R	×	T	=	W
Original	$12r$		8		$96r$
New	$\boxed{}$		6		$96r$

Therefore, there are $\frac{96r}{6} = 16r$ machines in total, or $16 - 12 = 4$ additional machines.

5. **(C):** You can rephrase the question as follows: What is the ratio of Al's driving distance to the entire distance driven? Alternatively, since the entire distance is the sum of only Al's distance and Barb's distance, you can simply find the ratio of Al's distance to Barb's distance:

(1): INSUFFICIENT. You have no rate information, and so you have no definitive distance relationships:

	R	$\times$	T	$=$	W
Al			$(3/4)t$		
Barb			t		
Total					

(2): INSUFFICIENT. As with Statement (1), you have no definitive distance relationships:

	R	$\times$	T	$=$	W
Al	$(4/5)r$				
Barb	r				
Total					

(1) & (2) TOGETHER: SUFFICIENT. Set up a chart like the one below:

	R	$\times$	T	$=$	W
Al	$(4/5)r$		$(3/4)t$		$(3/5)rt$
Barb	r		t		rt
Total					$(8/5)rt$

Al drove $\frac{\frac{3}{5}rt}{\frac{8}{5}rt} = \frac{3}{8}$ of the distance. (Alternatively, he drove $\frac{3}{5}$ as much as

Barb did, meaning that he drove $\frac{3}{8}$ of the trip.) Notice that you do not need the absolute time, nor the rate, of either driver's portion of the trip.

6. **(D) 30 seconds:** Initially, Nicky runs 36 meters at a rate of 3 meters per second. Therefore, Nicky runs for 36/3 = 12 seconds before Christina starts running.

Save time on this problem by dealing with the rate at which the distance between Cristina and Nicky changes. Nicky is originally 36 meters ahead of Cristina. If Nicky runs at a rate of 3 meters per second and Cristina runs at a rate of 5 meters per second, then the distance between the two runners is shrinking at a rate of 5 − 3 = 2 meters per second.

You can now figure out how long it will take for Cristina to catch Nicky using a single *RT* = *D* equation. The rate at which the distance between the two runners is shrinking is 2 meters per second, and the distance is 36 meters (because that's how far apart Nicky and Cristina are):

	R (meters/sec)	×	*T* (sec)	=	*D* (meters)
	2		*t*		36

$$2t = 36$$
$$t = 18$$

Nicky's total time is 30 seconds: 12 seconds + 18 seconds = 30 seconds.

7. **(D):** First, set up an *RTW* chart:

	R	×	*T*	=	*W*
Mary	1/*m*		*m*		1
Nancy	1/*n*		*n*		1

Recall that the question is “ < *n*?”

(1): SUFFICIENT. Find out how much time it would take for the task to be performed with both Mary and Nancy working:

	R	$\times$	T	$=$	W
Mary	$1/m$		m		1
Nancy	$1/n$		n		1
Total	$1/m + 1/n$		t		1

$$\left(\frac{1}{m}+\frac{1}{n}\right)t = 1$$

$$\left(\frac{m+n}{mn}\right)t = 1$$

$$t = \frac{mn}{m+n}$$

Now, set up the inequality described in the statement (that is, twice this time is greater than m):

$$2t > m$$

$$2\left(\frac{mn}{m+n}\right) > m$$

$$2mn > mn + m^2$$ You can cross multiply by $m+n$ because $m+n$ is positive.

$$mn > m^2$$

$$n > m$$ You can divide by m because m is positive.

Alternatively, you can rearrange the original inequality thus:

$$t > \frac{m}{2}$$

If both Mary and Nancy worked at Mary's rate, then together, they would complete the task in $\frac{m}{2}$ hours. Since the actual time is longer, Nancy must work more slowly than Mary, and thus $n > m$.

(2): SUFFICIENT. You can reuse the computation of t, the time needed for the task to be jointly performed:

$$2t < n$$

$$2\left(\frac{mn}{m+n}\right) < n$$

$$2mn < nm + n^2$$ Again, you can cross multiply by $m + n$ because $m + n$ is positive.

$$mn < n^2$$

$$m < n$$ You can divide by n because n is positive.

Alternatively, you can rearrange the original inequality thus:

$$t < \frac{n}{2}$$

If both Mary and Nancy worked at Nancy's rate, then together, they would complete the task in $\frac{n}{2}$ hours. Since the actual time is shorter, Mary must work faster than Nancy, and thus $m < n$.

Chapter 4
of
Word Problems

Strategy: Choose Smart Numbers

In This Chapter…

Chapter 4

Strategy: Choose Smart Numbers

Some algebra problems—problems that involve variables—can be turned into arithmetic problems, instead. You're better at arithmetic than algebra (everybody is!), so turning an annoying variable-based problem into one that uses real numbers can be a lifesaver on the GMAT.

Which of the below two problems is easier for you to solve?

How many miles can a car going x miles per hour travel in y hours?

(A) $\frac{x}{y}$ (B) $\frac{y}{x}$ (C) xy

How many miles can a car going 40 miles per hour travel in 3 hours?

(A) $\frac{40}{3}$ (B) $\frac{3}{40}$ (C) 120

You may think that the algebraic version is not difficult at all, but no matter how easy you think it is, it's still easier to work with real numbers.

If you compare the two problems, you'll see that the setup is identical—and this feature is at the heart of how you can turn algebra into arithmetic.

How Do Smart Numbers Work?

Here's how to solve the algebra version of the above problem using smart numbers:

Step 1: Choose smart numbers to replace the variables.

The problem has two variables, x and y. Are the two numbers tied to each other in some way? That is, if you choose a number for one, will that determine the number for the other?

In this case, no. The two variables are completely separate, so you can choose a smart number for each one. Keep these three factors in mind when choosing your smart numbers:

1. If you are picking for more than one variable, pick different numbers for each one. If possible, pick numbers with different characteristics (e.g., one even and one odd).
2. Follow any constraints given in the problem. In this problem, for example, neither *x* nor *y* can be negative; it's not possible to drive –3 miles per hour or to travel for –2 hours.
3. Choose numbers that work easily in the problem. They do not have to be realistic in the real world. For example, on this problem, you might say that the car is driving 2 miles per hour for 3 hours. Nobody's going to drive that way in the real world, but it doesn't matter!

On your scrap paper, write:

$$\boxed{\begin{array}{c} x = 2 \\ y = 3 \end{array}}$$

Always write down what you choose for your variables and box it off; you'll be coming back to these numbers repeatedly.

Step 2: Solve the problem using your chosen smart numbers.

Wherever the problem says *x*, it will now say 2 (since you picked *x* = 2). Wherever the problem says *y*, it will now say 3 (since you picked *y* = 3). Here's the rewritten problem:

How many miles can a car going 2 miles per hour travel in 3 hours?

If you drive 2 miles per hour for 3 hours, then you'll travel 2 × 3 = 6 hours. This is your answer; draw a circle around the answer on your scrap paper, as shown below.

6

Step 3: Find a match in the answers.

The same rules apply to the answers: replace all of the *x* variables with 2 and all of the *y* variables with 3. Check the answers:

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) 6

Only answer (C) equals 6; it is the correct answer.

When and How to Use Smart Numbers

It's crucial to know when you're allowed to use this technique. It's also crucial to know how *you* are going to decide whether to use textbook math or to choose smart numbers; you will typically have time to try just one of the two techniques during your two minutes on the problem.

The choose smart numbers technique can be used any time a problem contains only *unspecified* values. The easiest example of such a problem is one that contains variables throughout; it does not provide real numbers for those variables and it uses those same variables in the answer choices. The problem might also use only percentages or fractions; again, it does not provide real numbers, and the answers consist of percentages or fractions. Whenever a problem has these characteristics, you can choose your own smart numbers to turn the problem into arithmetic.

There is some cost to doing so: it can take extra time compared to the “pure” textbook solution. As a result, the technique is most useful when the problem is a hard one for you. If you find the abstract math involved to be very easy, then you may not want to take the time to transform the problem into arithmetic. As the math gets more complicated, however, the arithmetic form becomes comparatively easier and faster to use.

Try this problem. Solve it twice—once using textbook math and once using smart numbers:

A train travels at a constant rate. If the train takes 13 minutes to travel *m* kilometers, how long will the train take to travel *n* kilometers?

(A) $\frac{13m}{n}$

(B) $\frac{13n}{m}$

(C) $13mn$

(D) $\frac{n}{13}$

(E) $\frac{m}{13}$

First, how do you know that you can choose smart numbers on this problem? The problem actually does contain a real number: 13.

Think about what's going on. The train might take 13 minutes to travel 1 kilometer or 13 minutes to travel 62 kilometers. It could really be any distance; the problem doesn't depend on the exact number of kilometers. Another clue is the fact that the variables show up again in the answer choices. Put the two pieces together (unspecified amounts for which you could imagine many options and variables in the answer choices) and you know you can use smart numbers.

Step 1: Choose smart numbers to replace the variables.

The number 2 is often a great default number to pick when using the smart numbers technique.

Step 2: Solve the problem using your chosen smart numbers.

In this case, though, 2 is not such a great number. Why?

Sometimes, you have to solve a little bit to figure out what would be a good number to pick. Look at what you'll have to do with that number: the train takes 13 minutes to travel 2 kilometers, so it's going 2 km/13 min. Hmm. Can you think of a number that will make the math less annoying?

Try m = 26. If the train travels 26 kilometers in 13 minutes, then it is traveling $\frac{26 \text{ kilometers}}{13 \text{ minutes}}$ = 2 kilometers per minute.

Next, the problem asks how long the train will take to travel n kilometers. What number would work well for n? Try n = 4. If the train travels 4 kilometers and it is traveling 2 km/min, then it takes the train 2 minutes to travel 4 kilometers.

$$\boxed{\begin{array}{c} m = 26 \text{ km} \\ n = 4 \text{ km} \end{array}}$$

$$\text{min} = 2$$

Step 3: Find a match in the answers. Your goal is to find a match for 2. If you can tell that a certain answer will *not* equal 2, you can cross it off without calculating exactly what it does equal.

(A) $\frac{13m}{n} = \frac{13(26)}{4} =$ too big. Eliminate.

(B) $\frac{13n}{m} = \frac{13(4)}{26} =$ maybe. Simplify. $\frac{13(4)}{26} = \frac{4}{2} = 2.$ Match!

(C) $13mn$ = way too big. Eliminate.

(D) $\frac{n}{13} = \frac{4}{13}$ Eliminate.

(E) $\frac{m}{13} = \frac{26}{13} = 2.$ Wait a second—this one matches, too!

If your smart numbers aren't smart enough, then you could find yourself with two right answers. (Don't worry, this is rare, but we want you to know what to do if it happens.)

Now what? While the clock is still ticking, either guess between (B) and (E) or, if you have time, try a different set of numbers in the problem.

Afterwards, learn a valuable lesson about how to choose the best smart numbers.

Here's how the problem went:

> (choose)
> m = 26 km
> (calculate)
> Therefore, the train is traveling at a rate of 2 km/min.
> (choose)
> n = 4 km
> (calculate)
> Therefore, the train goes 4 km in 2 minutes.

Here's the flaw in the choice of numbers: the rate of the train, 2, matches the final answer, also 2. In general, avoid picking numbers that will give you the same numerical answers at different points in the problem; in both of your calculations, the result was 2!

To fix this as efficiently as possible, keep the first part of the calculation as is. For the second part, though, go back and choose something for n that will not again give you 2. For example:

(choose)	m = 26 km
(calculate)	Therefore, the train is traveling at a rate of 2 km/min.
(choose)	n = 6 km
(calculate)	Therefore, the train goes 6 km in 3 minutes.

Now try just those two answer choices again:

(A)

(B) $\frac{13n}{m} = \frac{13(6)}{26} =$ maybe. Simplify. $\frac{13(6)}{26} = \frac{6}{2} = 3$. Match!

(C)

(D)

(E) $\frac{m}{13} = \frac{26}{13} = 2$. Eliminate.

Now, only one answer matches. The correct answer is **(B)**.

Next time you're choosing numbers, you'll know to avoid picking values that return the same numerical outcome as another part of the problem. If you do accidentally find yourself in this situation and you're most of the way through the math, go ahead and try the answers; there may still be only one answer that works. If two answers work and you have the time, then go back, change one of your numbers, and do the math again. If you don't have time, just pick one of the two answers that did work.

It's also a good idea to avoid picking 0 or 1 or a number that was used elsewhere in the problem. For example, it wouldn't be a good idea to choose 13 on this problem.

Let's summarize the process:

Step 0: Recognize that you can choose smart numbers.
The problem talks about some values but doesn't provide real numbers for those values. Rather, it uses variables or only refers to fractions or percentages. The answer choices consist of variable expressions, fractions, or percentages.

Step 1: Choose smart numbers to replace the variables.
Follow all constraints given in the problem. If the problem says that x is odd, pick an odd number for x. If the problem says that $x + y = z$, then note that once you pick for x and y, you have to calculate z. Don't pick your own random number for z!

Avoid choosing 0 or 1. Avoid choosing numbers that show up elsewhere in the problem. If you have to choose more than one number, make sure you choose different numbers.

After considering all of the above, choose numbers that make the problem easier to tackle!

Step 2: Solve the problem using your chosen smart numbers.

Wherever the problem used to have variables or unknowns, it now contains the real numbers that you've chosen. Solve the problem arithmetically and find your target answer.

Step 3: Find a match in the answers.

Plug your smart numbers into the answer choices and look for the choice that matches your target. If, at any point, you can tell that a particular answer will *not* match your target, stop calculating that answer. Cross it off and move to the next answer.

Smart Numbers and Percentages or Fractions

As discussed earlier, percentages or fractions in a problem can also trigger you to use the smart numbers technique.

Try this problem:

The price of a certain computer is increased by 10%, and then the new price is increased by an additional 5%. The new price is what percentage of the original price?

(A) 120%
(B) 116.5%
(C) 115%
(D) 112.5%
(E) 110%

How do you know you can use smart numbers? The problem never gives a real price for the computer; the entire problem (including the answers) is in terms of percentages only.

Step 1: Choose smart numbers.

When working with percentages, 100 is a great number to choose. If the problem already uses 100 or 100% somewhere, then you may want to choose a different starting number. In this case, there are no warning signs to avoid 100, so say that the computer's initial price was $100.

Step 2: Solve.

First, the computer's price increased by 10%, so the new price is $100 + $10 = $110. Next, the *new* price is increased by a further 5%, so the price becomes $110 + (0.05)($110) = $110 + $5.50 = $115.50.

This is the beauty of choosing 100 as your starting number. The question asks for the new price as a percentage of the original price. The new price is $115.50 and the original price is $100. “Percentage” literally means “per 100.” In the same way that 80/100 equals 80%, 115.50/100 equals 115.5%. In other words, you can ignore the denominator; your final number already represents the desired percentage.

Step 3: Find a match.

(A) 120%
(B) 115.5% → match
(C) 115%
(D) 112.5%
(E) 110%

The correct answer is **(B)**.

As you can see, this technique is even better when working with percentages (or fractions) because you don't have that final step of replacing variables in the answer choices. When you're done with your calculations, you're done all around!

Let's say that, on the last problem, you weren't able to choose \$100 for some reason. Instead, you chose an initial price of \$20. How would the problem work? Try it out before continuing to read.

If the initial price is \$20, then the first increase of 10% would bring the price to \$20 + \$2 = \$22. The next increase of 5% would bring the price to \$22 + (0.05)(\$22) = \$22 + \$1.1 = \$23.10.

Tip: To calculate 5% quickly, first find 10% of the desired number, then halve the number. For example, to find 5% of 22, first find 10%: 2.2. Next, halve that number: $\frac{2.2}{2}$ = 1.1.

Now what? \$23.10 isn't in the answers. Remember that you didn't start with 100! The new price as a percentage of the original is $\frac{23.10}{20}$. How do you turn that into a percentage?

Here's how. First, remember that percentage means "per 100." Manipulate the fraction until you get 100 on the bottom:

$$\frac{23.10}{20} \times \frac{5}{5} = \frac{115.5}{100}$$

The answer is 115.5%.

That last calculation is annoying—you don't want to do it unless you have to. Therefore, if you can pick 100 on a percentage problem, do so.

How to Get Better at Smart Numbers

First, practice the problems at the end of this chapter. Try each problem two times: once using smart numbers and once using the "textbook" method. (Time yourself separately for each attempt.)

When you're done, ask yourself which way you prefer to solve *this* problem and why. Remember that, on the real test, you won't have time to try both methods; you'll have to make a decision and go with it. Learn *how* to make that strategy decision while studying; then, the next time a new problem pops up in front of you that could be solved by choosing smart numbers, you'll be able to make a quick (and good!) decision.

One important note: whenever you learn a new strategy, at first you may find yourself always choosing the textbook approach. You've practiced algebra for years, after all, and you've only been using the smart numbers technique for a short period of time. Keep practicing; you'll get better! Every high-scorer on the Quant section will tell you that choosing smart numbers is invaluable to getting through Quant on time and with a high enough performance to reach a top score.

When NOT to Use Smart Numbers

There are certain scenarios in which a problem contains some of the smart numbers characteristics but not all.

For example, why can't you use smart numbers on this problem from the work backwards strategy chapter?

> Four brothers are splitting a sum of money between them. The first brother receives 50% of the total, the second receives 25% of the total, the third receives 20% of the total, and the fourth receives the remaining $4. How many dollars are the four brothers splitting?
>
> (A) $50
> (B) $60
> (C) $75
> (D) $80
> (E) $100

The problem talks about a sum of money but, at first, tells you nothing concrete about this sum of money. Towards the end, though, it does give you one real value: $4. Because the “remaining” percentage has to equal $4 exactly, this problem has just one numerical answer. You can't pick any starting point that you want. (The answer to the above problem is (D), by the way. You can find the solution in Chapter 2, Strategy: Work Backwards.)

Chapter 5

of

Word Problems

Overlapping Sets

In This Chapter…

Chapter 5

Overlapping Sets

Translation problems that involve two or more given sets of data that partially intersect with each other are termed overlapping sets. For example:

> Of 30 integers, 15 are in set A, 22 are in set B, and 8 are in both sets A and B. How many of the integers are in NEITHER set A nor set B?

This problem involves two sets, A and B. The two sets overlap because some of the numbers are in both sets. Thus, these two sets can actually be divided into four categories:

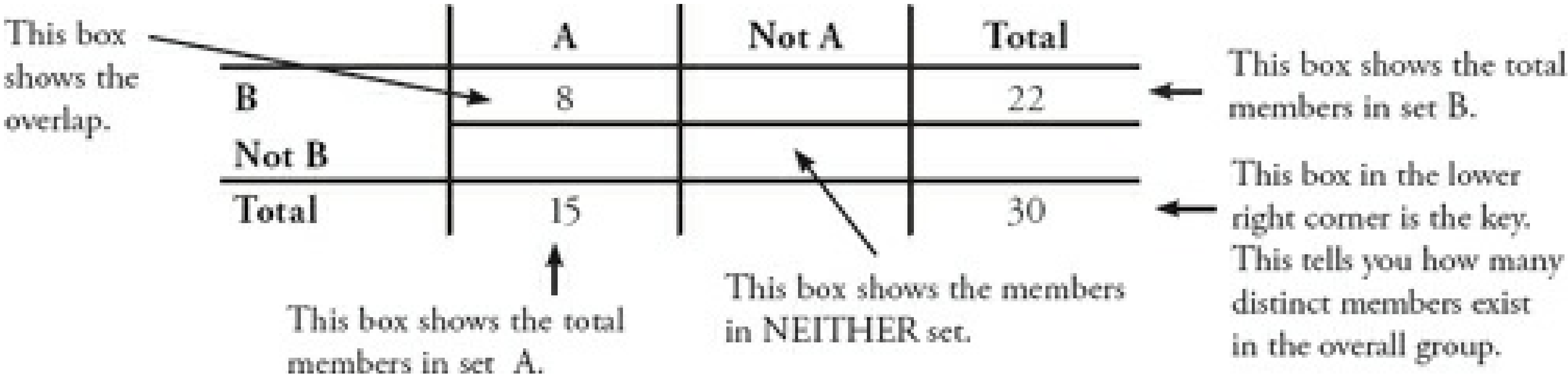
1. Numbers in set A
2. Numbers in set B
3. Numbers in both A and B
4. Numbers in neither A nor B

Solving double-set GMAT problems, such as in the example above, involves finding values for these four categories.

The Double-Set Matrix

For GMAT problems involving only *two* categorizations or decisions, the most efficient tool is the **double-set matrix**. Here's how to set one up, using the example from above:

> Of 30 integers, 15 are in set A, 22 are in set B, and 8 are in both set A and B. How many of the integers are in NEITHER set A nor set B ?

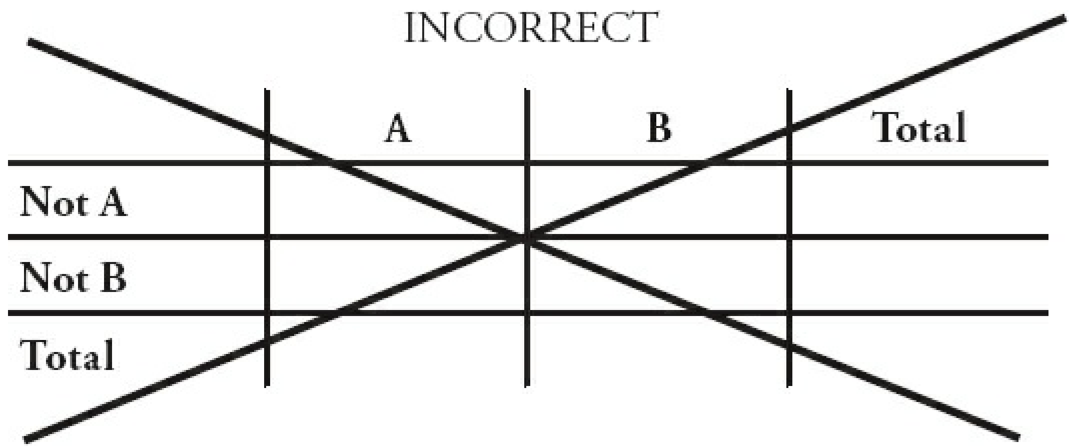


Once the information given in the problem has been filled in, as in the chart to the right, complete the chart, using the totals to guide you. Each row and each column sum to a total value.

	A	Not A	Total
B	8	**14**	22
Not B	**7**	**1**	**8**
Total	15	**15**	30

The question asks for the number of integers that are in *neither* set. Look at the chart to find the number of integers that are NOT A and NOT B; the answer is 1.

When you construct a double-set matrix, be careful! As mentioned above, the rows should correspond to the *mutually exclusive options* for one decision: you have A or you don't have A. Likewise, the columns should correspond to the mutually exclusive options for the other decision: you have B or you don't have B. For instance, do not draw the table this way:



Note: even if you are accustomed to using Venn diagrams for these problems, you should strongly consider switching to the double-set matrix for problems with only two sets of options. The double-set matrix conveniently displays *all* possible combinations of options, including totals, whereas the Venn diagram only displays a few of them easily.

Overlapping Sets and Percents

Many overlapping-sets problems involve *percents* or *fractions*. The double-set matrix is still effective on these problems, especially if you choose a smart number for the grand total. For problems involving percents, choose a total of 100. For problems involving fractions, choose a common denominator for the total. For example, choose 15 if the problem mentions categories that are $\frac{1}{3}$ and $\frac{2}{5}$ of the total.

> 70% of the guests at Company X's annual holiday party are employees of Company X. 10% of the guests are women who are not employees of Company X. If half the guests at the party are men, what percent of the guests are female employees of Company X?

First, set up your chart. The two groups are Men/Women and Employee/Not Employee. Because the problem uses only percentages, no real numbers, choose 100 for the total number of guests. Then, fill in the other information given in the problem: 70% of the guests are employees, and 10% are women who are not employees. You also know that half the guests are men. (Therefore, you also know that half the guests are women.)

	Men	Women	Total
Employee			70
Not Emp.		10	
Total	50	50	100

What does the question want? Calculate only what you need to answer the question. In this case, the question asks for female employees, so calculate the "Women + Employee" box: 50 – 10 = 40. The percentage of female employees is $\frac{40}{100}$, or 40%.

	Men	Women	TOTAL
Employee	**30**	**40**	70
Not Emp.	**20**	10	**30**
TOTAL	50	50	100

Note that the problem does not require you to complete the matrix with the number of male employees, since you have already answered the question asked in the problem. If you want to check your work, though, you can complete the matrix. (Make sure you have enough time!) The last box you fill in must work both vertically and horizontally.

As with other problems involving smart numbers, you can only assign a number to the total if the problem contains only fractions and/or percents, but no actual *numbers* of items or people. In that case, go ahead and pick a total of 100 (for percent problems) or a common denominator (for fraction problems). If actual quantities appear anywhere in the problem, though, then all the totals are already determined. In that case, you cannot assign numbers, but must solve for them instead.

Overlapping Sets and Algebraic Representation

When solving overlapping sets problems, pay close attention to the wording of the problem. For example, consider the problem below:

> A researcher estimates that 10% of the children in the world are between the ages of 8 and 18 and dislike soccer, and that 50% of the children who like soccer are between the ages of 8 and 18. If 40% of the children in the world are between the ages of 8 and 18, what percentage of children in the world are under age 8 and dislike the game of soccer? (Assume all children are between the ages of 0 and 18.)

It is tempting to fill in the number 50 to represent the percent of children aged 8 to 18 who like soccer. However, this approach is incorrect.

WRONG

	8 to 18	< 8	Total
Like	50		
Dislike	10		
Total			100

Notice that you are told that 50% of the children *who like soccer* are between the ages of 8 and 18. This is different from being told that 50% of the children *in the world* are between the ages of 8 and 18. In this problem, the information you have is a fraction of an unknown number. You do not yet know how many children like soccer. Therefore, you cannot yet write a number for the children aged 8 to 18 who like soccer. Instead, represent the unknown total number of children who like soccer with the variable x. Then, you can represent the number of children aged 8 to 18 who like soccer with the expression $0.5x$.

CORRECT

	8 to 18	< 8	Total
Like	$0.5x$		x
Dislike	10		
Total	40		100

From the relationships in the table, set up an equation to solve for *x*:

$$0.5x + 10 = 40$$
$$x = 60$$

With this information, you can fill in the rest of the table:

	8 to 18	< 8	Total
Like	**$0.5x = 30$**	**30**	**$x = 60$**
Dislike	10	**30**	**40**
Total	40	**60**	100

Therefore, 30% of the children are under age 8 and dislike soccer.

Problem Set

1. A is a set of even integers, while B is a set of integers that are multiples of 3. There are 16 integers in set A, 22 integers in set B, and 7 integers in both sets. How many integers are in exactly one of the two sets?

2. Of 28 people in a park, 12 are children and the rest are adults. 8 people have to leave at 3pm; the rest do not. If, after 3pm, there are 6 children still in the park, how many adults are still in the park?

3. 40% of all high school students hate roller coasters; the rest love them. 20% of those students who love roller coasters own chinchillas. What percentage of students love roller coasters but do not own a chinchilla?

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

4. Of 30 snakes at the reptile house, 10 have stripes, 21 are poisonous, and 5 have no stripes and are not poisonous. How many of the snakes have stripes AND are poisonous?

5. 10% of all aliens are capable of intelligent thought and have more than 3 arms, and 75% of aliens with 3 arms or less are capable of intelligent thought. If 40% of all aliens are capable of intelligent thought, what percent of aliens have more than 3 arms?

Solutions

1. **24 integers:** Use a Double-Set Matrix to solve this problem. First, fill in the numbers given in the problem: 16 integers in set A (first column total) and 22 integers in set B (first row total). There are 7 integers in both sets (first row, first column). Next, use subtraction to figure out that there are 9 integers in set A but not in set B and 15 integers in set B but not in set A. Finally, add those two numbers: 9 + 15 = 24..

	Set A	NOT Set A	Total
Set B	7	**15**	22
Not Set B	**9**		
Total	16		

2. **14 adults:** Use a double-set matrix to solve this problem. First, fill in the numbers given in the problem: 28 total people in the park, 12 children and the rest (16) adults; 8 leave at 3pm and the rest (20) stay. Then, you are told that there are 6 children left in the park after 3pm. Since you know there are a total of 20 people in the park after 3pm, the remaining 14 people must be adults.

	Children	Adults	Total
Leave at 3			8
Stay	6	**14**	**20**
Total	12	**16**	28

3. **48%:** Since all the numbers in this problem are given in percentages, assign a grand total of 100 students. You know that 40% of all high school students hate roller coasters, so fill in 40 for this total and 60 for the number of students who love roller coasters. You also know that 20% **of those students who love roller coasters** own chinchillas. It does not say that 20%

of all students own chinchillas. Since 60% of students love roller coasters, 20% of 60% own chinchillas. Therefore, fill in 12 for the students who both love roller coasters and own chinchillas. The other 48 roller coaster lovers do not own chinchillas.

	Love RCs	Do Not	Total
Chinchilla	12		
No Chinch.	48		
Total	60	40	100

4. **6:** Use a double-set matrix to solve this problem. First, fill in the numbers given in the problem: 30 snakes, 10 with stripes (and therefore 20 without), 21 that are poisonous (and therefore 9 that are not), and 5 that are neither striped nor poisonous. Use subtraction to fill in the rest of the chart. Thus, 6 snakes have stripes and are poisonous.

	Stripes	No Stripes	Total
Poisonous	6		21
Not Poison	4	5	9
Total	10	20	30

5. **60%:** Since all the numbers in this problem are given in percentages, assign a grand total of 100 aliens. You know that 10% of all aliens are capable of intelligent thought and have more than 3 arms. You also know that 75% **of aliens with 3 arms or less** are capable of intelligent thought. It does not say that 75% of all aliens are capable of intelligent thought. Therefore, assign the variable x to represent the percentage of aliens with three arms or less. Then, the percentage of aliens with three arms or less who are capable of intelligent thought can be represented by $0.75x$. Since you know that 40% of all aliens are capable of intelligent thought, you can write the equation $10 + 0.75x = 40$, or $0.75x = 30$. Solve for x: $x = 40$. Therefore, 40% of the aliens have three arms or less, and 60% of aliens have more than three arms.

	Thought	No Thought	Total
> 3 arms	10		**60**
≤ 3 arms	$0.75x =$ **30**		$x =$ **40**
Total	40		100

Chapter 6
of
Word Problems

Statistics

In This Chapter…

Chapter 6

Statistics

Averages

The average (or the **arithmetic mean**) of a set is given by the following formula:

$$\text{Average} = \frac{\text{Sum}}{\text{\# of terms}}, \text{ which is abbreviated as } A = \frac{S}{n}$$

The sum, *S*, refers to the sum of all the terms in the set.

The number, *n*, refers to the number of terms that are in the set.

The average, *A*, refers to the average value (arithmetic mean) of the terms in the set.

The language in an average problem will often refer to an “arithmetic mean.” However, occasionally, the concept is implied. “The cost per employee, if equally shared, is $20” means that the *average* cost per employee is $20. Likewise, the “per capita income” is the average income per person in an area.

Here's a commonly used variation of the average formula:

$$(\text{Average}) \times (\text{\# of terms}) = (\text{Sum}), \text{ or } A \times n = S$$

Using the Average Formula

Every GMAT problem dealing with averages can be solved using some form of the average formula. In general, if the average is unknown, the first formula, $A = \frac{S}{n}$, will solve the problem more directly. If the average is known, the second formula, $A \times n = S$, is better.

When you see any GMAT average problem, write down the average formula. Then, fill in any of the three variables (S, n, and A) that are given in the problem:

> The sum of 6 numbers is 90. What is the average term?
>
> $$A = \frac{S}{n}$$
>
> The sum, S, is given as 90. The number of terms, n, is given as 6. By plugging in, you can solve for the average: $\frac{90}{6} = 15$.

Notice that you do *not* need to know each term in the set to find the average!

Sometimes, using the average formula will be more involved. For example:

> If the average of the set {2, 5, 5, 7, 8, 9, x} is 6.1, what is the value of x?

Plug the given information into the average formula, and solve for x:

$$A \times n = S \qquad \begin{aligned}(6.1)(7\text{ terms}) &= 2+5+5+7+8+9+x\\ 42.7 &= 36 + x\\ 6.7 &= x\end{aligned}$$

More complex average problems involve setting up two average formulas. For example:

> Sam earned a \$2,000 commission on a big sale, raising his average commission by \$100. If Sam's new average commission is \$900, how many sales has he made?

To keep track of two average formulas in the same problem, you can set up a table. Sam's new average commission is \$900, and this is \$100 higher than his old average, so his old average was \$800.

Note that the Number and Sum columns add up to give the new cumulative values, but the values in the Average column do *not* add up:

	Average	$\times$	Number	$=$	Sum
Old Total	800	$\times$	n	$=$	$800n$
This Sale	2,000	$\times$	1	$=$	2,000
New Total	900	$\times$	$n + 1$	$=$	$900(n + 1)$

The right-hand column gives the equation you need:

$$800n + 2{,}000 = 900(n + 1)$$

$$800n + 2{,}000 = 900n + 900$$

$$1{,}100 = 100n$$

$$11 = n$$

Since you are looking for the new number of sales, which is $n + 1$, Sam has made a total of 12 sales.

Median: The Middle Number

Some GMAT problems feature another stats concept: the *median*, or middle value in a list of values place in increasing order. The median is calculated in one of two ways, depending on the number of data points in the set:

1. For sets containing an *odd* number of values, the median is the *unique middle value* when the data are arranged in increasing (or decreasing) order.
2. For sets containing an *even* number of values, the median is the *average (arithmetic mean) of the two middle values* when the data are arranged in increasing (or decreasing) order.

The median of the set {5, 17, 24, 25, 28} is the unique middle number, 24. The median of the set {3, 4, 9, 9} is the mean of the two middle values (4 and 9), or 6.5. Notice that the median of a set containing an *odd* number of values must be a value in the set. However, the median of a set containing an *even* number of values does not have to be in the set—and indeed will not be, unless the two middle values are equal.

Medians of Sets Containing Unknown Values

Unlike the arithmetic mean, the median of a set depends only on the one or two values in the middle of the ordered set. Therefore, you may be able to determine a specific value for the median of a set *even if one or more unknowns are present*.

For instance, consider the unordered set {x, 2, 5, 11, 11, 12, 33}. No matter whether x is less than 11, equal to 11, or greater than 11, the median of the resulting set will be 11. (Try substituting different values of x to see why the median does not change.)

By contrast, the median of the unordered set {x, 2, 5, 11, 12, 12, 33} depends on x. If x is 11 or less, the median is 11. If x is between 11 and 12, the median is x. Finally, if x is 12 or more, the median is 12.

Standard Deviation

The mean and median both give "average" or "representative" values for a set, but they do not tell the whole story. It is possible for two sets to have the same average but to differ widely in how spread out their values are. To describe the spread, or variation, of the data in a set, you use a different measure: the standard deviation (SD).

Standard deviation indicates how far from the average (mean) the data points typically fall. Therefore:

- A small SD indicates that a set is clustered closely around the average (arithmetic mean) value.
- A large SD indicates that the set is spread out widely, with some points appearing far from the mean.

Consider the sets {5, 5, 5, 5}, {2, 4, 6, 8}, and {0, 0, 10, 10}. These sets all have the same mean value of 5. You can see at a glance, though, that the sets are very different, and the differences are reflected in their SDs. The first set has a SD of 0 (no spread at all), the second set has a moderate SD, and the third set has a large SD.

	Set 1	Set 2	Set 3
	{5, 5, 5, 5}	{2, 4, 6, 8}	{0, 0, 10, 10}
Difference from the mean of 5 (in absolute terms)	{0, 0, 0, 0} average spread = 0 SD = 0 An SD of 0 means that all the numbers in the set are equal.	{3, 1, 1, 3} average spread = 2 SD = moderate (technically, SD = $\sqrt{5} \approx 2.24$)	{5, 5, 5, 5} average spread = 5 SD = large (technically, SD = 5) If every absolute difference from the mean is equal, then the SD equals that difference.

You might be asking where the $\sqrt{5}$ comes from in the technical definition of SD for the second set. The good news is that you do not need to know—the GMAT will not ask you to calculate a specific SD unless a shortcut exists, such as knowing that the SD is 0 if all of the numbers in the set are identical. If you just pay attention to what the *average spread* is doing, you'll be able to answer all GMAT standard deviation problems, which involve either 1) *changes* in the SD when a set is transformed, or 2) *comparisons* of the SDs of two or more sets. Just remember that the more spread out the numbers, the larger the SD.

If you see a problem focusing on changes in the SD, ask yourself whether the changes move the data closer to the mean, farther from the mean, or neither. If you see a problem requiring comparisons, ask yourself which set is more spread out from its mean.

Following are some sample problems to help illustrate SD properties.

1. Which set has the greater standard deviation: {1, 2, 3, 4, 6} or {441, 442, 443, 444, 445}?
2. If each data point in a set is increased by 7, does the set's standard deviation increase, decrease, or remain constant?
3. If each data point in a set is increased by a factor of 7, does the set's standard deviation increase, decrease, or remain constant? (Assume that the set consists of different numbers.)

Answers can be found on the following page.

Answer Key

1. **The first set has the greater SD.** One way to understand this is to observe that the gaps between its numbers are, on average, slightly bigger than the gaps in the second set (because the last two numbers are 2 units apart). Another way to resolve the issue is to observe that the set {441, 442, 443, 444, 445} would have the same standard deviation as {1, 2, 3, 4, 5}. Replacing 5 with 6, which is farther from the mean, will increase the SD of that set.
2. **The SD will not change.** "Increased by 7" means that the number 7 is *added* to each data point in the set. This transformation will not affect any of the gaps between the data points, and thus it will not affect how far the data points are from the mean. If the set were plotted on a number line, this transformation would merely slide the points 7 units to the right, taking all the gaps, and the mean, along with them.
3. **The SD will increase.** "Increased by a *factor* of 7" means that each data point is multiplied by 7. This transformation will make all the gaps between points 7 times as big as they originally were. Thus, each point will fall 7 times as far from the mean. The SD will increase by a factor of 7. Why did the problem specify that the set consists of different numbers? If each data point in the set was the same, then the SD would be 0. Multiplying each data point by 7 would still result in a set of identical numbers, and an identical SD of 0.

Problem Set

1. The average of 11 numbers is 10. When one number is eliminated, the average of the remaining numbers is 9.3. What is the eliminated number?

2. Given the set of numbers {4, 5, 5, 6, 7, 8, 21}, how much higher is the mean than the median?

3. The class mean score on a test was 60, and the standard deviation was 15. If Elena's score was within 2 standard deviations of the mean, what is the lowest score she could have received?

4. Matt gets a $1,000 commission on a big sale. This commission alone raises his average commission by $150. If Matt's new average commission is $400, how many sales has Matt made?

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

5. If the average of x and y is 50, and the average of y and z is 80, what is the value of $z - x$?

6. $S = \{1, 2, 5, 7, x\}$

   If x is a positive integer, is the mean of set S greater than 4?

   (1) The median of set S is greater than 2.
   (2) The median of set S is equal to the mean of set S.

7. {9, 12, 15, 18, 21}

   Which of the following pairs of numbers, when added to the set above, will increase the standard deviation of the set?

   I. 14, 16

II. 9, 21
III. 15, 100

(A) II only
(B) III only
(C) I and II
(D) II and III
(E) I, II, and III

Solutions

1. **17:** If the average of 11 numbers is 10, their sum is 11 × 10 = 110. After one number is eliminated, the average is 9.3, so the sum of the 10 remaining numbers is 10 × 9.3 = 93. The number eliminated is the difference between these sums: 110 – 93 = 17.

2. **2:** The mean of the set is the sum of the numbers divided by the number of terms: 56 ÷ 7 = 8. The median is the middle number: 6. Therefore, the mean to the median, 8, is 2 greater than 6.

3. **30:** Elena's score was within 2 standard deviations of the mean. Since one standard deviation is 15, her score is no more than 15 × 2 = 30 points from the mean. The lowest possible score she could have received, then, is 60 – 30, which is equal to 30.

4. **5:** Before the $1,000 commission, Matt's average commission was $250; you can express this algebraically with the equation $S = 250n$.

After the sale, the sum of Matt's commissions increased by $1,000, the number of sales made increased by 1, and his average commission was $400. You can express this algebraically with the equation:

$$S + 1{,}000 = 400(n + 1)$$

$$\begin{aligned} 250n + 1{,}000 &= 400(n + 1) \\ 250n + 1{,}000 &= 400n + 400 \\ 150n &= 600 \\ n &= 4 \end{aligned}$$

Before the big sale, Matt had made 4 sales. Including the big sale, Matt has made 5 sales.

5. **60:** The sum of two numbers is twice their average. Therefore:

$$x + y = 100 \qquad\qquad y + z = 160$$
$$x = 100 - y \qquad\qquad z = 160 - y$$

Substitute these expressions for *z* and *x*:

$$z - x = (160 - y) - (100 - y) = 160 - y - 100 + y = 160 - 100 = 60$$

Alternatively, pick smart numbers for *x* and *y*. Let *x* = 50 and *y* = 50 (this is an easy way to make their average equal to 50). Since the average of *y* and *z* must be 80, you have *z* = 110. Therefore, *z* – *x* = 110 – 50 = 60.

6. **(B):** Like any other statement or question about the mean of a fixed number of data points, the prompt question can be rephrased to a question about the sum of the numbers in the set. Plug the known values into the equation Sum = Average × Number: is $(1 + 2 + 5 + 7 + x)/5 > 4$?

is $15 + x > 20$?

is $x > 5$?

For reference below, note also that the mean of *S* is $\frac{1+2+5+7+x}{5} = \frac{15+x}{5} = 3 + \frac{x}{5}$.

(1) INSUFFICIENT: For the median of the set to be greater than 2, *x* must also be greater than 2. If *x* were less than or equal to 2, the median would be 2. If *x* is 3 or 4, then the average of the set will be less than 4. However, if *x* is greater than or equal to 5, the average of the set will be greater than 4. This statement is insufficient.

(2) SUFFICIENT: This statement is a bit trickier to deal with. You can express the mean as $3 + \frac{x}{5}$, but the median depends on the value of *x*. However, note that you know the median must be an integer. You know that *x* must be an integer, so all of the elements in the set are integers. There are an odd number of elements in the set, so one of the elements of the set will be the median.

If the mean equals the median, then you know that $3 + \frac{x}{5}$ must also equal an integer. For that to be the case, you know that *x* must be a multiple of 5. Now you can begin testing different values of *x*:

x	median	mean $\left(3+\frac{x}{5}\right)$
5	5	4
10	5	5

You know $x = 10$, and the average is greater than 4. Statement (2) is sufficient.

7. **(D) II and III:** Fortunately, you do not need to perform any calculations to answer this question. The mean of the set is 15. Take a look at each Roman Numeral:

I. The numbers 14 and 16 are both very close to the mean (15). Additionally, they are closer to the mean than four of the numbers in the set, and will reduce the spread around the mean. This pair of numbers will reduce the standard deviation of the set.

II. The numbers 9 and 21 are relatively far away from the mean (15). Adding them to the list will increase the spread of the set and increase the standard deviation.

III. While adding the number 15 to the set would actually decrease the standard deviation (because it is the same as the mean of the set), the number 100 is so far away from the mean that it will greatly increase the standard deviation of the set. This pair of numbers will increase the standard deviation.

Chapter 7
of
Word Problems

Weighted Averages

In This Chapter…

Chapter 7

Weighted Averages

The regular formula for averages applies only to sets of data consisting of individual values that are equally weighted—that is, all of the values “count” equally towards the average. Some averages, however, are weighted more heavily towards certain data points.

For example, imagine that your teacher tells you that your mid-term exam will count for 40% of your grade and your final exam will count for 60% of your grade. If you can score a perfect 100 on only one of those components, which one would you want it to be?

Your final exam, of course! It counts more heavily towards your final grade. Next, imagine that you score 100 on your final exam but only 80 on your mid-term exam. What is the weighted average of those two scores?

Any average has to be between the two starting points, in this case 100 and 80. The *regular* average would be 90. Can you tell whether the weighted average is higher or lower than the regular average of 90?

In this case, you do have enough information to tell. The final exam counts for more than 50% of your final score, so the weighted average should be closer to the final exam score of 100 than to 80. The weighted average must be between 90 and 100.

There are two ways to calculate the exact value: *algebraically* or via the *teeter-totter*.

The Algebraic Method

To solve algebraically, set up an equation in which you multiply each exam score by its weight. You scored 100 on your final exam and it has a 60%, or $\frac{3}{5}$, weighting. You scored 80 on your mid-term and it has a 40%, or $\frac{2}{5}$, weighting. Now add them together:

$$100\left(\frac{3}{5}\right)+80\left(\frac{2}{5}\right)=?$$
$$60+32=92$$

The weighted average is 92.

If you prefer just to memorize the formula, go ahead. If, on the other hand, you'd like to know how that formula works, read on—the below explanation will help you to remember how to calculate weighted averages algebraically.

Think of a regular average as one in which each item has exactly equal weight. If there are two items, both are weighted $\frac{1}{2}$. If your teacher weighted the two exams equally, then this would be the calculation:

$$100\left(\frac{1}{2}\right)+80\left(\frac{1}{2}\right)=50+40=90$$

That is, the average is 90, exactly halfway between 80 and 100. The initial equation could be rearranged in this way:

$$100\left(\frac{1}{2}\right)+80\left(\frac{1}{2}\right)=\frac{100+80}{2}$$

Is that starting to look familiar? That's the average formula: find the sum of the two numbers and divide by 2! Technically, regular averages all have these equal weightings, so you can always write the equation in the simplified form: $\frac{\text{sum}}{\text{\# of terms}}$.

Weighted averages, though, must include the individual weightings, so you'll always have a component multiplied by its weighting, and then the next component multiplied by its weighting, and so on.

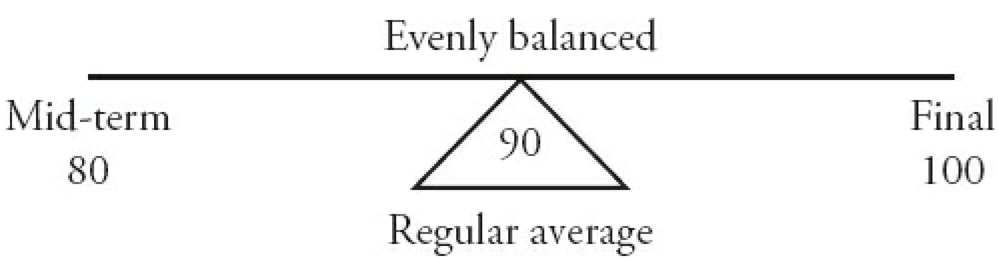
> Weighted Average = (component 1)(weighting 1) + (component 2)(weighting 2)

You can have more than two components, but the GMAT often sticks to just two.

The Teeter-Totter Method

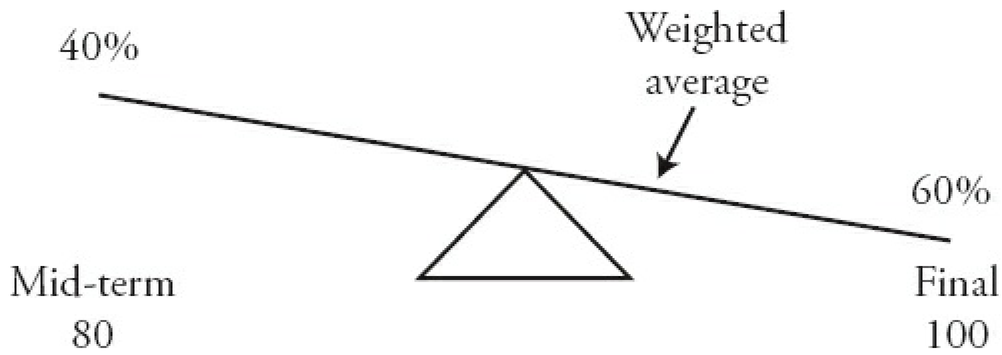
The Teeter-Totter method is very efficient as long as you do understand what a weighted average is and how the concept works in general. If you struggle with the concepts, then you may want to stick with the algebraic method.

The problem is the same: You scored 100 on your final exam and it has a $\frac{3}{5}$ weighting. You scored 80 on your mid-term and it has a $\frac{2}{5}$ weighting. What is your final grade?



If you did not have a weighted average, then the teeter-totter would be perfectly balanced and you would have a regular average, halfway between 80 and 100.

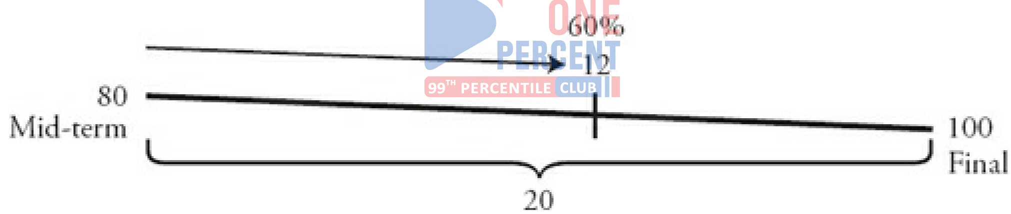
In this case, though, you do have a weighted average:



The weighted average “slips” down towards the heavier end of the teeter-totter, so you know that the weighted average must be between 90 and 100. On some Data Sufficiency questions, this is enough to determine that a statement is sufficient!

If you have to calculate the exact weighted average, here's what you do:

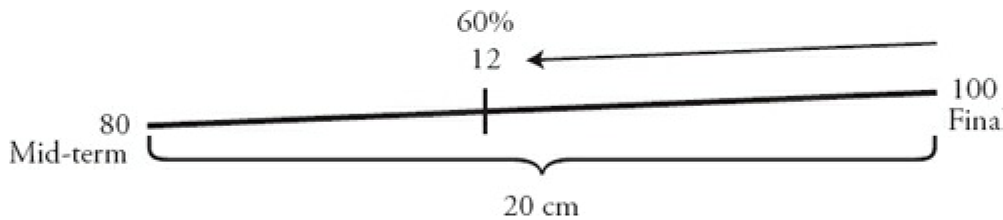
The two “ends” of the teeter-totter are 80 and 100; the difference is 20. The final exam is weighted more heavily, so it is responsible for 60% of that length of 20, and 60% of 20 is equal to 12. Start counting from the higher (lighter) end of the teeter-totter:



Imagine that the final exam weighs down his side of the teeter-totter by 12. Therefore, the average is 80 + 12, which sums to 92.

You don't need to draw out a full teeter-totter, but do draw at least the sloped line. Know which side is heavier. Calculate the “length” of the line and use it to calculate the value of the heavier weighting (in this case, 12); then start counting from the lighter side (80). In this case, you add: 80 + 12 = 92.

If the situation were reversed, and the score of 80 had the 60% weighting, then you would subtract at that last step instead:



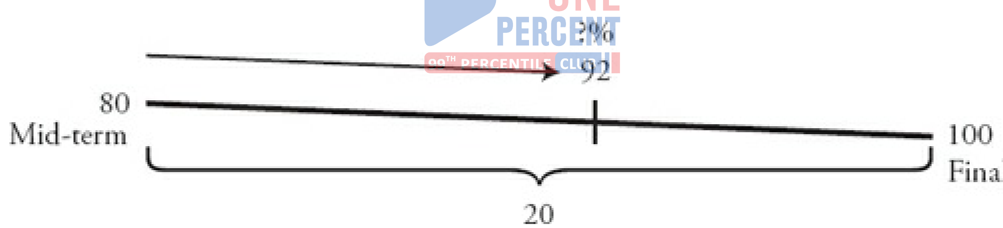
In this case, the mid-term exam is pulling the average 12 units towards him, so start counting from the higher (lighter) end, 100. Subtract from that end: 100 – 12 = 88.

In both cases, do draw the sloped line and place the final number in roughly the appropriate position. That step will tell you whether you should add from the smaller end (80 + 12) or subtract from the larger end (100 – 12).

What if the problem changed the given information? Try this:

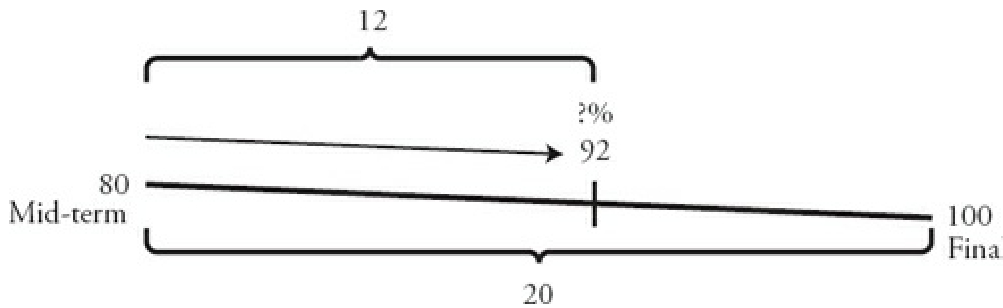
> You score 80 on your mid-term exam and 100 on your final exam. Only these two exams make up your final grade of 92. How heavily did your teacher weight the final exam?

First, draw your teeter-totter:



In this case, you still know the two end points (and the length), but now you're given the final average of 92 and you have to figure out the weighting that results in that average.

Because the weighted average is closer to 100 than to 80, you know that 100 is the heavier weight, so your final exam should be weighted more than 50%. Because the problem asks you to find that weight, find the longer of the two distances, between 80 and 92:



The weighting of the heavier (final exam) side is the fractional part 12 over the total length 20: $\frac{12}{20} = \frac{3}{5}$, which is equal to 60%.

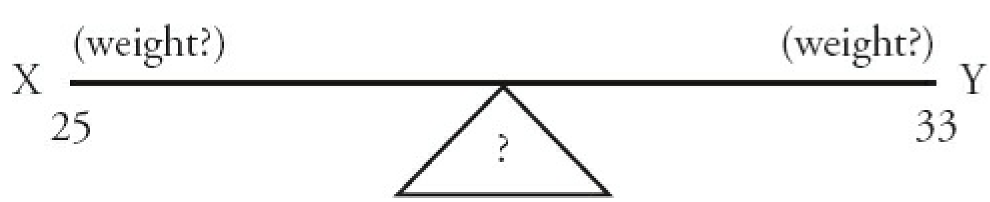
If the problem had asked you to calculate the weighting of the less-heavily weighted value, the mid-term exam, then you would find the shorter of the two distances, 8, and divide by the total distance, 20: $\frac{8}{20} = \frac{2}{5}$, which is equal to 40%.

Try this Data Sufficiency problem:

> The average number of students per class at School X is 25 and the average number of students per class at School Y is 33. Is the average number of students per class for both schools combined less than 29?
>
> (1) There are 12 classes in School X.
> (2) There are more classes in School X than in School Y.

Because this is a Data Sufficiency problem, there's a very good chance that you will not have to complete the calculations. In this case, try the teeter-totter method.

First, the question stem provides this information:



You don't know how to tilt the teeter-totter, because you don't know enough information yet. The question itself, though, implies something very intriguing.

If the two schools are equally weighted, then the “regular” average would be 29. If the average is less than 29, then what would you know?

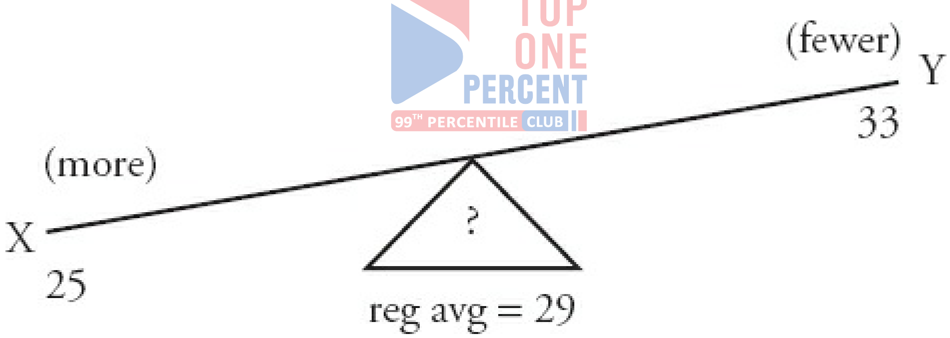
Then, the teeter-totter would have to be tilted down towards School X; this school would be weighted more heavily. Keep this in mind as you examine the statements:

(1) There are 12 classes in School X.

This statement doesn't provide any information about School Y, so it's impossible to tell whether one school is weighted more heavily. Statement (1) is not sufficient.

(2) There are more classes in School X than in School Y.

If there are more classes over at School X, then the weighted average has to tilt down towards this school.



As a result, the weighted average has to be less than the “regular” average of 29.

Statement (2) is sufficient; the correct answer is **(B)**.

On weighted average problems, you can choose whether to use the algebraic method or the teeter-totter method. Try both out on some OG problems and decide which method works better for you.

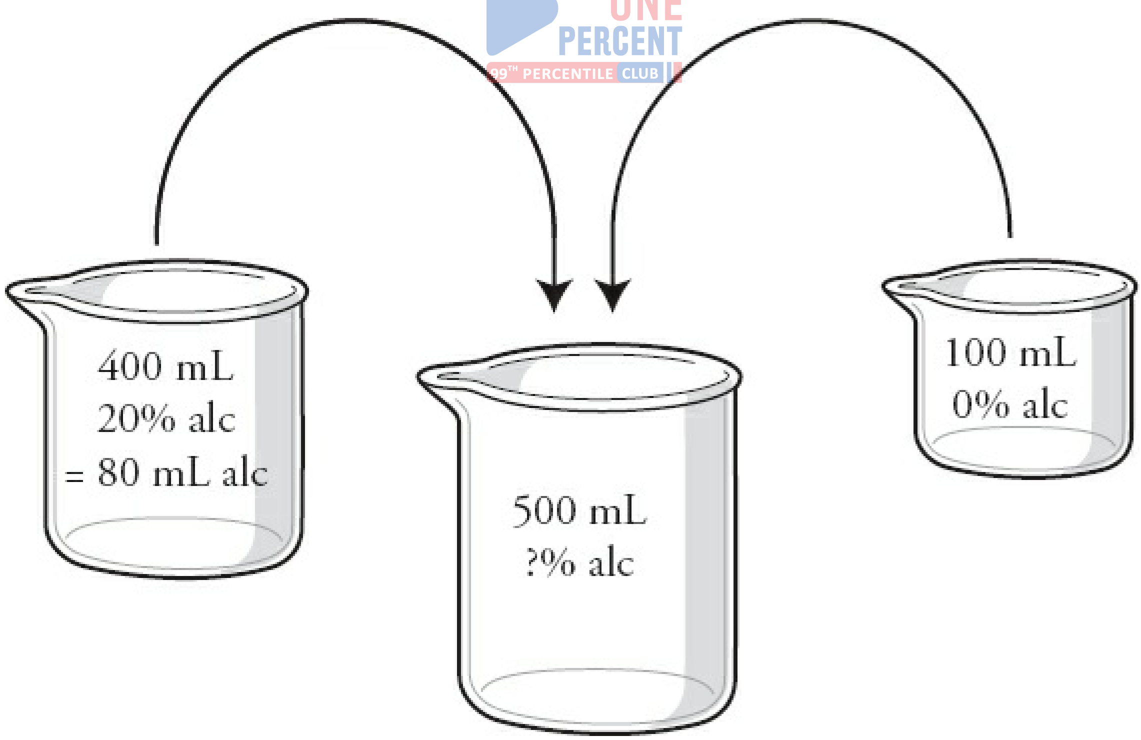
Mixtures, Percents, and Ratios

Percents and ratios can also show up in weighted average problems, particularly in the form of mixtures.

First, try this regular mixtures problem (you don't need to calculate a weighted average for this one):

> A 400 mL solution is 20% alcohol by volume. If 100 mL of water is added, what is the new concentration of alcohol, as a percent of volume?
>
> (A) 5%
> (B) 10%
> (C) 12%
> (D) 12.5%
> (E) 16%

To start, you have two liquid solutions: a 400 mL solution that is 20% alcohol and 80% something else, and a 100 mL solution that is 100% water (and therefore 0% alcohol).



You can actually calculate the mL of alcohol in the 400 mL beaker: 20% of

400 is 80 mL. The 100 mL beaker doesn't contribute any alcohol at all, so the 500 mL beaker contains a total of 80 mL of alcohol. The big beaker, then, is $\frac{80}{500} = \frac{8}{50} = \frac{16}{100} = 16\%$ alcohol. The correct answer is **(E)**.

In this case, only one of the two beakers contributed alcohol to the mixture. What happens when both parts of the problem contribute to the desired mixture?

Try this out:

> Kris-P cereal is 10% sugar by weight, whereas healthier but less delicious Bran-O cereal is 2% sugar by weight. To make a delicious and healthy mixture that is 4% sugar, what should be the ratio of Kris-P cereal to Bran-O cereal, by weight?
>
> (A) 1:2
> (B) 1:3
> (C) 1:4
> (D) 3:1
> (E) 4:1

You can use the algebraic method or the teeter-totter—your choice. Both solutions are shown below.

The question asks for a ratio. Note that you don't necessarily need to know the real values of something in order to find a ratio. Call the weight of Kris-P cereal k and the weight of Bran-O cereal b.

To set up an equation:

$$0.1k + 0.02b = 0.04(k + b)$$

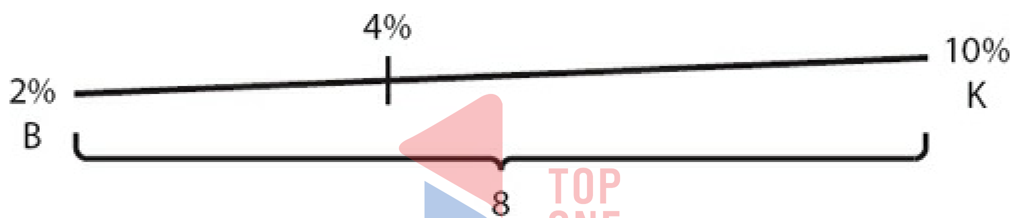
Kris-P is weighted 10% and Bran-O is weighted 2%. The final mixture is weighted 4% and note that the weight of the final mixture is the sum of the two components, k and b.

Because the question asks for the ratio of k to b, manipulate the equation to solve for $\frac{k}{b}$. First, multiply the whole equation by 100 to get rid of the decimals. Then simplify from there:

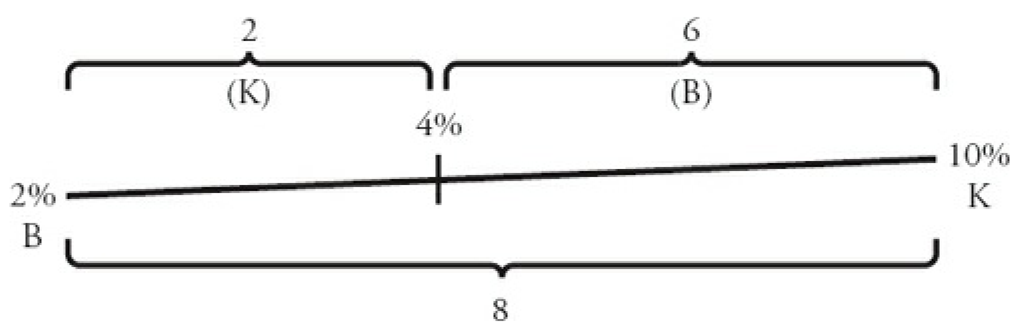
$$10k + 2b = 4(k + b)$$
$$10k + 2b = 4k + 4b$$
$$6k = 2b$$
$$\frac{k}{b} = \frac{2}{6} = \frac{1}{3}$$

The ratio of Kris-P to Bran-O is 1:3. The correct answer is **(B)**.

To use your teeter-totter, start drawing:



Because 4% is closer to 2%, you know that the Bran-O side is heavier. You also know the “length” is 10 – 2 = 8. Find the two smaller pieces of the line split by the weighted average:



The smaller number, 2, is associated with the less-heavily-weighted end (Kris-P, 10%). The larger number, 6, is associated with the more-heavily-weighted end (Bran-O, 2%). This will always be true: the larger portion always goes with the heavier end of the teeter-totter.

Therefore, the ratio of Kris-P to Bran-O is 2 : 6, or 1 : 3, which is **(B)**.

Again, you can choose whether to use algebra or the teeter-totter; try them both out and see what works best for you.

Problem Set

1. A professional gambler has won 40% of his 25 poker games for the week so far. If, all of a sudden, his luck changes and he begins winning 80% of the time, how many more games must he play to end up winning 60% of all his games for the week?

2. A charitable association sold an average of 66 raffle tickets per member. Among the female members, the average was 70 raffle tickets. The male to female ratio of the association is 1:2. What was the average number of tickets sold by the male members of the association?

3. Tickets to a play cost $10 for children and $25 for adults. If 90 tickets were sold, were more adult tickets sold than children's tickets?

    (1) The average revenue per ticket was $18.

    (2) The revenue from ticket sales exceeded $1,600

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

4. A feed store sells two varieties of birdseed: Brand A, which is 40% millet and 60% sunflower, and Brand B, which is 65% millet and 35% safflower. If a customer purchases a mix of the two types of birdseed that is 50% millet, what percent of the mix is Brand A?

5. On a particular exam, the boys in a history class averaged 86 points and the girls in the class averaged 80 points. If the overall class average was 82 points, what was the ratio of boys to girls in the class?

6. A mixture of “lean” ground beef (10% fat) and “super-lean” ground beef (3% fat) has a total fat content of 8%. What is the ratio of “lean” ground beef to “super-lean” ground beef?

Solutions

1. **25 more games:** This is a weighted averages problem. You can set up a table to calculate the number of games the gambler must play to obtain a weighted average win rate of 60%:

Poker Games	First 25 Games	Remaining Games	Total
Wins	$(0.4)25 = 10$	$(0.8)x$	$(0.6)(25 + x)$
Losses			
Total	**25**	x	$25 + x$

Thus, $10 + 0.8x = (0.6)(25 + x)$, $10 + 0.8x = 15 + 0.6x$, $0.2x = 5$, $x = 25$.

2. **58 tickets:** You can answer this question without doing a lot of calculation. Women sold an average of 70 raffle tickets, which is 4 higher than the total average of 66. Each woman is selling, therefore, an average of +4 tickets over the group average. You also know that the ratio of men to women is 1 : 2. There are twice as many women and the total amount of extra tickets that the women sell needs to be canceled out by lower-than-average sales from the men. The positive difference from the average for women multiplied by 2 must cancel with the negative difference from the average for men multiplied by 1. If you call the difference from the average for men *m* then:

$$\begin{aligned} 1 \times m + 2 \times (+4) &= 0 \\ m + 8 &= 0 \\ m &= -8 \end{aligned}$$

The men sold an average of 8 fewer tickets than the total average so subtract: 66 – 8 = 58.

3. **(D).** First things first: how would you recognize that this question is about weighted averages? The two different prices for tickets are like two different data points, and the number of tickets sold will act as the weight.

You can use weighted averages to work through the statements. Statement (1) may seem familiar by now: \$18 is closer to \$25 than to \$10. That means that there must have been more adult tickets sold than children's tickets. Statement 1 is sufficient. Cross off answers (B), (C), and (E) on your grid.

Statement (2) says that the total revenue from ticket sales exceeded \$1,750. Where did that number come from?

To figure that out, look at the information in the question stem again. Notice that the question actually told you that 90 tickets were sold. Consider two extreme scenarios:

> 90 children's tickets sold = 90 × \$10 = \$900 revenue
> 90 adult tickets sold = 90 × \$25 = (100 × \$25) – (10 × \$25) = \$2,500 – \$250 = \$2,250 revenue

If there were an equal number of adult and children's tickets sold, the revenue would be an average of \$900 and \$2,250, or \$1,575. That's the connection to weighted averages. If the revenue is greater than \$1,750, it is closer to \$2,500 than to \$1,000, which means more adult tickets must have been sold. Statement (2) is also sufficient. The correct answer is **(D)**.

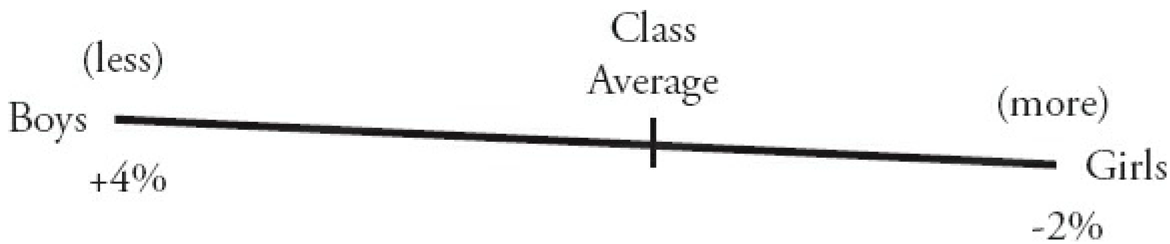
4. **60%:** This is a weighted averages problem. You can set up a table to calculate the answer, and assume that you purchased 100 lbs of Brand A:

Pounds (lbs)	Brand A	Brand B	Total
Millet	40	$0.65x$	$(0.5)(100 + x)$
Other stuff	60	$0.35x$	$(0.5)(100 + x)$
Total birdseed	100	x	$100 + x$

Thus, $40 + 0.65x = (0.5)(100 + x)$, $40 + 0.65x = 50 + 0.5x$, $0.15x = 10$, $x = \frac{1,000}{15}$.

$$\text{Therefore, Brand A is } \frac{100}{100+\frac{1,000}{15}} = \frac{100}{\frac{1,500}{15}+\frac{1,000}{15}} = \frac{1,500}{2,500} = 60\% \text{ of the total.}$$

5. **1:2:** The boys in the class scored 4 points higher on average than the entire class. Similarly, the girls scored 2 points lower on average than the class. You can draw a teeter-totter to answer the question. Set up the starting info:



There are more girls because a 2-points difference is smaller than a 4-point difference. What's the actual ratio?

The "length" of the line is 4 + 2 = 6. The girls side "pulls" the average away from boys by 4 points, and so girls are responsible for $\frac{4}{6}$ of the overall length.

But wait! That's a fraction, not a ratio! $\frac{4}{6}$ shows the part-to-whole relationship: 4 out of 6 points in the score spread are attributed to the girls. The boys are responsible for the other 2 out of 6 points in the spread. So the ratio of boys to girls is 2 to 4, or 1:2.

6. **5:2:** The question asks for the ratio of the two types of beef, so you don't need to worry about the actual amount of beef.

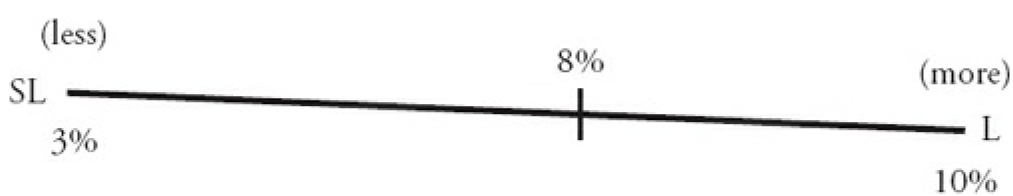
To set this problem up algebraically, first set up an equation, letting L be "lean" beef and S be "super-lean" beef:

$$0.1L + 0.03S = 0.08(L + S)$$

The question asks for the ratio of L to S, or. First, multiply the equation by 100 to get rid of the decimals, then solve for:

$$10L + 3S = 8(L + S)$$
$$10L + 3S = 8L + 8S$$
$$2L = 5S$$

Alternatively, you can draw a teeter-totter to answer the question. Set up the starting information:



There's more lean ground beef, because 8% is closer to 10%. What's the actual ratio?

The “length” of the line is 10 – 3 = 7. The lean side “pulls” the average a total of 8 – 3 = 5 units towards the lean side, so that side is responsible for $\frac{5}{7}$ of the overall average.

But wait! That's a fraction, not a ratio. A fraction is a part-to-whole relationship: 5 parts are lean out of the total 7 parts. The other 2 parts are super-lean. The ratio of lean to super-lean, then, is 5:2.

Chapter 8
of
Word Problems

Consecutive Integers

In This Chapter…

Chapter 8

Consecutive Integers

Consecutive integers are integers that follow one after another from a given starting point, without skipping any integers. For example, 4, 5, 6, and 7 are consecutive integers, but 4, 6, 7, and 9 are not. There are many other types of consecutive patterns. For example:

Consecutive even integers: 8, 10, 12, 14
(8, 10, 14, and 16 is incorrect, as it skips 12)

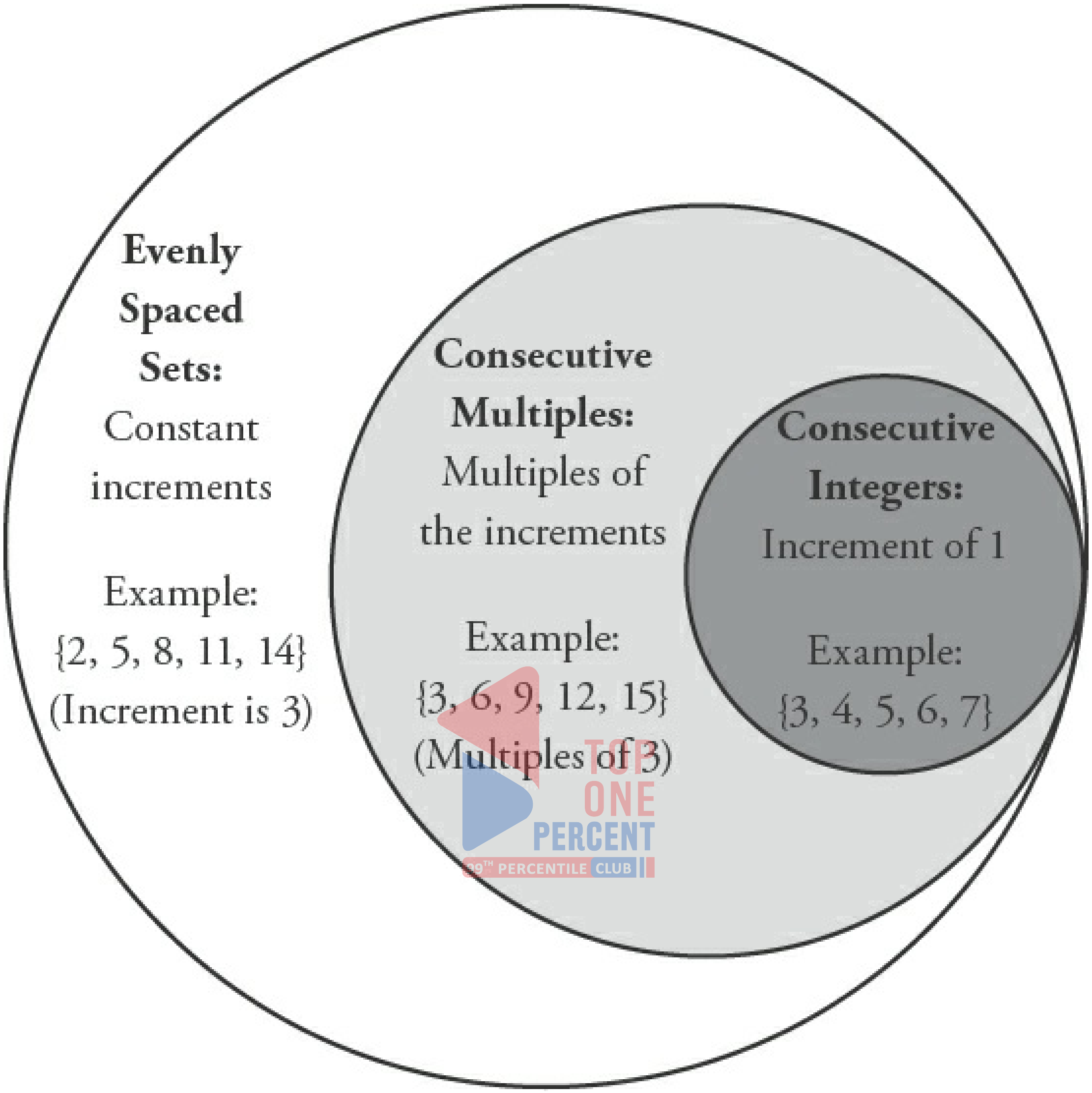
Consecutive primes: 11, 13, 17, 19
(11, 13, 15, and 17 is wrong, as 15 is not prime)

Evenly Spaced Sets

First, think about **evenly spaced sets** the values of the numbers in the set go up or down by the same amount (the **increment**) from one item in the sequence to the next. For instance, the set {4, 7, 10, 13, 16} is evenly spaced because each value increases by 3 over the previous value.

Sets of **consecutive multiples** are special cases of evenly spaced sets: all of the values in the set are multiples of the increment. For example, in the set {12, 16, 20, 24}, the values increase from one to the next by 4, and each element is a multiple of 4. Sets of consecutive multiples must be composed of integers.

Sets of **consecutive integers** are special cases of consecutive multiples: all of the values in the set increase by 1, and all integers are multiples of 1. For example, {12, 13, 14, 15, 16} is a set of consecutive integers.



Counting Integers: Add 1 Before You Are Done

How many integers are there from 6 to 10? Four, right? Wrong! There are actually five integers from 6 to 10. Count them: 6, 7, 8, 9, 10. It is easy to forget that you have to include (or, in GMAT lingo, be inclusive of) extremes. In this case, both extremes (the numbers 6 and 10) must be counted. When you merely subtract (10 − 6 = 4), you are forgetting to include the first extreme (6), as it has been subtracted away (along with 5, 4, 3, 2, and 1).

Do you have to methodically count each term in a long consecutive pattern? No. Just remember that if both extremes should be counted, you need to add 1 before you are done. For example:

> How many integers are there from 14 to 765, inclusive?

The formula is **(Last − First + 1):** 765 − 14 + 1 = 752.

This works easily enough if you are dealing with consecutive integers. Sometimes, however, the question will ask about consecutive multiples.

In this case, if you just subtract the largest number from the smallest and add one, you will be overcounting. For example, “All of the even integers between 12 and 24” yields 7 integers: 12, 14, 16, 18, 20, 22, and 24. However, (Last − First + 1) would yield (24 − 12 + 1) = 13, which is too large. How do you amend this? Since the items in the list are going up by increments of 2 (you are counting only the even numbers), you need to divide (Last − First) by 2. Then, add the 1 before you are done:

> (Last − First) ÷ Increment + 1 = (24 − 12) ÷ 2 + 1 = 6 + 1 = 7

For consecutive multiples, the formula is **(Last − First) ÷ Increment + 1**. The bigger the increment, the smaller the result, because there is a larger gap between the numbers you are counting.

Sometimes, it is easier to list the terms of a consecutive pattern and count them, especially if the list is short or if one or both of the extremes are omitted. For example:

> How many multiples of 7 are there between 100 and 150?

Here it may be easiest to list the multiples: 105, 112, 119, 126, 133, 140, 147. Count the number of terms to get the answer: 7. Alternatively, you could find that 105 is the first number, 147 is the last number, and 7 is the increment:

> Number of terms = (Last − First) ÷ Increment + 1 = (147 − 105) ÷ 7 + 1 = 6 + 1 = 7

Properties of Evenly Spaced Sets

The following properties apply to all evenly spaced sets:

1. The arithmetic mean (average) and median are equal to each other. For example:

   What is the arithmetic mean of 4, 8, 12, 16, and 20?

   In this example, the median is 12. Since this is an evenly spaced set, the arithmetic mean (average) is also 12.

   What is the arithmetic mean of 4, 8, 12, 16, 20, and 24?

   In this example, the median is the arithmetic mean (average) of the two middle numbers, or the average of 12 and 16. Thus, the median is 14. Since this is an evenly spaced set, the average is also 14.

2. The mean and median of the set are equal to the average of the FIRST and LAST terms. For example:

   What is the arithmetic mean of 4, 8, 12, 16, and 20?

   In this example, the arithmetic mean and median are both equal to (20 + 4) ÷ 2 = 12.

   What is the arithmetic mean of 4, 8, 12, 16, 20, and 24?

   In this example, the arithmetic mean and median are both equal to (24 + 4) ÷ 2 = 14.

   For all evenly spaced sets, the average equals **(First + Last) ÷ 2**.

The Sum of Consecutive Integers

Consider this problem:

What is the sum of all the integers from 20 to 100, inclusive?

Adding all those integers would take much more time than you have for a GMAT problem. Using the rules for evenly spaced sets, though, you can calculate more easily:

- The formula for the sum of an evenly spaced set is: Sum = Average × Number of Terms
- Average the first and last term to find the average of the set: $100 + 20 = 120$ and $120 \div 2 = 60$.
- Count the number of terms: $100 - 20 = 80$, plus 1 yields 81.
- Multiply the average by the number of terms to find the sum: $60 \times 81 = 4{,}860$.

Note some general facts about sums and averages of consecutive integers:

- The average of an odd number of consecutive integers (1, 2, 3, 4, 5) will always be an integer (3). This is because the "middle number" will be a single integer.
- On the other hand, the average of an even number of consecutive integers (1, 2, 3, 4) will never be an integer (2.5), because there is no true "middle number."

Problem Set

Solve these problems using the rules for consecutive integers.

1. How many terms are there in the set of consecutive integers from -18 to 33, inclusive?

2. What is the sum of all the positive integers up to 100, inclusive?

3. In a sequence of 8 consecutive integers, how much greater is the sum of the last four integers than the sum of the first four integers?

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

4. If the sum of the last 3 integers in a set of 6 consecutive integers is 624, what is the sum of the first 3 integers of the set?

5. If the sum of the last 3 integers in a set of 7 consecutive integers is 258, what is the sum of the first 4 integers?

6. The operation ==> is defined by x ==> $y = x + (x + 1) + (x + 2) \ldots + y$. For example, 3 ==> 7 = 3 + 4 + 5 + 6 + 7. What is the value of (100 ==> 150) − (125 ==> 150)?

Solutions

1. **52:** 33 − (−18) = 51. Then add 1 before you are done: 51 + 1 = 52.

2. **5,050:** There are 100 integers from 1 to 100, inclusive: (100 − 1) + 1. (Remember to add 1 before you are done.) The number exactly in the middle is 50.5. (You can find the middle term by averaging the first and last terms of the set.) Therefore, multiply 100 by 50.5 to find the sum of all the integers in the set: 100 × 50.5 = 5,050.

3. **16:** Think of the set of eight consecutive integers as follows: n, $(n + 1)$, $(n + 2)$, $(n + 3)$, $(n + 4)$, $(n + 5)$, $(n + 6)$, and $(n + 7)$.

First, find the sum of the first four integers:

$$n + (n + 1) + (n + 2) + (n + 3) = 4n + 6$$

Then, find the sum of the next four integers:

$$(n + 4) + (n + 5) + (n + 6) + (n + 7) = 4n + 22$$

The difference between these two partial sums is:

$$(4n + 22) - (4n + 6) = 22 - 6 = 16$$

Another way you could solve this algebraically is to line up the algebraic expressions for each number so that you can subtract one from the other directly:

$$\begin{array}{llcccccccl}
\text{Sum of the last four integers.} & & (n+4) & + & (n+5) & + & (n+6) & + & (n+7) & \\
\text{Less the sum of the first four integers.} & - & [n & + & (n+1) & + & (n+2) & + & (n+3)] & \\
\hline
 & & 4 & + & 4 & + & 4 & + & 4 & = 16
\end{array}$$

Yet another way to see this outcome is to represent the eight consecutive unknowns with eight lines:

— — — — ⋮ — — — —

Each of the first four lines can be matched with one of the second four lines, each of which is 4 greater:

$$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \vdots\ \underline{+4}\ \underline{+4}\ \underline{+4}\ \underline{+4}$$

So the sum of the last four numbers is 4 × 4 = 16 greater than the sum of the first four.

Finally, you could pick numbers to solve this problem. For example, assume you pick 1, 2, 3, 4, 5, 6, 7, and 8. The sum of the first four numbers is 10. The sum of the last four integers is 26. Again, the difference is 26 − 10 = 16.

4. **615:** Think of the set of integers as n, $(n + 1)$, $(n + 2)$, $(n + 3)$, $(n + 4)$, and $(n + 5)$. Thus, $(n + 3) + (n + 4) + (n + 5) = 3n + 12 = 624$. Therefore, $n = 204$.

The sum of the first three integers is: 204 + 205 + 206 = 615.

Alternatively, another way you could solve this algebraically is to line up the algebraic expressions for each number so that you can subtract one from the other directly:

$$\begin{array}{ll} \text{Sum of the last three integers.} & (n+3) + (n+4) + (n+5) \\ \text{Less the sum of the first three integers.} & \underline{-\ [n \quad + (n+1) + (n+2)]} \\ & \quad 3\ + \quad 3\ + \quad 3 = 9 \end{array}$$

Thus, the sum of the last three numbers is 9 greater than the sum of the first three numbers, so the sum of the first three numbers is 624 − 9 = 615.

Visually, you can represent the six consecutive unknowns with six lines:

$$\_\_\ \_\_\ \_\_\ \underbrace{\_\_\ \_\_\ \_\_}_{\text{Sum} = 624}$$

$$\text{Average} = 624 \div 3 = 208$$

$$\underbrace{\_\_\ \underline{205}\ \_\_}_{}\ \_\_\ \underline{208}\ \_\_$$

$$\text{Sum} = \text{Average} \times 3$$
$$= 205 \times 3 = 615.$$

5. **330:** Think of the set of integers as n, $(n + 1)$, $(n + 2)$, $(n + 3)$, $(n + 4)$, $(n + 5)$, and $(n + 6)$. $(n + 4) + (n + 5) + (n + 6) = 3n + 15 = 258$. Therefore, $n = 81$. The sum of the first four integers is $81 + 82 + 83 + 84 = 330$.

Alternatively, the sum of the first four integers is $4n + 6$. If $n = 81$, then $4n + 6 = 4(81) + 6 = 330$.

6. **2,800:** This problem contains two components: the sum of all the numbers from 100 to 150, and the sum of all the numbers from 125 to 150. Since you are finding the difference between these components, you are essentially finding just the sum of all the numbers from 100 to 124. You can think of this logically by solving a simpler problem: find the difference (1 ==> 5) − (3 ==> 5).

$$\begin{array}{rr} & 1+2+3+4+5 \\ - & 3+4+5 \\ \hline & 1+2 \end{array}$$

There are 25 numbers from 100 to 124 (124 − 100 + 1). Remember to add 1 before you are done! To find the sum of these numbers, multiply by the average term:

$$\frac{100+124}{2} = 112$$

$$25 \times 112 = 2{,}800$$

Chapter 9
of
Word Problems

Strategy: Draw It Out

In This Chapter…

Chapter 9

Strategy: Draw It Out

Numerous times throughout this book, you've learned how to draw out a story and find a more “real-world” approach to performing the necessary math. For example, many people find the teeter-totter method for weighted averages easier than the algebraic method.

This chapter contains additional examples of this **draw it out** approach.

Try this problem:

> Five identical pieces of wire are soldered together end-to-end to form one longer wire, with the pieces overlapping by 4 cm at each joint. If the wire thus made is exactly 1 meter long, how long, in centimeters, is each of the identical pieces? (1 meter = 100 cm)

(A) 21.2

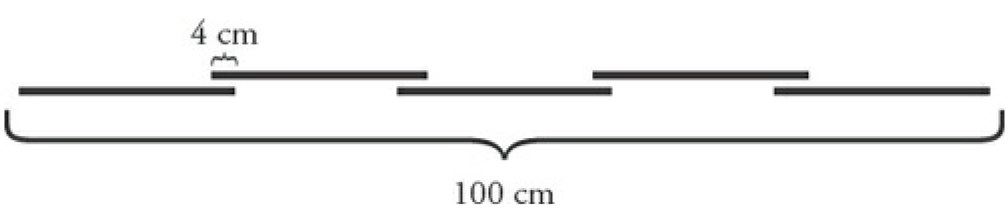
(B) 22

(C) 23.2

(D) 24

(E) 25.4

Draw out what the problem is describing:



That diagram is accurate but it might not exactly match what you'd been assuming in your head. Many people make the mistake of thinking that, because there are five pieces of wire, there are also five spots where the wires join. It turns out that there are only four joints!

The total length is 100 cm plus those extra amounts where the wires overlap. There are four overlaps of 4 cm each, or 16 cm of overlap. The total length shown in the picture, then, is 100 + 16 = 116 centimeters.

Because there are five wires, the length of each one is $\frac{116}{5} = 23.2$ cm. The correct answer is (C).

This problem can also be done algebraically: the relevant equation is $5x - 16 = 100$. Those who don't **draw it out**, though, are more likely to think that there are five joints and write the equation as $5x - 20 = 100$, which leads to answer (D) 24.

How Does Drawing It Out Work?

Essentially, there are multiple ways that you can avoid "textbook" math to get easier answers to many story problems on the GMAT. Whenever you find a problem that could actually be happening to someone in the real world, ask yourself: if I were in this situation right now, how would I try to figure out the answer?

You almost certainly wouldn't start writing equations. Instead, you'd sketch out the situation using a combination of logic, math, and just trying numbers out.

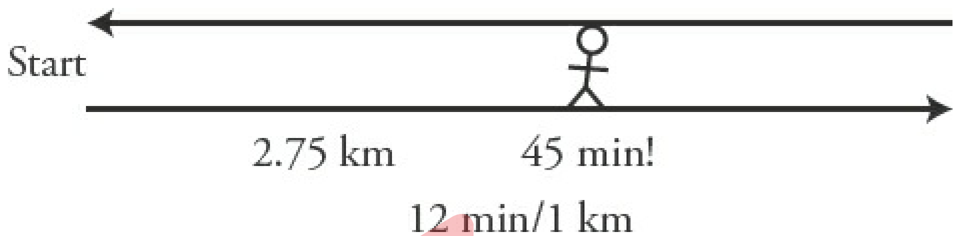
Try drawing out this problem:

> Annika hikes at a constant rate of 12 minutes per kilometer. She has hiked 2.75 kilometers east from the start of a hiking trail when she realizes that she has to be back at the start of the trail in 45 minutes. If Annika continues east, then turns around and retraces her path to reach the start of the trail in exactly 45 minutes, for how many kilometers total did she hike east?

(A) 2.25
(B) 2.75
(C) 3.25
(D) 3.75
(E) 4.25

This is a pretty nasty problem. There is an algebraic solution; you could also use an *RTD* chart to solve. The best way, though, is to draw it out.

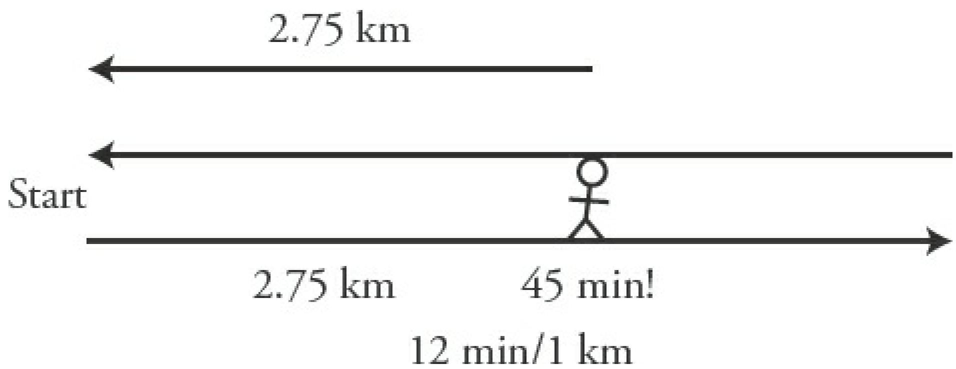
Here's Annika partway down her hiking trail, suddenly realizing that she's got 45 minutes till she needs to get back:



Pretend that isn't Annika at all—now, it's you. Are you going to whip out paper and pencil to start doing some algebra? No way. You're going to use real-world logic to figure out what to do.

What do you want to figure out? The question asks how far you traveled east. You know that part of the distance is 2.75 km, but you don't know how much further east you need to go before turning around. Glance at the answers. Hey, the answer can't be (A) and it can only be (B) if you have to turn around right now. Hmm.

First, if you didn't go a step further, how far would you have to go to get back?



You're going to need 2.75 kilometers to get back. How long is that going to take? You're going 1 kilometer every 12 minutes. Let's see, that would be 3 kilometers in (count it out) 12, 24, 36 minutes. How can you find the time for 2.75 kilometers?

If you hike 1 kilometer in 12 minutes, then you can hike a quarter of that distance in a quarter of the time: 0.25 kilometers in 3 minutes. Subtract from 3 kilometers in 36 minutes: 36 min − 3 min = 33 min.

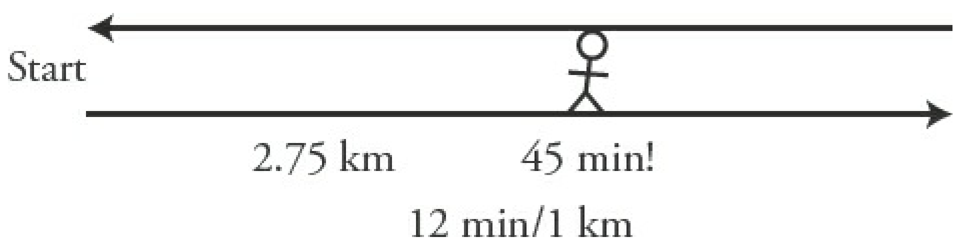
That takes care of 33 of your remaining 45 minutes. You have 12 minutes left.

How convenient! You already know that you can hike 1 kilometer in 12 minutes. You need to hike west *and* cover that same distance back, so you're going to continue for half a kilometer and then turn around.

Now, total: you'll hike 2.75 + 0.5 = 3.25 kilometers west before you turn around. The correct answer is (D).

Here's another way to draw it out:

Go back to the beginning.



Step back from the problem for a second—forget that you want 2.75 km plus some unknown distance. Look at your diagram. You're asking yourself how far you travel east before you turn around and exactly retrace your steps; that is, you want to know the distance of *half* of your trip. If you can calculate the total distance, you'll be done!

You know that you travel 2.75 kilometers to start. Then, you travel another 45 minutes at 12 minutes per kilometer. You can count it out again or do the straight math—$12 \times 4 = 48$—so it takes 48 minutes to go 4 kilometers. How can you find the distance for 45 minutes?

If you hike 1 kilometer in 12 minutes, then you hike 0.25 kilometers in 3 minutes. Subtract: you hike $4 - 0.25 = 3.75$ kilometers in $48 - 3 = 45$ minutes.

Therefore, you travel $2.75 + 3.75 = 6.5$ kilometers total. Half of that, 3.25 kilometers, is spent hiking east. The correct answer is **(D)**.

There are usually multiple ways to draw out the problem to get to the answer, so you aren't stuck trying to figure out the main textbook math method. Put yourself in the situation and ask yourself how you would go about this in the real world. You'd almost never start writing a bunch of equations; rather, you'd do "back of the envelope" arithmetic and estimation to get to the answer (or close enough!).

Rate and work problems, in particular, often lend themselves well to the draw it out method. Whenever you run across a problem that you think might work, try drawing out the problem to gain practice. At first, you may feel a bit slow, but you'll gain efficiency and accuracy with practice!

Write Out the Scenarios

Here's another variation on the draw it out method:

> During a week-long sale at a car dealership, the most number of cars sold on any one day was 12. If at least 2 cars were sold each day, was the average daily number of cars sold during that week more than 6?
>
> (1) During that week, the second smallest number of cars sold on

any one day was 4.
(2) During that week, the median number of cars sold was 10.

You're the manager of the car dealership and the owner has asked you to figure this out. How would you do so?

Start drawing out the scenarios. You know that the “highest” day was 12, but you don't know which day of the week that was. So how can you draw this out?

Glance at the statements. Neither statement provides information about a specific day of the week, either. Rather, they provide information about the least number of sales and the median number of sales.

The use of median is interesting. Maybe you should try organizing the number of sales from smallest to largest?

Draw out seven slots (one for each day) and add the information given in the question stem:

≥ 2	___	___	___	___	___	12

The question asks whether the average number of daily sales for the week is more than 6. Because this is a yes/no DS question, test each statement to see whether it can give you both a “Yes, the average is more than 6” answer and a “No, the average is not more than 6” answer. If so, then you'll know the statement is insufficient.

(1) During that week, the second smallest number of cars sold on any one day was 4.

Draw out a version of the scenario that includes statement (1):

≥ 2	4	___	___	___	___	12

Can you find a way to make the average less than 6? Keep the first day at 2 and make the other days as small as possible:

≥ 2	4	4	4	4	4	12

The sum of the numbers is 34. The average is $\frac{34}{7}$, which is a little less than 5.

Can you also make the average greater than 6? Try this:

$$\underline{3}\quad \underline{4}\quad \underline{12}\quad \underline{12}\quad \underline{12}\quad \underline{12}\quad \underline{12}$$

You may be able to eyeball that and tell it will be greater than 6. If not, calculate: the sum is 66, so the average is just less than 10.

Statement (1) is not sufficient because the average might be greater than or less than 6.

Cross off answers (A) and (D) and move to the statement (2).

> (2) During that week, the median number of cars sold was 10.

Again, draw out the scenario (using only the second statement this time!).

$$\underline{\geq 2}\quad \underline{\quad}\quad \underline{\quad}\quad \underline{10}\quad \underline{\quad}\quad \underline{\quad}\quad \underline{12}$$

Can you make the average less than 6? The three lowest days could each be 2. Then, the next three days could each be 10.

$$\underline{2}\quad \underline{2}\quad \underline{2}\quad \underline{10}\quad \underline{10}\quad \underline{10}\quad \underline{12}$$

The sum is 6 + 30 + 12 = 48. The average is $\frac{48}{7}$ just less than 7, but bigger than 6. The numbers cannot be made any smaller—you have to have a minimum of 2 a day. Once you hit the median of 10 in the middle slot, you have to have something greater than or equal to the median for the remaining slots to the right.

The smallest possible average is still bigger than 6, so this statement is sufficient to answer the question. The correct answer is (B).

If a problem talks about a set of numbers but doesn't give you the value of all of those numbers, try drawing out slots to represent each number in the set and stepping through the allowed possibilities. Make sure to test the extreme cases, depending upon what parameters the problem allows.

If a problem includes information about the median, you will probably want to order the numbers from least to greatest.

Maximizing and Minimizing

In other cases, a story problem might ask you to find the minimum or maximum possible value of something.

For example:

> There are enough available spaces on a school team to select at most $\frac{1}{3}$ of 50 students trying out for the team. What is the greatest number of students that could be rejected while still filling all available spaces for the team?
>
> (A) 32
> (B) 33
> (C) 34
> (D) 35
> (E) 36

You're asked to maximize the number of rejected students. You first have to fill all available spaces on the team, though. If at most $\frac{1}{3}$ of the students can be selected, then at most $\frac{50}{3}$, or $16\frac{2}{3}$, students can be selected. It's impossible to select $\frac{2}{3}$ of a person, of course, so the maximum possible is actually 16.

The maximum number of rejected students, then, is 50 − 16 = 34. The correct answer is **(C)**.

This problem has a hidden *integer constraint*: you have to assume that the numbers can only be integers, since you can't split a person. Many Word Problems have similar hidden constraints. Notice also that you have to be careful to round in the right direction—not up, but down. If the maximum number of available spaces is $16\frac{2}{3}$, then you cannot select 17 students. The maximum is 16.

Try another. How would you draw this problem out?

> Orange Computers is breaking up its conference attendees into groups. Each group must have exactly 1 person from Division A, 2 people from Division B, and 3 people from Division C. There are 20 people from Division A, 30 people from Division B, and 40 people from Division C at the conference. What is the smallest number of people who will not be able to be assigned to a group?
>
> (A) 12
> (B) 5
> (C) 2
> (D) 1
> (E) 0

You're in charge of the conference and you have to figure this out. First, jot down the given information on your scrap paper. Then, think about how you would figure this out in the real world.

Div.	**Total People**	**Per Group**
A	20	1
B	30	2
C	40	3

Try out some scenarios. If you start with Division A, you can make 20 groups.

Wait! Then you'd need 40 people from B and you have only 30. Hmm.

Okay, if you start with Division B, you can make 15 groups. Oh, but now you don't have enough people in C. Division C is the limiting factor—start there.

From Division C, you can make 13 groups of 3, using a total of 39 people. One person is left over.

Do you have enough Division B people? You need $13 \times 2 = 26$. Okay, there are 4 Division B people left over.

You'll also take 13 people from Division A, leaving 7 left over.

If you have 13 groups, then there are $1 + 4 + 7 = 12$ people without a group. The correct answer is **(A)**.

Some max/min problems will be more like the first one, where the path of the math is fairly straightforward, but you have to make decisions along the way about maximizing or minimizing other pieces in order to get to your desired answer.

In others, the starting point won't be so obvious. As with the second problem, you'll test a couple of cases until you find the limiting factor, and then you'll follow the math from there.

In both cases, make sure to pay attention to any constraints, especially those not explicitly stated. People and cars and rabbits cannot be split into fractional parts.

When in Doubt, Draw It Out

Drawing out a problem, doing “back of the envelope” rough calculations, and using logic to get through the math are fantastic ways to get through some especially annoying story problems—and not just when you're in doubt! As you get better at working in this way, you'll find that these methods are very effective even when you do know how to do the “textbook” version of the math.

Your steps are straightforward: put yourself in the problem. Pretend that you have to figure this out in the real world. Then ask yourself what you would do in order to find the answer—even if just to estimate and narrow down the answers.

Problem Set

1. A bookshelf holds both paperback and hardcover books. The ratio of paperback books to hardcover books is 22 to 3. How many paperback books are on the shelf?

   (1) The number of books on the shelf is between 202 and 247, inclusive.

   (2) If 18 paperback books were removed from the shelf and replaced with 18 hardcover books, the resulting ratio of paperback books to hardcover books on the shelf would be 4 to 1.

2. a, b, and c are integers in the set $\{a, b, c, 51, 85, 72\}$. Is the median of the set greater than 70?

   (1) $b > c > 69$

   (2) $a < c < 71$

3. Velma has exactly one week to learn all 71 Japanese hiragana characters. If she can learn at most a dozen of them on any one day and will only have time to learn four of them on Friday, what is the least number of hiragana that Velma will have to learn on Saturday?

4. An eccentric casino owner decides that his casino should only use chips in \$5 and \$7 denominations. Which of the following amounts cannot be paid out using these chips?

   (A) \$31 (B) \$29 (C) \$26 (D) \$23 (E) \$21

Solutions

1. **(D):** The question stem states that the ratio of paperback books to hardcover books is 22 to 3. What does this tell you? Try writing out scenarios:

# of Paperbacks	# of Hardcovers	Total # of Books
22	3	25
44	6	50
66	9	75

Although the total number of books changes, it is always a multiple of 25. This makes sense because the number of paperbacks will be in multiples of 22, the number of hardbacks will be in multiples of 3, and 22 + 3 sums to 25.

(1) SUFFICIENT: There is only one multiple of 25 between 202 and 247, so the total number of books must be 225. You could stop here, because only one possible value for the total implies only one possible value for the number of paperback books.

(2) SUFFICIENT: Hmm. Since this statement gives very specific information, writing out scenarios seems tricky since it would be hard to find a scenario that fits. It's more direct to set up and solve algebraically.

Let *p* be the number of paperbacks and *h* be the number of hardbacks. From the question set, $\frac{p}{h}=\frac{22}{3}$. From this statement, you get: $\left(\frac{p-18}{h+18}\right)=\frac{4}{1}$. This gives you two equations and two unknowns, so it is possible to solve for *p*, and you could stop here.

However, for the sake of completeness, the calculation follows:

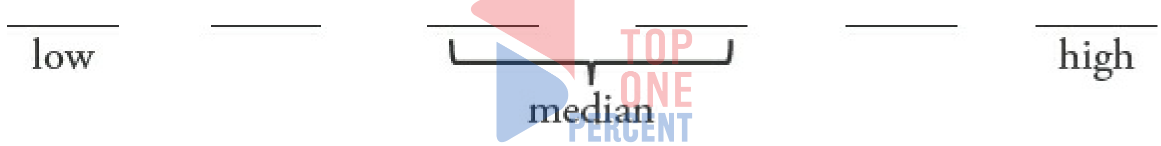
Cross-multiply both equations: $3p = 22h$, or $h=\frac{3p}{22}$, and $(p - 18) = 4(h + 18)$.

Substitute for p into the statement equation:

$$(p - 18) = 4((3p/22) + 18)$$
$$p - 18 = 6p/11 + 72$$
$$p - 6p/11 = 90$$
$$11p/11 - 6p/11 = 90$$
$$5p/11 = 90$$
$$p = (90)(11)/5$$
$$p = (18)(11) = 198$$

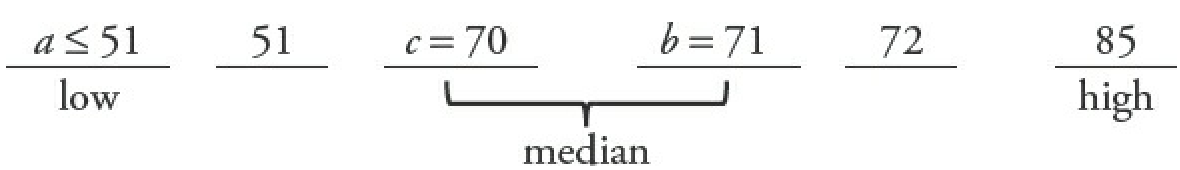
The correct answer is **(D)**.

2. (**A**)**:** Draw out six spaces and imagine they contain values ordered from low to high. What would the median be? Since there are an even number of numbers, the median of a set of six integers is the average of the two middle terms (the 3rd and 4th) when the terms are placed in order from low to high.

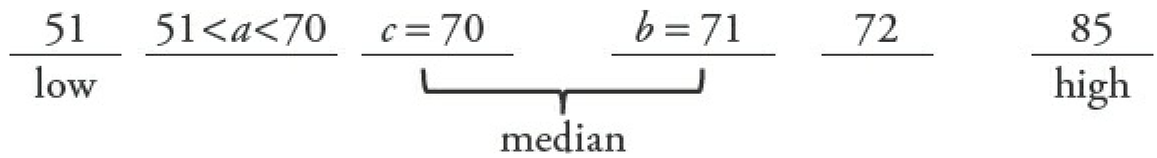


(1) SUFFICIENT: Look at the minimum case. If c is an integer greater than 69, the smallest c can be is 70. By similar logic, the smallest b could be is 71. In this case, the set is {51, 70, 71, 72, 85, a}. The only unknown is the value of a:

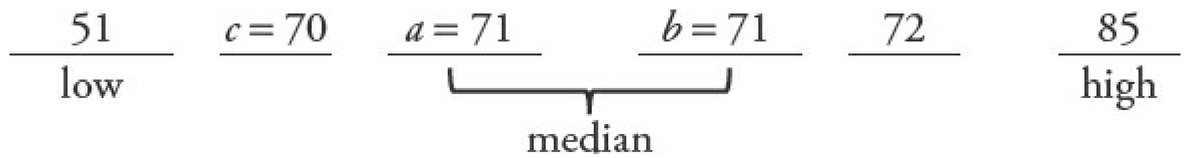
If $a \leq 51$, the ordered set is {a, 51, 70, 71, 72, 85}; median = (70 + 71)/2 = 70.5.



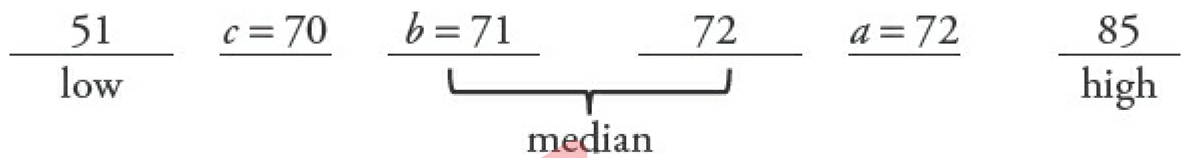
If $51 < a \leq 70$, the ordered set is {51, a, 70, 71, 72, 85}; median = (70 + 71)/2 = 70.5.



If a = 71, the ordered set is {51, 70, a, 71, 72, 85}; median = (71 + 71)/2 = 71.



If a = 72, the ordered set is {51, 70, 71, 72, a, 85}; median = (71 + 72)/2 = 71.5.



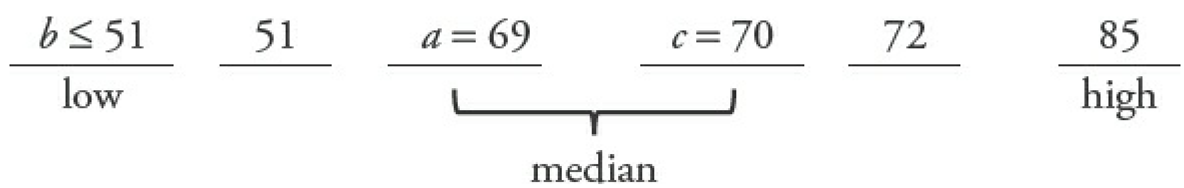
If a > 72, the order of the first four terms doesn't change from the line above, so the median is 71.5.

In all cases, the median is greater than 70, so the answer is a definite "yes."

Furthermore, if c and b are larger than the minimum case you tested, say c = 71 and b = 72 or c = 100 and b = 150, a quick check reveals that there is no a value that would make the median less than or equal to 70.

(2) INSUFFICIENT: Look at the maximum case. If c is an integer less than 71, the greatest c can be is 70. By similar logic, the greatest a could be is 69. In this case, the set is {b, 51, 69, 70, 72, 85}. The only unknown is the value of b:

If $b \leq 51$, the ordered set is {b, 51, 69, 70, 72, 85}; median = (69 + 70)/2 = 69.5.



If b = 70, the ordered set is {51, 69, 70, 70, 72, 85}; median = (70 + 70)/2 = 70.

51	$a = 69$	$b = 70$	$c = 70$	72	85
low		median			high

If b = 71, the ordered set is {51, 69, 70, 71, 72, 85}; median = (70 + 71)/2 = 70.5.

51	$a = 69$	$c = 70$	$b = 71$	72	85
low		median			high

In some cases, the median is greater than 70, but in other cases, it isn't. The answer is "maybe."

Thus, the correct answer is **(A)**.

3. **7:** Draw it out! Draw seven slots and label them for the days of the week. The problem states that Velma will learn 4 hiragana on Friday and at most 12 on any other day. Finally, you want the least possible number that she will need to learn on Saturday:

≤12	≤12	≤12	≤12	≤12	4	least?
Sun	M	Tu	W	Th	F	Sat

To minimize the number of hiragana that Velma will have to learn on Saturday, consider the extreme case in which she learns *as many* hiragana *as possible* on the other days. If Velma learns the maximum of 12 hiragana on the other five days (besides Saturday), then she will have 67 − 5(12) = 7 left for Saturday:

12	12	12	12	12	4	7
Sun	M	Tu	W	Th	F	Sat

4. **(D):** The payouts will have to be in the sum of some integer number of $5 chips and some integer number of $7 chips. Which of the answer choices cannot be the sum? One efficient way to eliminate choices is first to cross off any multiples of 7 and/or 5, which eliminates (E). Now, any other possible sums must have at least one 5 and one 7 in them. So you can subtract off 5's

one at a time until you reach a multiple of 7. (It is easier to subtract 5's than 7's, because our number system is base-10.) Thus:

> Answer choice (A): $31 - 5 = 26$; $26 - 5 = 21$, a multiple of 7; this eliminates (A). (In other words, $31 = 3 \times 7 + 2 \times 5$.)
>
> Answer choice (B): $29 - 5 = 24$; $24 - 5 = 19$; $19 - 5 = 14$, a multiple of 7; this eliminates (B).
>
> Answer choice (C): $26 - 5 = 21$, a multiple of 7; this eliminates (C).

So the answer must be **(D)**, 23. You check by successively subtracting 5 and looking for multiples of 7: $23 - 5 = 18$, not a multiple of 7; $18 - 5 = 13$, also not a multiple of 7; $13 - 5 = 8$, not a multiple of 7; and no smaller result will be a multiple of 7 either.

Chapter 10
of
Word Problems

Extra Overlapping Sets and Consecutive Integers

In This Chapter…

Chapter 10

Extra Overlapping Sets and Consecutive Integers

Two Sets, Three Choices: Still Double-Set Matrix

Very rarely, you might need to consider more than two options for one or both of the dimensions of your chart. As long as each set of distinct options is complete and has no overlaps, you can extend the chart.

For instance, if respondents can answer “Yes,” “No,” or “Maybe” to a survey question, and the question specifies whether the respondents are male or female, then you might set up the following matrix:

	Yes	No	Maybe	Total
Female				
Male				
Total				

The set of three answer choices is complete (there are no other options). Also, the choices do not overlap (no respondent can give more than one response). So this extended chart is fine.

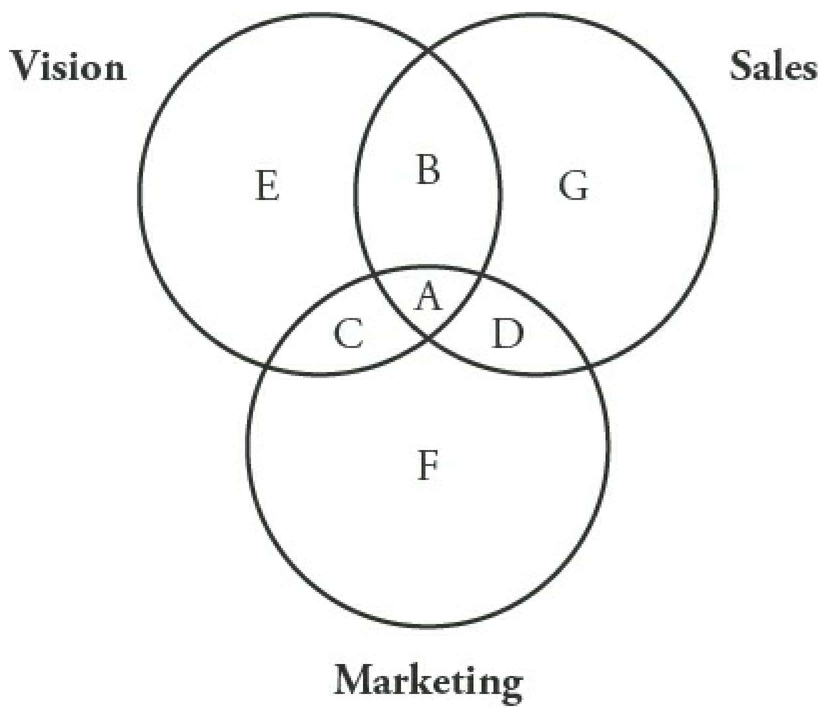
You rarely need to do real computation, but setting up an extended chart such as this can be helpful on certain Data Sufficiency problems, so that you can see what information is or is not sufficient to answer the given question.

Three-Set Problems: Venn Diagrams

Problems that involve three overlapping sets can be solved by using a Venn diagram. The three overlapping sets are usually three teams or clubs, and each person is either *on* or *not on* any given team or club. That is, there are only two choices for any club: member or not. For example:

> Workers are grouped by their areas of expertise and are placed on at least one team. There are 20 workers on the Marketing team, 30 on the Sales team, and 40 on the Vision team. Five workers are on both the Marketing and Sales teams, 6 workers are on both the Sales and Vision teams, 9 workers are on both the Marketing and Vision teams, and 4 workers are on all three teams. How many workers are there in total?

In order to solve this problem, use a Venn diagram instead of a double-set matrix.



Begin your Venn diagram by drawing three overlapping circles.

Notice that there are seven different sections in a Venn diagram. There is one innermost section **(A)** where all three circles overlap. This contains individuals who are on all three teams.

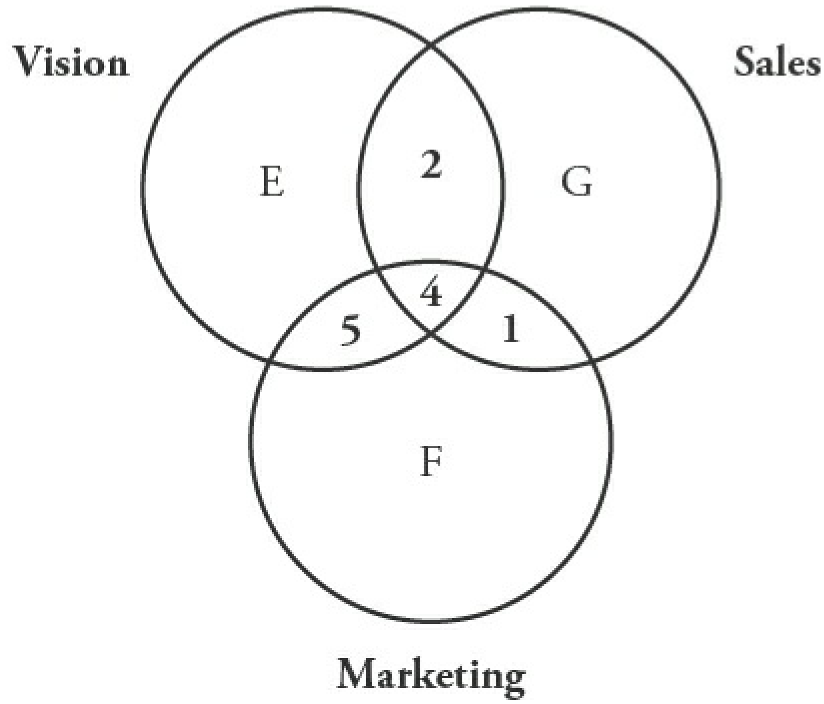
There are three sections **(B, C, and D)** where two circles overlap. These contain individuals who are on two teams. There are three non-overlapping sections **(E, F, and G)** that contain individuals who are on only one team.

Venn diagrams are easier to work with if you remember one simple rule: work from the inside out.

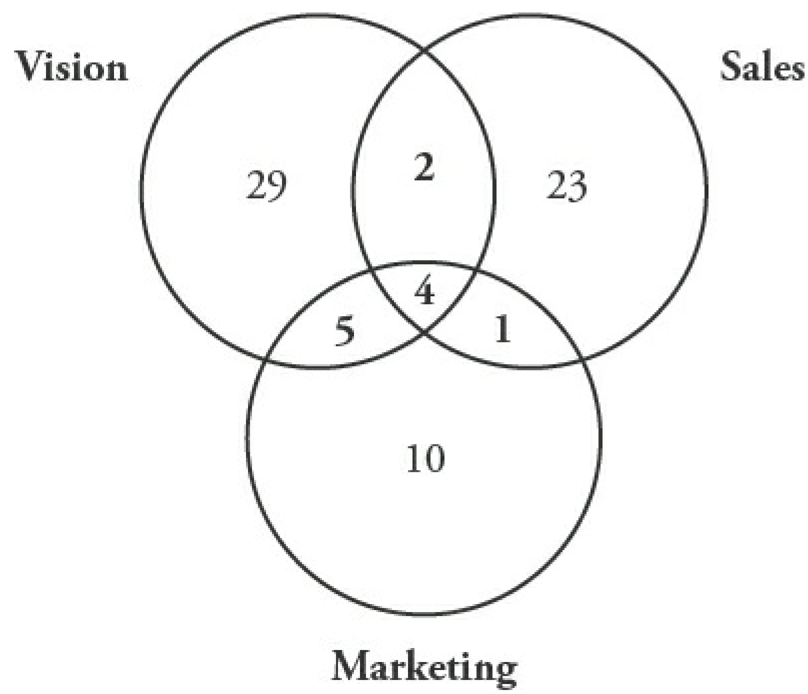
That is, it is easiest to begin by filling in a number in the innermost section **(A)**. Then, fill in numbers in the middle sections **(B, C, and D)**. Fill in the outermost sections **(E, F, and G)** last.

First: *Workers on all three teams.* Fill in the innermost circle. This is given in the problem as 4.

Second: *Workers on two teams.* Here you must remember to subtract those workers who are on all three teams. For example, the problem says that there are 5 workers on the Marketing and Sales teams. However, this includes the 4 workers who are on all three teams.



Therefore, in order to determine the number of workers who are on the Marketing and Sales teams exclusively, subtract the 4 workers who are on all three teams. You are left with 5 − 4 = 1. The number of workers on the Marketing and Vision teams exclusively is 9 − 4 = 5. The number of workers on the Sales and Vision teams exclusively is 6 − 4 = 2.



Third: *Workers on one team only*. Here you must remember to subtract those workers who are on two teams and those workers who are on three teams. For example, the problem says that there are 20 workers on the Marketing team. But this includes the 1 worker who is on the Marketing and Sales teams, the 5 workers who are on the Marketing and Vision teams, and the 4 workers who are on all three teams. Subtract all of these workers to find that there are $20 - 1 - 5 - 4 = 10$ people who are on the Marketing team exclusively. There are $30 - 1 - 2 - 4 = 23$ people on the Sales team exclusively. There are $40 - 2 - 5 - 4 = 29$ people on the Vision team exclusively.

In order to determine the total, add all seven numbers together: $29 + 5 + 4 + 2 + 1 + 23 + 10$. Thus, there are 74 total workers.

Products of Consecutive Integers and Divisibility

Can you come up with a series of three consecutive integers in which none of the integers is a multiple of 3? Go ahead, try it! You will find that any set of three consecutive integers must contain one multiple of 3. The result is that the product of any set of three consecutive integers is divisible by 3, as shown below:

$$1 \times 2 \times 3 = 6 \qquad 4 \times 5 \times 6 = 120$$
$$2 \times 3 \times 4 = 24 \qquad 5 \times 6 \times 7 = 210$$
$$3 \times 4 \times 5 = 60 \qquad 6 \times 7 \times 8 = 336$$

According to the Factor Foundation Rule, every number is divisible by all the factors of its factors. If there is always a multiple of 3 in a set of three consecutive integers, the product of three consecutive integers will always be divisible by 3. Additionally, there will always be at least one multiple of 2 (an even number) in any set of three consecutive integers. Therefore, the product of three consecutive integers will also be divisible by 2. Thus, the product of three consecutive integers will always be divisible by 3!: $3 \times 2 \times 1 = 6$.

The same logic applies to a set of four consecutive integers, five consecutive integers, and any other number of consecutive integers. For instance, the product of any set of 4 consecutive integers will be divisible by 4!—$4 \times 3 \times 2 \times 1 = 24$—since that set will always contain one multiple of 4, at least one multiple of 3, and another even number (a multiple of 2).

This rule applies to any number of consecutive integers: The product of k consecutive integers is always divisible by k factorial (k!).

Sums of Consecutive Integers and Divisibility

Find the sum of any five consecutive integers:

$4 + 5 + 6 + 7 + 8 = 30$	Notice that both sums are multiples of 5.
$13 + 14 + 15 + 16 + 17 = 75$	In other words, both sums are divisible by 5.

You can generalize this observation. For any set of consecutive integers with an ODD number of items, the sum of all the integers is ALWAYS a multiple of the number of items. This is true because the sum equals the average multiplied by the number of items. The average of {13, 14, 15, 16, 17} is 15, so $15 \times 5 = 13 + 14 + 15 + 16 + 17$.

Find the sum of any four consecutive integers:

$1 + 2 + 3 + 4 = 10$	Notice that NEITHER sum is a multiple of 4.
$8 + 9 + 10 + 11 = 38$	In other words, both sums are NOT divisible by 4.

For any set of consecutive integers with an EVEN number of items, the sum of all the items is NEVER a multiple of the number of items. This is true because the sum equals the average multiplied by the number of items. For an even number of integers, the average is never an integer, so the sum is never a multiple of the number of items. The average of {8, 9, 10, 11} is 9.5, so $9.5 \times 4 = 8 + 9 + 10 + 11$. That is, $8 + 9 + 10 + 11$ is NOT a multiple of 4.

Consider this Data Sufficiency problem:

Is k^2 odd?

(1) $k - 1$ is divisible by 2.
(2) The sum of k consecutive integers is divisible by k.

Statement (1) tells you that $k - 1$ is even. Therefore, k is odd, so k^2 will be odd. SUFFICIENT.

Statement (2) tells you that the sum of k consecutive integers is divisible by k. Therefore, this sum divided by k is an integer. Moreover, the sum of k consecutive integers divided by k is the average (arithmetic mean) of that set of k integers. As a result, statement (2) is telling you that the average of the k consecutive integers is itself an integer:

$$\frac{(\text{Sum of } k \text{ integers})}{k} = (\text{Average of } k \text{ integers}) = \text{Integer}$$

If the average of this set of consecutive integers is an integer, then k must be odd. SUFFICIENT.

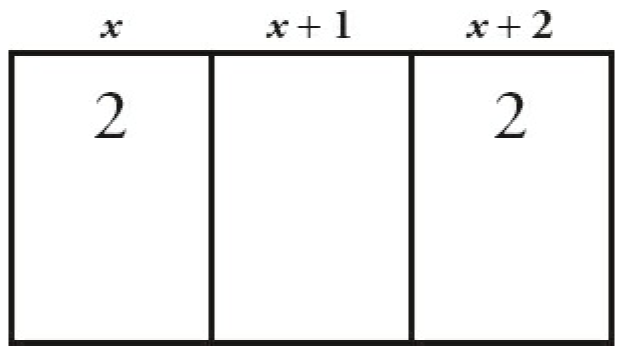
The correct answer is **(D)**.

Consecutive Integers and Divisibility

You can use prime boxes to keep track of factors of consecutive integers. (Refer to the Divisibility and Primes chapter of the *Number Properties* Strategy Guide for more information on prime boxes.) Consider the following problem:

If x is an even integer, is $x(x + 1)(x + 2)$ divisible by 4?

You know $x(x + 1)(x + 2)$ is the product of three consecutive integers, because x is an integer. If there is one even integer in a series of consecutive integers, the product of the series is divisible by 2. If there are two even integers in a series of consecutive integers, the product of the series is divisible by 4. Set up prime boxes:



If x is even, then $x + 2$ is even, so 2 is a factor of $x(x + 1)(x + 2)$ twice. Therefore, the product $2 \times 2 = 4$ is a factor of the product of the series. The answer to the question then is “yes.”

Scheduling

Scheduling problems, which require you to determine possible schedules satisfying a variety of constraints, can usually be tackled by careful consideration of **extreme possibilities**, usually the earliest and latest possible time slots for the events to be scheduled. Consider the following problem:

> How many days after the purchase of Product X does its standard warranty expire? (1997 is not a leap year.)

(1) When Mark purchased Product X in January 1997, the warranty did not expire until March 1997.

(2) When Santos purchased Product X in May 1997, the warranty expired in May 1997.

Rephrase the two statements in terms of extreme possibilities:

(1) Shortest possible warranty period: Jan. 31 to Mar. 1 (29 days later) Longest possible warranty period: Jan. 1 to Mar. 31 (89 days later) Note that 1997 was not a leap year.

(2) Shortest possible warranty period: May 1 to May 2, or similar (1 day later) Longest possible warranty period: May 1 to May 31 (30 days later)

Even taking both statements together, there are still two possibilities—29 days and 30 days—so both statements together are still insufficient.

Note that, had the given year been a leap year, the two statements together would have become sufficient! Moral of the story: *Read the problem very, very carefully*.

Problem Set

1. If r, s, and t are consecutive positive multiples of 3, is rst divisible by 27, 54, or both?

2. Is the sum of the integers from 54 to 153, inclusive, divisible by 100?

3. When it is 2:01pm Sunday afternoon in Nullepart, it is Monday in Eimissaan. When it is 1:00pm Wednesday in Eimissaan, it is also Wednesday in Nullepart. When it is noon Friday in Nullepart, what is the possible range of times in Eimissaan?

4. Is the average of n consecutive integers equal to 1?

   (1) n is even.

   (2) If S is the sum of the n consecutive integers, then $0 < S < n$.

5. Students are in clubs as follows: Science–20, Drama–30, and Band–12. No student is in all three clubs, but 8 are in both Science and Drama, 6 are in both Science and Band, and 4 are in Drama and Band. How many different students are in at least one of the three clubs?

6. If x, y, and z are consecutive integers, is $x + y + z$ divisible by 3?

7. A list kept at Town Hall contains the town's average daily temperature in Fahrenheit, rounded to the nearest integer. A particular completed month has either 30 or 31 days. How many days does the month have?

   (1) The median temperature is 73.5.

   (2) The sum of the average daily temperatures is divisible by 3.

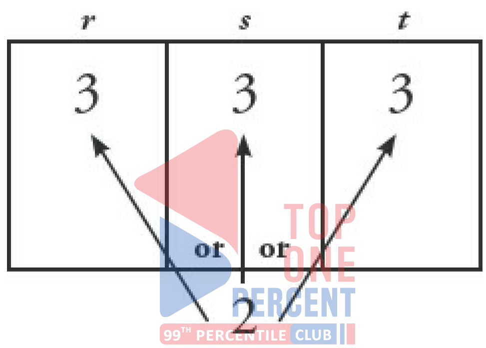
8. The 38 movies in the video store fall into the following three categories: 10 action, 20 drama, and 18 comedy. However, some movies are classified under more than one category: 5 are both action and drama, 3 are both

action and comedy, and 4 are both drama and comedy. How many action–drama–comedy movies are there?

9. Of 60 children, 30 are happy, 10 are sad, and 20 are neither happy nor sad. There are 20 boys and 40 girls. If there are 6 happy boys and 4 sad girls, how many boys are neither happy nor sad?

Solutions

1. **Both:** Because r, s, and t are all multiples of 3, the product rst must have THREE 3's as factors. Additionally, at least one of the integers must be even, so the product will have a 2 as a factor, because every other multiple of 3 is even (for example, 3, **6**, 9, **12**, etc.). $27 = 3 \times 3 \times 3$ can be constructed from the known prime factors and is therefore a factor of the product rst. $54 = 2 \times 3 \times 3 \times 3$ can also be constructed from the known prime factors and therefore is also factor of the product rst.



2. **No:** There are 100 integers from 54 to 153, inclusive. For any even number of consecutive integers, the sum of all the integers is *never* a multiple of the number of integers. Thus, the sum of the integers from 54 to 153 will not be divisible by 100.

3. **Anywhere from 10pm Friday to 1am Saturday:** The first statement tells you that the time in Eimissaan is at least 10 hours ahead of the time in Nullepart; given this information, the second statement tells you that the time in Eimissaan is at most 13 hours ahead of the time in Nullepart. (The second statement *by itself* could allow Nullepart time to be ahead of Eimissaan time, but that situation is already precluded by the first statement.) Therefore, the time in Eimissaan is between 10 and 13 hours ahead of the time in Nullepart.

4. **(D):** (1) SUFFICIENT: Statement (1) states that there is an even number of consecutive integers. This statement tells you nothing about the actual values of the integers, but the average of an even number of consecutive integers

will never be an integer. Therefore, the average of the *n* consecutive integers cannot equal 1.

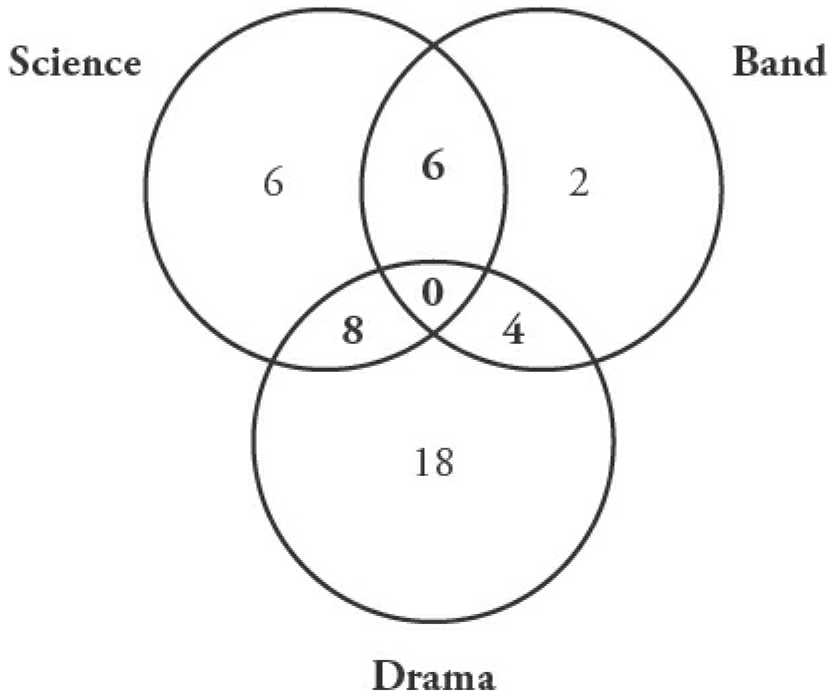
(2) SUFFICIENT: You know that the sum of the *n* consecutive integers is positive, but smaller than *n*. Perhaps the most straightforward way to interpret this statement is to express it in terms of the average of the *n* numbers, rather than the sum. Using the formula Average = Sum ÷ Number, you can reinterpret the statement by dividing the compound inequality by *n*:

$$0 < S < n \qquad \frac{0}{n} < \frac{S}{n} < \frac{n}{n} \qquad 0 < \frac{S}{n} < 1$$

This tells you that the average integer in set *S* is larger than 0 but less than 1. Therefore, the average number in the set does NOT equal 1, so the statement is sufficient. The correct answer is (D).

As a footnote, this situation can happen ONLY when there is an even number of integers, and when the "middle numbers" in the set are 0 and 1. For example, the set of consecutive integers {0, 1} has a median number of 0.5. Similarly, the set of consecutive integers {-3, -2, -1, 0, 1, 2, 3, 4} has a median number of 0.5.

5. **44 different students:** There are three overlapping sets here. Therefore, use a Venn diagram to solve the problem. First, fill in the numbers given in the problem, working from the inside out: no students in all three clubs, 8 in Science and Drama, 6 in Science and Band, and 4 in Drama and Band. Then, use the totals for each club to determine how many students are in only one club. For example, you know that there are 30 students in the Drama club. So far, you have placed 12 students in the circle that represents the Drama club (8 who are in Science and Drama, and 4 who are in Band and Drama). Therefore, 30 − 12 = 18, the number of students who are in only the Drama Club. Use this process to determine the number of students in just the Science and Band clubs as well. To find the number of students in at least one of the clubs, sum all the numbers in the diagram:



$6 + 18 + 2 + 6 + 8 + 4 = 44.$

6. **Yes:** For any odd number of consecutive integers, the sum of those integers is divisible by the number of integers. There are three consecutive integers (x, y, and z), so the rule applies in this case.

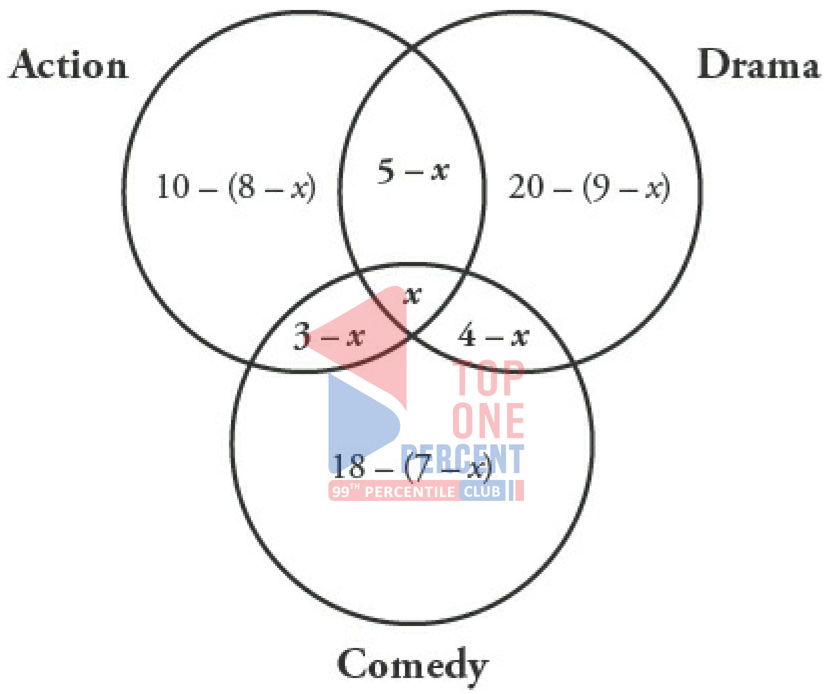
7. (**A):** This question is really about evens and odds. A list of values contains either 30 or 31 elements. Does the list have an even or an odd number of elements?

(1) SUFFICIENT: Since every item in the list is an integer, the only way for the median to be a non-integer is if there is an even number of items in the list (and therefore no middle term—in this case, the median is calculated as the average of the two middle terms). Therefore, the month must have an even number of days, so it must contain 30 days.

(2) INSUFFICIENT: The sum of either 30 or 31 values can be divisible by 3. Since there are no constraints on what the temperatures might be, it is perfectly possible to have a list of 30 values or a list of 31 values that add up to a multiple of 3. For example, if the temperature every day was 60 degrees, the sum of the temperatures would be divisible by 3 no matter how many days the month contained.

Therefore, the correct answer is **(A)**.

8. **2:** There are three overlapping sets here; therefore, use a Venn diagram to solve the problem. First, fill in the numbers given in the problem, working from the inside out. Assign the variable *x* to represent the number of action–drama–comedy movies. Then, create variable expressions, using the totals given in the problem, to represent the number of movies in each of the other categories. You know that there is a total of 38 movies; therefore, you can write the following equation to represent the total number of movies in the store:



$$
\begin{array}{rr}
 & 10-8+x \\
 & 20-9+x \\
 & 18-7+x \\
 & 5-x \\
 & 4-x \\
 & 3-x \\
+ & x \\
\hline
 & 36+x=38 \\
 & x=\ \ 2
\end{array}
$$

If you are unsure of the algebraic solution, you can also guess a number for *x* and fill in the rest of the diagram until the total number of movies reaches 38.

9. **8 boys:** Use a double-set matrix to solve this problem, with the “mood” set divided into three categories instead of only two. First, fill in the numbers given in the problem: of 60 children, 30 are happy, 10 are sad, and 20 are neither happy nor sad; 20 are boys and 40 are girls. You also know there are 6 happy boys and 4 sad girls. Therefore, by subtraction, there are 6 sad boys and there are 8 boys who are neither happy nor sad.

	Happy	Sad	Neither	Total
Boys	6	**6**	**8**	20
Girls		4		40
Total	30	10	20	60

Chapter 1
of
Geometry

Geometry Strategy

In This Chapter…

Chapter 1

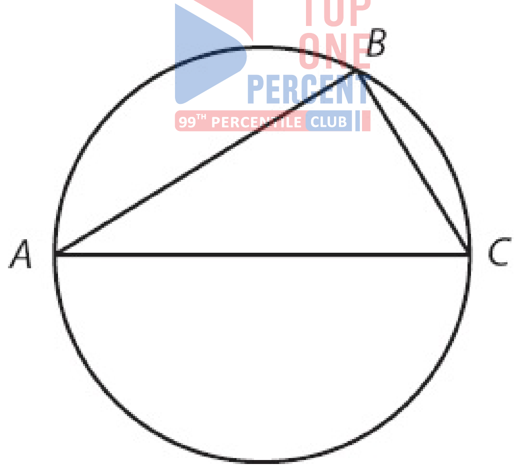
Geometry Strategy

Welcome to your *Geometry Strategy Guide*—and everything you ever wanted to know about geometry (as it's tested on the GMAT, anyway). Before diving into the rules and formulas, take a few minutes to learn some important strategies that will help you on every Geometry problem you will do on the real test.

The Three Principles

Use three general principles to succeed on Geometry problems:

1. If they don't tell you, don't assume.



> Points *A*, *B*, and *C* lie on the circle and form a triangle. Is line segment *AC* a diameter of the circle?

Line segment *AC* does look like a diameter of the circle, but it could be just slightly off and not a diameter at all. Don't make any assumptions; just *looking* like a diameter doesn't make *AC* a diameter on the GMAT.

Vocab Lesson: when a triangle lies inside a circle and the "points" (or *vertices*) of a triangle touch the circle, then the triangle is said to be *inscribed* in the circle.

2. If they give you a piece of information, use it.

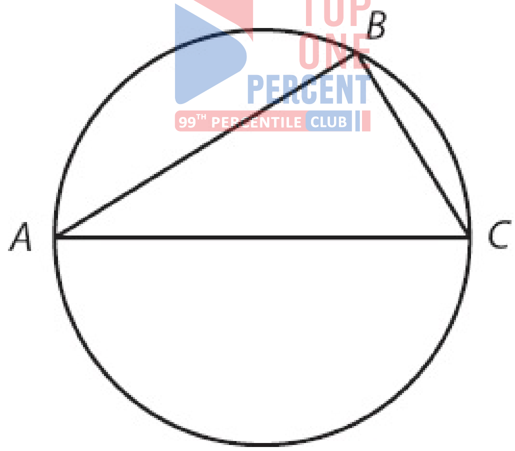
Given: line segment *AC* passes through the center of the circle.

What can you infer from that piece of information?

If a line segment passes from one side to the other of a circle through the center, then that line segment must be a diameter of the circle. Now, you've got a connection between the triangle and the circle: the longest side of the triangle is also a diameter of the circle.

How does that help? Read on—but note that, any time you're given multiple shapes, the trick to solving the problem usually revolves around finding connections between those shapes.

3. Know your rules and formulas.



Rule:

If one of the sides of a triangle inscribed in a circle is a diameter of the circle, then the triangle must be a right triangle.

Translation necessary! If you inscribe a triangle in a circle (as in the figure shown above), and one side of that triangle is also a diameter of the circle, then angle *B* has to be a right angle. It doesn't matter where you place *B* on the circle; it will still be a right angle. (Well, if you place *B* right on *A* or *C*, then *B* won't be a right angle. In that case, though, *ABC* also won't be a triangle!)

In short, it is not enough just to memorize a bunch of rules. The test writers are going to “cut up” the rules and give them to you in pieces. You need to know the rules well enough that you can put those pieces back together.

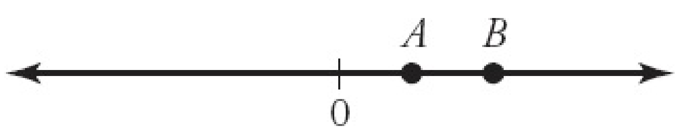
To recap:

1. If they don't tell you, don't assume.
2. If they give you a piece of information, use it.
3. Know your rules and formulas.

One last thing: it turns out, thankfully, that there are a few small things you *can* take for granted on GMAT Geometry.

If the problem describes a shape as a triangle, then it really is a triangle. If the problem discusses a line, then you really do have a 180° straight line. In other words, you can take the test at its word—it will use the word *line* in the official geometry sense—but you can't add in any extra assumptions.

Figures on Problem-Solving questions will be drawn to scale unless noted. Any points shown on a figure or number line do appear in the order shown. For instance, consider the figure below:

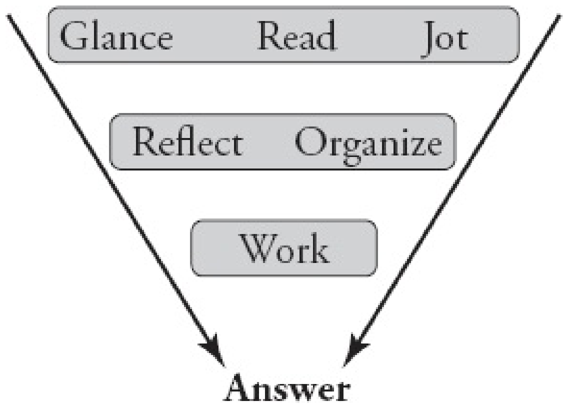


You can trust that both *A* and *B* are positive (because both are shown to the right of 0) and that *B* is greater than *A* (because *B* is to the right of *A*).

Figures on Data-Sufficiency questions, however, are *not* necessarily drawn to scale (and they will *not* be noted accordingly). You can trust that lines are lines, and that intersecting lines or shapes do intersect, including the relative positions of points, angles, and regions. Other than that, anything goes.

The Three-Step Approach

On all Quant problems, you're going to use a three-step approach to solving, as depicted in the figure below:



Your first task is to figure out what you've got. This will include a couple of special steps for Geometry problems.

1. Glance, read, jot: draw everything on your scrap paper.

Glance at the problem: is it problem solving or data sufficiency? Are there any figures or obvious formulas?

As you read, jot down any obvious information. If the problem gives you a formula, write it down. If it gives you a figure, redraw it on your scrap paper. If it doesn't, but it's a geometry problem, draw one anyway.

You have graph paper, so make the figure decently precise. Don't waste time or space, of course, but make the figure big enough that you can see what you're doing and accurate enough to prevent careless mistakes. For instance, if you know one side of a triangle is larger than another, draw the figure so that the longer side *looks* longer.

Add to the figure as you work. Every time you infer something new, write or draw it in. (Make sure, when you first draw the figure, that you give yourself enough space to draw and write additional information on it!)

Also, off to the side, write down any formulas that are mentioned in the problem. For example, if the problem mentions the area of a circle, immediately write $A = \pi r^2$ next to the figure.

2. Reflect, organize: identify the "wanted" element.

Don't dive into the calculations quite yet. Figure out what you need, first. Perhaps the question asks you to find the measure of angle x, which has already been labeled on the figure. Put a symbol, such as a star, next to the *x* to remind yourself that this is your goal. (You can use any symbol you want, as long as you use the same symbol consistently and as long as you use a symbol that will never be used by the test writers themselves.)

Perhaps the question asks you to find the perimeter of a rectangle. It would be tough to show that on the figure, so instead, write the formula for perimeter and put a star next to the P:

$$\star P = 2l + 2w$$

Alternatively, write something like:

$$P = \underline{\hspace{3em}}\,?$$

You have flexibility in terms of how you decide to show this information, however, you should develop a consistent method for noting what the question wants.

Finally, take a look at what you've been given. What possible solution paths come to mind? Choose one and…

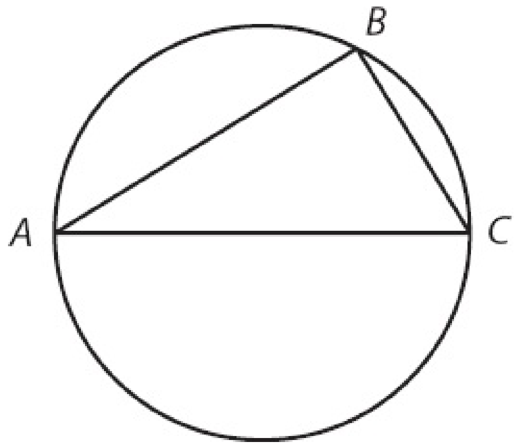
3. Work: infer from the given information.

The “givens,” or starting information, will allow you to deduce certain other things that must be true. Your task is to figure out the path from those givens to the answer.

Do you remember doing geometry proofs in school? You were given two or three starting points and had to prove, in a certain number of steps, that some other piece of information was true. People generally don't like proofs because it feels as though there are a million different ways that you could try to complete the proof.

GMAT questions can feel that way, too, but don't let that feeling demoralize you. On the vast majority of Geometry questions, you won't have to take more than three or four steps to find your way to the answer.

Try out the three-step process on this problem:



Triangle *ABC* is inscribed in the circle and line *AC* passes through the center of the circle. If the length of line segment *AB* is 3 and the length of line segment *AC* is 5, then what is the length of line segment *BC*?

(A) 2

(B) 3

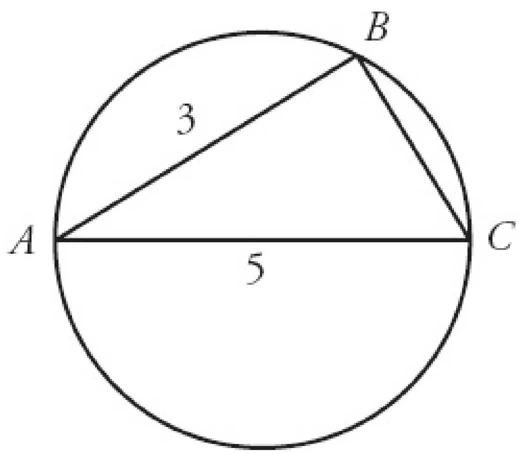
(C) 4

(D) 6

(E) 8

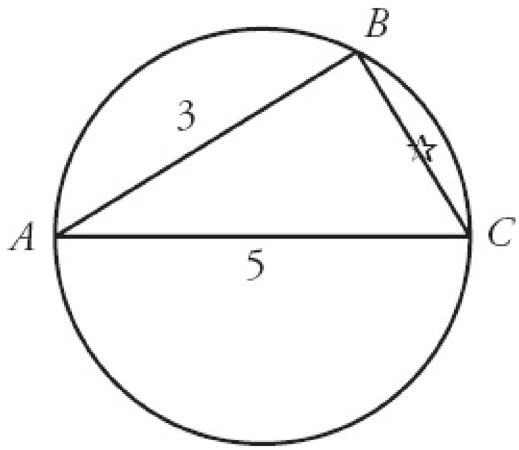
1. Glance, read, jot

Draw the figure on your scrap paper and add the given lengths:



2. Reflect, organize

Time to start thinking! What do they want? The length of line *BC*—add that to your figure:



What else should you think about? Usually, when the figure has multiple shapes, the solution will hinge upon some connection between those shapes.

3. Work

Bingo! Since line segment *AC* passes through the center of the circle, line segment *AC* must be a diameter of the circle. The two shapes are connected. What else can you infer?

Given that *AC* is a diameter, *ABC* must be a right triangle and angle *B* must be the right angle. Great! You can use the Pythagorean theorem to solve!

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
3^2 + b^2 &= 5^2 \\
9 + b^2 &= 25 \\
b^2 &= 16^2 \\
b &= 4
\end{aligned}
$$

The correct answer is 3.

Now, don't worry if you've completely forgotten about the Pythagorean theorem or any of the other math needed to answer this question. You'll re-learn how to do it all while working through this book.

Estimation

You can estimate your way to an answer on problems with certain characteristics; this technique is often helpful on Geometry problems in particular.

First, it's important that the problem gives you either a figure drawn to scale or enough information to draw a figure reasonably to scale yourself. Remember that your scrap paper will be graph paper, so you can draw right angles, squares, and other dimensions reasonably accurately.

Second, the answers need to be spread far enough apart that estimating an answer will still keep you in the range of the one correct answer.

For instance, say you are given these answer choices:

(A) 25°

(B) 45°

(C) 60°

(D) 90°

(E) 110°

You might not know how to calculate the correct answer, but you might be able to tell, for example, that the desired angle is less than 90°, which will eliminate (D) and (E). Alternatively, you might be able to tell that the answer is close to 90°, allowing you to chop out (A) and (B), and possibly (C).

This technique can be used on any Problem Solving problem (not just Geometry!) in which the answers are spread far apart, though on Geometry, you also benefit from a figure that's reasonably to scale.

Try this problem, inspired by one from *The Official Guide for GMAT Review 2015*:

> A square has a 10-centimeter diagonal. What is the area of the square, in centimeters?
>
> (A) 50

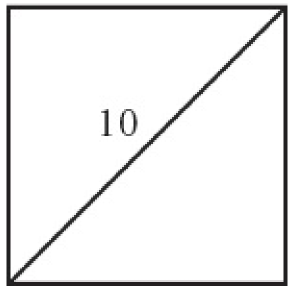
(B) 64

(C) 100

(D) 144

(E) 200

First, draw a square on your scrap paper. Remember, you'll have graph paper, so you can make a true square. Draw a diagonal and label it 10:



The area of a square is s^2, where s is the length of one side. Because the diagonal is 10, the length of one side must be less than 10. If the length of a side were 10, then the area would be 10^2, which equals 100, so if the length is less than 10, then the area must also be less than 100. Eliminate answers (C), (D), and (E). There are only two answers left!

You might be thinking, “that's too good to be true…the real test won't do that.” It does; in fact, this problem is based on a real question from a past official exam. If you have a copy of the *Official Guide 2015*, feel free to try Problem Solving question #104 right now.

You may also be thinking: I can already answer this problem; why would I estimate to make a guess?

First, you learn how to estimate on harder problems by practicing the skill on easier ones, so if you find this problem easy, don't dismiss the idea of estimating.

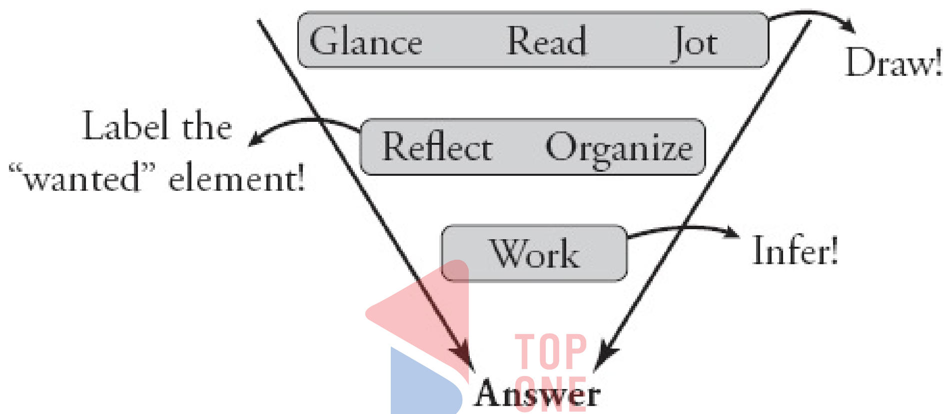
Second, you can use the rough estimation to check your work. Say that you made a calculation error and called the length of one side $10\sqrt{2}$. (There is a specific reason why someone might be susceptible to that particular mistake! If you're not sure what it is, look at this problem again after you've studied Chapter 4, Triangles & Diagonals.)

If you call one side $10\sqrt{2}$, then you're going to calculate the area as 200, which is answer (E). If you then double-check your work via estimation, you'll realize that 200 is too big.

The online *Official Guide* Problem Sets that come with this guide offer some additional *OG* problems on which you can test your estimation skills.

Get Started!

Start using the following ahree-step Approach on all Quant problems:



Remember to draw figures, label what you're looking for, and only then think about solving. Infer from the information given.

While you're working, avoid making assumptions. Make sure to use the information they give you. Finally, if you don't know the rules and formulas, it will be very tough to solve; make sure you know your rules!

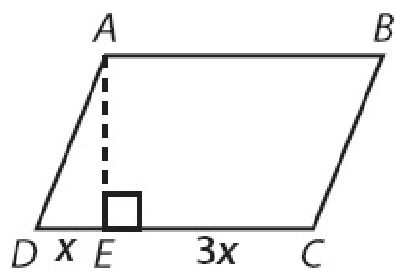
You're ready to dive into Geometry. Good luck!

Problem Set

If you think you remember some (or many!) geometry rules, use this problem set as a quiz to see where you need to review. If, on the other hand, you've totally forgotten all of your geometry rules, skip this set for now and come back to the problems after working through the relevant chapters in this book.

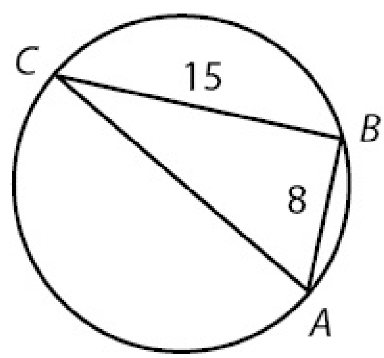
1. If the length of an edge of cube A is one-third the length of an edge of cube B, what is the ratio of the volume of cube A to the volume of cube B?

2.



ABCD is a parallelogram (see figure above). The ratio of *DE* to *EC* is 1 : 3. Height *AE* has a length of 3. If quadrilateral *ABCE* has an area of 21, what is the area of *ABCD*?

3.

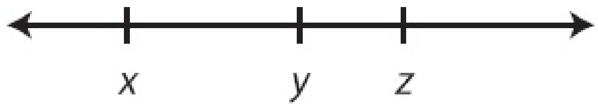


Triangle *ABC* is inscribed in a circle, such that *AC* is a diameter of the circle (see figure above). If *AB* has a length of 8 and *BC* has a length of 15, what is the circumference of the circle?

4. Triangle *ABC* is inscribed in a circle, such that *AC* is a diameter of the circle and angle *BAC* is 45°. If the area of triangle *ABC* is 72 square units, how much

larger is the area of the circle than the area of triangle *ABC*?

5.

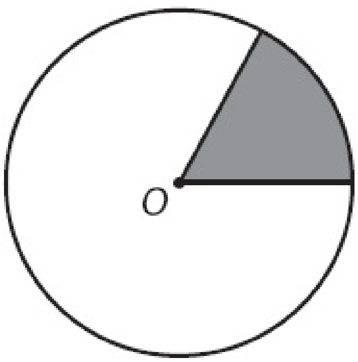


On the number line above, is $xy < 0$?

(1) Zero is to the left of y on the number line above.

(2) xy and yz have opposite signs.

6.

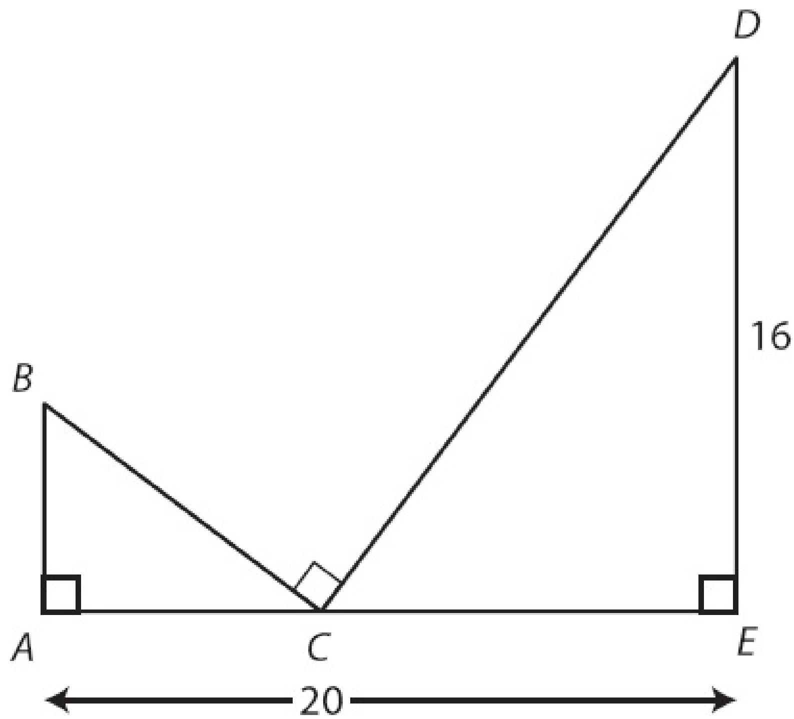


In the figure above, if *O* represents the center of a circular clock and the point of the clock hand is on the circumference of the circle, does the shaded sector of the clock represent more than 10 minutes?

(1) The clock hand has a length of 10.

(2) The area of the sector is greater than 16π.

7.



What is the area of triangle *ABC* (see figure above)?

(1) $DC = 20$

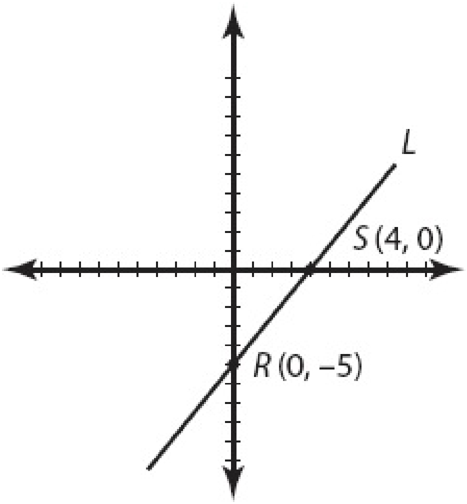
(2) $AC = 8$

8. The side of an equilateral triangle has the same length as the diagonal of a square. What is the area of the square?

(1) The height of the equilateral triangle is equal to $6\sqrt{3}$.

(2) The area of the equilateral triangle is equal to $36\sqrt{3}$.

9.



Line L passes through points R (0, −5) and S (4, 0) (see figure above). Point P with coordinates (x, y) is a point on line L. Is $xy > 0$?

(1) $x > 4$

(2) $y > -5$

Solutions

1. **1 to 27:** There are no specified amounts in this question, so pick numbers. You can say that cube A has sides of length 1 and cube B has sides of length 3:

Volume of cube A = $1 \times 1 \times 1 = 1$
Volume of cube B = $3 \times 3 \times 3 = 27$

Therefore, the ratio of the volume of cube A to the volume of cube B is $\frac{1}{27}$, or 1 : 27.

2. **24:** First, break quadrilateral *ABCE* into two pieces: a 3 by $3x$ rectangle and a right triangle with a base of x and a height of 3. Therefore, the area of quadrilateral *ABCE* is given by the following equation:

$$(3 \times 3x) + \frac{3 \times x}{2} = 9x + 1.5x = 10.5x$$

If *ABCE* has an area of 21, then $21 = 10.5x$, which reduces to $x = 2$. Quadrilateral *ABCD* is a parallelogram, so you can use the formula for area: $A = (\text{base}) \times (\text{height})$, or $4x \times 3$. Substitute the known value of 2 for x and simplify:

$A = 4(2) \times 3 = 24$

3. **17π:** If line segment *AC* is a diameter of the circle, then inscribed triangle *ABC* is a right triangle, with *AC* as the hypotenuse. Therefore, you can apply the Pythagorean Theorem to find the length of *AC*:

$$8^2 + 15^2 = c^2$$
$$64 + 225 = c^2$$
$$289 = c^2$$
$$c = 17$$

You might also have recognized the common 8–15–17 right triangle.

The circumference of the circle is πd, or 17π.

4. **72π – 72:** If *AC* is a diameter of the circle, then angle *ABC* is a right angle. Therefore, triangle *ABC* is a 45–45–90 triangle, and the base and the height are equal. Assign the variable *x* to represent both the base and height:

$$A = \frac{bh}{2}$$
$$72 = \frac{(x)(x)}{2}$$
$$144 = x^2$$
$$x = 12$$

Because this is a 45–45–90 triangle, and the two legs are equal to 12, the common ratio tells you that the hypotenuse, which is also the diameter of the circle, is $12\sqrt{2}$. Therefore, the radius is equal to $6\sqrt{2}$ and the area of the circle, πr^2, equals 72π. The area of the circle is $72\pi - 72$ square units larger than the area of triangle *ABC*.

5. **(C):** First, note that this is a Yes/No Data Sufficiency question.

For *xy* to be negative, *x* and *y* need to have opposite signs. On the number line shown, this would only happen if 0 falls between *x* and *y*. If 0 is to the left of *x* on the number line shown, both *x* and *y* would be positive, so $xy > 0$. If 0 is to the right of *y* on the number line shown, both *x* and *y* would be negative, so $xy > 0$.

(1) INSUFFICIENT: If zero is to the left of *y* on the number line, zero could be between *x* and *y*. Thus, $xy < 0$ and the answer to the question is yes. However, if 0 is to the left of *x*, both *x* and *y* would be positive, so $xy > 0$ and the answer is no.

(2) INSUFFICIENT: *xy* and *yz* having opposite signs implies that one of the three variables has a different sign than the other two. If *x*, *y*, and *z* all have the same sign, *xy* and *yz* would have the same sign. Thus, this statement implies that 0 does not fall to the left of *x* (which would make all three variables, as well as *xy* and *yz*, positive) nor to the right of *z* (which would make all three variables negative, and both *xy* and *yz* positive). The only two cases this statement allows are:

Zero is between x and y: yz is positive and xy is negative (the answer is “yes”).
Zero is between y and z: yz is negative and xy is positive (the answer is “no”).

(1) AND (2) SUFFICIENT: Statement (1) restricts 0 to left of y on the number line. This rules out one of the two cases allowed by statement (2), leaving only the case in which 0 is between x and y. Thus, xy is negative, and the answer is a definite yes.

The correct answer is **(C)**.

6. **(E):** First, note that this is a Yes/No Data Sufficiency question.

The question “Does the shaded sector of the clock represent more than 10 minutes?” is really asking you about the area of a sector of a circle.

Since 10 minutes is $\frac{1}{6}$ of an hour, you are being asked if the shaded region is equal to more than $\frac{1}{6}$ of the area of the circle.

(1) INSUFFICIENT: The “clock hand” is equal to the radius. Knowing that the radius equals 10 is enough to tell you that the entire area of the circle is equal to 100π. You can rephrase the question as, “Is the area of the shaded region more than one-sixth of 100π?” You can simplify $\frac{1}{6}$ of 100π as such:

$$\frac{100\pi}{6} = \frac{50\pi}{3} = 16.\overline{6}\pi$$

Thus, the question can be rephrased as, “Is the area of the shaded region more than $16.\overline{6}\pi$?” However, you don't know anything about the area of the shaded region from this statement alone.

(2) INSUFFICIENT: The area of the sector is more than 16π. By itself, this does not tell you anything about whether the area of the sector is more than $\frac{1}{6}$ the area of the circle, since you do not know the area of the entire circle.

(1) AND (2) INSUFFICIENT: The area of the entire circle is 100π, and the area of the sector is “more than 16π.”

Since $\frac{1}{6}$ of the area of the circle is actually $16.\overline{6}\pi$, knowing that the area of the sector is “more than 16π” is still insufficient—the area of the sector could be 16.1π or something much larger.

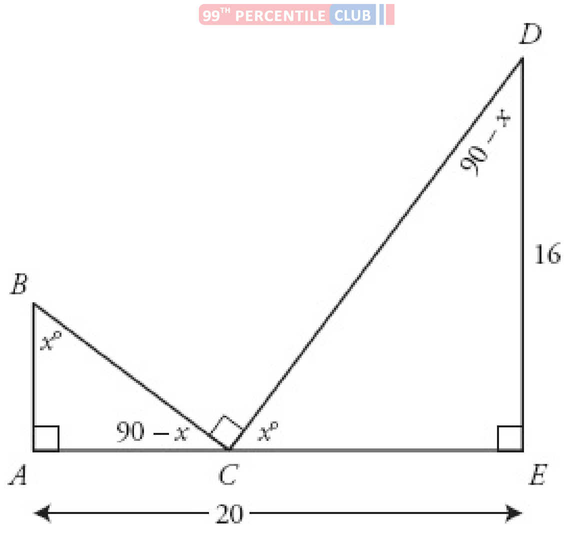
The correct answer is **(E)**.

7. **(D):** First, note that this is a Value Data Sufficiency question.

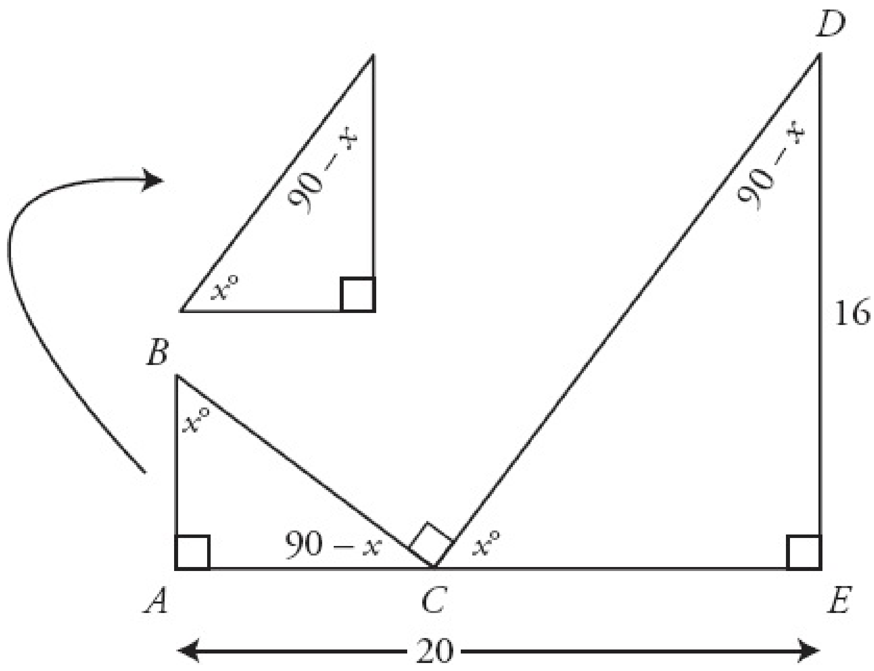
A big mistake in this problem would be to plunge into the statements without fully analyzing and exploiting the figure. You've got two right triangles that share a 90° span on either side of point *C*. What's going on here?

As it turns out, *these triangles are similar*.

Any time two triangles *each* have a right angle and *also* share an additional right angle (or, in this case, the 90° span at point *C*), they will be similar. But if you didn't know that, you could easily uncover that fact by labeling any angle as *x* and labeling the others in terms of *x*:



Once you determine that both triangles have the angles 90°, *x*, and 90 − *x*, you may wish to redraw one or both of them in order to get them facing in the same direction.



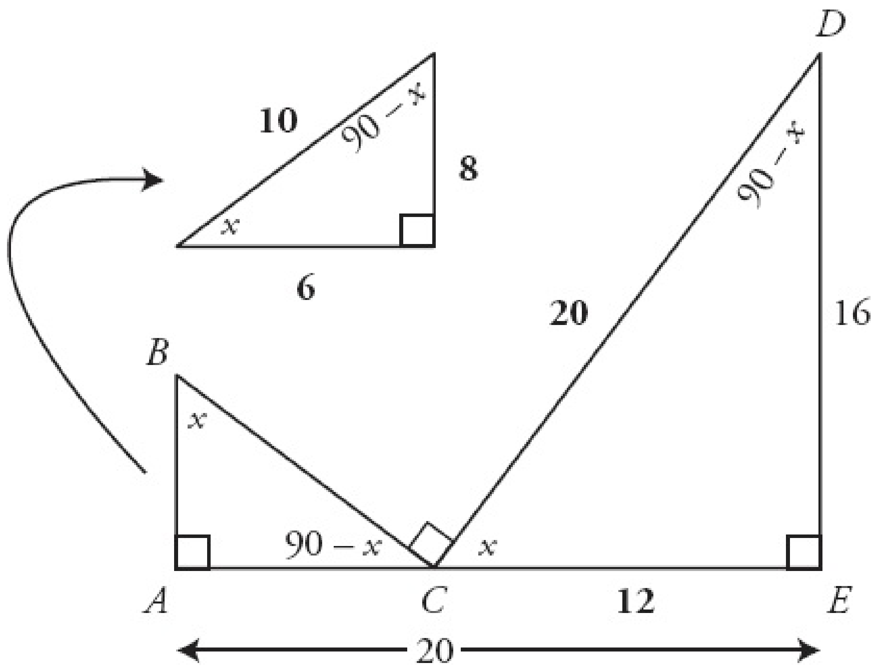
Now, decide exactly what the question is asking. You need the area of triangle *ABC*. In order to get that, you need the base and height of that triangle.

Since the two triangles are right triangles, if you had any two sides of triangle *ABC*, you could get the third. Because the two triangles are similar, you could use any two sides of triangle *CDE* (note that you already have that side *DE* = 16), as well as the ratio of one triangle's size to the other, to get the third side of *CDE* as well as all three sides of *ABC*.

Thus, the rephrased question is, "What are any two sides of *ABC*, or what is any additional side of *CDE* plus the ratio of the size of each triangle to the other?"

(1) SUFFICIENT: Side *DC* equals 20. Use the 20 and the 16 to get, via the Pythagorean theorem, that side *CE* equals 12 (or simply recognize that you have a multiple of a 3–4–5 triangle). If *CE* equals 12 then *AC* equals 8. Thus, you have all three sides of *CDE*, plus the ratio of one triangle to the other (side *AC*, which equals 8, matches up with side *DE*, which equals 16; thus the smaller triangle is one-half the side of the larger).

Note that it is totally unnecessary to calculate further (once you have correctly rephrased the question, don't waste time doing more than is needed to answer the rephrase!), but if you are curious:



(2) SUFFICIENT: Side *AC* equals 8. Note that this gives you the same information as Statement 1. If *AC* equals 8, then *CE* equals 12 and you can calculate all three sides of *CDE*. Once you know that side *AC* equals 8 and that *AC* matches up with *DE*, which is equal to 16, you can know all three sides of *ABC*, as above.

The correct answer is **(D)**.

8. **(D):** No calculation is needed to solve this problem. Both equilateral triangles and squares are *regular figures*—those that can change size, but never shape.

Regular figures (squares, equilaterals, circles, spheres, cubes, 45–45–90 triangles, 30–60–90 triangles, and others) are those for which you only need one measurement to know *every* measurement. For instance, if you have the radius of a circle, you can get the diameter, circumference, and area. If you have a 45–45–90 or 30–60–90 triangle, you only need *one* side to get all three. In this problem, if you have the side of an equilateral, you could get the height, area, and perimeter. If you have the side of a square, you could get the diagonal, area, and perimeter.

If you have *two* regular figures, as you do in this problem, and you know how they are related numerically ("the side of an equilateral triangle has the same length as the diagonal of a square"), then you can safely conclude that *any* measurement for *either* figure will give you *any* measurement for either figure.

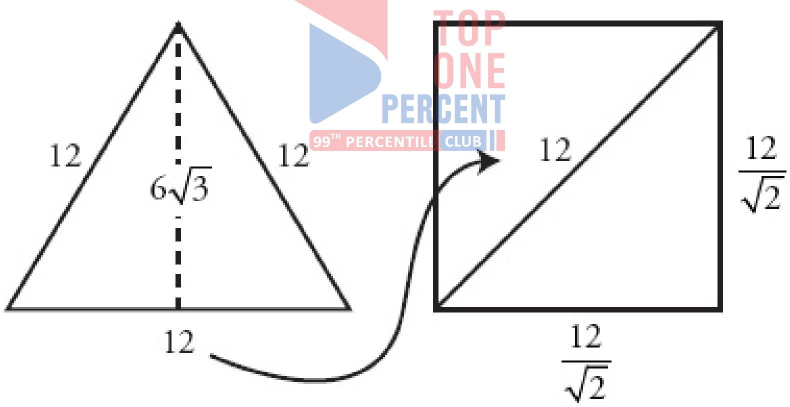
The question can be rephrased as, *"What is the length of any part of either figure?"*

(1) This gives you the height of the triangle. SUFFICIENT.

(2) This gives you the area of the triangle. SUFFICIENT.

If you really wanted to "prove" that the answer is (D), you could waste a lot of time:

From statement 1, if the height of the equilateral is $6\sqrt{3}$, then the side equals 12, because heights and sides of equilaterals always exist in that ratio (the height is always one-half the side times $\sqrt{3}$). Then you would know that the diagonal of the square was also equal to 12, and from there you could use the 45–45–90 formula to conclude that the side of the square was $\frac{12}{\sqrt{2}}$, and therefore that the area was $\frac{144}{2}$, or 72.



Similarly, from statement 2, you could conclude that if the area of the triangle is $36\sqrt{3}$, then the base times the height is $72\sqrt{3}$, and that since the side and height of an equilateral always exist in a fixed ratio (as above, the height is always one-half the side times $\sqrt{3}$), that the side is 12 and the height is $6\sqrt{3}$. Then, as above, you would know that the diagonal of the square was also equal to 12, and from there you could use the 45–45–90 formula to conclude that the side of the square was $\frac{12}{\sqrt{2}}$, and therefore that the area was $\frac{144}{2}$, or 72.

Who's got the time? This is a logic problem more than it is a math problem. If you understand the logic behind *regular figures*, you can answer this question in under 30 seconds with no math whatsoever.

The correct answer is (D).

9. **(A):** First, note that this is a Yes/No Data Sufficiency question.

Line L passes through three quadrants:

1. Quadrant I, where x and y are both positive, so $xy > 0$ and the answer is yes.
2. Quadrant III, where x and y are both negative, so $xy > 0$ and the answer is yes.
3. Quadrant IV, where x is positive and y is negative, so $xy < 0$ and the answer is no.

If you can determine what quadrant point P is in, you will have sufficient information to answer the question. Also, if you know that point P is in either Quadrant I or Quadrant III, that would also be sufficient.

(1) SUFFICIENT: If $x > 4$, then point P is in Quadrant I, so $xy > 0$ and the answer is "yes."

(2) INSUFFICIENT: If $y > -5$, then point P could be in either Quadrant I ($xy > 0$) or Quadrant IV ($xy < 0$).

The correct answer is (A).

Chapter 2
of
Geometry

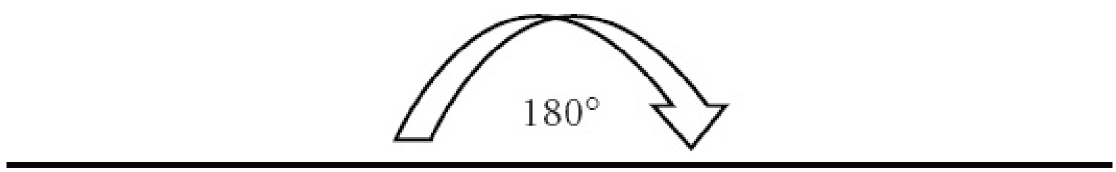
Lines & Angles

In This Chapter…

Chapter 2

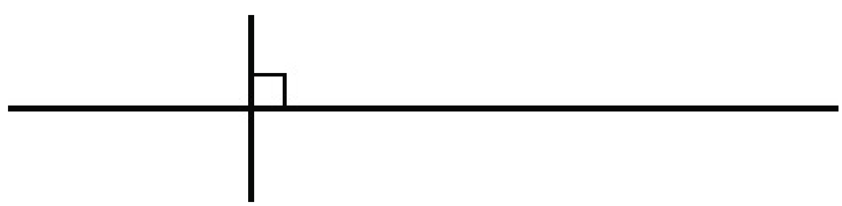
Lines & Angles

A straight line is the shortest distance between two points. As an angle, a line measures 180°.



Parallel lines are lines that lie in a plane and that never intersect. No matter how far you extend the lines, they never meet. Two parallel lines are shown below:

Perpendicular lines are lines that intersect at a 90° angle. Two perpendicular lines are shown below:



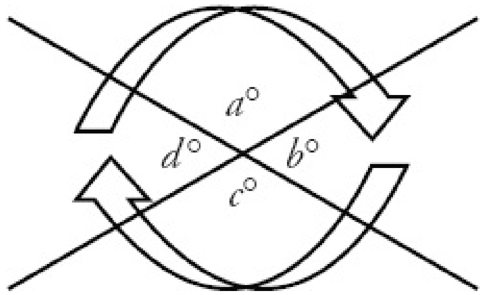
There are two major line–angle relationships to know for the GMAT. You'll learn about both in this chapter:

1. The angles formed by any intersecting lines
2. The angles formed by parallel lines cut by a transversal

Intersecting Lines

Intersecting lines have three important properties.

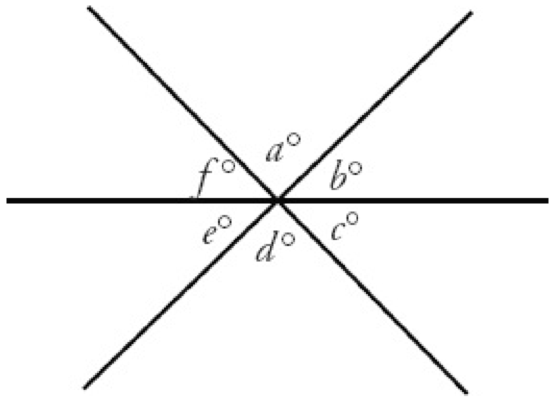
First, the interior angles formed by intersecting lines form a circle, so the sum of these angles is 360°. In the figure to the right: $a + b + c + d = 360$.



Second, interior angles that combine to form a line sum to 180°. Thus, in the figure shown, $a + b = 180$, because angles a and b form a line together. Other pairs of angles are $b + c = 180$, $c + d = 180$, and $d + a = 180$.

Third, angles found opposite each other where two lines intersect are equal. These are called **vertical angles**. Thus, in the figure above, $a = c$, because these angles are opposite each other and are formed from the same two lines. Additionally, $b = d$ for the same reason.

Note that these rules apply to more than two lines that intersect at a point, as shown to the right. In this figure, $a + b + c + d + e + f = 360$, because these angles combine to form a circle. In addition, $a + b + c = 180$, because these three angles combine to form a line. Finally, $a = d$, $b = e$, and $c = f$, because they are pairs of vertical angles.

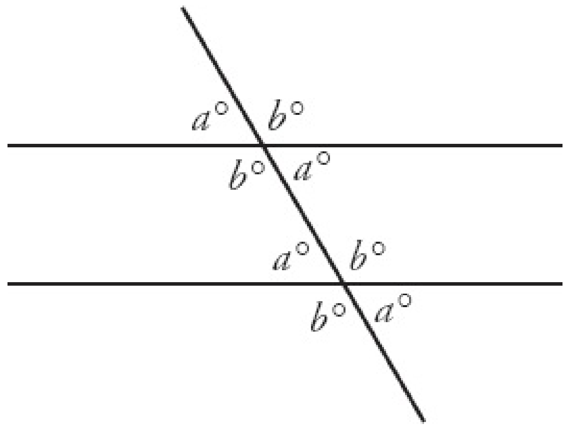


Parallel Lines Cut By a Transversal

The GMAT makes frequent use of figures that include parallel lines cut by a

transversal.

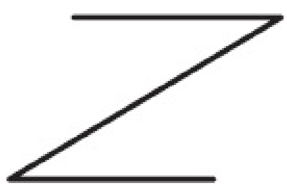
Notice that there are eight angles formed by this construction, but there are only two *different* angle measures (a and b). All the **acute** angles (less than 90°) in this figure are equal. Likewise, all the **obtuse** angles (greater than 90° but less than 180°) are equal. Any acute angle plus any obtuse angle equals 180°.



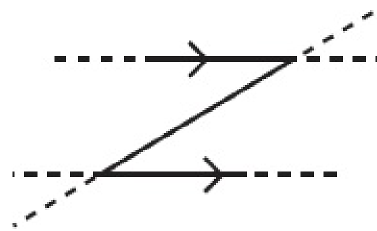
Thus, $a + b = 180$.

When you see a transversal cutting two lines that you know to be parallel, fill in all the a (acute) and b (obtuse) angles, as in the figure on the previous page.

Sometimes the GMAT disguises the parallel lines and the transversal so that they are not readily apparent, as in the figure to the right. In these disguised cases, extend the lines so that you can more easily see the parallel lines and the transversal, as in the second figure. Label the acute and obtuse angles. You might also mark the parallel lines with arrows, as shown in the figure, in order to indicate that the two lines are parallel.



The GMAT uses the symbol || to indicate in text that two lines or line segments are parallel. For instance, if you see $MN \parallel OP$ in a problem, you know that line segment MN is parallel to line segment OP.

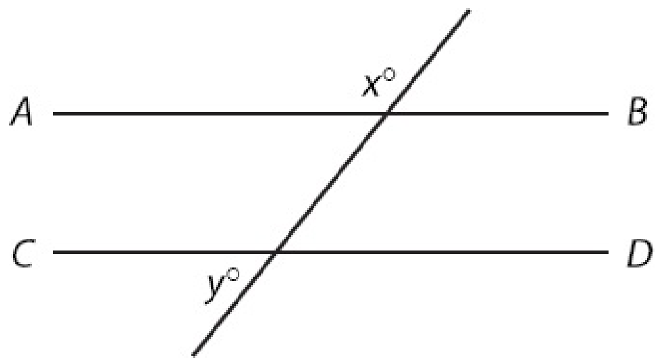


Problem Set

Problems 1–2 refer to the figure to the right, where line *AB* is parallel to line *CD*.

1. If $x - y = 10$, what is x?

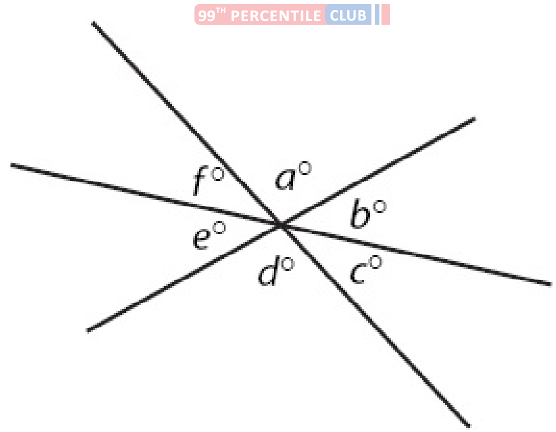
2. If $x + (x + y) = 320$, what is x?



Problems 3–4 refer to the figure to the right.

3. If a is 95, what is $b + d - e$?

4. If $c + f = 70$, and $d = 80$, what is b?

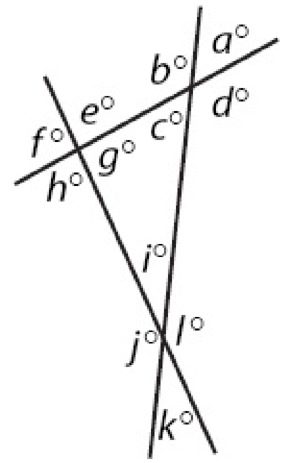


Problems 5–7 refer to the figure to the right.

5. If $c + g = 140$, find k.

6. If $g = 90$, what is $a + k$?

7. If $f + k = 150$, find b.


a°
b°
d°
c°
e°
f°
g°
h°
i°
j°
l°
k°

Solutions

1. **95°**: You know that $x + y = 180$, since the figure is a transversal cutting across two parallel lines, in which any acute angle plus any obtuse angle equals 180. Add the two equations together to eliminate the *y* variable and solve for *x*:

$$\begin{array}{rr} & x + y = 180 \\ + & x - y = 10 \\ \hline & 2x = 190 \\ & x = 95 \end{array}$$

2. **140°**: Subtract the equation $x + y = 180$ from $2x + y = 320$ to eliminate *y* and solve for *x*:

$$\begin{array}{rr} & 2x + y = 320 \\ - & (x + y = 180) \\ \hline & x \qquad = 140 \end{array}$$

Don't forget to *subtract* each element in the second line.

Alternatively, because you know that $x + y = 180$, you can substitute this into the given equation of $x + (x + y) = 320$ to solve for *x*:

$$\begin{aligned} x + 180 &= 320 \\ x &= 140 \end{aligned}$$

3. **95°**: Because *a* and *d* are vertical angles, they have the same measure: $a = d = 95$. Likewise, since *b* and *e* are vertical angles, they have the same measure: $b = e$. Therefore, $b + d - e = b + d - b = d = 95$.

4. **65°**: Because *c* and *f* are vertical angles, they have the same measure: $c + f = 70$, so $c = f = 35$. Notice that *b*, *c*, and *d* form a straight line: $b + c + d = 180$. Substitute the known values of *c* and *d* into this equation:

$$\begin{aligned} b + 35 + 80 &= 180 \\ b + 115 &= 180 \\ b &= 65 \end{aligned}$$

5. **40°**: If $c + g = 140$, then $i = 40$, because there are 180° in a triangle. Since k is vertical to i, k is also equal to 40.

6. **90°**: If $g = 90$, then the other two angles in the triangle, c and i, sum to 90. Since a and k are vertical angles to c and i, they sum to 90 as well.

7. **150°**: Angles f and k are vertical to angles g and i. The latter two angles, then, must also sum to 150. Therefore, the third angle in the triangle must be 180 – 150, so $c = 30$. Because $c + b = 180$, $30 + b = 180$, and $b = 150$.

Chapter 3
of
Geometry

Polygons

In This Chapter…

Chapter 3

Polygons

A polygon is defined as a closed shape formed by line segments. The polygons tested on the GMAT include the following:

- Three-sided shapes (triangles)
- Four-sided shapes (quadrilaterals)
- Other polygons with *n* sides (where *n* is five or more)

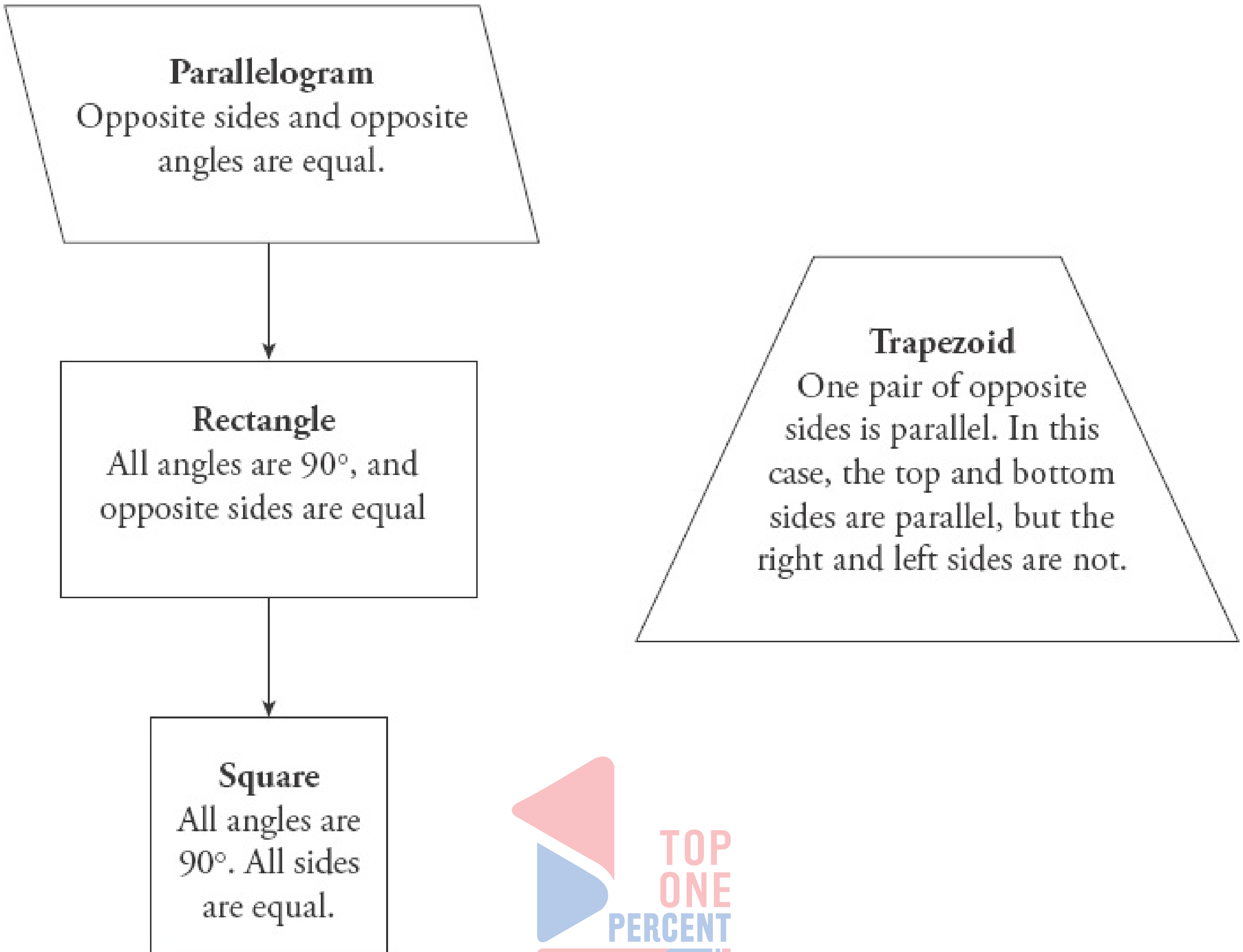
This section will focus on polygons of four or more sides. In particular, the GMAT emphasizes quadrilaterals—or four-sided polygons—such as squares, rectangles, and less common shapes including trapezoids and parallelograms.

Polygons are two-dimensional shapes—they lie in a plane. The GMAT tests your ability to work with different measurements associated with polygons. In this book, you will learn how to work with angles, lengths, perimeter, and area.

The GMAT also tests your knowledge of three-dimensional shapes formed from polygons, particularly rectangular solids and cubes. You will learn how to calculate surface area and volume.

Quadrilaterals: An Overview

The most common polygon tested on the GMAT, aside from the triangle, is the quadrilateral (any four-sided polygon). Almost all GMAT polygon problems involve the special types of quadrilaterals shown below:



Polygons and Interior Angles

The sum of the interior angles of a given polygon depends only on the **number of sides in the polygon**. The following table displays the relationship between the type of polygon and the sum of its interior angles.

The sum of the interior angles of a polygon follows a specific pattern that depends on *n*, the number of sides that the polygon has.

Polygon	# of Sides	Sum of Interior Angles
Triangle	3	180°
Quadrilateral	4	360°
Pentagon	5	540°

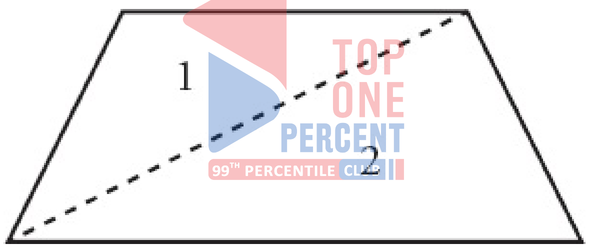
Hexagon	6	720°

This pattern can be expressed with the following formula:

$$(\boldsymbol{n} - \mathbf{2}) \times \mathbf{180} = \textbf{Sum of Interior Angles of a Polygon}$$

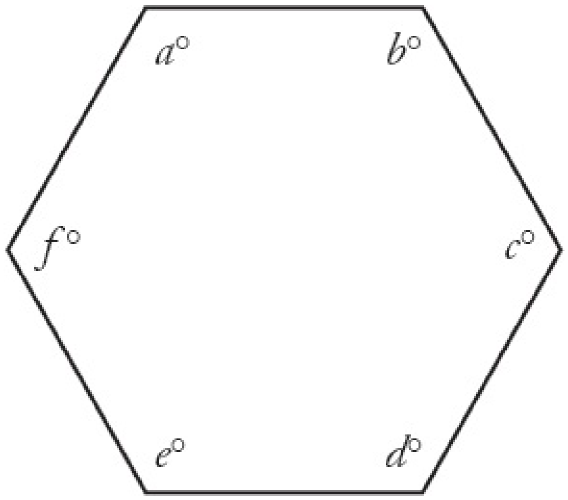
Since this polygon has four sides, the sum of its interior angles is $(4 - 2)180 = 2(180) = 360°$.

Alternatively, note that a quadrilateral can be cut into two triangles by a line connecting opposite corners. Thus, the sum of the angles is $2(180) = 360°$.

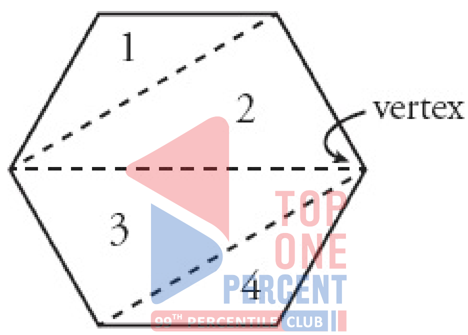


The interior angles of all four-sided polygons will always sum to 360°.

Since the next polygon shown to the right has six sides, the sum of its interior angles is $(6 - 2)180 = 4(180) = 720°$.



Alternatively, note that a hexagon can be cut into four triangles by three lines connecting corners:



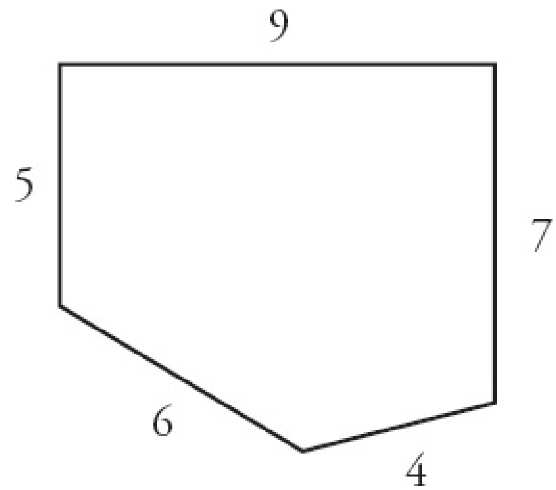
Thus, the sum of the angles is 4(180) = 720°.

By the way, the corners of polygons are also known as vertices (singular: vertex).

Polygons and Perimeter

The perimeter refers to the distance around a polygon, or the sum of the lengths of all the sides. The amount of fencing needed to surround a yard would be equivalent to the perimeter of that yard (the sum of all the sides).

The perimeter of the pentagon to the right is: 9 + 7 + 4 + 6 + 5 = 31.

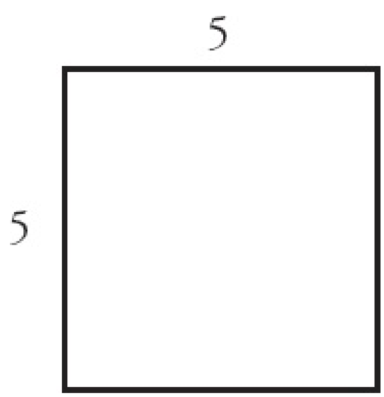


Polygons and Area

The area of a polygon refers to the space inside the polygon. Area is measured in square units, such as cm^2 (square centimeters), m^2 (square meters), or ft^2 (square feet). For example, the amount of space that a garden occupies is the area of that garden.

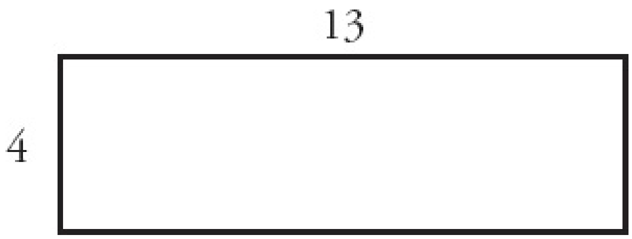
On the GMAT, you must know at least the first two area formulas:

1. Area of a Square = **Side** × **Side** = **Side**^{2}



The side length of this square is 5. Therefore, the area is $5^2 = 25$.

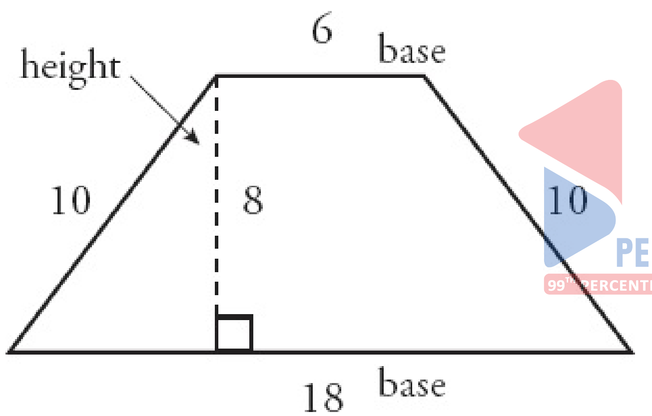
2. Area of a Rectangle = **Length** × **Width**



The length of this rectangle is 13, and the width is 4. Therefore, the area is 13 × 4 = 52.

The GMAT will occasionally ask you to find the area of a polygon more complex than a square or rectangle. The following formulas can be used to find the areas of other types of quadrilaterals:

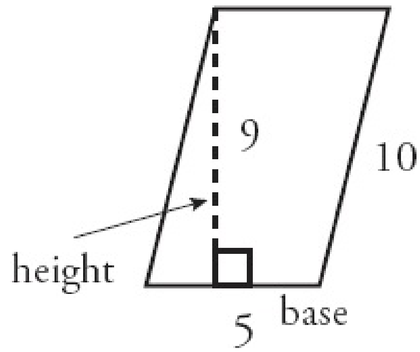
3. Area of a Trapezoid = $\dfrac{(\mathbf{Base_1} + \mathbf{Base_2}) \times \mathbf{Height}}{\mathbf{2}}$



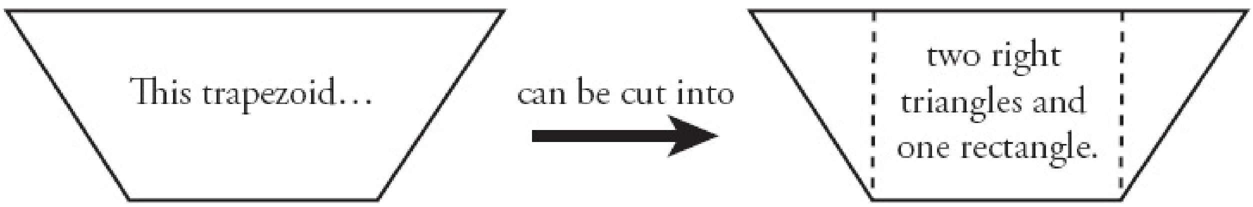
Note that the height refers to a line perpendicular to the two bases, which are parallel. (You often have to draw in the height, as in this case.) In the trapezoid shown, $base_1 = 18$, $base_2 = 6$, and the height = 8. The area is (18 + 6) × 8 ÷ 2 = 96. Another way to think about this is to take the *average* of the two bases and multiply it by the height.

4. Area of any Parallelogram = **Base** × **Height**

Note that the height refers to the line perpendicular to the base. (As with the trapezoid, you often have to draw in the height.) In the parallelogram shown, the base is 5 and the height is 9. Therefore, the area is 5 × 9 = 45.



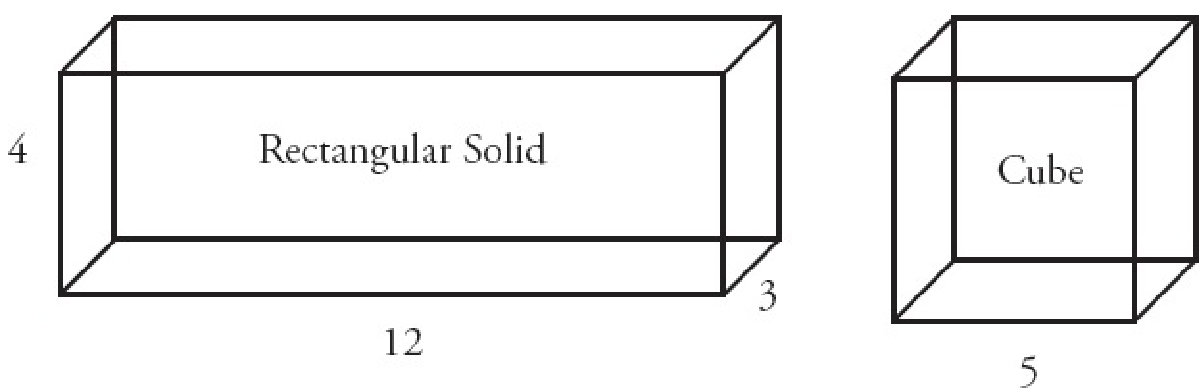
Note that some more complex shapes can be divided into a combination of rectangles and right triangles. For example:



Solving in this way will take longer than using the real formula, but trapezoids are infrequent enough that you might be willing to take that risk in order to avoid having to memorize yet another formula.

Three Dimensions: Surface Area

The GMAT tests two particular three-dimensional shapes formed from polygons: the rectangular solid and the cube.



The surface area of a three-dimensional shape is the amount of space on the surface of that particular object. For example, the amount of paint that it would take to fully cover a rectangular box could be determined by finding the surface area of that box. As with simple area, surface area is measured in square units such as in^2 (square inches) or ft^2 (square feet).

Surface Area = the *sum* of *all* of the faces

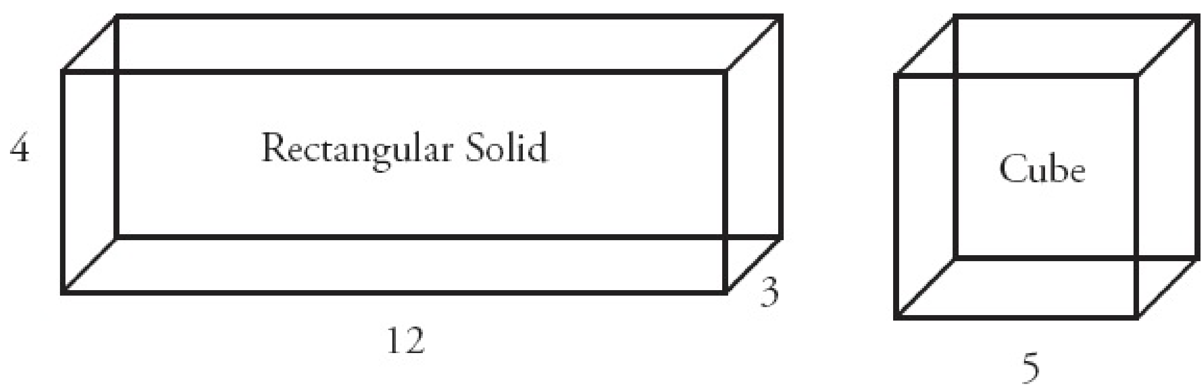
Both a rectangular solid and a cube have **six faces**.

To determine the surface area of a rectangular solid, you must find the area of each face. Notice, however, that in a rectangular solid, the front and back faces have the same area, the top and bottom faces have the same area, and the two side faces have the same area. In the solid on the previous page, the area of the front face is equal to 12 × 4 = 48. Thus, the back face also has an area of 48. The area of the bottom face is equal to 12 × 3 = 36, so the top face also has an area of 36. Finally, each side face has an area of 3 × 4 = 12. Therefore, the surface area, or the sum of the areas of all six faces equals 48(2) + 36(2) + 12(2) = 192.

To determine the surface area of a cube, you only need the length of one side. First, find the area of one face: 5 × 5 = 25. Then, multiply by six to account for all of the faces: 6 × 25 = 150.

Three Dimensions: Volume

The volume of a three-dimensional shape is the amount of “stuff” it can hold. “Capacity” is another word for volume. For example, the amount of liquid that a rectangular milk carton holds can be determined by finding the volume of the carton. Volume is measured in cubic units such as in^3 (cubic inches), ft^3 (cubic feet), or m^3 (cubic meters).



Volume = Length × Width × Height

The length of the rectangular solid above is 12, the width is 3, and the height is 4. Therefore, the volume is $12 \times 3 \times 4 = 144$.

In a cube, all three of the dimensions—length, width, and height—are identical. Therefore, knowing the measurement of just one side of the cube is sufficient to find the volume. In the cube above, the volume is $5 \times 5 \times 5 = 125$.

Beware of a GMAT volume trick, as in this example:

> How many books, each with a volume of 100 in^3, can be packed into a crate with a volume of 5,000 in^3?

It is tempting to answer "50 books" (since $50 \times 100 = 5{,}000$). However, this is incorrect, because you do not know the exact dimensions of each book! One book might be $5 \times 5 \times 4$, while another book might be $20 \times 5 \times 1$. Even though both have a volume of 100 in^3, they have different rectangular shapes. Without knowing the exact shapes of all the books, you cannot tell whether they would all fit into the crate or whether there would be empty space because the 50 books don't fill the crate perfectly. Remember, when you are fitting three-dimensional objects into other three-dimensional objects, knowing the respective volumes is not enough. You must know the specific dimensions (length, width, and height) of each object to determine whether the objects can fit without leaving gaps.

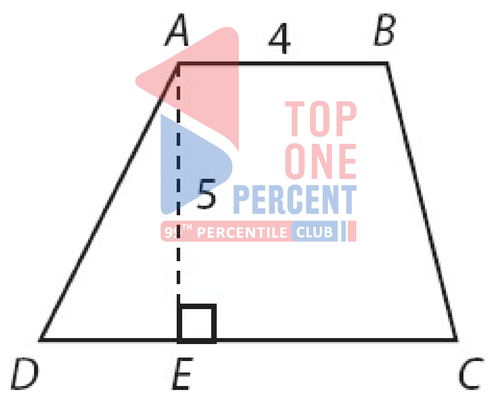
Problem Set

Note: Figures are not drawn to scale.

1. 40 percent of Andrea's living room floor is covered by a carpet that is 4 feet by 9 feet. What is the area of her living room floor?

2. A pentagon has three sides with length x, and two sides with the length $3x$. If x is $\frac{2}{3}$ of an inch, what is the perimeter of the pentagon?

3. *ABCD* is a quadrilateral, with *AB* parallel to *CD* (see figure). Point *E* is between *C* and *D* such that *AE* represents the height of *ABCD*, and *E* is the midpoint of *CD*. If *AB* is 4 inches long, *AE* is 5 inches long, and the area of triangle *AED* is 12.5 square inches, what is the area of *ABCD*? (Note: figure not drawn to scale.)



4. A rectangular swimming pool has a length of 30 meters, a width of 10 meters, and an average depth of 2 meters. If a hose can fill the pool at a rate of 0.5 cubic meters per minute, how many hours will it take the hose to fill the pool?

5. A rectangular solid has a square base, with each side of the base measuring 4 meters. If the volume of the solid is 112 cubic meters, what is the surface area of the solid?

Save the problem set below for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

6. Frank the Fencemaker needs to fence in a rectangular yard. He fences in three of the four sides of the yard. The unfenced side of the yard is 40 feet

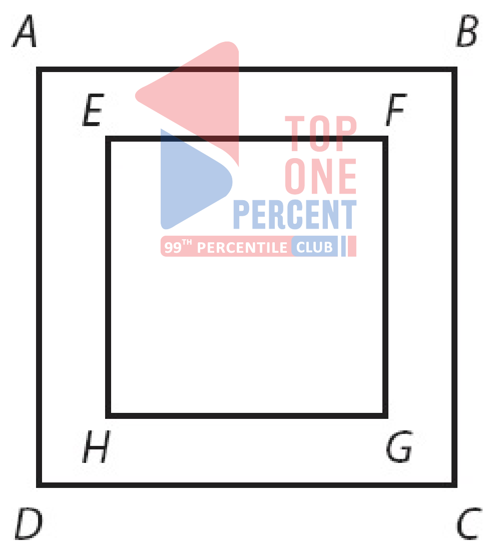
long. The yard has an area of 280 square feet. What is the length, in feet, of the fence that Frank installs?

7. If the perimeter of a rectangular flower bed is 30 feet, and its area is 44 square feet, what is the length of each of its shorter sides?

8. A rectangular tank needs to be coated with insulation. The tank has dimensions of 4 feet, 5 feet, and 2.5 feet. Each square foot of insulation costs $20. How much will it cost to cover the surface of the tank with insulation?

9. There is a rectangular parking lot with a length of $2x$ and a width of x. What is the ratio of the perimeter of the parking lot to the area of the parking lot, in terms of x?

10. *ABCD* is a square picture frame (see figure). *EFGH* is a square inscribed within *ABCD* as a space for a picture. The area of *EFGH* (for the picture) is equal to the area of the picture frame (the area of *ABCD* minus the area of EFGH). If $AB = 6$, what is the length of EF?



Solutions

1. **90 ft^2**: The area of the carpet is equal to $l \times w$, or $4 \times 9 = 36$ ft^2. Set up a proportion to find the area of the whole living room floor:

$$\begin{aligned} \frac{40}{100} &= \frac{36}{x} \qquad \text{Cross-multiply to solve.} \\ 40x &= 3{,}600 \\ x &= 90 \text{ ft}^2 \end{aligned}$$

2. **6 inches:** The perimeter of a pentagon is the sum of its five sides: $x + x + x + 3x + 3x = 9x$. If x is $\frac{2}{3}$ of an inch, the perimeter is $9\left(\frac{2}{3}\right)$, or 6 inches.

3. **35 in^2:** If E is the midpoint of C, then $CE = DE = x$. You can determine the length of x by using what you know about the area of triangle AED:

$$\begin{aligned} A &= \frac{b \times h}{2} \\ 12.5 &= \frac{5x}{2} \\ 25 &= 5x \\ x &= 5 \end{aligned}$$

Therefore, the length of CD is $2x$, or 10.

To find the area of the trapezoid, use the formula:

$$\begin{aligned} A &= \frac{b_1 + b_2}{2} \times h \\ &= \frac{4 + 10}{2} \times 5 \\ &= 35 \text{ in}^2 \end{aligned}$$

4. **20 hours:** The volume of the pool is (length) × (width) × (height), or 30 × 10 × 2 = 600 cubic meters. Use a standard work equation, $RT = W$, where W represents the total work of 600 m^3:

$$0.5t = 600$$
$$t = 1{,}200 \text{ minutes}$$

Convert this time to hours by dividing by 60: 1,200 ÷ 60 = 20 hours.

Alternatively, you could convert first: $\frac{0.5\text{m}^3}{\text{min}} \times \frac{60\text{ min}}{\text{hr}} = \frac{30\text{m}^3}{\text{hr}}$ Next, use the standard work equation:

$$30t = 600$$
$$t = 20 \text{ hours}$$

5. **144 m^2**: The volume of a rectangular solid equals (length) × (width) × (height). If you know that the length and width are both 4 meters long, you can substitute values into the formulas as shown:

$$112 = 4 \times 4 \times h$$
$$h = 7$$

To find the surface area of a rectangular solid, sum the individual areas of all six faces:

	Area of One Face		**Total Area of Two Identical Faces**
Top and Bottom:	$4 \times 4 = 16$	→	$2 \times 16 = 32$
Sides:	$4 \times 7 = 28$	→	$4 \times 28 = 112$
	All 6 faces	→	$32 + 112 = 144 \text{ m}^2$

6. **54 feet:** You know that one side of the yard is 40 feet long; call this the length. You also know that the area of the yard is 280 square feet. In order to determine the perimeter, you must know the width of the yard:

$$A = l \times w$$
$$280 = 40w$$
$$w = 280 \div 40 = 7 \text{ feet}$$

Frank fences in the two 7-foot sides and one of the 40-foot sides. Thus, he needs

54 feet of fence: 40 + 7 + 7 = 54.

7. **4 feet:** Set up equations to represent the area and perimeter of the flower bed:

$A = l \times w \qquad\qquad P = 2(l + w)$

Then, substitute the known values for the variables *A* and *P*:

$44 = l \times w \qquad\qquad 30 = 2(l + w)$

Solve the two equations using the substitution method:

$$l = \frac{44}{w}$$

$$30 = 2\left(\frac{44}{w} + w\right)$$

Multiply the entire equation by $\frac{w}{2}$.

$$15w = 44 + w^2$$

$$w^2 - 15w + 44 = 0$$

$$(w - 11)(w - 4) = 0$$

$$w = \{4, 11\}$$

Solving the quadratic equation yields two solutions: 4 and 11. Each represents a possible side length. Since you were asked to find the length of the shorter side, the answer is the smaller of the two possible values, 4.

Alternatively, you can arrive at the correct solution by picking numbers. What length and width add up to 15 (half of the perimeter) and multiply to produce 44 (the area)? Some experimentation will demonstrate that the longer side must be 11 and the shorter side must be 4.

8. **$1,700:** To find the surface area of a rectangular solid, sum the individual areas of all six faces:

	Area of One Face		**Total Area of Two Identical Faces**
Top and Bottom:	$5 \times 4 = 20$	→	$20 \times 2 = 40$
Side 1:	$5 \times 2.5 = 12.5$	→	$12.5 \times 2 = 25$
Side 2:	$4 \times 2.5 = 10$	→	$10 \times 2 = 20$
	All 6 faces	→	$40 + 25 + 20 = 85$ ft^2

Thus, covering the entire tank will cost 85 × \$20 which equals \$1,700.

9. $\mathbf{\frac{3}{x}}$ or **3 : *x*:** If the length of the parking lot is $2x$ and the width is x, you can set up a fraction to represent the ratio of the perimeter to the area as follows:

$$\frac{\text{perimeter}}{\text{area}} = \frac{2(2x + x)}{(2x)(x)} = \frac{6x}{2x^2} = \frac{3}{x}$$

10. $\mathbf{3\sqrt{2}}$**:** The area of the frame and the area of the picture sum to the total area of the image, which is 6^2, or 36. Therefore, the area of the frame and the picture are each equal to half of 36, or 18. Since *EFGH* is a square, the length of *EF* is $\sqrt{18}$, or $3\sqrt{2}$.

Chapter 4
of
Geometry

Triangles & Diagonals

In This Chapter…

Chapter 4

Triangles & Diagonals

The polygon most commonly tested on the GMAT is the triangle.

Right triangles (those with a 90° angle) require particular attention, because they have special properties that are useful for solving many GMAT Geometry problems.

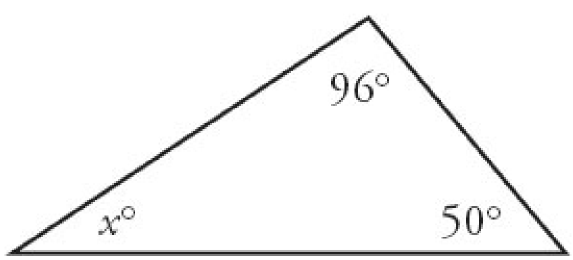
The most important property of a right triangle is the unique relationship of the three sides. Given the lengths of any two of the sides of a right triangle, you can determine the length of the third side using the Pythagorean theorem. There are even two special types of right triangles—the 30–60–90 triangle and the 45–45–90 triangle—for which you only need the length of *one* side to determine the lengths of the other two sides.

Finally, right triangles are essential for solving problems involving other polygons. For instance, you might cut more complex polygons into right triangles.

The Angles of a Triangle

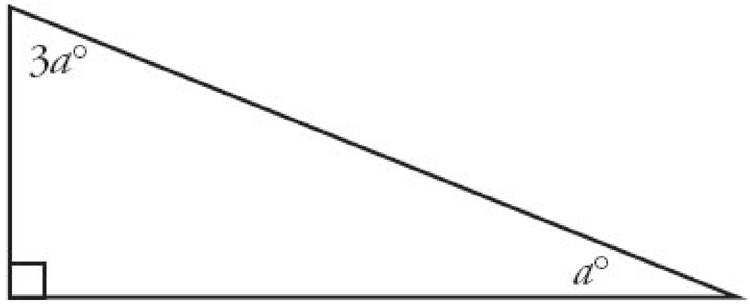
The angles in any given triangle have two key properties:

1. The sum of the three angles of a triangle equals 180°.



What is x? Since the sum of the three angles must be 180°, you can solve for x as follows:

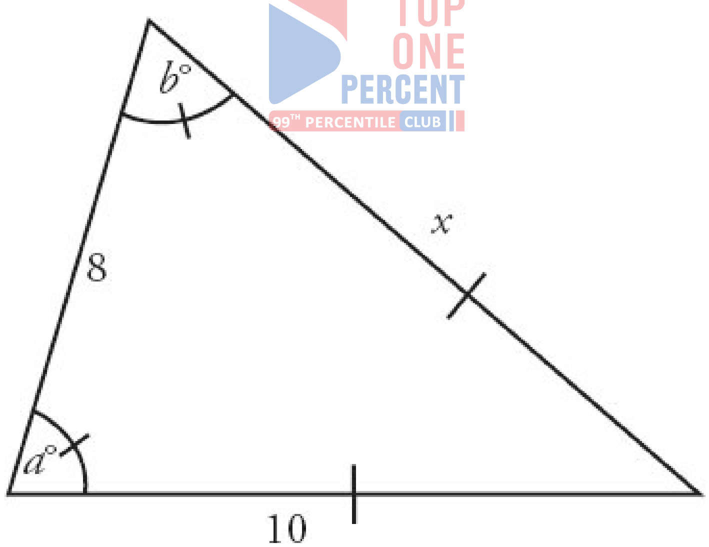
$$x = 180 - 96 - 50 = 34$$



What is a? Since the sum of the three angles must be 180°, you can solve for x as follows:

$$90 + 3a + a = 180 \longrightarrow a = 22.5.$$

2. Angles correspond to their opposite sides. This means that the largest angle is opposite the longest side, while the smallest angle is opposite the shortest side. Additionally, **if two sides are equal, their opposite angles are also equal.**



If $a = b$, what is the length of side x?

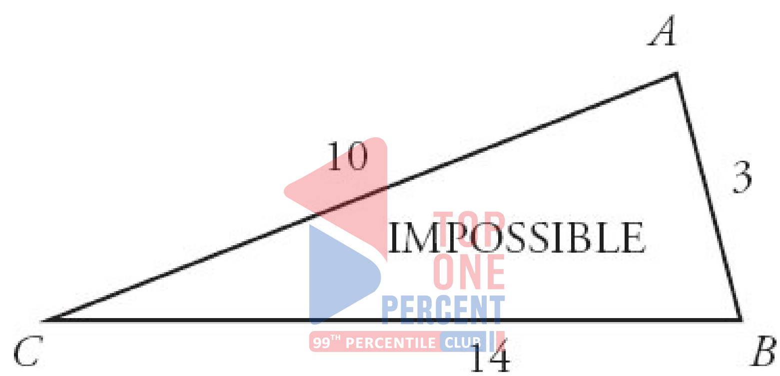
Since the side opposite angle b has a length of 10, the side opposite angle a must have the same length. Therefore, x is equal to 10.

Mark equal angles and equal sides with a slash, as shown. Also don't hesitate to redraw; if a figure doesn't match the dimensions you were given, redraw the triangle closer to scale.

The Sides of a Triangle

Consider the following "impossible" triangle *ABC* and what it reveals about the relationship between the three sides of any triangle.

The triangle to the right could never be drawn with the given measurements. Why? Consider that the shortest distance between any two points is a straight line. According to the triangle shown, the direct straight line distance between point *C* and point *B* is 14; however, the indirect path from point *C* to *B* (the path that goes from *C* to *A* to *B*) is 10 + 3, or 13, which is shorter than the direct path! This is impossible.



The example above leads to the following Triangle Inequality Law:

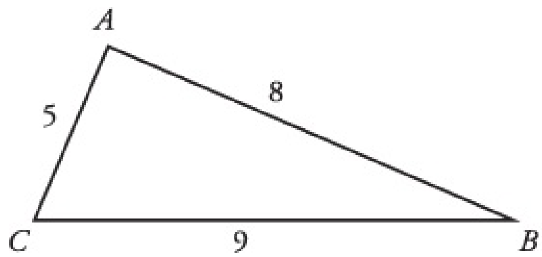
The sum of any two sides of a triangle must be *greater than* the third side.

Therefore, the maximum integer distance for side *BC* in the triangle above is 12. If the length of side *BC* is not restricted to integers, then this length has to be *less than* 13.

Note that the length cannot be smaller than a certain length, either. It must be *greater than* the difference between the lengths of the other two sides. In this case, side *BC* must be longer than 10 − 3, or 7. This is a consequence of the same idea.

Consider the following triangle and the proof that the given measurements are possible:

Test each combination of sides to prove that the measurements of this triangle are possible.



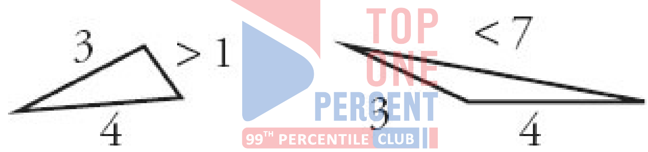
$8 + 5 > 9$ $\quad$ $8 - 5 < 9$

$9 + 5 > 8$ $\quad$ $9 - 5 < 8$

$9 - 8 > 5$ $\quad$ $9 - 8 < 5$

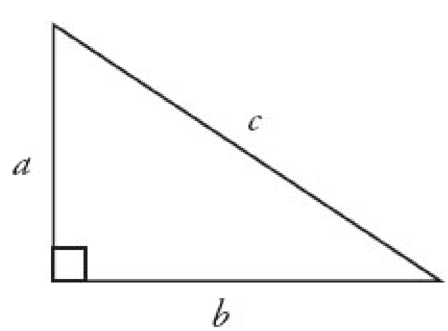
Note that the sum of two sides cannot be equal to the third side. The sum of two sides must always be **greater than** the third side. Likewise, the difference cannot be equal to the third side. The difference between two sides must be **less than** the third side.

If you are given two sides of a triangle, the length of the third side must lie between the difference and the sum of the two given sides. For instance, if you are told that two sides are of length 3 and 4, then the length of the third side must be between 4 - 3 = 1 and 4 + 3 = 7.



The Pythagorean Theorem

A right triangle is a triangle with one right angle (90°). Every right triangle is composed of two **legs** and a **hypotenuse**. The hypotenuse is the side opposite the largest angle (in this case, the right angle) and is often assigned the letter *c*. The two legs that form the right angle are often called *a* and *b* (it does not matter which leg is *a* and which leg is *b*).

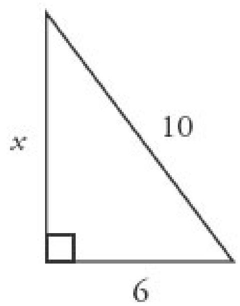


Given the lengths of two sides of a right triangle, how can you determine the length of the third side? Use the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

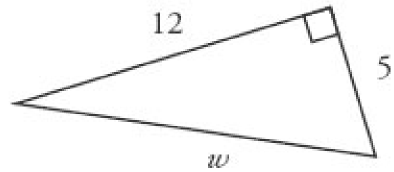
What is x?

$$\begin{aligned} a^2 + b^2 &= c^2 \\ x^2 + 6^2 &= 10^2 \\ x^2 + 36 &= 100 \\ x^2 &= 64 \\ x &= 8 \end{aligned}$$



What is w?

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 12^2 &= w^2 \\ 25 + 144 &= w^2 \\ 169 &= w^2 \\ 13 &= w \end{aligned}$$



Common Right Triangles

Certain right triangles appear over and over on the GMAT. It pays to memorize these common combinations in order to save time on the exam. Instead of using the Pythagorean theorem to solve for the lengths of the sides of these common right triangles, memorize the following Pythagorean triples:

Common Combinations	Key Multiples
3–4–5 The most popular of all right triangles $3^2 + 4^2 = 5^2$ (9 + 16 = 25)	6–8–10 9–12–15 12–16–20
5–12–13 Also quite popular on the GMAT $5^2 + 12^2 = 13^2$ (25 + 144 = 169)	10–24–26
8–15–17 This one appears less frequently. $8^2 + 15^2 = 17^2$ (64 + 225 = 289)	None

Watch out for impostor triangles! A non-right triangle with one side equal to 3 and another side equal to 4 does not have a third side of length 5.

Isosceles Triangles and the 45–45–90 Triangle

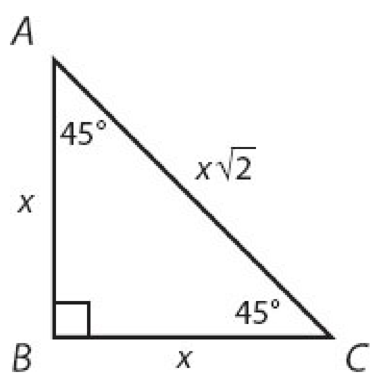
An isosceles triangle is one in which two of the three sides are equal. The two angles opposite those two sides will also be equal. The most important isosceles triangle on the GMAT is the isosceles right triangle.

An isosceles right triangle has one 90° angle (opposite the hypotenuse) and two 45° angles (opposite the two equal legs). This triangle is called the 45–45–90 triangle.

The lengths of the legs of every 45–45–90 triangle have a set ratio; memorize this:

leg	leg	hypotenuse
45°	45°	90°
x	x	$x\sqrt{2}$

Try an example:



If the length of side *AB* is 5, are the lengths of sides *BC* and *AC*?

The question tells you that *AB* is 5, so $x = 5$. Use the ratio $x:x:x\sqrt{2}$ for sides *AB*:*BC*:*AC* to determine that the sides of the triangle have lengths $5:5:5\sqrt{2}$. Therefore, the length of side *BC*=5 and the length of side *AC* = $5\sqrt{2}$.

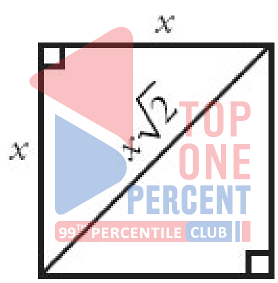
For the same triangle, if the length of side *AC* is $\sqrt{18}$, what are the lengths of sides *AB* and *BC*?

Since the hypotenuse *AC* is $\sqrt{18}$:

$$x\sqrt{2} = \sqrt{18}$$
$$x = \frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}}$$
$$x = \sqrt{9} = \mathbf{3}$$

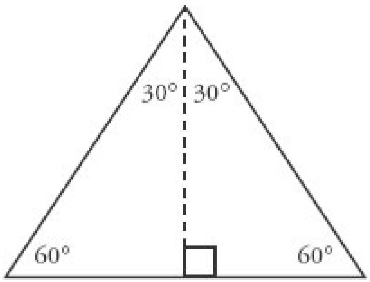
Thus, the sides *AB* and *BC* are each equal to *x*, or 3.

Interestingly, the 45–45–90 triangle is exactly half of a square! That is, two 45–45–90 triangles put together make up a square. Thus, if you are given the diagonal of a square, you can use the 45–45–90 ratio to find the length of a side of the square.

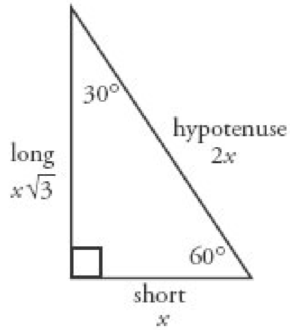


Equilateral Triangles and the 30–60–90 Triangle

An equilateral triangle is one in which all three sides (and all three angles) are equal. Each angle of an equilateral triangle is 60° (because all three angles must sum to 180°). A close relative of the equilateral triangle is the 30–60–90 triangle. Notice that two of these triangles, when put together, form an equilateral triangle:



Equilateral Triangle



30–60–90 Triangle

The lengths of the legs of every 30–60–90 triangle have a set ratio; memorize this:

leg	leg	hypotenuse
30°	60°	90°
x	$x\sqrt{3}$	$2x$

Try some examples:

> If the short leg of a 30–60–90 triangle has a length of 6, what are the lengths of the long leg and the hypotenuse?

The question tells you that the short leg, which is opposite the 30° angle, is 6. Use the ratio x:$x\sqrt{3}$:2x to determine that the sides of the triangle have lengths 6:$6\sqrt{3}$:12. The long leg measures $6\sqrt{3}$ and the hypotenuse measures 12.

> If an equilateral triangle has a side of length 10, what is its height?

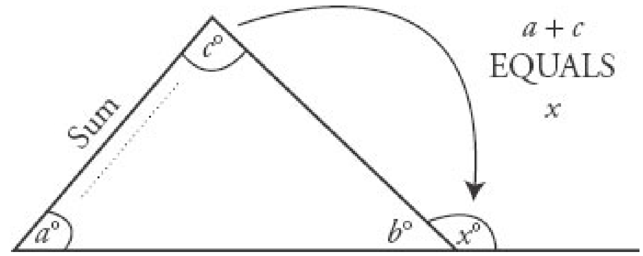
The side of an equilateral triangle is the hypotenuse of a 30–60–90 triangle created when the height of the equilateral triangle is drawn. Additionally, the height of an equilateral triangle is the same as the long leg of a 30–60–90 triangle. Since you are told that the hypotenuse is 10, use the ratio x:$x\sqrt{3}$:2x to set $2x = 10$ and determine that the multiplier x is 5. Therefore, the sides of the 30–60–90 triangle have lengths 5 : $5\sqrt{3}$: 10. The long leg has a length of $5\sqrt{3}$, which is the height of the equilateral triangle.

If you get tangled up on a 30–60–90 triangle, try to find the length of the short leg. The other legs will then be easier to figure out.

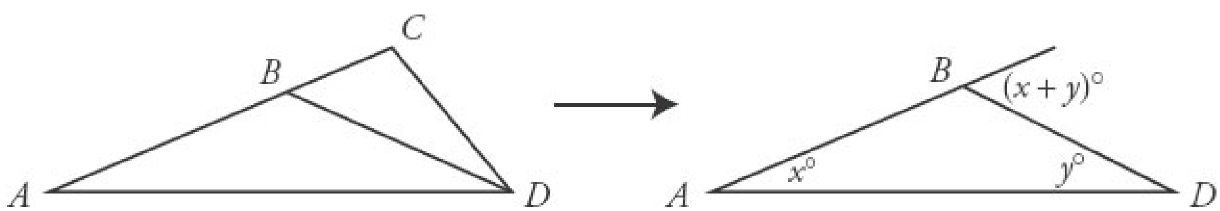
Exterior Angles of a Triangle

An **exterior angle** of a triangle is equal in measure to the sum of the two non-adjacent (opposite) **interior angles** of the triangle. For example:

$a + b + c = 180$ (sum of angles in a triangle).
$b + x = 180$ (form a straight line).
Therefore, $x = a + c$.



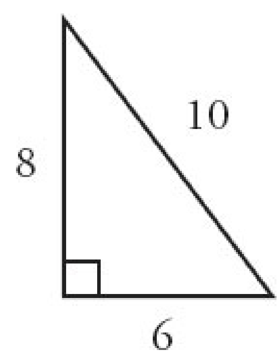
In particular, look for exterior angles within more complicated figures. You might even redraw the figure with certain lines removed to isolate the triangle and exterior angle you need:



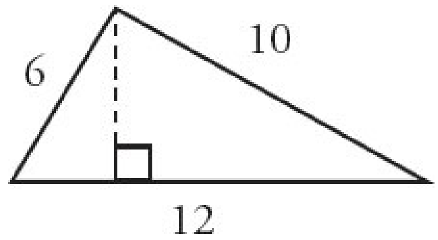
Triangles and Area

$$\text{Area of a Triangle} = \frac{\textbf{Base} \times \textbf{Height}}{\textbf{2}}$$

The base refers to the bottom side of the triangle. The height *always* refers to a line drawn from the opposite vertex to the base, creating a 90° angle.

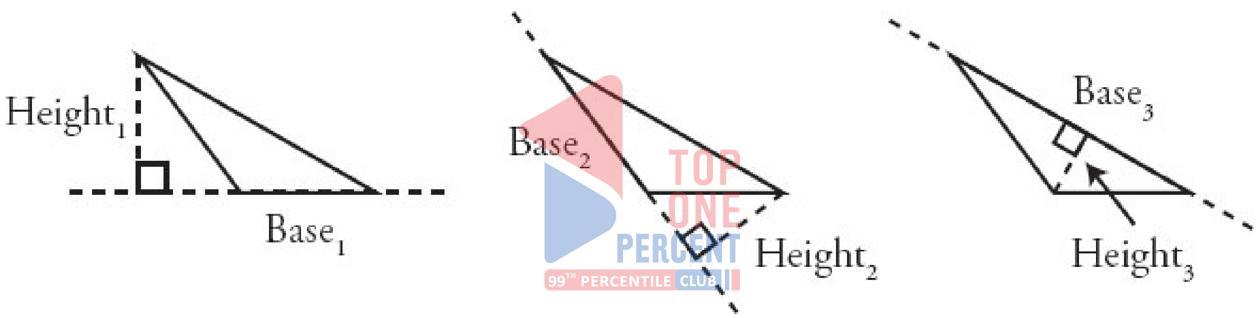


In the triangle on the left, the base is 6 and the height (perpendicular to the base) is 8. Therefore, the area is $(6 \times 8) \div 2 = 48 \div 2 = 24$.



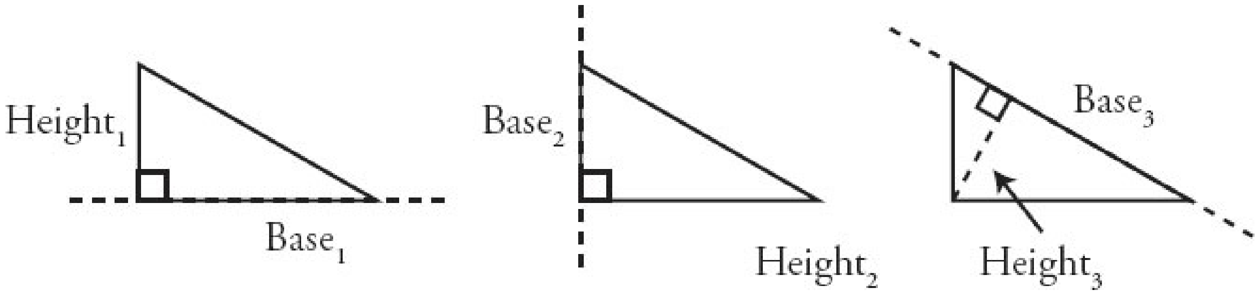
In this triangle, the base is 12, but the height is not shown. Neither of the other two sides of the triangle is perpendicular to the base. In order to find the area of this triangle, you would first need to determine the height, which is represented by the dotted line.

Although you may commonly think of “the base” of a triangle as whichever side is drawn horizontally, you can designate any side of a triangle as the base. For example, the following three figures show the same triangle, with each side in turn designated as the base:



Since a triangle has only one area, the area must be the same regardless of the side chosen as the base. You can choose any pair of height and base that you like, as long as the height is a perpendicular line drawn from the opposite vertex to the base that you've chosen.

Right triangles have three possible bases just as other triangles do, but they are special because their two legs are perpendicular. Therefore, if one of the legs is chosen as the base, then the other leg is the height. Of course, you can also choose the hypotenuse as the base.



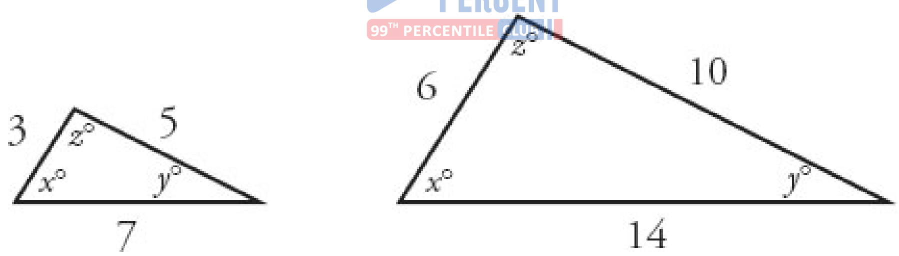
Thus, the area of a right triangle is given by the following formulas:

$$A = \frac{1}{2} \times (\text{One leg}) \times (\text{Other leg}) = \frac{1}{2}\ (\text{Hypotenuse}) \times (\text{Height from hypotenuse})$$

Similar Triangles

One final tool that you can use for GMAT triangle problems is the similar triangle strategy. Often, looking for similar triangles can help you solve complex problems.

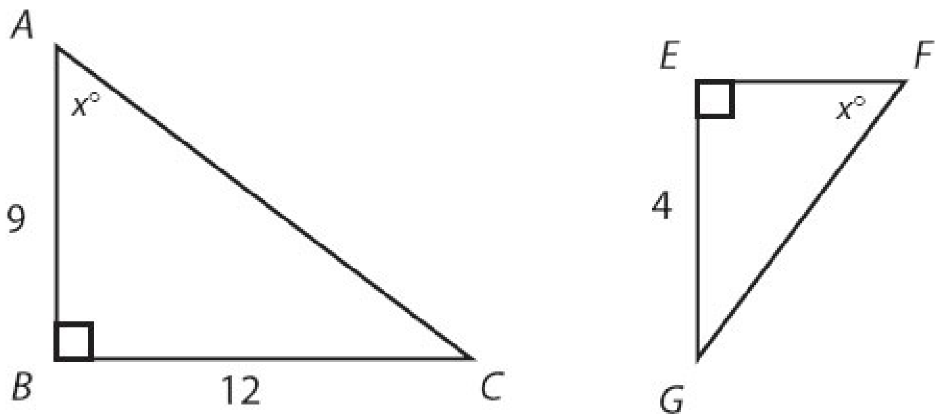
Triangles are defined as similar if all of their **corresponding angles are equal** and their **corresponding sides are in proportion**, as in the triangles below:



Once you find that two triangles have two pairs of equal (or congruent) angles, you know that the triangles are similar. If two sets of angles are congruent, then the third set of angles must be congruent, since the sum of the angles in any triangle is 180°.

Try an example:

What is the length of side *EF*?



The two triangles above are similar because they have two angles in common (x and the right angle). Since they are similar triangles, their corresponding sides must be in proportion.

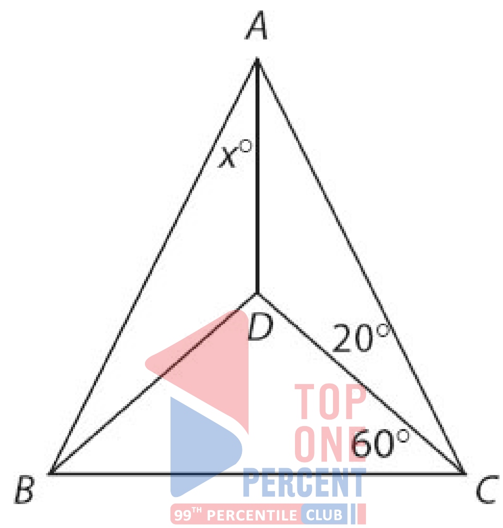
Side BC corresponds to side EG (since they both are opposite angle x). Because these sides are in the ratio of 12 : 4, you can determine that the large triangle is three times bigger than the smaller one. That is, the triangles are in the ratio of 3 : 1. Since side AB corresponds to side EF, and AB has a length of 9, you can conclude that side EF has a length of 3.

Problem Set

Note: Figures are not drawn to scale.

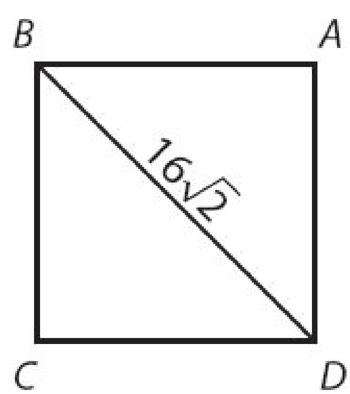
1. Two sides of a triangle have lengths 4 and 10. If the third side has a length of integer *x*, how many possible values are there for *x*?

2. In triangle *ABC*, *AD* = *DB* = *DC* (see figure). If angle *DCB* is 60° and angle *ACD* is 20°, what is the value of *x*?



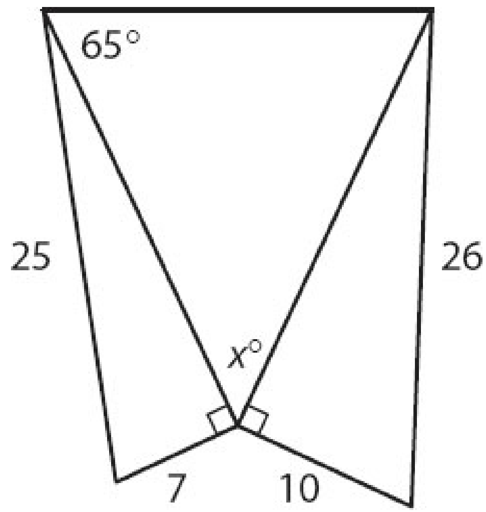
3. Beginning in Town A, Biker Bob rides his bike 10 miles west, 3 miles north, 5 miles east, and then 9 miles north, to Town B. How far apart are Town A and Town B? (Assume perfectly flat terrain.)

4. A square is bisected into two equal triangles (see figure). If the length of *BD* is $16\sqrt{2}$ inches, what is the area of the square?



Save the problem set below for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

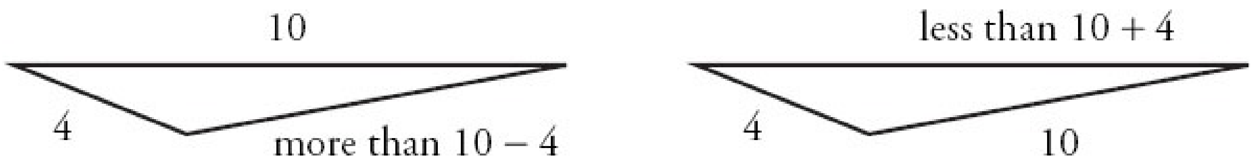
5. What is the value of *x* in the figure below?



6. The size of a square computer screen is measured by the length of its diagonal. How much bigger is the visible area of a square 24-inch screen than the area of a square 20-inch screen?

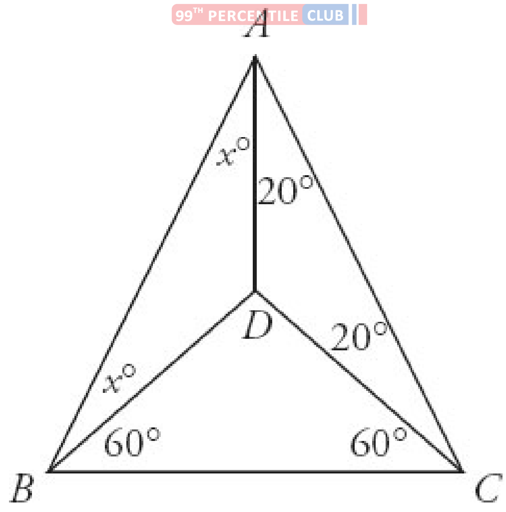
Solutions

1. **Seven:** If two sides of a triangle are 4 and 10, the third side must be greater than 10 − 4 and smaller than 10 + 4. Therefore, the possible values for x are {7, 8, 9, 10, 11, 12, and 13}. You can draw a sketch to convince yourself of this result:

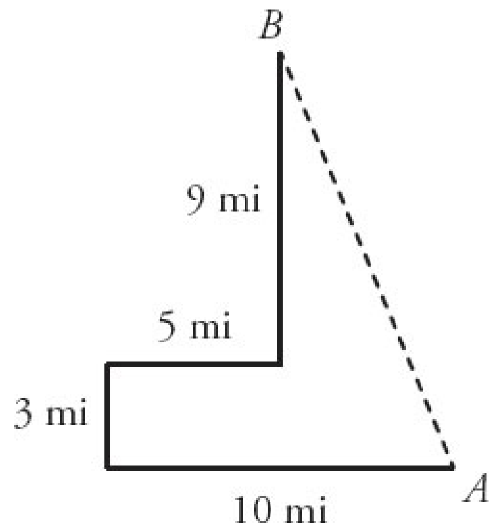


2. **10°**: If $AD = DB = DC$, then the three triangular regions in this figure are all isosceles triangles. Therefore, you can fill in some of the missing angle measurements as shown to the right. Since you know that there are 180° in the large triangle ABC, you can write the following equation:

$$x + x + 20 + 20 + 60 + 60 = 180$$
$$2x + 160 = 180$$
$$x = 10$$



3. **13 miles:** If you draw a rough sketch of the path Biker Bob takes, as shown to the right, you can see that the direct distance from A to B forms the hypotenuse of a right triangle. The short leg (horizontal) is 10 − 5 = 5 miles, and the long leg (vertical) is 9 + 3 = 12 miles. Therefore, you can use the Pythagorean theorem to find the direct distance from A to B:



$$5^2 + 12^2 = c^2$$
$$25 + 144 = c^2$$
$$c^2 = 169$$
$$c = 13$$

You might recognize the common right triangle: 5–12–13. If so, you don't need to use the Pythagorean theorem to calculate the value of 13.

4. **256 in^2:** The diagonal of a square is $s\sqrt{2}$ and the given length of the diagonal in the problem is $16\sqrt{2}$. therefore, the side length of square *ABCD* is $s = 16$ inches. The area of the square is s^2, or $16^2 = 256$.

5. **50°:** Use the Pythagorean theorem to establish the missing lengths of the two right triangles on the right and left sides of the figure:

$$7^2 + b^2 = 25^2 \qquad 10^2 + b^2 = 26^2$$
$$49 + b^2 = 625 \qquad 100 + b^2 = 676$$
$$b^2 = 576 \qquad b^2 = 576$$
$$b = 24 \qquad b = 24$$

Alternatively, if you have the common right triangles memorized, notice that the second triangle (10–*x*–26) is the 5–12–13 triangle multiplied by 2. The missing length, therefore, is $12 \times 2 = 24$.

The inner triangle is isosceles. Therefore, both angles opposite the equal sides measure 65°. Since there are 180° in a right triangle, $x = 180 - 2(65) = 50$.

6. **88 in^2**: If the diagonal of the larger screen is 24 inches, and it is always true for a square that $d = s\sqrt{2}$, then:

$$s = \frac{d}{\sqrt{2}} = \frac{24}{\sqrt{2}} = \frac{24(\sqrt{2})}{(\sqrt{2})(\sqrt{2})} = \frac{24\sqrt{2}}{2} = 12\sqrt{2}$$

By the same reasoning, the side length of the smaller screen is

$$\frac{20}{\sqrt{2}} = \frac{20(\sqrt{2})}{(\sqrt{2})(\sqrt{2})} = 10\sqrt{2}\,.$$

The areas of the two screens are:

Large screen: $A = 12\sqrt{2} \times 12\sqrt{2} = 288$

Small screen: $A = 10\sqrt{2} \times 10\sqrt{2} = 200$

The visible area of the larger screen is 88 square inches bigger than the visible area of the smaller screen.

Chapter 5
of
Geometry

Circles & Cylinders

In This Chapter…

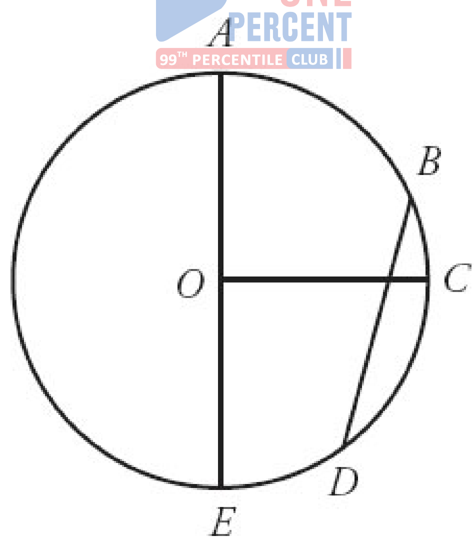
Chapter 5

Circles & Cylinders

A circle is defined as the set of points in a plane that are equidistant from a fixed center point. A circle contains 360° (360 degrees).

Any line segment that connects the center point to a point on the circle is termed a **radius** of the circle. If point *O* is the center of the circle shown to the right, then segment *OC* is a radius.

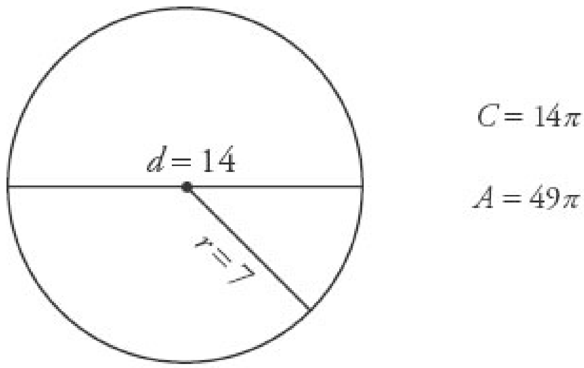
Any line segment that connects two points on a circle is called a **chord**. Any chord that passes through the center of the circle is called a **diameter**. Notice that the diameter is two times the length of the radius. Line segment *BD* is a chord of the circle shown to the right. Line segment *AE* is a diameter of the circle.

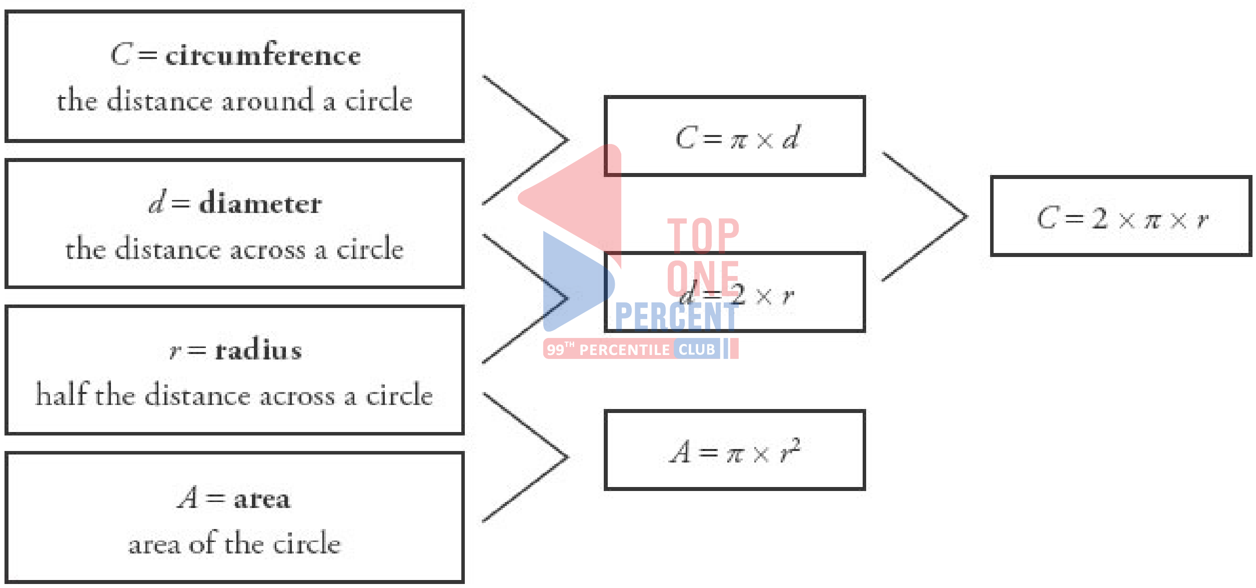


The GMAT tests your ability to find the circumference and the area of whole and partial circles. In addition, some advanced problems may test cylinders, which are three-dimensional shapes made, in part, of circles. The GMAT may test your ability to find the volume of cylinders.

Radius, Diameter, Circumference, and Area

The relationships between the radius, diameter, circumference, and area remain constant for every circle.





If you know *any* one of these values, you can determine the rest.

For Problem Solving questions, you will often need to use one of these values to solve for one of the other three. For Data Sufficiency questions, a little information goes a long way. If you know that you are able to solve for each of these values, you do not actually have to perform the calculation.

The space inside a circle is termed the area of the circle. This area is just like the area of a polygon. As with circumference, the only information you need to find the area of a circle is the radius of that circle. The formula for the area of any circle is:

$$\boxed{A = \pi r^2}$$

where A is the area, r is the radius, and π is a number that is approximately 3.14.

What is the area of a circle with a circumference of 16π?

In order to find the area of a circle, all you must know is its radius. If the circumference of the circle is 16π (and $C = 2\pi r$), then the radius must be 8. Plug this into the area formula:

$$A = \pi r^2 = \pi(8^2) = 64\pi$$

Area of a Sector

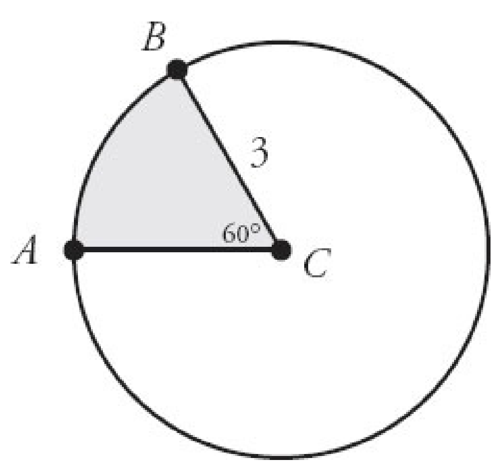
The GMAT may ask you to solve for the area of a sector of a circle, instead of the area of the entire circle. You can find the area of a sector by determining the fraction of the entire area that the sector occupies. Try an example:

What is the area of sector ACB (the shaded region) below?

First, find the area of the entire circle:

$$A = \pi r^2 = \pi(3^2) = 9\pi$$

Then, use the central angle to determine what fraction of the entire circle is represented by the sector. Since the sector is defined by the central angle of 60°, and the entire circle is 360°, the sector occupies $\frac{60°}{360°}$, or one-sixth, of the area of the circle.

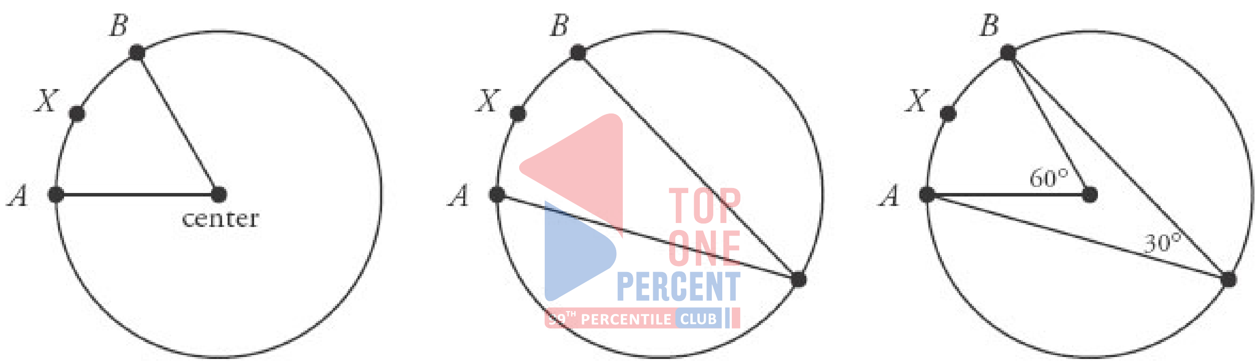


Therefore, the area of sector *ACB* is $\left(\frac{1}{6}\right)(9\pi) = 1.5\pi.$.

Inscribed vs. Central Angles

Thus far, in dealing with arcs and sectors, you have learned about the **central angle**. A central angle is defined as an angle whose vertex lies at the center point of a circle. As discussed earlier, a central angle defines both an arc and a sector of a circle.

Another type of angle is termed an **inscribed angle**. An inscribed angle has its vertex on the circle itself. The following figures illustrate the difference between a central angle and an inscribed angle.

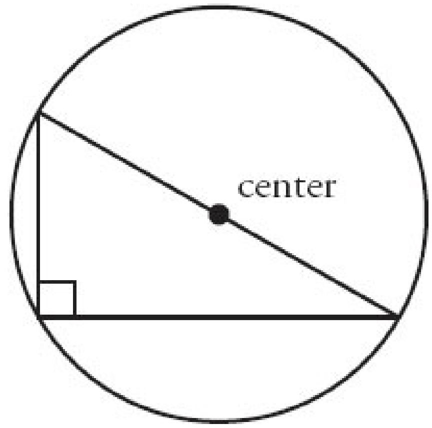


Central Angle   Inscribed Angle

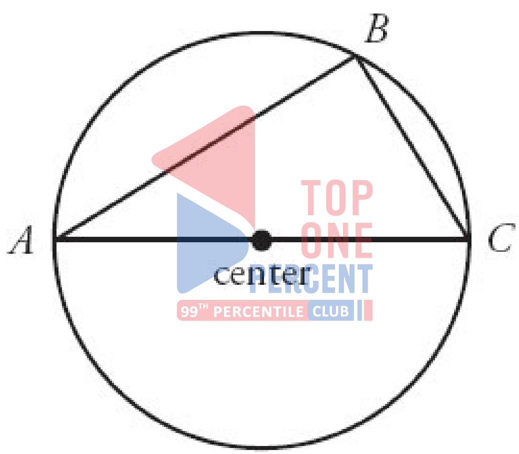
Notice that, in the circle at the far right, there is a central angle and an inscribed angle, both of which intercept arc *AXB*. The arc is 60° (or one-sixth of the complete 360° circle). **An inscribed angle is equal to half of the arc it intercepts**, in degrees. In this case, the inscribed angle is 30°, which is half of 60°.

Inscribed Triangles

Related to this idea of an inscribed angle is that of an **inscribed triangle**. A triangle is said to be inscribed in a circle if all of the vertices of the triangle are points on the circle. The important rule to remember is: **if one of the sides of an inscribed triangle is a *diameter* of the circle, then the triangle *must* be a right triangle**. Conversely, any right triangle inscribed in a circle must have the diameter of the circle as one of its sides (thereby splitting the circle in half).



In the inscribed triangle to the right, triangle *ABC* must be a right triangle, since *AC* is a diameter of the circle.



Cylinders and Volume

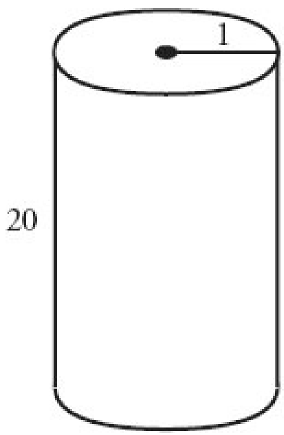
The volume of a cylinder measures how much “stuff” it can hold inside. In order to find the volume of a cylinder, use the following formula:

$$\boxed{V = \pi r^2 h}$$

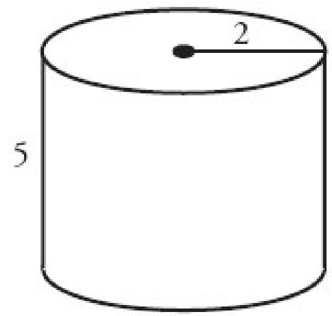
V is the volume, r is the radius of the cylinder, and h is the height of the cylinder.

Determining the volume of a cylinder requires two pieces of information: (1) the radius of the cylinder and (2) the height of the cylinder.

The figures below show that two cylinders can have the same volume but different shapes (and therefore each fits differently inside a larger object):



$$V = \pi r^2 h$$
$$= \pi(1)^2 20$$
$$= 20\pi$$

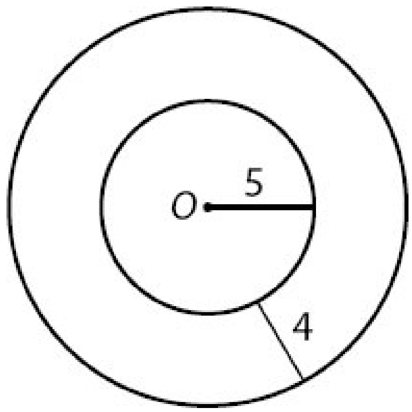


$$V = \pi r^2 h$$
$$= \pi(2)^2 5$$
$$= 20\pi$$

Problem Set

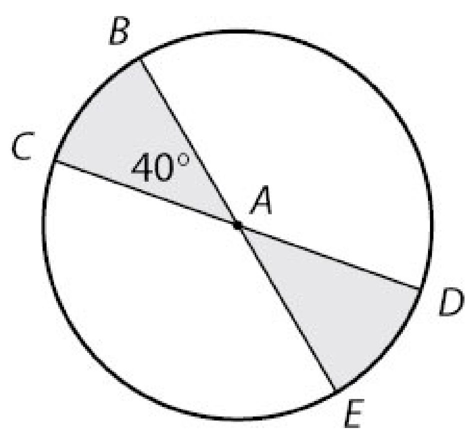
Note: Figures are not drawn to scale.

1. A circular lawn with a radius of 5 meters is surrounded by a circular walkway that is 4 meters wide (see figure). What is the area of the walkway?



2. Randy can run π meters every 2 seconds. If the circular track has a radius of 75 meters, how many minutes does it take Randy to run twice around the track?

3. *BE* and *CD* are both diameters of circle with center A (see figure). If the area of the circle is 180 units^2, what is the total area of the shaded regions?



4. A cylindrical water tank has a diameter of 14 meters and a height of 20 meters. A water truck can fill π cubic meters of the tank every minute. How long will it take the water truck to fill the water tank from empty to half full?

5. A Hydrogenator water gun has a cylindrical water tank, which is 30

centimeters long. Using a hose, Jack fills his Hydrogenator with π cubic centimeters of his water tank every second. If it takes him 8 minutes to fill the tank with water, what is the diameter of the circular base of the gun's water tank?

Solutions

1. **56π m^2**: The area of the walkway is the area of the entire image (walkway + lawn) minus the area of the lawn. To find the area of each circle, use the formula:

Large circle: $A = \pi r^2 = \pi(9)^2 = 81\pi$

Small circle: $A = \pi r^2 = \pi(5)^2 = 25\pi$

Thus, $81\pi - 25\pi = 56\pi$ m^2.

2. **10 minutes:** The distance around the track is the circumference of the circle:

$$\begin{aligned} C &= 2\pi r \\ &= 150\pi \end{aligned}$$

Running twice around the circle would equal a distance of 300π meters. If Randy can run π meters every 2 seconds, he runs 30π meters every minute. Therefore, it will take him 10 minutes to run around the circular track twice.

3. **40 units^2**: The two central angles of the shaded sectors include a total of 80°. Simplify the fraction to find out what fraction of the circle this represents:

$$\frac{80}{360} = \frac{2}{9} \qquad \frac{2}{9} \text{ of } 180 \text{ units}^2 \text{ is } 40 \text{ units}^2.$$

4. **490 minutes**, or **8 hours and 10 minutes:** First find the volume of the cylindrical tank:

$$\begin{aligned} V &= \pi r^2 \times h \\ &= \pi(7)^2 \times 20 \\ &= 980\pi \end{aligned}$$

If the water truck can fill π cubic meters of the tank every minute, it will take 980 minutes to fill the tank completely; therefore, it will take $980 \div 2 = 490$ minutes to fill the tank halfway. This is equal to 8 hours and 10 minutes.

5. **8 cm:** In 8 minutes, or 480 seconds, 480π cm^3 of water flows into the tank. Therefore, the volume of the tank is 480π. You are given a height of 30, so you can solve for the radius:

$$V = \pi r^2 \times h$$
$$480\pi = 30\pi r^2$$
$$r^2 = 16$$
$$r = 4$$

Therefore, the diameter of the tank's base is 8 centimeters.

Chapter 6
of
Geometry

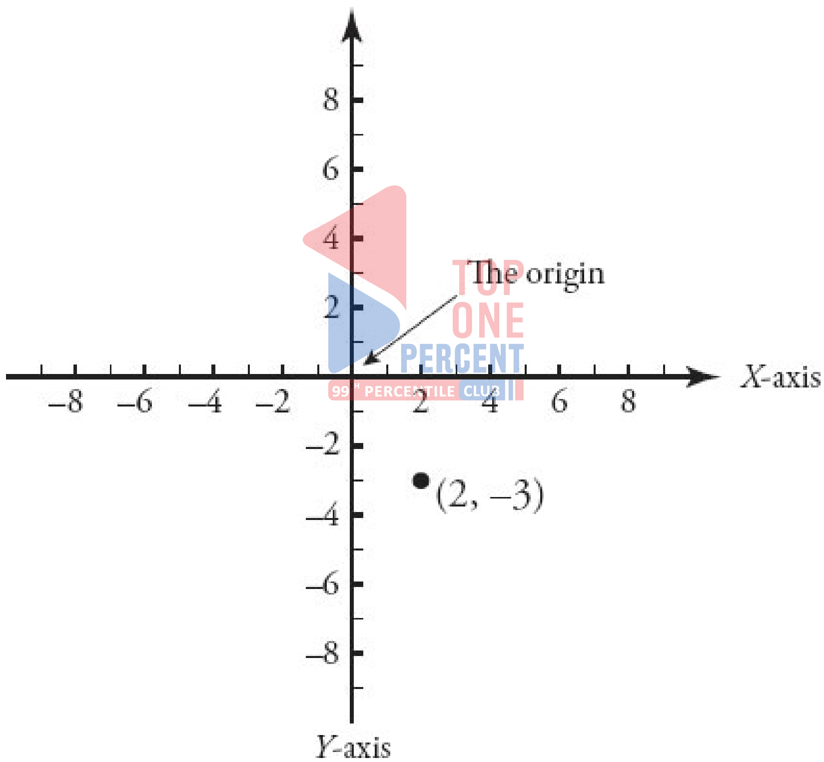
Coordinate Plane

In This Chapter…

Chapter 6

Coordinate Plane

The coordinate plane is formed by a horizontal axis or reference line (the ***“x-axis”***) and a vertical axis (the ***“y-axis”***), as shown here. These axes are each marked off like a number line, with both positive and negative numbers. The axes cross at right angles at the number zero on both axes.



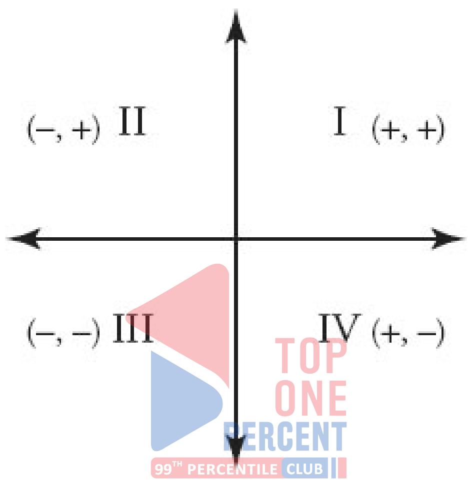
Points in the plane are identified by using an ordered pair of numbers, such as the point to the left, which is written as (2, −3). The first number in the ordered pair (2) is the ***x*-coordinate**, which corresponds to the point's horizontal location, as measured by the *x*-axis. The second number in the ordered pair (−3) is the ***y*-**coordinate, which corresponds to the point's vertical location, as indicated by the *y*-axis. The point (0, 0), where the axes cross, is called the **origin**.

A line in the plane is formed by the connection of two or more points. Notice that along the *x*-axis, the *y*-coordinate is 0. Likewise, along the *y*-axis, the *x*-coordinate is 0.

If the GMAT gives you coordinates with other variables, match them to *x* and *y*. For instance, if you have point (*a*, *b*), *a* is the *x*-coordinate and *b* is the *y*-coordinate.

Positive and Negative Quadrants

There are four quadrants in the coordinate plane, as shown in the figure below:



Quadrant I contains only those points with a **positive** *x*-coordinate and a **positive** *y*-coordinate.

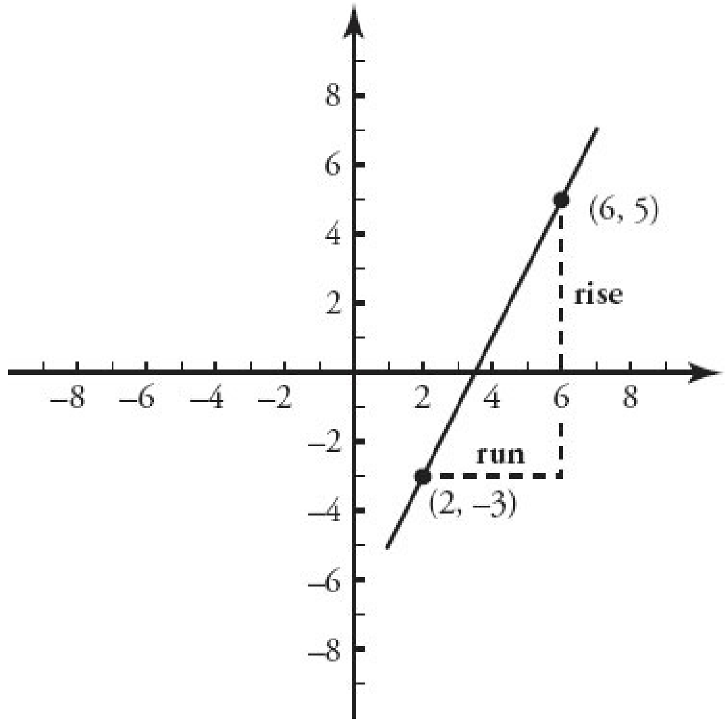
Quadrant II contains only those points with a **negative** *x*-coordinate and a **positive** *y*-coordinate.

Quadrant III contains only those points with a **negative** *x*-coordinate and a **negative** *y*-coordinate.

Quadrant IV contains only those points with a **positive** *x*-coordinate and a **negative** *y*-coordinate.

The Slope of a Line

The slope of a line is defined as “rise over run”—that is, how much the line *rises* vertically divided by how much the line *runs* horizontally.



The slope of a line can be determined by taking any two points on the line and (1) determining the "rise," or difference between their *y*-coordinates, and (2) determining the "run," or difference between their *x*-coordinates. Thus:

$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$

For example, in the graph on the left, the line rises vertically from −3 to +5. To find the distance, subtract the *y*-coordinates: 5 − (−3) = 8. Thus, the line rises 8 units. The line also runs horizontally from 2 to 6. To find the distance, subtract the *x*-coordinates: 6 − 2 = 4. Thus, the line runs 4 units.

Put the results together to find the slope of the line: $\frac{\text{rise}}{\text{run}} = \frac{8}{4} = 2$.

Two other points on the line may have a different rise and run, but the slope would be the same. The "rise over run" would always be 2 because a line has a constant slope.

The slope of a line is equal to $\frac{y_2 - y_1}{x_2 - x_1}$

For example, if you are given the two points (2, 3) and (4, -1), then the slope would be:

$$\frac{-1-3}{4-2} = \frac{-4}{2} = -2$$

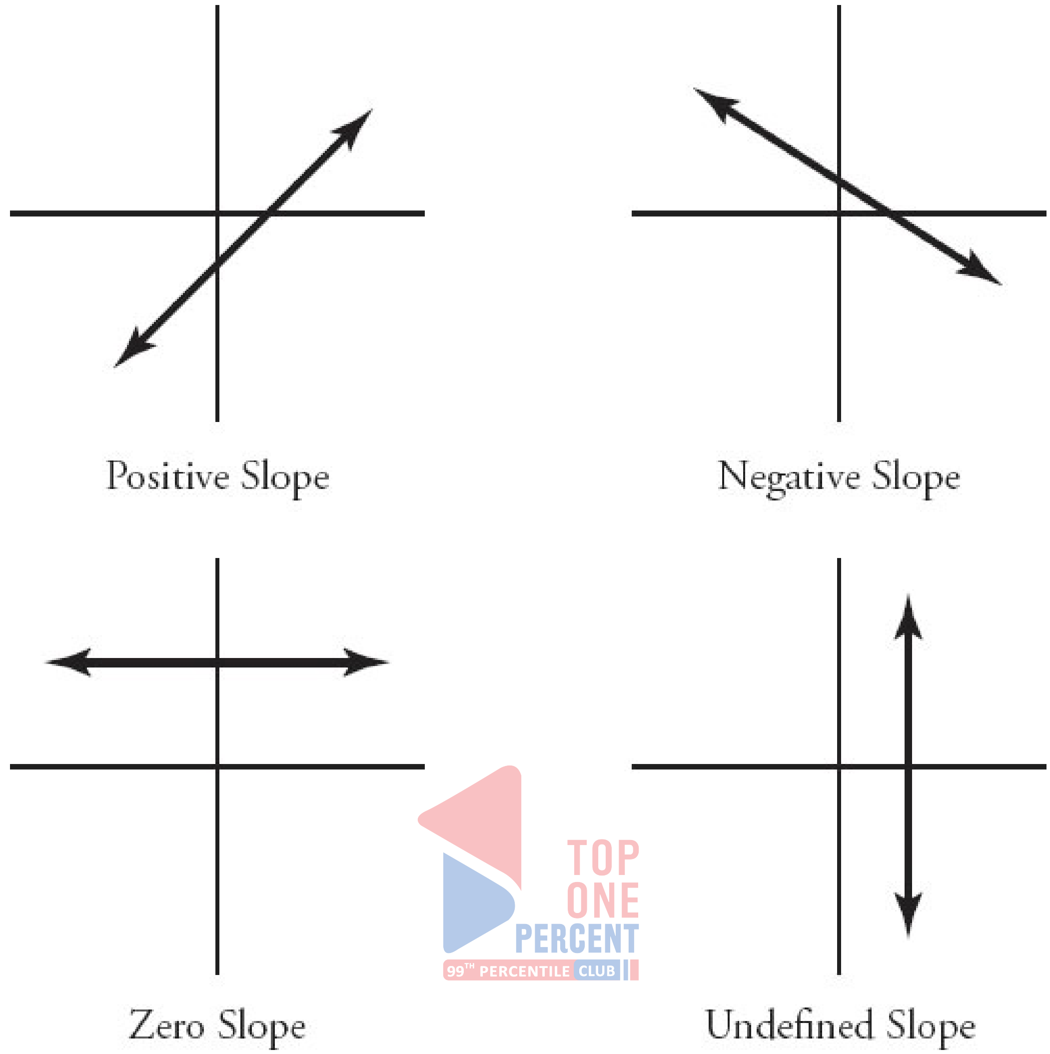
You can choose to reorder the two points, but make sure that y_2 and x^2 always come from the same point (and that y_1 and x_1 always come from the same point). For example:

$$\frac{3-(-1)}{2-4} = \frac{4}{-2} = -2$$

Note that, either way, the slope is the same.

The Four Types of Slopes

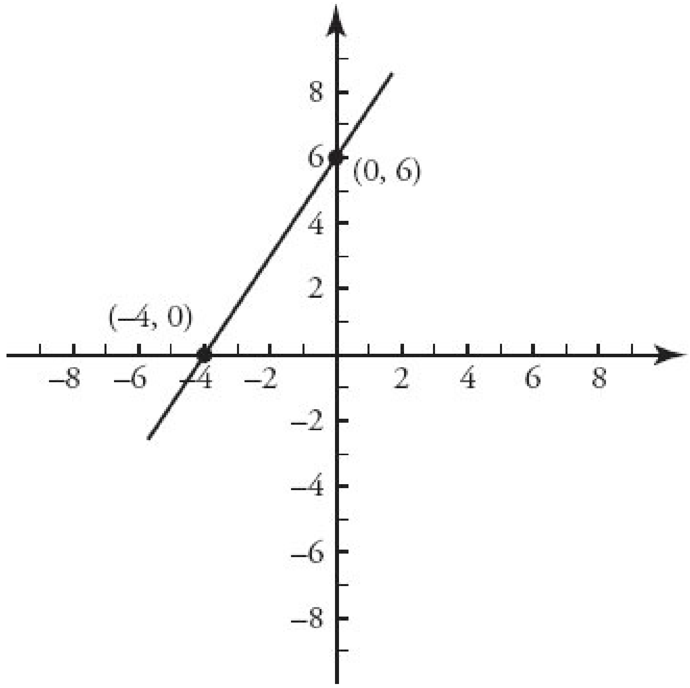
A line can have one of four types of slopes:



A line with positive slope rises upward from left to right. A line with negative slope falls downward from left to right. A horizontal line has zero slope. A vertical line has undefined slope. Notice that the *x*-axis has zero slope, while the *y*-axis has undefined slope.

The Intercepts of a Line

A point where a line intersects a coordinate axis is called an **intercept.** There are two types of intercepts: the *x*-intercept, where the line intersects the *x*-axis, and the *y*-intercept, where the line intersects the *y*-axis.



The *x*-intercept is expressed using the ordered pair (x, 0), where x is the point where the line intersects the *x*-axis. **The *x*-intercept is the point on the line at which $y = 0$.** In this graph, the *x*-intercept is −4, as expressed by the ordered pair (−4, 0).

The *y*-intercept is expressed using the ordered pair (0, y), where y is the point where the line intersects the *y*-axis. **The *y*-intercept is the point on the line at which $x = 0$.** In this graph, the *y*-intercept is 6, as expressed by the ordered pair (0, 6).

Slope-Intercept Equation: $y = mx + b$

Linear equations represent lines in the coordinate plane. Linear equations often look like this: $Ax + By = C$, where A, B, and C are numbers. For instance, $6x + 3y = 18$ is a linear equation. Linear equations never involve terms such as x^2, $\sqrt{x}$, or xy.

In coordinate plane problems, it is useful to write linear equations in the slope-intercept form:

$$y = mx + b$$

In this equation, *m* represents the slope of the line and *b* represents the *y*-intercept of the line, or the point at which the line crosses the *y*-axis. When you want to graph a linear equation, rewrite the equation in the slope-intercept form. Try this example:

> What is the slope-intercept form for a line with the equation $6x + 3y = 18$?

Rewrite the equation by solving for *y* as follows:

$6x + 3y = 18$

$3y = 18 - 6x$ Subtract $6x$ from both sides

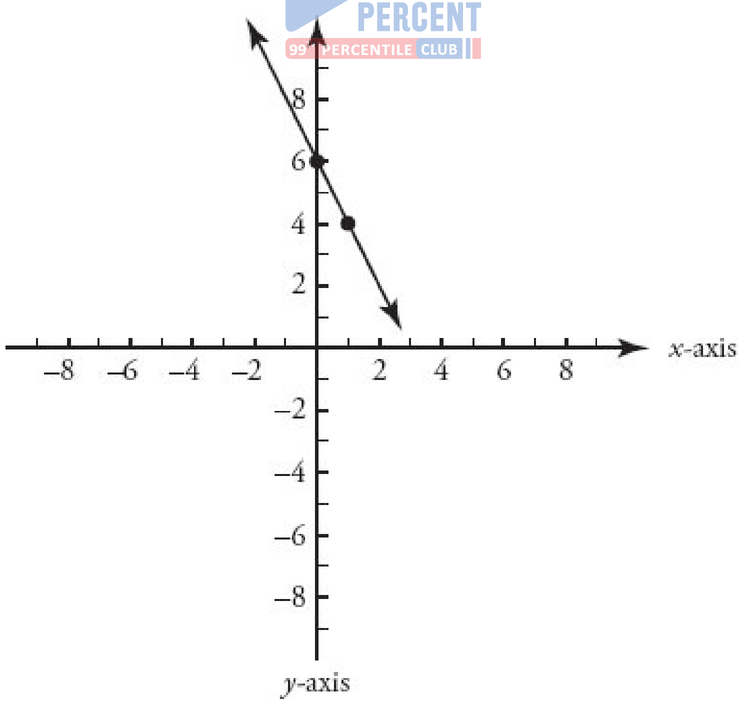
$y = 6 - 2x$ Divide both sides by 3

$y = -2x + 6$ Thus, the *y*-intercept is (0, 6), and the slope is -2.

To graph this line, first put a point at +6 on the *y*-axis (because the *y*-intercept, *b*, equals 6).

Then count down 2 units (because the slope is negative) and to the right 1 unit. Place another point.

Draw a line between the two points.

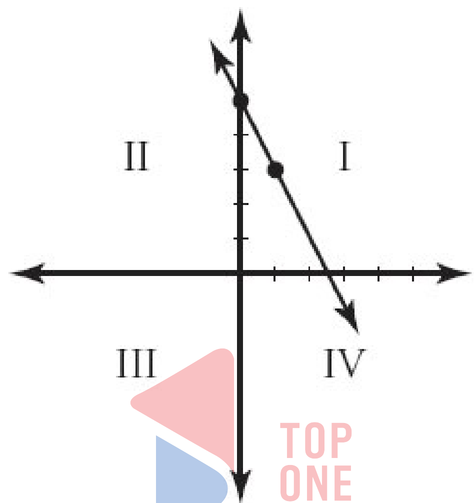


The GMAT sometimes asks you to determine which quadrants a given line passes through. For example:

Which quadrants does the line $2x + y = 5$ pass through?

First, rewrite the line in the form $y = mx + b$:

$$\begin{aligned} 2x + y &= 5 \\ y &= 5 - 2x \\ y &= -2x + 5 \end{aligned}$$



Next, sketch the line. Since $b = 5$, the y-intercept is the point (0, 5). The slope is -2, so the line slopes downward to the right from the y-intercept. A slope of -2 is the equivalent of $\frac{-2}{1}$. Count one place to the right of the intercept (the run) and two places down (the “rise” of a negative slope). Draw a second point, then connect the two points with a line. Although you do not know exactly where the line intersects the x-axis, you can now see that the line passes through quadrants I, II, and IV.

Alternatively, find two points on the line by setting x and y equal to 0 in the original equation. In this way, you find the x- and y-intercepts:

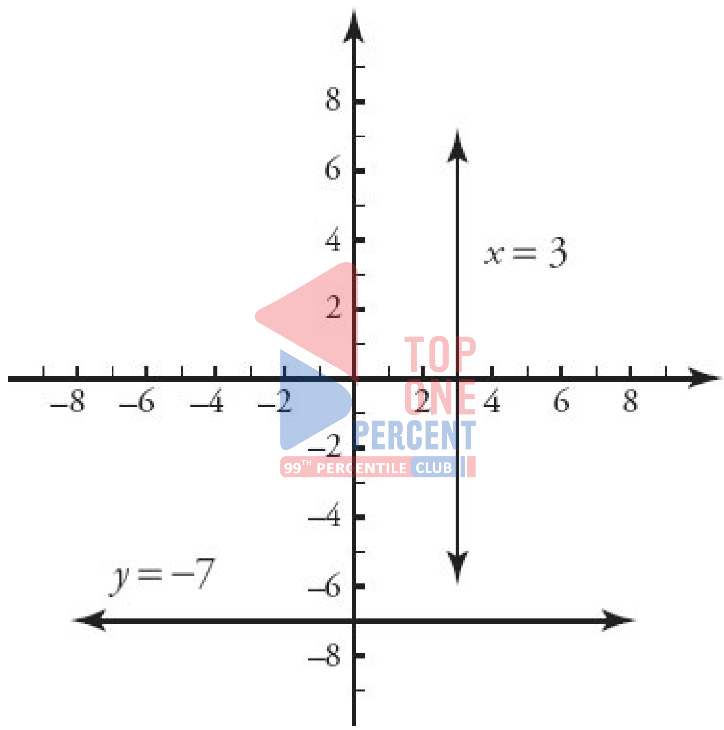
$$\begin{aligned} x &= 0 \\ 2x + y &= 5 \\ 2(0) + y &= 5 \\ y &= 5 \end{aligned} \qquad \begin{aligned} y &= 0 \\ 2x + y &= 5 \\ 2x + (0) &= 5 \\ x &= 2.5 \end{aligned}$$

The points (0, 5) and (2.5, 0) are both on the line.

Now sketch the line, using the points you have identified. If you plot (0, 5) and (2.5, 0) on the coordinate plane, you can connect them to see the position of the line. Again, the line passes through quadrants I, II, and IV.

Horizontal and Vertical Lines

Horizontal and vertical lines are not expressed in the $y = mx + b$ form. Instead, they are expressed as simple, one-variable equations.



Horizontal lines are expressed in the form:

y = *some number*, such as $y = 2$ or $y = -7$

Vertical lines are expressed in the form:

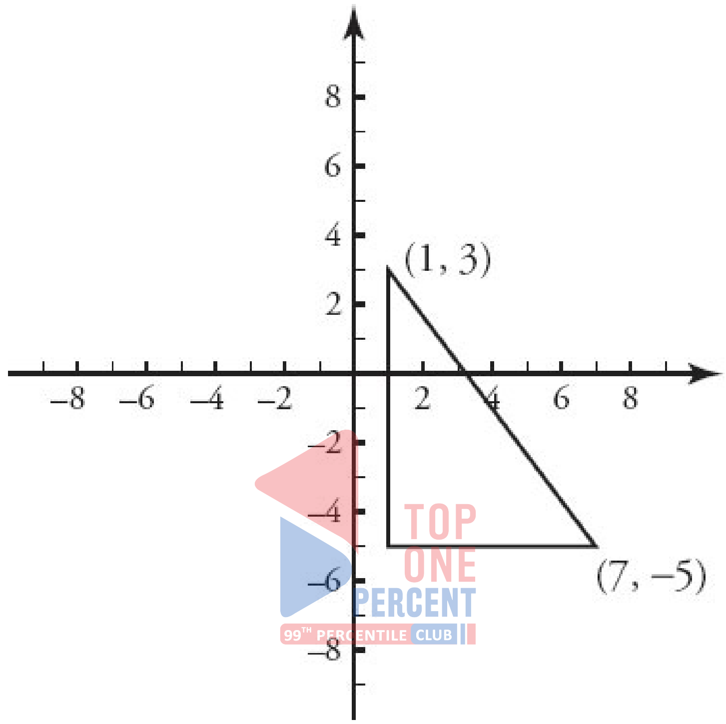
x = *some number*, such as $x = 3$ or $x = 5$

All the points on a vertical line have the same x-coordinate. This is why the equation of a vertical line is defined only by x. The y-axis itself corresponds to the equation $x = 0$. Likewise, all the points on a horizontal line have the same y-coordinate. This is why the equation of a horizontal line is defined only by y.

The x-axis itself corresponds to the equation $y = 0$.

The Distance Between Two Points

The distance between any two points in the coordinate plane can be calculated by using the Pythagorean theorem. For example:



> What is the distance between the points (1, 3) and (7, −5)?

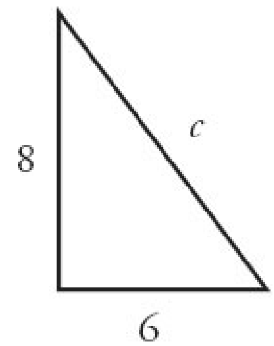
Start by drawing a right triangle connecting the points, as shown.

Next, find the lengths of the two legs of the triangle by calculating the rise and the run.

> The y-coordinate changes from 3 to −5, a difference of 8 (the vertical leg).
>
> The x-coordinate changes from 1 to 7, a difference of 6 (the horizontal leg).

Now, use the Pythagorean theorem to calculate the length of the diagonal, which is the distance between the points.

$$6^2 + 8^2 = c^2$$
$$36 + 64 = c^2$$
$$100 = c^2$$
$$c = 10$$



The distance between the two points is 10 units.

Alternatively, to find the hypotenuse, you might have recognized this triangle as a multiple of the 3−4−5 triangle (specifically, a 6−8−10 triangle).

Problem Set

1. A line has the equation $y = 3x + 7$. At which point does this line intersect the y-axis?
2. A line has the equation $x = -2y + z$. If (3, 2) is a point on the line, what is z?
3. Which quadrants, if any, do NOT contain any points on the line represented by $x - y = 18$?
4. A line has a slope of $\frac{1}{6}$ and intersects the x-axis at (−24, 0). At which point does this line intersect the y-axis?

Save the problem set below for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

5. A line has the equation $x = \frac{y}{80}$. At which point does this line intersect the x-axis?
6. Which quadrants, if any, do NOT contain any points on the line represented by $x = 10y$?
7. Which quadrants, if any, contain points on the line represented by $x + 18 = 2y$?
8. A line has a slope of $\frac{3}{4}$ and intersects the point (−12, −39). At which point does this line intersect the x-axis?

Solutions

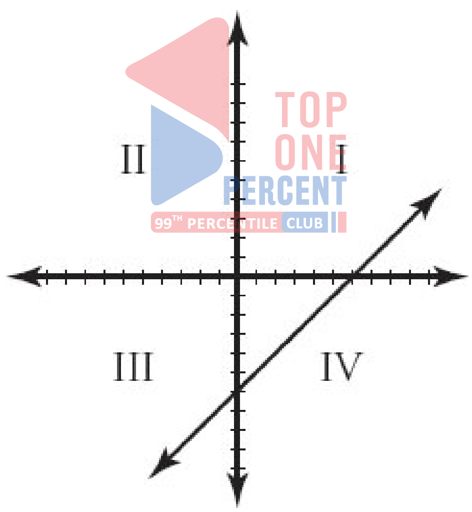
1. (**0, 7**): A line intersects the *y*-axis at the *y*-intercept. Since this equation is written in slope-intercept form, the *y*-intercept is easy to identify: 7. Thus, the line intersects the *y*-axis at the point (0, 7).

2. **7**: Substitute the coordinates (3, 2) for x and y and solve for z:

$$3 = -2(2) + z$$
$$3 = -4 + z$$
$$z = 7$$

3. **II**: First, rewrite the line in slope-intercept form:

$$y = x - 18$$



Find the intercepts by setting x equal to 0 and y equal to 0:

$$y = 0 - 18 \qquad 0 = x - 18$$

$$y = -18 \qquad x = 18$$

Plot the points: (0, −18), and (18, 0). From the sketch, you can see that the line does not pass through quadrant II.

4. (**0, 4**): Use the information given to find the equation of the line:

$$y = \frac{1}{6}x + b$$
$$0 = \frac{1}{6}(-24) + b$$
$$0 = -4 + b$$
$$b = 4$$

The variable b represents the *y*-intercept. Therefore, the line intersects the *y*-axis at (0, 4).

5. (**−20, 0**): A line intersects the *x*-axis at the *x*-intercept, or when the *y*-coordinate is equal to 0. Substitute 0 for *y* and solve for *x*:

$$x = 0 - 20$$

$$x = -20$$

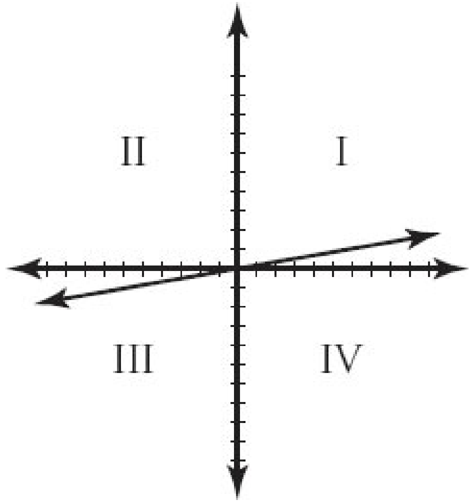
The line intersects the *x*-axis when $y = 0$. Set *y* equal to 0 and solve for *x*:

$$0 = \frac{3}{4}x - 30$$
$$\frac{3}{4}x = 30$$
$$x = 40$$

The line intersects the *x*-axis at (40, 0).

6. **II and IV:** First, rewrite the line in slope-intercept form:

$$y = \frac{x}{10}$$



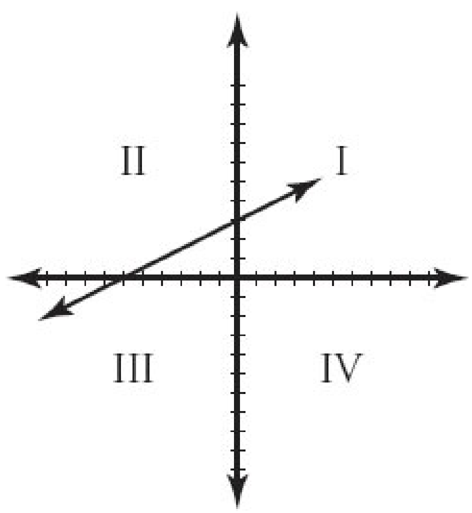
Notice from the equation that the *y*-intercept of the line is (0,0). This means that the line crosses the *y*-intercept at the origin, so the *x*- and *y*-intercepts are the same. To find another point on the line, substitute any convenient number for *x*; in this case, 10 would be a Smart Number to choose.

$$y = \frac{10}{10} = 1$$ The point (10, 1) is on the line.

Plot the points: (0, 0) and (10, 1). From the sketch, you can see that the line does not pass through quadrants II and IV.

7. **I, II, and III:** First, rewrite the line in slope-intercept form:

$$y = \frac{x}{2} + 9$$



Find the intercepts by setting *x* equal to 0 and *y* equal to 0:

$$0 = \frac{x}{2} + 9 \qquad\qquad y = \frac{0}{2} + 9$$
$$x = -18 \qquad\qquad y = 9$$

Plot the points: (−18, 0) and (0, 9). From the sketch, you can see that the line passes through quadrants I, II, and III.

8. (**40, 0**): Use the information given to find the equation of the line:

$$y = \frac{3}{4}x + \text{b}$$
$$-39 = \frac{3}{4}(-12) + \text{b}$$
$$-39 = -9 + \text{b}$$
$$\text{b} = -30$$

Chapter 7
of
Geometry

Extra Geometry

In This Chapter…

Chapter 7

Extra Geometry

Some difficult Geometry problems draw on the same geometric principles as easier problems. The GMAT makes these problems more difficult by adding steps. Other problems test more esoteric geometry rules, which you'll find in this chapter. For instance, to solve Problem Solving #145 in *The Official Guide for GMAT Quantitative Review 2015*, you have to complete several steps, using both Triangle concepts and Circle concepts. However, once you have labeled the figure appropriately, each step is itself straightforward. Likewise, Problem Solving #228 in *The Official Guide for GMAT Review 2015* does not contain fundamentally difficult Coordinate Plane Geometry. What makes #228 hard is its hybrid nature: it is a Combinatorics problem in a Coordinate Plane disguise.

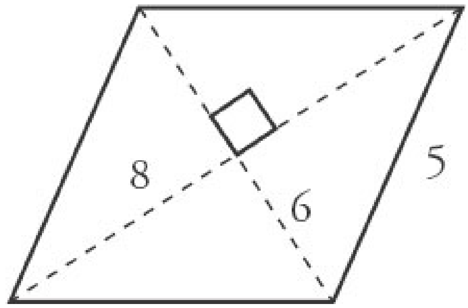
The topics in this chapter rarely appear on the GMAT. If you want an exceptionally high Quant score, then study this chapter. Otherwise, you have our permission to skip this material and guess quickly if you do happen to get a problem testing one of these concepts on the real test.

Polygons and Area

Rarely, the GMAT might test someone on the area of a rhombus, so you should know the formula:

$$\text{Area of a rhombus} = \frac{\textbf{Diagonal}_1 \times \textbf{Diagonal}_2}{\textbf{2}}$$

Note that the diagonals of a rhombus are *always* perpendicular bisectors (meaning that they cut each other in half at a 90° angle).



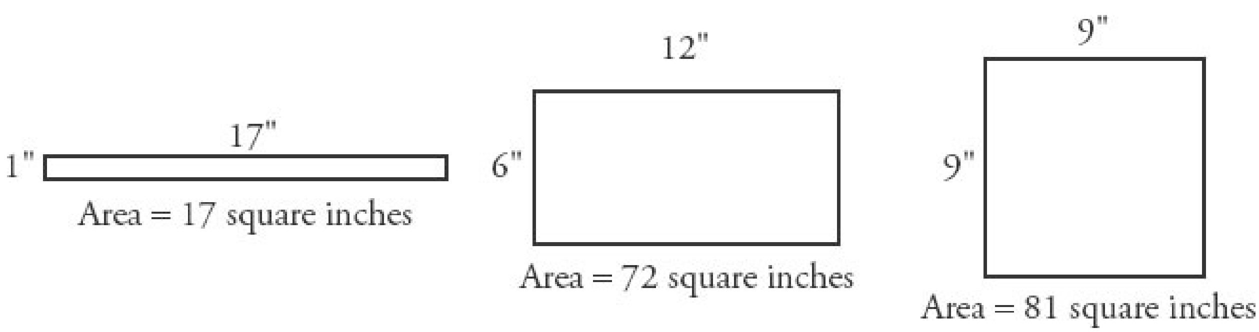
The area of the rhombus on the right is $\frac{6\times 8}{2}=\frac{48}{2}=24$.

Maximum Area of Polygons

In some problems, the GMAT may ask you to determine the maximum or minimum area of a given figure. This condition could be stated *explicitly*, as in Problem Solving questions (“What is the maximum area of…?”), or *implicitly*, as in Data Sufficiency questions (“Is the area of rectangle *ABCD* less than 30?”). Following are two shortcuts that can help you optimize certain problems quickly.

Maximum Area of a Quadrilateral

Perhaps the most common maximum area problem is to maximize the area of a quadrilateral (usually a rectangle) with a *fixed perimeter*. If a quadrilateral has a fixed perimeter, say, 36 inches, it can take a variety of shapes:



Of these figures, the one with the largest area is the square. This is a general rule: **of all quadrilaterals with a given perimeter, the *square* has the largest area**. This is true even in cases involving non-integer lengths. For instance, of all quadrilaterals with a perimeter of 25 feet, the one with the largest area is a square with $\frac{25}{4} = 6.25$ feet per side.

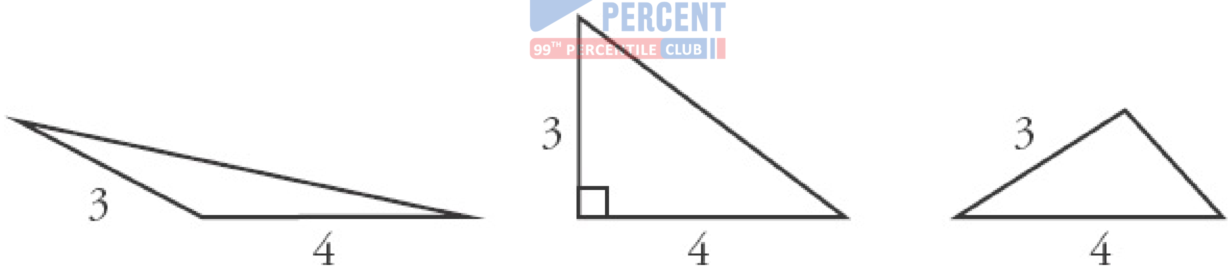
This principle can also be turned around to yield the following corollary: **of all quadrilaterals with a given area, the square has the minimum *perimeter***.

Both of these principles can be generalized for *n* sides: a regular polygon with all sides equal will maximize area for a given perimeter and minimize perimeter for a given area.

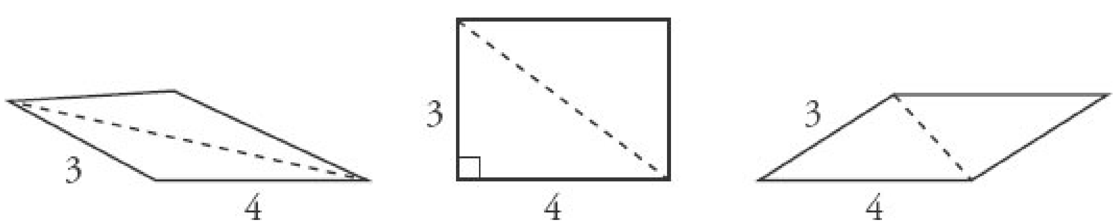
Maximum Area of a Parallelogram or Triangle

Another common optimization problem involves maximizing the area of a *triangle or parallelogram with given side lengths*.

For instance, there are many triangles with two sides 3 and 4 units long. Imagine that the two sides of length 3 and 4 are on a hinge. The third side can have various lengths:



There are many corresponding parallelograms with two sides 3 and 4 units long:



The area of a triangle is given by $A = \frac{1}{2}bh$ and the area of a parallelogram is given by $A = bh$. Because both of these formulas involve the perpendicular height h, the maximum area of each figure is achieved when the 3-unit side is perpendicular to the 4-unit side, so that the height is 3 units. All the other figures have lesser heights. (Note, that in this case, the triangle of maximum area is the famous 3–4–5 right triangle.) If the sides are not perpendicular, then the figure is squished, so to speak.

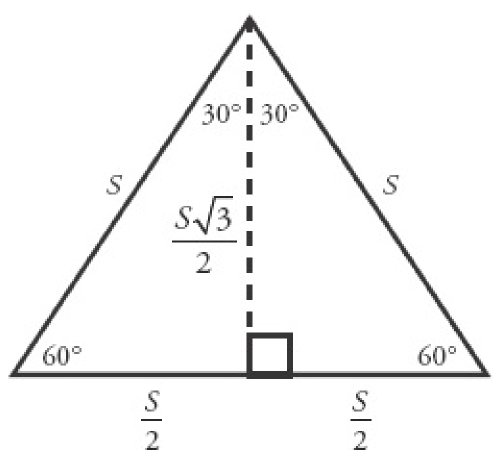
The general rule is this: **if you are given two sides of a triangle or parallelogram, you can maximize the area by placing those two sides *perpendicular* to each other**.

Since the rhombus is a special case of a parallelogram, this rule holds for rhombuses as well. All sides of a rhombus are equal. Thus, you can maximize the area of a rhombus with a given side length by making the rhombus into a square.

Triangles and Area

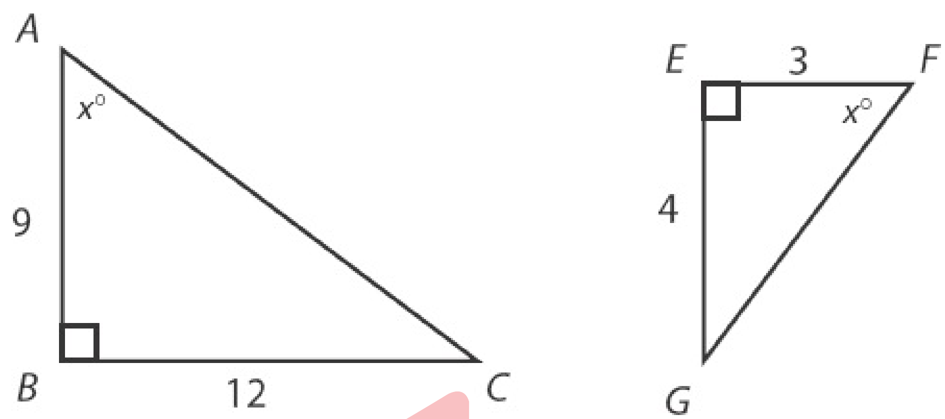
Because an **equilateral triangle** can be split into two 30–60–90 triangles, a useful formula can be derived for its area. If the side length of the equilateral triangle is S, then S is also the hypotenuse of each of the 30–60–90 triangles, so their sides are as shown in the figure below.

The equilateral triangle has base of length S and a height of length $\frac{S\sqrt{3}}{2}$.



Therefore, the **area of an equilateral triangle with a side of length *S* is equal** to $\frac{1}{2}(S)\left(\frac{S\sqrt{3}}{2}\right)=\frac{S^2\sqrt{3}}{4}$.

This formula can save you time, but it is not always needed on the test. If you find it easy to memorize formulas, then memorize this one. If you don't, then don't memorize it and take the risk that you won't see a problem like this on the test. If you do see one, you can always solve the long way.



If you compute the areas of these two similar triangles, you get the following results:

$$\begin{aligned}\text{Area of } ABC &= \frac{1}{2}bh\\ &= \frac{1}{2}(9)(12)\\ &= 54\end{aligned}$$

$$\begin{aligned}\text{Area of } EFG &= \frac{1}{2}bh\\ &= \frac{1}{2}(3)(4)\\ &= 6\end{aligned}$$

These two areas are in the ratio of 54 : 6, or 9 : 1. Notice the connection between this 9 : 1 ratio of the areas and the 3 : 1 ratio of the side lengths. The 9 : 1 ratio is the 3 : 1 ratio *squared*.

This observation can be generalized:

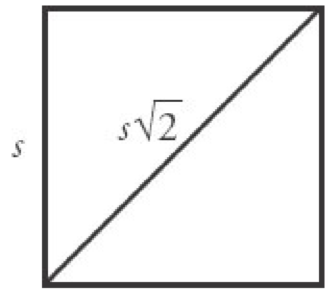
> **If two similar triangles have corresponding side lengths in ratio a : b, then their areas will be in ratio a^2 : b^2.**

The lengths being compared do not have to be sides—they can represent heights or perimeters. In fact, the figures do not have to be triangles. The principle holds true for *any* similar polygons, quadrilaterals, pentagons, etc.

Diagonals of Other Polygons

Right triangles are useful for more than just triangle problems. They are also helpful for finding the diagonals of other polygons, specifically squares, cubes, rectangles, and rectangular solids.

The diagonal of a square can be found using this formula:



$d = s\sqrt{2}$, where *s* is a side of the square.

This is also the face diagonal of a cube.

The main diagonal of a cube can be found using this formula:

$d = s\sqrt{3}$, where *s* is an edge of the cube.

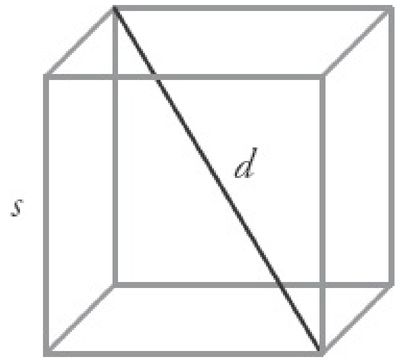
Try an example:

For a square with a side of length 5, what is the length of the diagonal?

To solve, plug 5 into the formula for a square, $d = s\sqrt{2}$. Thus, the length of the diagonal of the square is $5\sqrt{2}$.

What is the measure of an edge of a cube with a main diagonal of length $\sqrt{60}$?

To solve, use the formula for a cube, $d = s\sqrt{3}$, and plug in the information you know as follows:

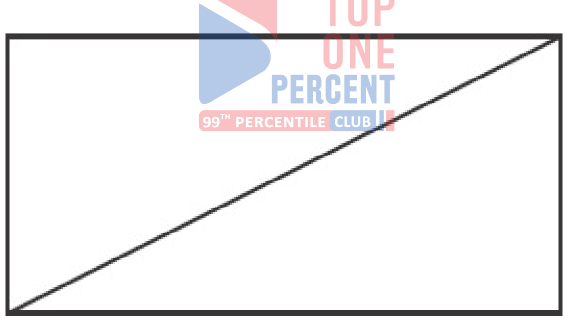


$$\sqrt{60} = s\sqrt{3} \rightarrow s = \frac{\sqrt{60}}{\sqrt{3}} = \sqrt{20}$$

The length of the edge of the cube is $\sqrt{20}$.

To find the diagonal of a rectangle, you must know either the length and the width or one dimension and the proportion of one to the other. For example:

> If the rectangle to the right has a length of 12 and a width of 5, what is the length of the diagonal?



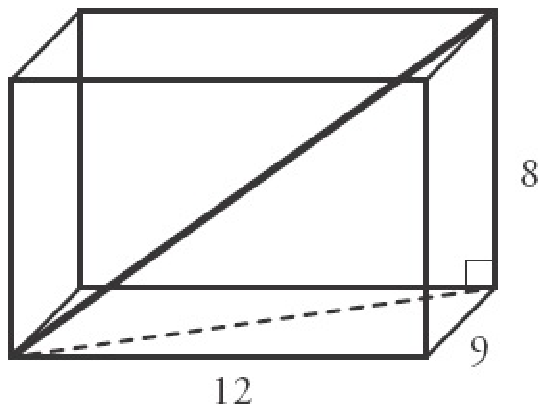
Using the Pythagorean theorem, solve:

$$5^2 + 12^2 = c^2 \rightarrow 25 + 144 = c^2 \rightarrow c = 13$$

The diagonal length is 13. Alternatively, note that this is a right triangle and you know two of the sides are 5 and 12. If you have memorized this common right triangle, then you know the length of the hypotenuse is 13. For example:

> If the rectangle above has a width of 6, and the ratio of the width to the length is 3 : 4, what is the diagonal?

Use the ratio to find the length: $\frac{3}{4} = \frac{6}{x}$. Therefore, $x = 8$. Then use the Pythagorean theorem or recognize that this is a 6–8–10 triangle. Either way, the diagonal length is 10.



What is the length of the main diagonal of the rectangular solid in the figure to the right?

To find the diagonal of a rectangular solid, use the Pythagorean theorem twice.

First, find the diagonal of the bottom face: $9^2 + 12^2 = c^2$ yields $c = 15$ (this is a multiple of a 3–4–5 triangle), so the bottom (dashed) diagonal is 15. This bottom diagonal of length 15 is the base leg of another right triangle with a height of 8. Now use the Pythagorean theorem a second time: $8^2 + 15^2 = c^2$ yields $c = 17$, so the main diagonal is 17.

Alternatively, memorize the “Deluxe” Pythagorean theorem: $d^2 = x^2 + y^2 + z^2$, where x, y, and z are the sides of the rectangular solid and d is the main diagonal. In this case, $9^2 + 12^2 + 8^2 = d^2$, which yields $d = 17$.

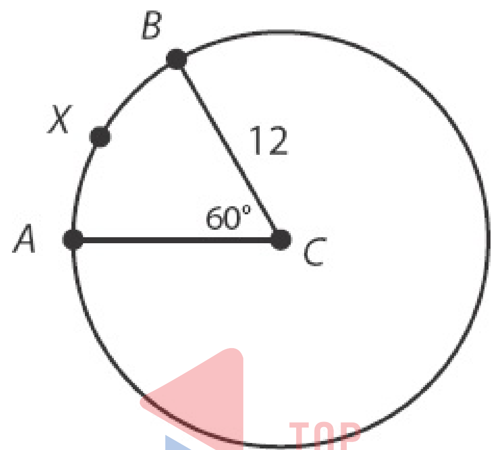
Revolution = Circumference

The GMAT occasionally asks about a wheel or spinning circle. A full revolution, or turn, of a spinning wheel is equivalent to the wheel going around once. If you were to place a point on the edge of the wheel, it would travel one full circumference in one revolution. For example, if a wheel spins at 3 revolutions per second, a point on the edge travels a distance equal to 3 circumferences per second. If the wheel has a diameter of 4 feet, then the point travels at a rate of 3

$\times 4\pi = 12\pi$ feet per second.

Circumference and Arc Length

Often, the GMAT will ask you to solve for a portion of the distance on a circle, instead of the entire circumference. This portion is termed an **arc**. Arc length can be found by determining what fraction the arc is of the entire circumference. For example:



In the circle to the right, what is the length of arc *AXB*?

Arc *AXB* is the arc from *A* to *B*, passing through the point *X*. To find its length, first find the circumference of the circle. The radius is given as 12. To find the circumference, use the formula $C = 2\pi r$. Thus, $2\pi(12) = 24\pi$.

Next, use the central angle, the angle at the center of the circle when two lines are drawn to points *A* and *B*, to determine what fraction the arc is of the entire circle. Since the arc is defined by the central angle of 60°, and the entire circle is 360°, then the arc is $\frac{60}{360} = \frac{1}{6}$ of the circle.

Therefore, the measure of arc *AXB* is $\left(\frac{1}{6}\right)(24\pi) = 4\pi$.

Perimeter of a Sector

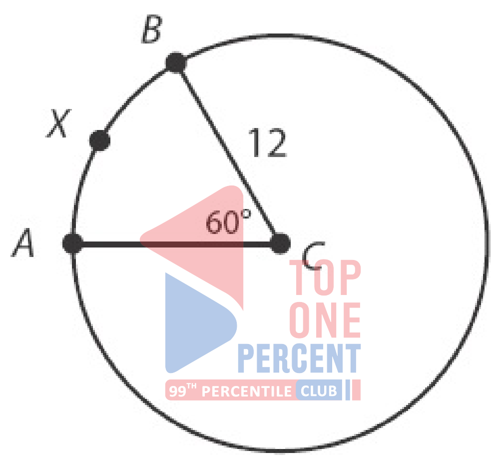
The boundaries of a **sector** of a circle are formed by the arc and two radii. Think of a sector as a slice of pizza. The arc corresponds to the crust, and the center of the circle corresponds to the tip of the slice.

If you know the length of the radius and the central angle, you can find the perimeter of the sector. For example:

> What is the perimeter of sector *ABC*?

In the previous example, you found the length of arc *AXB*, which was 4π. Therefore, the perimeter of the sector is:

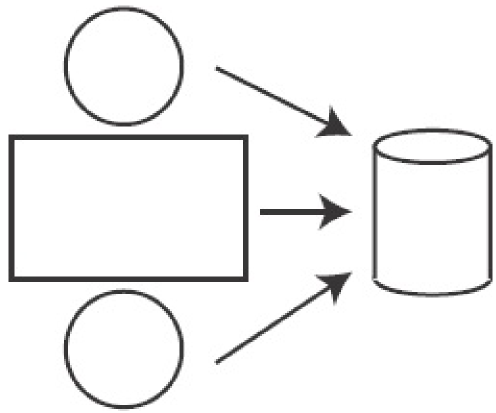
$$4\pi + 12 + 12 = 24 + 4\pi$$



Cylinders and Surface Area

Two circles and a rectangle combine to form a three-dimensional shape called a right circular cylinder (referred to from now on simply as a **cylinder**). The top and bottom of the cylinder are circles, while the middle of the cylinder is formed from a rolled-up rectangle, as shown in the figure below:

In order to determine the surface area (SA) of a cylinder, sum the areas of the three surfaces: the area of each circle is πr^2, while the area of the rectangle is length × width. The length of the rectangle is equal to the circumference of the circle ($2\pi r$), and the width of the rectangle is equal to the height of the cylinder (h). Therefore, the area of the rectangle is $2\pi r \times h$. To find the total surface area of a cylinder, add the area of the circular top and bottom, as well as the area of the rectangle that wraps around the outside.



$$SA = 2 \text{ circles} + \text{rectangle} = 2(\pi r^2) + 2\pi rh$$

The only information you need to find the surface area of a cylinder is the radius of the cylinder and the height of the cylinder.

Step by Step: From Two Points to a Line

If you are given any two points on a line, you can write an equation for that line in the form $y = mx + b$. Here is an example showing the step-by-step method:

> Find the equation of the line containing the points (5, -2) and (3, 4).

First, find the slope of the line by calculating the rise over the run.

The rise is the difference between the *y*-coordinates, while the run is the difference between the *x*-coordinates. The sign of each difference is important, so subtract the *x*-coordinates and the *y*-coordinates in the same order:

$$\frac{\text{rise}}{\text{run}} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-2-4}{5-3} = \frac{-6}{2} = -3 \qquad \text{The slope of the line is } -3.$$

Second, plug in the slope for *m* in the slope-intercept equation:

$y = -3x + b$

Third, solve for *b*, the *y*-intercept, by plugging the coordinates of one point into the equation. Either point's coordinates will work.

Plugging the point (3, 4) into the equation (3 for x and 4 for y) yields the following:

$4 = -3(3) + b$

$4 = -9 + b$

$b = 13$

The y-intercept of the line is 13.

Fourth, write the equation in the form $y = mx + b$:

$y = -3x + 13$ This is the equation of the line.

Note that sometimes the GMAT will only give you one point on the line, along with the y-intercept. This is the same thing as giving you two points on the line, because the y-intercept is a point! A y-intercept of 4 is the same as the ordered pair (0, 4).

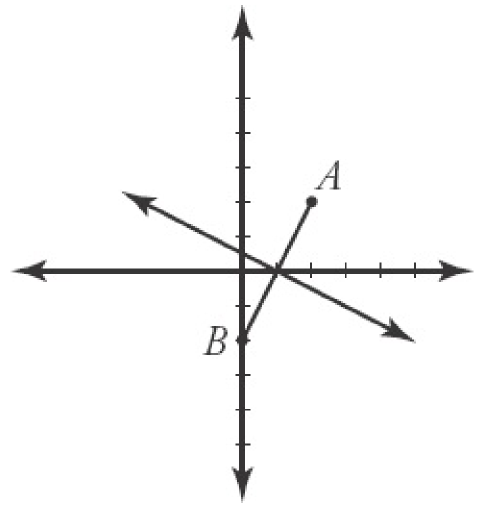
Perpendicular Bisectors

The perpendicular bisector of a line segment forms a 90° angle with that line segment and divides the segment exactly in half. Questions about perpendicular bisectors are rare on the GMAT, but they do appear occasionally. For example:

> If the coordinates of point A are (2, 2) and the coordinates of point B are (0, −2), what is the equation of the perpendicular bisector of line segment AB?

The key to solving perpendicular bisector problems is to use this property: the perpendicular bisector has the **negative reciprocal slope** of the line segment it bisects. That is, the product of the two slopes is −1. (The only exception occurs when one line is horizontal and the other line is vertical, since vertical lines have undefined slopes.)

To start, find the slope of segment AB.



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{2-(-2)}{2-0} = \frac{4}{2} = 2$$

The slope of *AB* is 2.

Next, find the slope of the perpendicular bisector of *AB*.

Since perpendicular lines have negative reciprocal slopes, flip the fraction and change the sign to find the slope of the perpendicular bisector. The slope of *AB* is 2, or $\frac{2}{1}$. Therefore, the slope of the perpendicular bisector of *AB* is $-\frac{1}{2}$.

Now you know that the equation of the perpendicular bisector has the following form:

$$y = -\frac{1}{2}x + b$$

However, you still need to find the value of *b* (the *y*-intercept). To do this, you first need to find the point where the line bisects line segment *AB*. Then you plug the coordinates of this point into the equation above.

Now you can find the midpoint of *AB*.

The perpendicular bisector passes through the midpoint of *AB*. Thus, if you find the midpoint of *AB*, you will have found a point on the perpendicular bisector. Organize a table such as the one shown below to find the coordinates of the midpoint. Write the *x*- and *y*-coordinates of *A* and *B*. The coordinates of the midpoint will be the numbers halfway between each pair of *x*- and *y*-coordinates. In other words, the *x*-coordinate of the midpoint is the *average* of the *x*-coordinates of *A* and *B*. Likewise, the *y*-coordinate of the midpoint is the *average* of the *y*-coordinates of *A* and *B*. This process will yield the midpoint of any line segment.

	x	y
A	2	2
Midpoint	**1**	**0**
B	0	–2

Finally, put the information together.

To find the value of *b* (the *y*-intercept), substitute the coordinates of the midpoint into the line equation for *x* and *y*:

$$0 = -\frac{1}{2}(1) + b$$

$$b = \frac{1}{2}$$

The perpendicular bisector of segment *AB* has the equation: $y = -\frac{1}{2}x + \frac{1}{2}$.

In summary, the following rules can be given:

- **Parallel lines have equal slopes:** $m_1 = m_2$.

- **Perpendicular lines have negative reciprocal slopes.** $\frac{-1}{m_1} = m_2$, **or** $m_1 \cdot m_2 = -1$.

- The **midpoint** between point $A(x_1, y_1)$ and point $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The Intersection of Two Lines

Recall that a line in the coordinate plane is defined by a linear equation relating *x* and *y*. That is, if a point (*x*, *y*) lies on the line, then those values of *x* and *y* satisfy the equation. For instance, the point (3, 2) lies on the line defined by the equation $y = 4x - 10$, since the equation is true when you plug in $x = 3$ and $y = 2$:

$$y = 4x - 10$$
$$2 = 4(3) - 10 = 12 - 10$$
$$2 = 2 \text{ TRUE}$$

On the other hand, the point (7, 5) does not lie on that line, because the equation is false when you plug in $x = 7$ and $y = 5$:

$$y = 4x - 10$$
$$5 = 4(7) - 10 = 28 - 10$$
$$5 = 18 \text{ FALSE}$$

So what does it mean when two lines intersect in the coordinate plane? It means that at the point of intersection, *both* equations representing the lines are true. That is, the pair of numbers (*x*, *y*) that represents the point of intersection solves *both* equations. Finding this point of intersection is equivalent to solving a system of two linear equations. You can find the intersection by using algebra more easily than by graphing the two lines. Here's an example:

> At what point does the line represented by $y = 4x - 10$ intersect the line represented by $2x + 3y = 26$?

Since $y = 4x - 10$, replace *y* in the second equation with $4x - 10$ and solve for *x*:

$$2x + 3(4x - 10) = 26$$
$$2x + 12x - 30 = 26$$
$$14x = 56$$
$$x = 4$$

Now solve for y. You can use either equation:

$$y = 4x - 10$$
$$y = 4(4) - 10$$
$$y = 16 - 10 = 6$$
$$y = 6$$

Thus, the point of intersection of the two lines is (4, 6).

If two lines in a plane do not intersect, then the lines are parallel. If this is the case, there is *no* pair of numbers (x, y) that satisfies both equations at the same time.

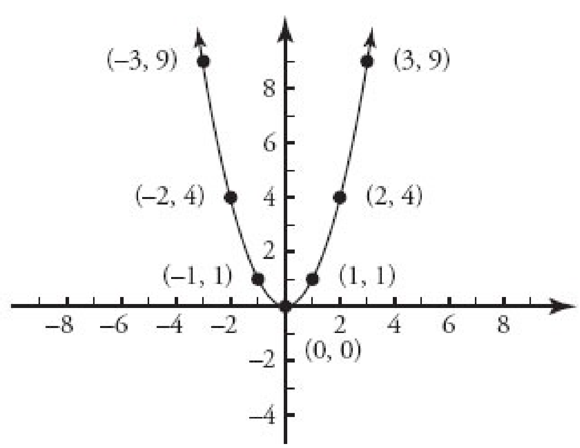
There is one other possibility: the two equations might represent the same line. In this case, infinitely many points (x, y) along the line satisfy the two equations (which must actually be the same equation in two different forms).

Function Graphs and Quadratics

You can think of the slope-intercept form of a linear equation as a function: $y = f(x) = mx + b$. That is, you input the x-coordinate into the function $f(x) = mx + b$, and the output is the y-coordinate of the point that you plot on the line.

You can apply this process more generally. For instance, imagine that $y = f(x) = x^2$. You can generate the graph for $f(x)$ by plugging in a variety of values for x and getting values for y. The points (x, y) that you find lie on the graph of $y = f(x) = x^2$.

x	$f(x) = y$	Point
−3	$(-3)^2 = 9$	(−3, 9)
−2	$(-2)^2 = 4$	(−2, 4)
−1	$(-1)^2 = 1$	(−1, 1)
0	$0^2 = 0$	(0, 0)
1	$1^2 = 1$	(1, 1)
2	$2^2 = 2$	(2, 4)
3	$3^2 = 9$	(3, 9)



This curved graph is called a **parabola**. Any function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are constants, is called a **quadratic function** and can be plotted as a parabola in the coordinate plane. Depending on the value of a, the curve will have different shapes:

Positive value for a	Curve opens upward
Negative value for a	Curve opens downward
Large $\lvert a \rvert$ (absolute value)	Narrow curve
Small $\lvert a \rvert$	Wide curve

The parabola will always open upward or downward.

The most important questions you will be asked about the parabola are these:

1. How many times does the parabola touch the x-axis?
2. If the parabola does touch the x-axis, where does it touch?

In other words, how many x-intercepts are there and what are they?

These questions are important because the x-axis is the line representing $y = 0$. In other words, the parabola touches the x-axis at those values of x that make $f(x) = 0$. Therefore, these values solve the quadratic equation given by $f(x) = ax^2 + bx + c = 0$.

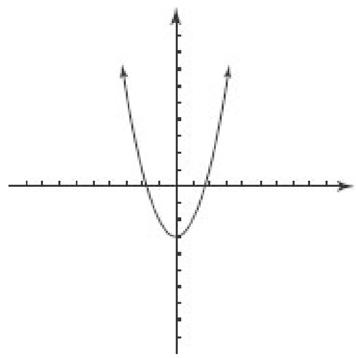
You can solve for 0 by factoring and solving the equation directly. Alternatively, you might plug in points and draw the parabola. Finally, for some very difficult problems, you could use the quadratic formula (though note that you will almost certainly never need to use this on the test, so this may not be worth memorizing):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

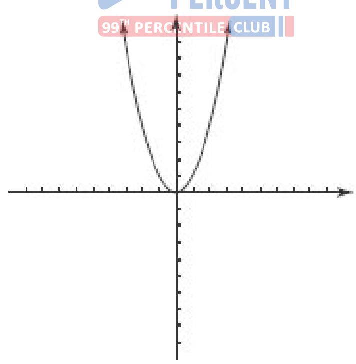
One solution is $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$, and the other is $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

The vast majority of GMAT Quadratic problems can be solved *without* using the quadratic formula. If you do apply this formula, the advantage is that you can quickly tell how many solutions the equation has by looking at just one part: the expression under the radical sign, b^2 - 4ac. This expression is known as the **discriminant,** because it discriminates or distinguishes three cases for the number of solutions to the equation, as follows:

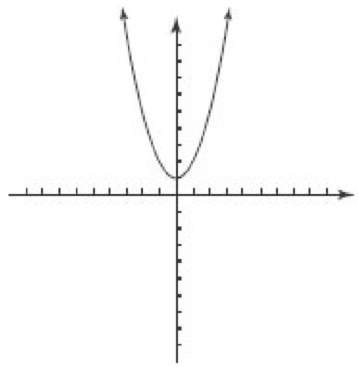
1. If $b^2 - 4ac > 0$, then the square root operation yields a positive number. The quadratic formula produces *two roots* of the quadratic equation. This means that the parabola crosses the x-axis twice and has two x-intercepts.



2. If $b^2 - 4ac = 0$, then the square root operation yields 0. The quadratic formula only produces *one root* of the quadratic equation. This means that the parabola touches the x-axis only once and has just one x-intercept.



3. If $b^2 - 4ac < 0$ then the square root operation cannot be performed. This means that the quadratic formula produces *no roots* of the quadratic equation and the parabola never touches the x-axis (it has no x-intercepts).

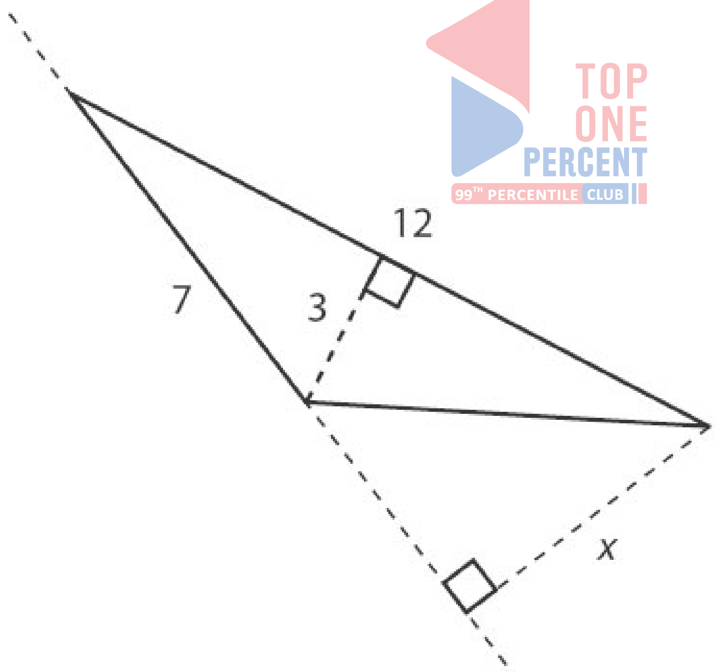


It is possible for the GMAT to ask you to graph other nonlinear functions of *x*. The following statements lie at the heart of all problems involving graphs of other nonlinear functions, as well as lines and parabolas:

1. If a point lies on the graph, then you can plug its coordinates into the equation $y = f(x)$. Conversely, if a value of x and a value of y satisfy the equation $y = f(x)$, then the point (x, y) lies on the graph of $f(x)$.
2. To find *x*-intercepts, find the values of x for which $y = f(x) = 0$.
3. To find *y*-intercepts, set $x = 0$ and find $y = f(0)$.

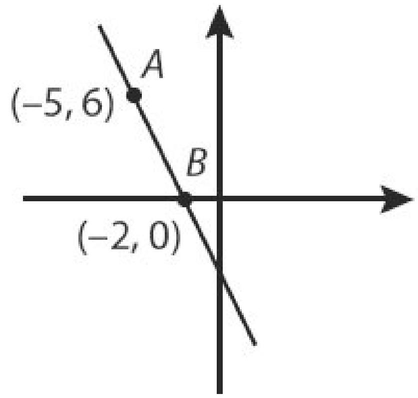
Problem Set

1. What is the maximum possible area of a quadrilateral with a perimeter of 80 centimeters?
2. What is the minimum possible perimeter of a quadrilateral with an area of 1,600 square feet?
3. What is the maximum possible area of a parallelogram with one side of length 2 meters and a perimeter of 24 meters?
4. What is the maximum possible area of a triangle with a side of length 7 units and another side of length 8 units?
5. The lengths of the two shorter legs of a right triangle add up to 40 units. What is the maximum possible area of the triangle?
6. What is x in the figure below?



7. The line represented by the equation $y = -2x + 6$ is the perpendicular bisector of the line segment AB. If A has the coordinates (7, 2), what are the coordinates for B?
8. How many x-intercepts does $f(x) = x^2 + 3x + 3$ have?

9. The line represented by the equation $y = x$ is the perpendicular bisector of line segment *AB*. If *A* has the coordinates $(-3, 3)$, what are the coordinates of *B*?

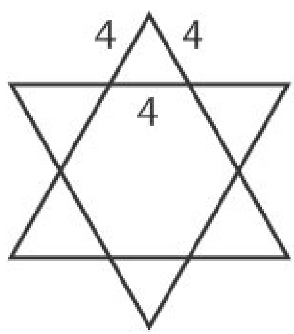


10. What are the coordinates for the point on line *AB* (see figure at right) that is three times as far from *A* as from *B*, and that is in between points *A* and *B*?

11. Triangle *A* has a base of x and a height of $2x$. Triangle *B* is similar to triangle *A*, and has a base of $2x$. What is the ratio of the area of triangle *A* to triangle *B*?

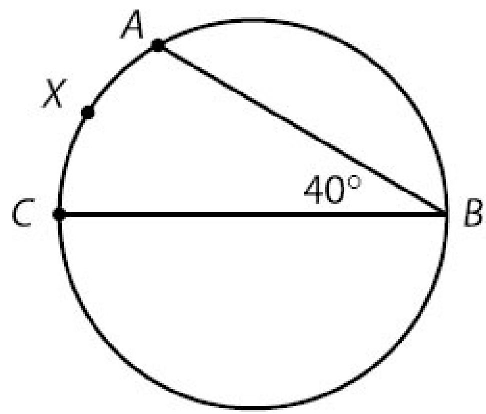
12. What is the longest diagonal of a rectangular box that is 120 inches long, 90 inches wide, and 80 inches tall?

13. The points of a six-pointed star consist of six identical equilateral triangles, with each side 4 cm (see figure). What is the area of the entire star, including the center?



14. A cylinder has a surface area of 360π units square, and is 3 units tall. What is the diameter of the cylinder's circular base?

15. Angle *ABC* is 40° (see figure) and the area of the circle is 81π. If *CB* is a diameter of the circle, how long is arc *AXC*?


A
X
40°
C
B

Solutions

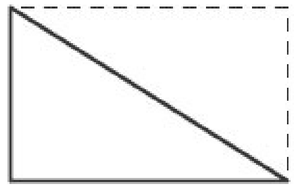
1. **400 cm^2:** The quadrilateral with maximum area for a given perimeter is a square, which has four equal sides. Therefore, the square that has a perimeter of 80 centimeters has sides of length 20 centimeters each. Since the area of a square is the side length squared, the area is: (20 cm)(20 cm) = 400 cm^2.

2. **160 ft:** The quadrilateral with minimum perimeter for a given area is a square. Since the area of a square is the side length squared, you can solve the equation x^2 = 1,600 ft^2 for the side length x, yielding x = 40 ft. The perimeter, which is four times the side length, is (4)(40 ft), which equals 160 ft.

3. **20 m^2**: If one side of the parallelogram is 2 meters long, then the opposite side must also be 2 meters long. You can solve for the unknown sides, which are equal in length, by writing an equation for the perimeter: 24 = 2(2) + 2x, with x as the unknown side. Solving, you get x = 10 meters. The parallelogram with these dimensions and maximum area is a *rectangle* with 2 meter and 10 meter sides. Thus, the maximum possible area of the figure is: (2 m)(10 m) = 20 m^2.

4. **28 square units:** A triangle with two given sides has maximum area if these two sides are placed at right angles to each other. For this triangle, one of the given sides can be considered the base, and the other side can be considered the height (because they meet at a right angle). Thus, plug these sides into the formula $A = \frac{1}{2}bh$: $A = \frac{1}{2}(7)(8) = 28$.

5. **200 square units:** You can think of a right triangle as half of a rectangle. Constructing this right triangle with legs adding to 40 is equivalent to constructing the rectangle with a perimeter of 80. Since the area of the triangle is half that of the rectangle, you can use the previously mentioned technique for maximizing the area of a rectangle: of all rectangles with a given perimeter, the *square* has the greatest area. The desired rectangle is thus a 20 by 20 square, and the right triangle has an area of $\left(\frac{1}{2}\right)(20)(20) = 200$ units.



6. $\mathbf{\frac{36}{7}}$**:** You can calculate the area of the triangle using the side of length 12 as the base:

$$\left(\frac{1}{2}\right)(12)(3) = 18$$

Next, use the side of length 7 as the base and write the equation for the area:

$$\left(\frac{1}{2}\right)(7)(x) = 18$$

Now solve for *x*, the unknown height:

$$7x = 36$$

$$x = \frac{36}{7}$$

Alternatively, the large overall triangle is similar to the small triangle on the left side of the picture because they have the same angle measurements (both are right triangles, and they also share one angle in the far left tip of the figure). Draw these two triangles side by side and match up the sides. The hypotenuse of 7 in the smaller triangle "matches up" with the hypotenuse of 12 in the larger triangle, so the ratio of the two triangles is $\frac{12}{7}$. Multiply the known leg 3 in the smaller triangle by the ratio multiplier $\frac{12}{7}$ to get $3 \times \frac{12}{7} = \frac{36}{7}$. The value of *x* is $\frac{36}{7}$.

7. (−1, −2): If *y* = −2*x* + 6 is the perpendicular bisector of segment *AB*, then the line containing segment *AB* must have a slope of 0.5 (the negative inverse of −2). You can represent this line with the equation *y* = 0.5*x* + *b*. Substitute the

coordinates (7, 2) into the equation to find the value of b.

	x	y
A	7	2
Midpoint	3	0
B	**-1**	**-2**

$2 = 0.5(7) + b.$
$b = -1.5$

The line containing AB is $y = 0.5x - 1.5$.

Find the point at which the perpendicular bisector intersects AB by setting the two equations, $y = -2x + 6$ and $y = 0.5x - 1.5$, equal to each other:

$$-2x + 6 = 0.5x - 1.5$$
$$2.5x = 7.5$$
$$x = 3;\ y = 0$$

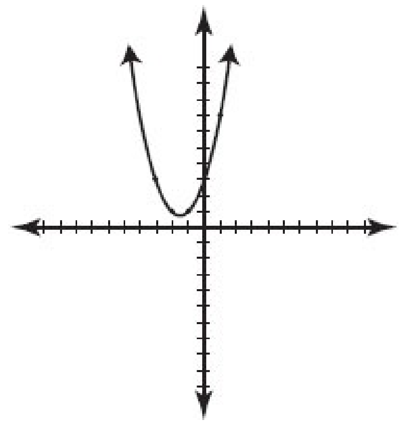
The two lines intersect at (3, 0), which is the midpoint of AB.

Use a table to find the coordinates of B. The x- and y- coordinates of the midpoint are the averages of the x- and y- coordinates, respectively, of A and B.

8. **None:** There are three ways to solve this equation. The first is to attempt to factor the quadratic equation to find solutions. Since no two integers multiply to 3 and add to 3, this strategy fails.

The second approach is to pick numbers for x, solve for $f(x)$ (plotted as y in the coordinate plane), and plot these (x, y) pairs to determine the shape of the parabola. An example of this technique is displayed to the right.

x	$x^2 + 3x + 3 = y$	Point
−3	$9 - 9 + 3 = 3$	(−3, 3)
−2	$4 - 6 + 3 = 1$	(−2, 1)
−1	$1 - 3 + 3 = 1$	(−1, 1)
0	$0 + 0 + 3 = 3$	(0, 3)
1	$1 + 3 + 3 = 7$	(1, 7)



This approach demonstrates that the parabola never touches the *x*-axis. There are no *x*-intercepts.

The third method is to use the discriminant of the quadratic equation to count the number of *x*-intercepts.

First, identify the coefficients of each term. The function is $f(x) = x^2 + 3x + 3$. Matching this up to the definition of the standard quadratic equation, $f(x) = ax^2 + bx + c$, you have $a = 1$, $b = 3$, and $c = 3$.

Next, write the discriminant from the quadratic formula (the expression that is under the radical sign in the quadratic formula):

$$\begin{aligned} b^2 - 4ac &= 3^2 - 4(1)(3) \\ &= 9 - 12 \\ &= -3 \end{aligned}$$

Since the discriminant is less than 0, you cannot take its square root. This means that there is no solution to the equation $f(x) = x^2 + 3x + 3 = 0$, so the function's graph does not touch the *x*-axis. There are no *x*-intercepts.

9. **(3, −3):** Perpendicular lines have negative inverse slopes. Therefore, if $y = x$ is perpendicular to segment *AB*, you know that the slope of the perpendicular bisector is 1, and therefore the slope of segment *AB* is −1. The line containing segment *AB* takes the form of $y = -x + b$. To find the value of *b*, substitute the coordinates of *A*, (−3, 3), into the equation:

$$\begin{aligned} 3 &= -(-3) + b \\ b &= 0 \end{aligned}$$

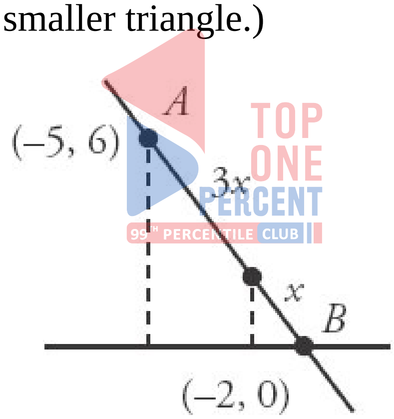
The line containing segment AB is $y = -x$.

Find the point at which the perpendicular bisector intersects AB by setting the two equations, $y = x$ and $y = -x$, equal to each other:

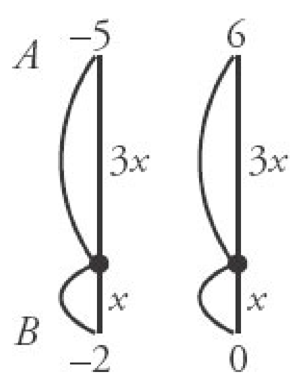
$$x = -x$$
$$x = 0;\ y = 0$$

The two lines intersect at (0, 0), which is the midpoint of AB. Use a table to find the coordinates of B.

10. **(−2.75, 1.5):** The point in question is 3 times farther from A than it is from B. You can represent this fact by labeling the point $3x$ units from A and x units from B as shown, giving a total distance of $4x$ between the two points. If you drop vertical lines from the point and from A to the x-axis, you get two similar triangles, the smaller of which is a quarter of the larger. (You can get this relationship from the fact that the larger triangle's hypotenuse is 4 times larger than the hypotenuse of the smaller triangle.)



The horizontal distance between points A and B is 3 units (from −2 to −5). Therefore, $4x = 3$, so $x = 0.75$. The horizontal distance from B to the point is x, or 0.75 units. The x-coordinate of the point is 0.75 away from −2, or −2.75.



The vertical distance between points A and B is 6 units (from 0 to 6).

Therefore, $4x = 6$, so $x = 1.5$. The vertical distance from B to the point is x, or 1.5 units. The y-coordinate of the point is 1.5 away from 0, or 1.5.

11. **1 to 4:** Since you know that triangle B is similar to triangle A, you can set up a proportion to represent the relationship between the sides of both triangles:

$$\frac{\text{base}}{\text{height}} = \frac{x}{2x} = \frac{2x}{?}$$

By proportional reasoning, the height of triangle B must be $4x$. Calculate the area of each triangle with the area formula:

Triangle A: $$A = \frac{b \times h}{2} = \frac{(x)(2x)}{2} = x^2$$

Triangle B: $$A = \frac{b \times h}{2} = \frac{(2x)(4x)}{2} = 4x^2$$

The ratio of the area of triangle A to triangle B is 1 to 4. Alternatively, you can simply square the base ratio of 1 : 2.

12. **170 inches:** Using the Deluxe Pythagorean theorem, calculate the length of the diagonal:

$$120^2 + 90^2 + 80^2 = d^2.$$

Note: to make the math easier, drop a 0 from each number—but don't forget to put it back in later!

$$12^2 + 9^2 + 8^2 = d^2$$
$$144 + 81 + 64 = d^2$$
$$289 = d^2$$
$$17 = d$$
$$d = 170 \text{ inches}$$

Don't forget to put the zero back in!

Alternatively, you can find the diagonal of this rectangular solid by applying the Pythagorean Theorem twice. First, find the diagonal across the bottom of the box:

$$120^2 + 90^2 = c^2$$
$$14{,}400 + 8{,}100 = c^2$$
$$c^2 = 22{,}500$$
$$c = 150$$

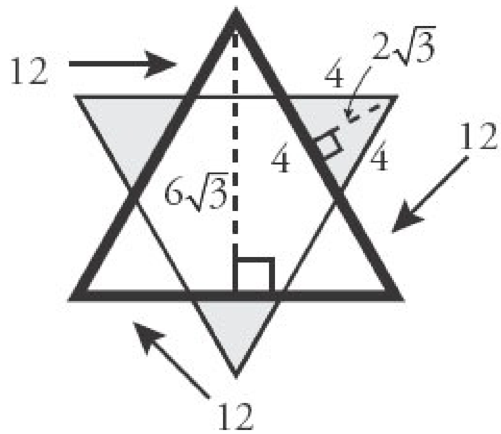
You might recognize this as a multiple of the common 3–4–5 right triangle.

Next, find the diagonal of the rectangular box:

$$80^2 + 150^2 = c^2$$
$$6{,}400 + 22{,}500 = c^2$$
$$c^2 = 28{,}900$$
$$c = 170$$

You might recognize this as a multiple of the common 8–15–17 right triangle.

13. $48\sqrt{3}$ **cm²:** You can think of this star as a large equilateral triangle with sides 12 centimeters long, and three additional smaller equilateral triangles (shaded in the figure to the right) with sides 4 centimeters long. Using the same 30–60–90 logic discussed in chapter 4, note that the height of the larger equilateral triangle is $6\sqrt{3}$ and the height of the smaller equilateral triangle is $2\sqrt{3}$.



Therefore, the areas of the triangles are as follows:

Large triangle: $A=\frac{b\times h}{2}=\frac{12\times 6\sqrt{3}}{2}=36\sqrt{3}$

Small triangles: $A=\frac{b\times h}{2}=\frac{4\times 2\sqrt{3}}{2}=4\sqrt{3}$

The total area of three smaller triangles and one large triangle is:

$36\sqrt{3}+3(4\sqrt{3})=48\sqrt{3}\ \text{cm}^2$

Alternatively, you can apply the formula $A=\frac{S^2\sqrt{3}}{4}$:

Large triangle: $A=\frac{12^2\sqrt{3}}{4}=\frac{144\sqrt{3}}{4}=36\sqrt{3}$

Small triangle: $A=\frac{4^2\sqrt{3}}{4}=\frac{16\sqrt{3}}{4}=4\sqrt{3}$

Next, add the area of the large triangle and the area of three smaller triangles, as above.

14. **24 units:** The surface area of a cylinder is the area of the circular top and bottom, plus the area of its wrapped-around rectangular third face. You can express this in formula form as:

$SA = 2(\pi r^2) + 2\pi rh$

Substitute the known values into this formula to find the radius of the circular base:

$$360\pi = 2(\pi r^2) + 2\pi r(3)$$
$$360\pi = 2\pi r^2 + 6\pi r$$
$$r^2 + 3r - 180 = 0$$
$$(r + 15)(r - 12) = 0$$

$r + 15 = 0$ OR $r - 12 = 0$

$r = \{-15, 12\}$

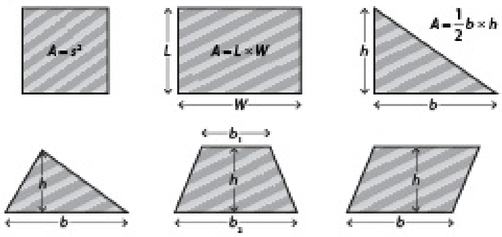
Use only the positive value of r: 12. If $r = 12$, the diameter of the cylinder's circular base is 24.

15. **4*π***: If the area of the circle is 81π, then the radius of the circle is 9 ($A = \pi r^2$). Therefore, the total circumference of the circle is 18π ($C = 2\pi r$). Angle *ABC*, an inscribed angle of 40°, corresponds to a central angle of 80°. Thus, arc *AXC* is equal to $\frac{80}{360} = \frac{2}{9}$ of the total circumference. Therefore, arc *AXC* equals $\frac{2}{9}(18\pi) = 4\pi$.

Geometry Cheat Sheet

Shown on the next page.

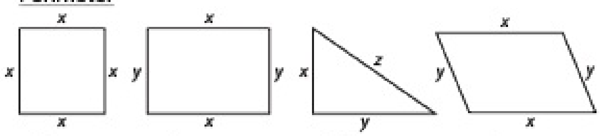
Area



$A = s^2$ $\quad$ $A = L \times W$ $\quad$ $A = \frac{1}{2} b \times h$

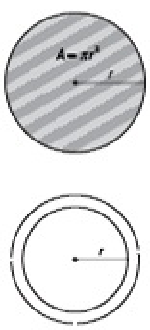
$A = \frac{1}{2} b \times h$ $\quad$ $A = \left(\frac{b_1 + b_2}{2}\right) \times h$ $\quad$ $A = b \times h$

Perimeter



$P = 4x$ $\quad$ $P = 2x + 2y$ $\quad$ $P = x + y + z$ $\quad$ $P = 2x + 2y$

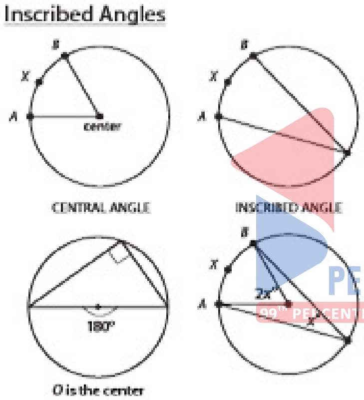
Circles

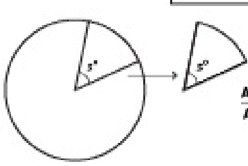


$A = \pi r^2$

$C = 2\pi r$

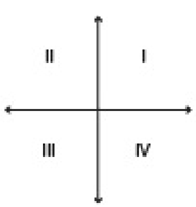
Inscribed Angles

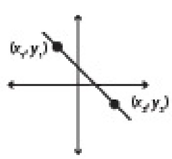




$$\frac{\text{Area}_{\text{sector}}}{\text{Area}_{\text{circle}}} = \frac{s}{360}$$

Coordinate Plane

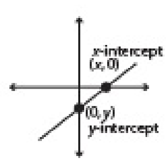




slope $= \frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{y_1 - y_2}{x_1 - x_2}$

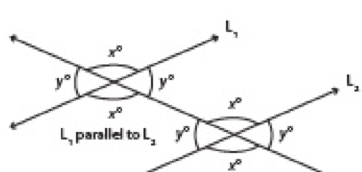
$y = mx + b$

slope $\quad$ y-intercept



slopes of perpendicular lines: negative reciprocals

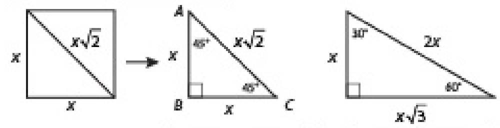
ex. l_1 slope $= \frac{1}{2}$, l_2 slope $= -2$



L_1 parallel to L_2

$x + y = 180$

Special Right Triangles



leg	leg	hypotenuse
45°	45°	90°
x	x	$x\sqrt{2}$

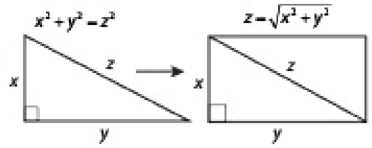
leg	leg	hypotenuse
30°	60°	90°
x	$x\sqrt{3}$	$2x$

Triangles

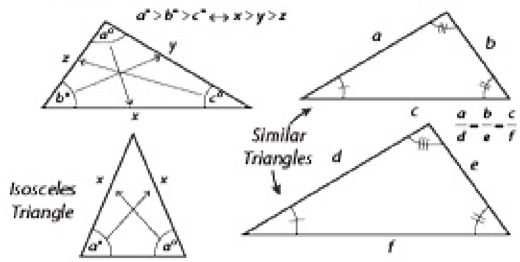
$x^2 + y^2 = z^2$ $\quad$ $z = \sqrt{x^2 + y^2}$

Pythagorean Triplets

3–4–5
6–8–10
5–12–13
8–15–17



$a° > b° > c° \leftrightarrow x > y > z$

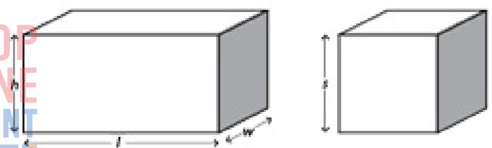


Isosceles Triangle

Similar Triangles

$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

3-D Shapes

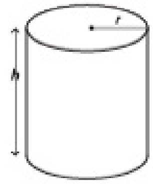


Volume $= l \times w \times h$

Surface Area $= 2 \times (l \times w + l \times h + w \times h)$

Volume $= s^3$

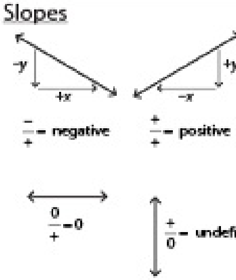
Surface Area $= 6s^2$



Volume $= \pi r^2 h$

Surface Area $= 2(\pi r^2) + 2\pi rh$

Slopes



$\frac{-}{+}$ = negative $\quad$ $\frac{+}{+}$ = positive

$\frac{0}{+} = 0$ $\quad$ $\frac{+}{0}$ = undefined

Chapter 1
of
Number Properties

Divisibility & Primes

In This Chapter…

Chapter 1

Divisibility & Primes

Integers are “whole” numbers, such as 0, 1, 2, and 3, that have no fractional part. Integers include positive numbers (1, 2, 3…), negative numbers (−1, −2, −3…), and the number 0.

The GMAT uses the term integer to mean a non-fraction or a non-decimal. The special properties of integers form the basis of most Number Properties problems on the GMAT.

Arithmetic Rules

Most arithmetic operations on integers will result in an integer. For example:

$4 + 5 = 9$	$(-2) + 1 = -1$	The sum of two integers is always an integer.
$4 - 5 = -1$	$(-2) - (-3) = 1$	The difference of two integers is always an integer.
$4 \times 5 = 20$	$(-2) \times 3 = -6$	The product of two integers is always an integer.

Division, however, is different. Sometimes the result is an integer, and sometimes it is not:

$8 \div 2 = 4$, but $2 \div 8 = \frac{1}{4}$

$(-8) \div 4 = -2$, but $(-8) \div (-6) = \frac{4}{3}$

The result of dividing two integers is *sometimes* an integer.
(This result is called the **quotient.**)

An integer is said to be **divisible** by another number if the integer can be divided by that number with an integer result (there is no remainder).

For example, 21 is divisible by 3 because 21 divided by 3 results in an integer (21 ÷ 3 = 7). However, 21 is not divisible by 4 because 21 divided by 4 results in a non-integer (21 ÷ 4 = 5.25).

You can also talk about divisibility in terms of remainders. For example, 21 is divisible by 3 because 21 divided by 3 yields 7 with a remainder of 0. On the other hand, 21 is not divisible by 4 because 21 divided by 4 yields 5 with a remainder of 1.

Here are some more examples:

$8 \div 2 = 4$	Therefore, 8 is divisible by 2.
	You can also say that 2 is a **divisor** or **factor** of 8.
$2 \div 8 = 0.25$	Therefore, 2 is *not* divisible by 8.
$(-6) \div 2 = -3$	Therefore, −6 is divisible by 2.
$(-6) \div (-4) = 1.5$	Therefore, −6 is *not* divisible by −4.

Rules of Divisibility by Certain Integers

The **divisibility rules** are very useful shortcuts to determine whether an integer is divisible by 2, 3, 4, 5, 6, 8, 9, and 10.

An integer is divisible by:

2 if the integer is even.
12 is divisible by 2, but 13 is not. Integers that are divisible by 2 are called *even* and integers that are not are called *odd*. You can tell whether a number is even by checking to see whether the units (ones) digit is 0, 2, 4, 6, or 8. For example, 1,234,567 is odd, because 7 is odd, whereas 2,345,678 is even, because 8 is even.

3 if the sum of the integer's digits is divisible by 3.
72 is divisible by 3 because the sum of its digits is 7 + 2 = 9, which is divisible by 3. By contrast, 83 is not divisible by 3, because the sum of its digits is 11, which is not divisible by 3.

4 if the integer is divisible by 2 *twice*, or if the last two digits are divisible by 4.

28 is divisible by 4 because you can divide it by 2 twice and get an integer result ($28 \div 2 = 14$, and $14 \div 2 = 7$). For larger numbers, check only the last two digits. For example, 23,456 is divisible by 4 because 56 is divisible by 4, but 25,678 is not divisible by 4 because 78 is not divisible by 4.

5 if the integer ends in 0 or 5.
75 and 80 are divisible by 5, but 77 and 83 are not.

6 if the integer is divisible by *both* 2 and 3.
48 is divisible by 6 since it is divisible by 2 (it ends with an 8, which is even) AND by 3 ($4 + 8 = 12$, which is divisible by 3).

8 if the integer is divisible by 2 three times, or if the last three digits are divisible by 8.
32 is divisible by 8 since you can divide it by 2 three times and get an integer result ($32 \div 2 = 16$, $16 \div 2 = 8$, and $8 \div 2 = 4$). For larger numbers, check only the last three digits. For example, 23,456 is divisible by 8 because 456 is divisible by 8, whereas 23,556 is not divisible by 8 because 556 is not divisible by 8.

9 if the sum of the integer's digits is divisible by 9.
4,185 is divisible by 9 since the sum of its digits is $4 + 1 + 8 + 5 = 18$, which is divisible by 9. By contrast, 3,459 is not divisible by 9, because the sum of its digits is 21, which is not divisible by 9.

10 if the integer ends in 0.
670 is divisible by 10, but 675 is not.

The GMAT can also test these divisibility rules in reverse. For example, if you are told that a number has a ones digit equal to 0, you can infer that that number is divisible by 10. Similarly, if you are told that the sum of the digits of x is equal to 21, you can infer that x is divisible by 3 but *not* by 9.

Note also that there is no rule listed for divisibility by 7. The simplest way to check for divisibility by 7, or by any other number not found in this list, is to perform long division.

Factors and Multiples

Factors and multiples are essentially opposite terms.

A **factor** is a positive integer that divides evenly into an integer. For example, 1, 2, 4, and 8 are all the factors (also called divisors) of 8. A factor of an integer is smaller than or equal to that integer.

A **multiple** of an integer is formed by multiplying that integer by any integer, so 8, 16, 24, and 32 are some of the multiples of 8. On the GMAT, multiples of an integer are equal to or larger than that integer.

Note that an integer is always both a factor and a multiple of itself, and that 1 is a factor of *every* integer.

An easy way to find all the factors of a *small* integers is to use **factor pairs**. Factor pairs for any integer are the pairs of factors that, when multiplied together, yield that integer. For instance, the factor pairs of 8 are (1, 8) and (2, 4).

To find the factor pairs of a number such as 72, start with the automatic factors: 1 and 72 (the number itself). Then, "walk upwards" from 1, testing to see whether different numbers are factors of 72. Once you find a number that is a factor of 72, find its partner by dividing 72 by the factor. Keep walking upwards until all factors are exhausted.

Small	Large
1	72
2	36
3	24
4	18
6	12
8	9

Step by step:

1. Make a table with two columns labeled *Small* and *Large*.
2. Start with 1 in the Small column and 72 in the Large column.
3. Test the next possible factor of 72 (which is 2); 2 is a factor of 72, so write 2 underneath the 1 in your table. Divide 72 by 2 to find the factor pair: 36. Write 36 in the Large column.
4. Repeat this process until the numbers in the Small and the Large columns run into each other. In this case, once you have tested 8 and

found that 9 is its paired factor, you can stop.

Fewer Factors, More Multiples

It can be easy to confuse factors and multiples. The mnemonic “Fewer Factors, More Multiples” can help you remember the difference. Every positive integer has a limited number of factors. Factors divide into the integer and are therefore less than or equal to the integer. For example, there are only four factors of 8: 1, 2, 4, and 8.

By contrast, every positive integer has infinite multiples. These multiply out from the integer and are therefore greater than or equal to to the integer. For example, the first five multiples of 8 are 8, 16, 24, 32, and 40, but you could go on listing multiples of 8 forever.

Factors, multiples, and divisibility are very closely related concepts. For example, 3 is a factor of 12. This is the same as saying that 12 is a multiple of 3, or that 12 is divisible by 3.

On the GMAT, this terminology is often used interchangeably in order to make the problem seem harder than it actually is. Be aware of the different ways that the GMAT can phrase information about divisibility. Moreover, try to convert all such statements to the same terminology. For example, all of the following statements *say exactly the same thing*:

- 12 is divisible by 3.
- 12 is a multiple of 3.
- $\frac{12}{3}$ is an integer.
- 12 is equal to 3*n*, where *n* is an integer.
- 12 items can be shared among 3 people so that each person has the same number of items.
- 3 is a divisor of 12, or 3 is a factor of 12.
- 3 divides 12.
- $\frac{12}{3}$ yields a remainder of 0.

- 3 “goes into” 12 evenly.

Divisibility and Addition/Subtraction

If you add two multiples of 7, you get another multiple of 7. Try it: 35 + 21 = 56. This is always mathematically valid (5 × 7) + (3 × 7) = (5 + 3) × 7 = 8 × 7.

Likewise, if you subtract two multiples of 7, you get another multiple of 7. Try it: 35 − 21 = 14. Again, you can see why: (5 × 7) − (3 × 7) = (5 − 3) × 7 = 2 × 7.

This pattern holds true for the multiples of any integer N. If you add or subtract multiples of N, the result is a multiple of N. You can restate this principle using any of the disguises above: for instance, if N is a divisor of x and of y, then N is a divisor of $x + y$.

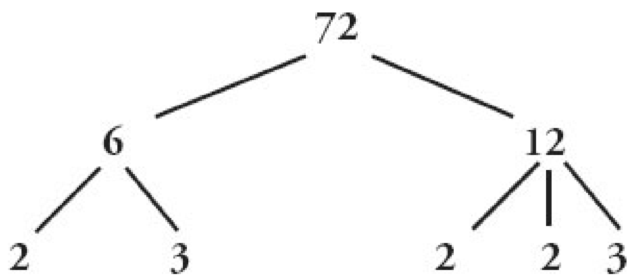
Primes

Prime numbers are a very important topic on the GMAT. A prime number is any positive integer with exactly two different factors: 1 and itself. In other words, a prime number has *no* factors other than 1 and itself. For example, 7 is prime because the only factors of 7 are 1 and 7. However, 8 is not prime because it is divisible by 2 and 4.

Note that the number 1 is not prime, as it has only one factor (itself). The first prime number is 2, which is also the only even prime. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. Memorizing these primes will save you time on the test.

Prime Factorization

Breaking a number down to its prime factors can be very useful on the GMAT. Create a prime factor tree, as shown below with the number 72. Test different numbers to see which ones “go into” 72 without leaving a remainder. Once you find such a number, then split 72 into factors. For example, 72 is divisible by 6, so it can be split into 6 and 72 ÷ 6, or 12. Then repeat this process on 6 and 12 until every branch on the tree ends at a prime number. Once you have only primes, stop, because you cannot split prime numbers into two smaller factors. In this example, 72 splits into 5 total prime factors (including repeats): $2 \times 3 \times 2 \times 2 \times 3$.

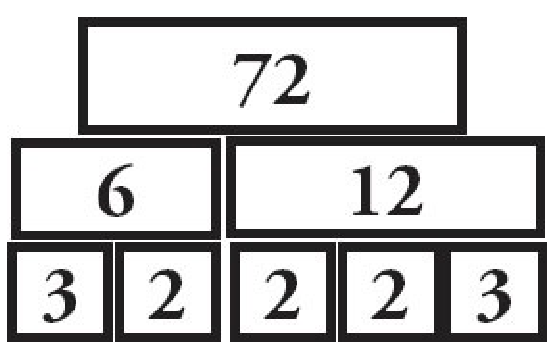


Prime factorization is an extremely important tool to use on the GMAT. Once you know the prime factors of a number, you can determine *all* the factors of that number, even large numbers. The factors can be found by building all the possible products of the prime factors.

Factor Foundation Rule

The GMAT expects you to know the **factor foundation rule**: **if *a* is a factor of *b*, and *b* is a factor of *c*, then *a* is a factor of *c*.** In other words, any integer is divisible by all of its factors—and it is also divisible by all of the *factors* of its factors.

For example, if 72 is divisible by 12, then 72 is also divisible by all the factors of 12 (1, 2, 3, 4, 6, and 12). Written another way, if 12 is a factor of 72, then all the factors of 12 are also factors of 72. The factor foundation rule allows you to conceive of factors as building blocks in a foundation; for example, 12 and 6 are factors, or building blocks, of 72 (because 12×6 builds 72).



The number 12, in turn, is built from its own factors; for example, 4 × 3 builds 12. Thus, if 12 is part of the foundation of 72 and 12 in turn rests on the foundation built by its prime factors (2, 2, and 3), then 72 is also built on the foundation of 2, 2, and 3.

Going further, you can build almost any factor of 72 out of the bottom level of the foundation. For instance, 8 is a factor of 72, because you can build 8 out of the three 2's in the bottom row (8 = 2 × 2 × 2).

It is *almost* any factor, however, because one of the factors cannot be built out of the building blocks in the foundation: the number 1. Remember that the number 1 is not prime, but it is still a factor of every integer. Except for the number 1, every factor of 72 can be built out of the lowest level of 72 building blocks.

The Prime Box

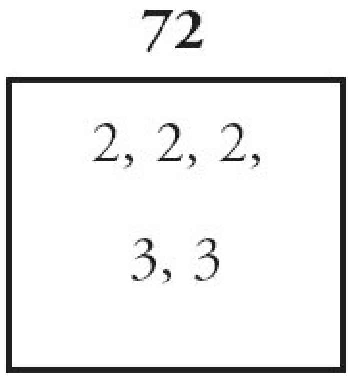
The easiest way to work with the factor foundation rule is with a tool called a prime box. A **prime box** is exactly what its name implies: a box that holds all the prime factors of a number (in other words, the lowest-level building blocks). Here are prime boxes for 72, 12, and 125:

72	12	125
2, 2, 2, 3, 3	2, 2, 3	5, 5, 5

Notice that you must repeat copies of the prime factors if the number has multiple copies of that prime factor. You can use the prime box to test whether or not a specific number is a factor of another number. For example:

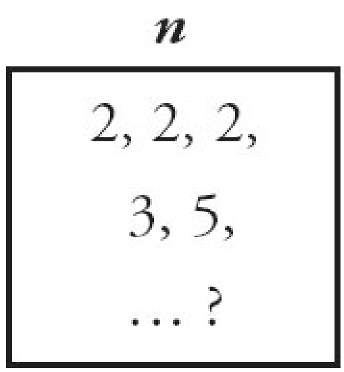
Is 27 a factor of 72?

For instance, 27 = 3 × 3 × 3, but 72 only has



two 3's in its prime box. Therefore, you cannot make 27 from the prime factors of 72; 27 is not a factor of 72.

Given that the integer *n* is divisible by 8 and 15, is *n* divisible by 12?



First, factor both numbers: $8 = 2 \times 2 \times 2$ and $15 = 3 \times 5$. Although you don't know what *n* is, *n* has to be divisible by any number made up of those primes.

Because $12 = 2 \times 2 \times 3$, then yes, *n* is also divisible by 12.

Notice the ellipses and question mark ("…?") in the prime box of *n*. This indicates that you have created a **partial prime box** of *n*. Whereas the *complete* set of prime factors of 72 can be calculated and put into its prime box, you only have a *partial* list of prime factors of *n*, because *n* is an unknown number. You know that *n* is divisible by 8 and 15, but you do *not* know what additional primes, if any, *n* has in its prime box.

Most of the time, when building a prime box for a *variable*, you will use a partial prime box, but when building a prime box for a *number*, you will use a complete prime box.

Remainders

Most of this chapter has focused on numbers that are divisible by other numbers (factors). This section, however, discusses what happens when a number, such as 8, is divided by a *non*-factor, such as 5.

Every division has four parts:

1. The **dividend** is the number being divided. In 8 ÷ 5, the dividend is 8.
2. The **divisor** is the number that is dividing. In 8 ÷ 5, the divisor is 5.
3. The **quotient** is the number of times that the divisor goes into the dividend *completely*. The quotient is always an integer. In 8 ÷ 5, the quotient is 1 because 5 goes into 8 one (1) time completely.
4. The **remainder** is what is left over if the dividend is not divisible by the divisor. In 8 ÷ 5, the remainder is 3 because 3 is left over after 5 goes into 8 once.

Putting it all together, you have 8 ÷ 5 = 1, with a remainder of 3.

As another example, the number 17 is not divisible by 5. When you divide 17 by 5 using long division, you get 3 with a remainder of 2:

$$\begin{array}{r} 3 \\ 5\overline{)17} \\ -15 \\ \hline 2 \end{array}$$

The quotient is 3 because 15 is the largest multiple of 5 smaller than 17, and 15 ÷ 5 = 3. The remainder is 2 because 17 is 2 more than a multiple of 5 (15).

You can also express this relationship as a general formula:

Dividend = Quotient × Divisor + Remainder
(or, Dividend = Multiple of Divisor + Remainder)

Problem Set

For questions #1–6, answer each question Yes, No, or Cannot Be Determined. If your answer is Cannot Be Determined, use two numerical examples to show

how the problem could go either way. All variables in problems #1–6 are assumed to be integers.

1. If *a* is divided by 7 or by 18, an integer results. Is $\frac{a}{42}$ an integer?

2. If 80 is a factor of *r*, is 15 a factor of *r*?

3. Given that 7 is a factor of *n* and 7 is a factor of *p*, is *n* + *p* divisible by 7?

4. If *j* is divisible by 12 and 10, is *j* divisible by 24?

5. Given that 6 is a divisor of *r* and *r* is a factor of *s*, is 6 a factor of *s*?

6. If *s* is a multiple of 12 and *t* is a multiple of 12, is 7*s* + 5*t* a multiple of 12?

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

7. A skeet shooting competition awards prizes for each round as follows: the first-place winner receives 11 points, the second-place winner receives 7 points, the third-place finisher receives 5 points, and the fourth-place finisher receives 2 points. No other prizes are awarded. Johan competes in several rounds of the skeet shooting competition and receives points every time he competes. If the product of all of the points he receives equals 84,700, in how many rounds does he participate?

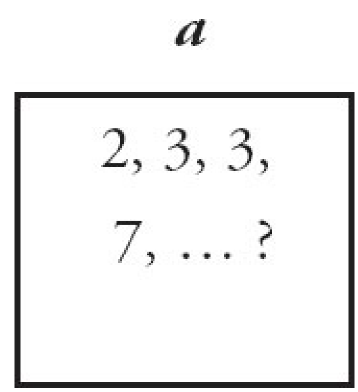
8. If *x*, *y*, and *z* are integers, is *x* even?

   (1) $10^x = (4^y)(5^z)$

   (2) $3^{x+5} = 27^{y+1}$

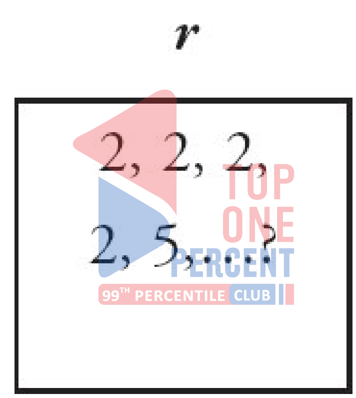
Solutions

1. **Yes:**



If *a* is divisible by 7 and by 18, its prime factors include 2, 3, 3, and 7, as indicated by the prime box to the left. Therefore, any integer that can be constructed as a product of any of these prime factors is also a factor of *a*. Factoring 42, you get = 2 × 3 × 7. Therefore, 42 is also a factor of *a*.

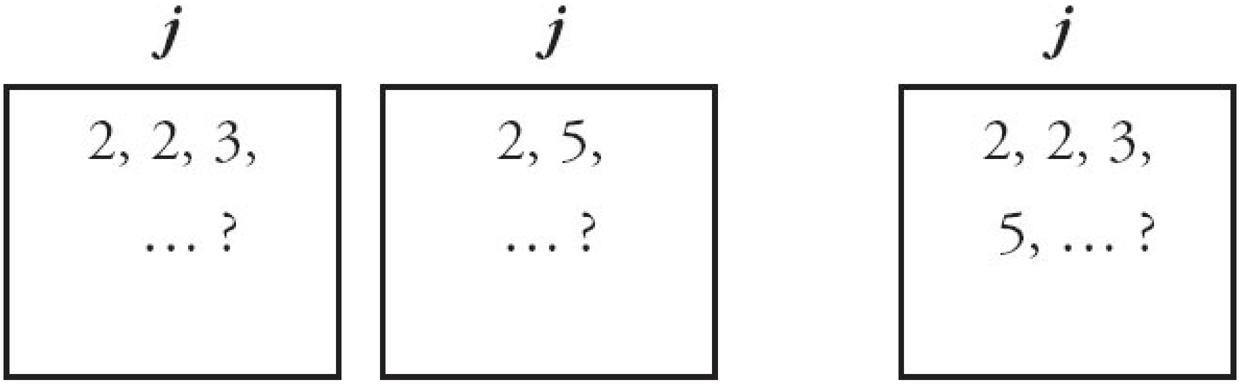
2. **Cannot Be Determined:**



If *r* is divisible by 80, its prime factors include 2, 2, 2, 2, and 5, as indicated by the prime box to the left. Therefore, any integer that can be constructed as a product of any of these prime factors is also a factor of *r*. Factoring 15, you get 3 × 5. Since you don't know whether the prime factor 3 is in the prime box, you cannot determine whether 15 is a factor of *r*. As numerical examples, you could take *r* = 80, in which case 15 is *not* a factor of *r*, or *r* = 240, in which case 15 *is* a factor of *r*.

3. **Yes:** If two numbers are both multiples of the same number, then their *sum* is also a multiple of that same number. Since *n* and *p* share the common factor 7, the sum of *n* and *p* must also be divisible by 7.

4. **Cannot Be Determined:**



If j is divisible by 12 and by 10, its prime factors include 2, 2, 3, and 5, as indicated by the prime box to the left. What is the minimum number of 2's necessary to create 12 or 10? You need two 2's to create 12. You could use one of those same 2's to create the 10. Therefore, there are only *two* 2's that are definitely in the prime factorization of j, because the 2 in the prime factorization of 10 may be *redundant*—that is, it may be the *same* 2 as one of the 2's in the prime factorization of 12.

Factoring 24, you get $2 \times 2 \times 2 \times 3$. The prime box of j contains at least two 2's and could contain more. The number 24 requires three 2's. Therefore, you may or may not be able to create 24 from j's prime box; 24 is not necessarily a factor of j.

As another way to prove that you cannot determine whether 24 is a factor of j, consider 60. The number 60 is divisible by both 12 and 10. However, it is *not* divisible by 24. Therefore, j could equal 60, in which case it is not divisible by 24. Alternatively, j could equal 120, in which case it *is* divisible by 24.

5. **Yes:** By the factor foundation rule, if 6 is a factor of r and r is a factor of s, then 6 is a factor of s.

6. **Yes:** If s is a multiple of 12, then so is $7s$. If t is a multiple of 12, then so is $5t$. Since $7s$ and $5t$ are both multiples of 12, then their sum ($7s + 5t$) is also a multiple of 12.

7. **7 rounds:** Notice that the values for scoring first, second, third, and fourth place in the competition are all prime numbers. Notice also that the *product* of all of the scores Johan received is known. Therefore, if you simply take the prime factorization of the product of his scores, you can determine what scores he received (and how many scores he received):

$84{,}700 = 847 \times 100 = 7 \times 121 \times 2 \times 2 \times 5 \times 5 = 7 \times 11 \times 11 \times 2 \times 2 \times 5 \times 5$

Thus, Johan received first place twice (11 points each), second place once (7 points each), third place twice (5 points each), and fourth place twice (2 points each.) He received a prize 7 times, so he competed in 7 rounds.

8. **(A):** (1) SUFFICIENT: Statement (1) tells you that $10^x = (4^y)(5^z)$. You can break the bases down into prime factors: $(2 \times 5)^x = (2^2)^y \times 5^z = 2^{2y} \times 5^z$. This tells you that $x = 2y$ and $x = z$. (You need the same number of 2's and the same number of 5's on either side of the equation.) Since you know y is an integer, x must be even, because $x = 2y$.

(2) INSUFFICIENT: Statement (2) tells you that $3^{x+5} = 27^{y+1}$. You can again break the bases down into prime factors: $3^{x+5} = (3^3)^{y+1}$, so $3^{x+5} = 3^{3(y+1)}$. This tells you that $x + 5 = 3y + 3$, so $x = 3y - 2$. (Again, you need the same number of 3's on either side of the equation.) Since y is an integer, x must be 2 smaller than a multiple of 3, but that does not tell you whether x is even. If $y = 1$, then $x = 1$ (odd), but if $y = 2$, then $x = 4$ (even).

Chapter 2
of
Number Properties

Odds, Evens, Positives, & Negatives

In This Chapter…

Chapter 2

Odds, Evens, Positives, & Negatives

Even numbers are integers that are divisible by 2. Odd numbers are integers that are not divisible by 2. All integers are either even or odd. For example:

Evens: 0, 2, 4, 6, 8, 10, 12…   Odds: 1, 3, 5, 7, 9, 11…

Consecutive integers alternate between even and odd:	9, 10, 11, 12, 13…
	O, E, O, E, O…

Negative integers are also either even or odd:

Evens: −2, −4, −6, −8, −10, −12…   Odds: −1, −3, −5, −7, −9, −11…

Arithmetic Rules of Odds & Evens

The GMAT tests your knowledge of how odd and even numbers combine through addition, subtraction, multiplication, and division. Rules for adding, subtracting, multiplying, and dividing odd and even numbers can be derived by testing out simple numbers, but it pays to memorize the following rules for operating with odds and evens, as they are extremely useful for certain GMAT math questions.

Addition and subtraction:

Even ± Even = Even	8 + 6 = 14
Odd ± Odd = Even	7 + 9 = 16

Even ± Odd = Odd	$7 + 8 = 15$

If they're the same, the sum (or difference) will be even. If they're different, the sum (or difference) will be odd.

Multiplication:

Even × Even = Even	$2 \times 4 = 8$
Even × Odd = Even	$4 \times 3 = 12$
Odd × Odd = Odd	$3 \times 5 = 15$

If one even number is present, the product will be even. If you have only odd numbers, the product will be odd.

If you multiply together several even integers, the result will be divisible by higher and higher powers of 2 because each even number will contribute at least one 2 to the factors of the product.

For example, if there are two even integers in a set of integers being multiplied together, the result will be divisible by 4:

$\mathbf{2} \times 5 \times \mathbf{6} = 60$	(divisible by 4)

If there are three even integers in a set of integers being multipled together, the result will be divisible by 8:

$\mathbf{2} \times 5 \times \mathbf{6} \times \mathbf{10} = 600$	(divisible by 8)

Division:
There are no guaranteed outcomes in division, because the division of two integers may not yield an integer result. In these cases, you'll have to try the actual numbers given. The divisibility tools outlined in Chapter 1 can help you determine the outcome.

Representing Odds & Evens Algebraically

Try this problem:

Is positive integer *m* odd?

(1) $m = 2k + 1$, where k is an integer.

(2) m is a multiple of 3.

Statement (2) is easier to attack. The variable m could be 3, which is odd, or 6, which is even. This statement is NOT sufficient.

Statement (1) is a bit trickier. An even number is a multiple of 2, so any even number can be represented as $2n$, where n is an integer. An odd number is always one more than an even number, so the subsequent odd number could be written $2n + 1$. This notation is a signal that you have an odd integer!

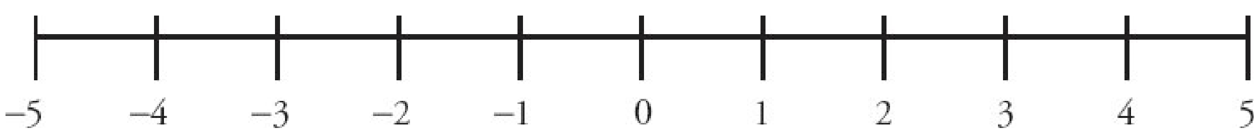
If $m = 2k + 1$, where k is an integer, then m must be odd. Statement (1) is sufficient to answer the question.

The answer is **(A)**.

The GMAT will sometimes use this notation to disguise information about odds and evens; keep an eye out for it!

Positives & Negatives

Numbers can be either positive or negative (except the number 0, which is neither):



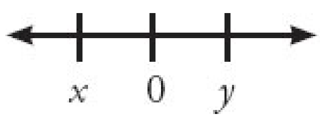
Negative numbers are all to the left of the number 0. Positive numbers are all to the right of the number 0.

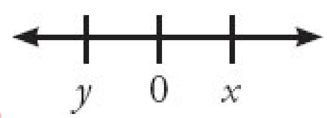
Note that a variable (such as x) can have either a positive or a negative value, unless there is evidence otherwise. The variable x is not necessarily positive, nor is $-x$ necessarily negative. For example, if $x = -3$, then $-x = 3$.

Absolute Value: Absolutely Positive

The absolute value of a number answers this question: How far away is the number from 0 on the number line? For example, the number 5 is exactly 5 units away from 0, so the absolute value of 5 equals 5. Mathematically, this is written using the symbol for absolute value: $|5| = 5$. To find the absolute value of −5, look at the number line above: −5 is also exactly 5 units away from 0. Thus, the absolute value of −5 equals 5, or, in mathematical symbols, $|-5| = 5$. Notice that absolute value is always positive, because it disregards the direction (positive or negative) from which the number approaches 0 on the number line. When you interpret a number in an absolute value sign, just think: Absolutely positive! (Except, of course, for 0, because $|0| = 0$, which is the smallest possible absolute value.)

Note that 5 and −5 are the same distance from 0, which is located halfway between them. In general, if two numbers are opposites of each other, then they have the same absolute value, and 0 is halfway between. If $x = -y$, then you have either of the below:





(You cannot tell which variable is positive and which is negative without more information.)

A Double Negative = A Positive

A double negative occurs when a minus sign is in front of a negative number (which already has its own negative sign). For example:

What is $7 - (-3)$?

As you learned in English class, two negatives yield a positive:

$7 - (-3) = 7 + 3 = 10$

This is a very easy step to miss, especially when the double negative is somewhat hidden. For example:

What is $7 - (12 - x)$?

Many people will make the mistake of computing this as $7 - 12 - x$. However, notice that the second term in the expression in parentheses has a double negative. Therefore, this expression should be simplified as $7 - 12 + x$.

Multiplying & Dividing Signed Numbers

When you multiply or divide numbers, positive or negative, follow one simple rule:

If you have an even number of negative signs, the answer is positive:	$7 \times 8 = 56$ & $(-7) \times (-2) \times 3 = 42$ $56 \div 7 = 8$ & $-42 \div (-7) = 6$

If you have an odd number of negative signs, the answer is negative:	$(-7) \times 8 = -56$ & $7 \times (-2) \times 3 = -42$ $56 \div (-7) = -8$ & $-42 \div 7 = -6$

Try this Data Sufficiency problem:

Is the product of all of the elements in Set *S* negative?

(1) All of the elements in Set *S* are negative.

(2) There are 5 negative numbers in Set *S*.

This is a tricky problem. Based on what you have learned so far, it would seem that statement (2) tells you that the product must be negative. (However, 5 is an odd number, and when the GMAT says “there are 5” of something, you *can* conclude there are *exactly* 5 of that thing.) While it's true that the statement indicates that there are exactly 5 negative numbers in the set, it does not tell you that there are not other numbers in the set. For instance, there could be 5 negative numbers as well as a few other numbers.

If 0 is one of those numbers, then the product will be 0, and 0 is not negative.

Therefore, statement (2) is NOT sufficient.

Statement (1) indicates that all of the numbers in the set are negative. If there is an even number of negatives in Set *S*, the product of these numbers will be positive; if there is an odd number of negatives, the product will be negative. This also is NOT sufficient.

Combined, you know that Set *S* contains 5 negative numbers and nothing else, so this statement is sufficient. The product of the elements in Set *S* must be negative. The correct answer is **(C)**.

Disguised Positives & Negatives Questions

Some positives and negatives questions are disguised as inequalities. This generally occurs whenever a question tells you that a quantity is greater than or less than 0, or asks you whether a quantity is greater than or less than 0. For example:

If $\frac{a-b}{c} < 0$, is $a > b$?

(1) $c < 0$

(2) $a + b < 0$

The fact that $\frac{a-b}{c} < 0$ indicates that $a - b$ and c have *different* signs. That is, one of the expressions is positive and the other is negative.

Therefore, statement (1) establishes that c is negative. Therefore, $a - b$ must be positive:

$$a - b > 0$$
$$a > b$$

Statement (1) is sufficient.

Statement (2) tells you that the sum of a and b is negative. This does not indicate whether a is larger than b, so this statement is NOT sufficient.

The correct answer is **(A)**.

Generally speaking, whenever you see inequalities with the number 0 on either side of the inequality, consider testing positive and negative cases to help solve the problem.

The Sum of Two Primes

All prime numbers are odd, except the number 2. (All larger even numbers are divisible by 2, so they cannot be prime.) Thus, the sum of any two primes will be even (odd + odd = even), unless one of those primes is the number 2. So, if you see a sum of two primes that is odd, one of those primes must be the number 2. Conversely, if you know that 2 *cannot* be one of the primes in the sum, then the sum of the two primes must be even. Try an example:

> If a and b are both prime numbers greater than 10, which of the following CANNOT be true?
>
> I. ab is an even number.
>
> II. The difference between a and b equals 117.
>
> III. The sum of a and b is even.
>
> (A) I only
>
> (B) I and II only
>
> (C) I and III only
>
> (D) II and III only
>
> (E) I, II, and III

Since a and b are both prime numbers greater than 10, they must both be odd. Therefore, ab must be an odd number, so statement I cannot be true. Similarly, if a and b are both odd, then $a - b$ cannot equal 117 (an odd number). This difference must be even. Therefore, statement II cannot be true. Finally, since a and b are both odd, $a + b$ must be even, so statement III will always be true. Since statements I and II CANNOT be true, but statement III IS true, the correct answer is **(B)**.

Try the following Data Sufficiency problem:

If x is an integer greater than 1, what is the value of x?

(1) There are x unique factors of x.

(2) The sum of x and any prime number larger than x is odd.

Statement (1) indicates that there are x unique factors of x. In order for this to be true, *every* integer between 1 and x, inclusive, must be a factor of x. Try some numbers. This property holds for 1 and for 2, but not for 3 or for 4. In fact, this property does not hold for any higher integer, because no integer x greater than 2 is divisible by $x - 1$. Therefore, x is equal to 1 or 2. However, the question stem indicates that $x > 1$, so x must equal 2. Thus, statement (1) is sufficient.

Statement (2) indicates that x plus any prime number larger than x is odd. Since $x > 1$, x must equal at least 2, so the prime number in question must be larger than 2. Therefore, the prime number is odd. Because the rule is Odd + Even = Odd, then x must be even. However, this is not enough information to indicate the value of x. Therefore, statement (2) is insufficient.

The correct answer is **(A)**.

Problem Set

For questions #1–6, answer each question Odd, Even, or Cannot Be Determined. Try to explain each answer using the rules you learned in this section. All variables in questions #1–6 are assumed to be integers.

1. If $x \div y$ yields an odd integer, what is x?
2. If $a + b$ is even, what is ab?
3. If c, d, and e are consecutive integers, what is cde?
4. If h is even, j is odd, and k is odd, what is $k(h + j)$?
5. If n, p, q, and r are consecutive integers, what is their sum?
6. If xy is even and z is even, what is $x + z$?
7. Simplify $\frac{-30}{5} - \frac{18-9}{-3}$
8. Simplify $\frac{20 \times (-7)}{-35 \times (-2)}$

9. If x, y, and z are prime numbers and $x < y < z$, what is the value of x?

(1) xy is even.

(2) xz is even.

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

10. If c and d are integers, is $c - 3d$ even?

(1) c and d are odd.

(2) $c - 2d$ is odd.

11. Is the integer x odd?

(1) $2(y + x)$ is an odd integer.

(2) $2y$ is an odd integer.

Solutions

1. **Cannot Be Determined:** There are no guaranteed outcomes in division.

2. **Cannot Be Determined:** If $a + b$ is even, a and b are either both odd or both even. If they are both odd, ab is odd. If they are both even, ab is even.

3. **Even:** At least one of the consecutive integers, c, d, and e, must be even. Therefore, the product cde must be even.

4. **Odd:** $h + j$ must be odd (E + O = O). Therefore, $k(h + j)$ must be odd (O × O = O).

5. Even: If n, p, q, and r are consecutive integers, two of them must be odd and two of them must be even. You can pair them up to add them: O + O = E and E + E = E. Adding the pairs, you will see that the sum must be even: E + E = E.

6. **Cannot Be Determined:** If xy is even, then either x or y (or both x and y) must be even. Given that z is even, $x + z$ could be O + E or E + E. Therefore, you cannot determine whether $x + z$ is odd or even.

7. **−3:** This is a two-step subtraction problem. First, simplify each fraction:

$\frac{-30}{5} = -6$, and the second fraction simplifies to $\frac{9}{-3}$, which equals −3. The final answer is −6 − (−3) = −3.

8. **−2:** The sign of the first product, 20 × (−7), is negative. The sign of the second product, −35 × (−2), is positive. Therefore, −140 divided by 70 is −2.

9. **(D):** (1) SUFFICIENT: If xy is even, then x is even or y is even. Since $x < y$, x must equal 2, because 2 is the smallest and only even prime number.

(2) SUFFICIENT: Similarly, if xz is even, then x is even or z is even. Since $x < z$, x must equal 2, because 2 is the smallest and only even prime number.

10. **(A):** (1) SUFFICIENT: If both c and d are odd, then $c - 3d$ equals O − (3 × O) = O − O = E.

(2) INSUFFICIENT: If $c - 2d$ is odd, then c must be odd, because $2d$ will always be even. However, this tells you nothing about d.

Therefore, the correct answer is **(A)**.

11. **(E):** (1) INSUFFICIENT: $2(y + x)$ is an odd integer. How is it possible that 2 multipled by something could yield an odd integer? The value in the parentheses must not be an integer itself. For example, the decimal 1.5 times 2 yields the odd integer 3. List some other possibilities:

$$2(y + x) = 1, 3, 5, 7, 9, \text{ etc.}$$
$$(y + x) = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \text{ etc.}$$

You know that x is an integer, so y must be a fraction in order to get such a fractional sum. Say that $y = \frac{1}{2}$. In that case, x = 0, 1, 2, 3, 4, etc. Thus, x can be either odd (“yes”) or even (“no”).

(2) INSUFFICIENT: This statement tells you nothing about x. If $2y$ is an odd integer, this implies that $y = \frac{\text{odd}}{2} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc.

(1) AND (2) INSUFFICIENT: Statement (2) fails to eliminate the case you used in statement (1) to determine that x can be either odd or even. Thus, you still cannot answer the question with a definite yes or no.

But, just to combine the statements another way:

> Statement (1) says that $2(y + x) = 2y + 2x$ = an odd integer.
> Statement (2) says that $2y$ = an odd integer. By substitution, odd + $2x$ = odd, so $2x$ =
> odd − odd = even. $2x$ would be even regardless of whether x is even or odd.

The correct answer is **(E)**.

Chapter 3
of
Number Properties

Strategy: Test Cases

In This Chapter…

Chapter 3

Strategy: Test Cases

Certain problems allow for multiple possible scenarios, or cases. When you **test cases**, you try different numbers in a problem to see whether you have the same outcome or different outcomes.

The strategy plays out a bit differently for Data Sufficiency (DS) than for Problem Solving (PS); both will be covered in this chapter. (If you have not yet studied Data Sufficiency, please see Appendix A.)

Try this problem, using any solution method you like:

> Is integer x odd?
>
> (1) $2x + 1$ is odd.
>
> (2) $\frac{x}{2}$ is even.

How to Test Cases

Here's how to test cases to solve the above problem:

Step 1: What possible cases are allowed?

> The problem indicates that x is an integer; it could be positive, negative, or the number 0, but it is not a fraction (or decimal).

Step 2: Choose numbers that work for the statement.

Before you dive into the work, remember this crucial rule:

When choosing numbers to test cases, ONLY choose numbers that are allowed by that statement.

If you inadvertently choose numbers that make the statement false, discard that case and try again.

Step 3: Try to prove the statement *insufficient*.

Here's how:

(1) $2x + 1$ is odd.

What numbers would make this statement true?

Case 1: $x = 6$.

Statement true? $2x + 1$ is odd.	Is x odd?
$2(6) + 1 = 13$ ✓	No

First, ensure that the value you've chosen does make the statement true. In this case, plugging in 6 for x does produce an odd result, so $x = 6$ is a valid number to test.

Second, answer the question asked. If $x = 6$, then x is not odd.

Next, ask yourself: Is there another case that would make the statement true but give you a *different* outcome?

Case 2: $x = 3$.

Statement true? $2x + 1$ is odd.	Is x odd?
$2(3) + 1 = 7$ ✓	Yes

Because *x* could be odd or even, this statement is NOT sufficient to answer the question; $\frac{\cancel{AD}}{BCE}$ cross off answers (A) and (D). Try statement (2) next. Whenever possible, reuse the numbers that you tried for the first statement.

(2) $\frac{x}{2}$ is even.

Case 1: *x* = 6.

Statement true? $\left(\frac{x}{2} \text{ is even}\right)$	Is *x* odd?
$\frac{6}{2} = 3$ ✗	

Careful! The result is not even; it's odd. You have to pick a value that makes statement (2) true. Discard this case. (Literally cross it off on your scrap paper.)

What kind of number will make statement (2) true?

Case 2: *x* = 8.

Statement true? $\left(\frac{x}{2} \text{ is even}\right)$	Is *x* odd?
$\frac{8}{2} = 4$ ✓	No

You've got a “no” answer. Can you think of a value that would return a “yes” answer?

Case 3: *x* = 3.

Statement true? $\left(\frac{x}{2} \text{ is even}\right)$	Is *x* odd?

$\frac{3}{2} = 1.5$ ×	

Nope, that won't work either. In fact, you can't divide an odd number by 2 and get an integer result. Therefore, no odd number will ever make statement (2) true.

In other words, this statement only allows even numbers. You can answer the question: ~~AD~~ BCE Is *x* odd? No, never.

The correct answer is **(B)**.

When you test cases in Data Sufficiency, your ultimate goal is to try to prove the statement insufficient, if you can. The first case you try will give you one outcome. For the next case, think about what numbers would be likely to give you a *different* outcome.

As soon as you do find two different outcomes, as in statement (1) above, you know the statement is not sufficient, and you can cross off some answer choices and move on.

If you cannot find two different outcomes, then you may be able to prove to yourself why you will always get the same outcome, as in statement (2) above. If you have tried several times to prove the statement insufficient but you keep getting the same outcome, then that statement is probably sufficient.

Try a Problem Solving example:

If $ab > 0$, which of the following must be negative?

(A) $a + b$ (B) $|a| + b$ (C) $b - a$ (D) $\frac{a}{b}$ (E) $-\frac{a}{b}$

Step 1: What possible cases are allowed?

On Problem Solving problems, *must be* or *could be* language is a signal that you can test cases.

When you see > 0 (or < 0), know that you've got a positive/negative problem in disguise. In this case, if the product of the two variables is positive, then *a* and *b* must have the same sign: either both are positive or both are negative.

Step 2: Choose numbers that work for the given information.

On DS problems, your task is to choose numbers that are valid for the given statements. On PS problems, your task is to choose numbers that are valid for any given information. In this case, you can only choose values that make $ab > 0$.

Step 3: Try to prove each answer choice *wrong*.

The question asks which answer must be negative. If you can find one circumstance in which a particular answer choice is *not* negative, then you can cross that answer off.

Test each answer choice, using a chart to keep track of your work. Take the time to write out each answer; if you work off of the screen, you're much more likely to make a careless mistake. As soon as you find a positive or 0 scenario, cross off that answer and move to the next row.

Answer	$a = 2, b = 1$	$a = -1, b = -2$	
(A) $a + b$	$2 + 1 = 3$		Eliminate.
(B) $\lvert a\rvert + b$	$2 + 1 = 3$		Eliminate.
(C) $b - a$	$1 - 2 = -1$	$-2 - (-1) = -1$	
(D) $\frac{a}{b}$	$\frac{2}{1}$		Eliminate.
(E) $-\frac{a}{b}$	$-\frac{2}{1}$	$-\left(\frac{-1}{-2}\right) = -\frac{1}{2}$	

Three wrong answers down, one more to go. As you're testing the cases, you may realize that if you change the numbers just a bit, you'll get a different response. If that's the case, feel free to test a different set of numbers right away; you don't have to get through all five answer choices first.

In this case, if you flip around the values ($a = 1$, $b = 2$), then answer (C) turns positive; eliminate it.

The correct answer is **(E)**. If $ab > 0$, then $\frac{a}{b}$ also has to be greater than 0. Adding a negative sign to a positive number will always turn it negative.

To sum up, when you are asked to test cases, follow three main steps:

Step 1: What possible cases are allowed?

Before you start solving, make sure you know what restrictions have been placed on the basic problem in the question stem. You may be told that the particular number is positive, or odd, and so on. Follow these restrictions when choosing numbers to test.

Step 2: Choose numbers that work for the given information.

Pause for a moment to remind yourself that you are only allowed to choose numbers that are valid for the given information in that problem. On both PS and DS problems, the question stem may contain givens. On DS problems, the statements are always givens.

With enough practice, this will become second nature. If you answer a testing cases problem incorrectly but aren't sure why, see whether you accidentally tested cases that weren't allowed because they weren't valid options for the statement or given information.

Step 3: Try to prove the statement *insufficient* or the answer wrong.

On DS questions, when the problem discusses abstract characteristics rather than real numbers, test cases to try to prove the statement insufficient:

Value

Sufficient: single numerical answer

Insufficient: two or more possible answers

Yes/No

Sufficient: Always Yes *or* Always No

Insufficient: Maybe or Sometimes Yes, Sometimes No

On PS questions, look for *must be* or *could be* language, such as:

- What must be true?
- What could be true?
- Which of the following must be even?
- Which of the following could equal 5?

This language will be your signal to try to disprove the four wrong answers.

The Theory Shortcut

The last problem can also be solved using a more theoretical approach, as long as you feel very comfortable with the concepts being tested. For example:

If $ab > 0$, which of the following must be negative?

(A) $a + b$ (B) $|a| + b$ (C) $b - a$ (D) $\frac{a}{b}$ (E) $-\frac{a}{b}$

As always, your goal is to try to prove the answers wrong. When you find a positive or 0 scenario, cross off that answer immediately. Think your way through each answer using your knowledge of positive and negative rules:

Answer	a +, b +	
(A) $a + b$	+	If a and b are +, the sum is +.
(B) $\lvert a\rvert + b$	+	Both are +, so the sum is +.
(C) $b - a$	can be +	As long as b is larger, the difference will be +. But if b is smaller, it will be −.
(D) $\frac{a}{b}$	+	If product is +, so is quotient.
(E) $-\frac{a}{b}$	−	If product is +, so is quotient. Adding a − sign makes this one always −.

Try the theory approach on this problem:

Is $pqr > 0$?

(1) $pq > 0$

(2) $\frac{q}{r} < 0$

The problem asks about characteristics of numbers in general; it doesn't ask about specific numbers. Time to test some cases!

Step 1: What possible cases are allowed?

Is $pqr > 0$?

The question stem does not contain any restrictions. The numbers can be positive, negative, fractions, or 0—anything goes.

Note that the question stem does ask whether the product is greater than 0. A quick glance at the statements confirms that they contain similar information. This is a positive/negative question.

Before trying numbers, think about what would need to be true to answer yes vs. no. Use a table to keep track of your thinking:

p	q	r	$pqr > 0$?	
+	+	+	Yes	
+	–	–	Yes	Yes if any 2 are negative
+	+	–	No	No if any 1 is negative
–	–	–	No	No if all 3 are negative
0	+	–	No	No if any are 0

Step 2: Choose numbers that work for the given statements.

Step 3: Try to prove the statements *insufficient*.

(1) $pq > 0$

p	q	$pq > 0$?	$pqr > 0$?	
+	+	Yes	Maybe	Depends on r

If r is positive, then the product is positive, but if r is negative, then the product is negative. This statement is insufficient.

(2) $\frac{q}{r} < 0$

q	r	$\frac{q}{r} < 0$?	$pqr > 0$?	
+	–	Yes	Maybe	Depends on p

If p is positive, then the product is negative. If p is negative, then the product is positive. This statement is insufficient.

Try the two statements together:

(1) $pq > 0$

(2) $\frac{q}{r} < 0$

p	q	r	Statements true?	$pqr > 0$?
+	+	–	Yes	No
–	–	+	Yes	Yes

If p is positive, then q must also be positive and r must be negative. In this case, pqr is less than 0.

On the other hand, if p is negative, then q must also be negative and r must be positive. In this case, pqr is greater than 0.

Even when used together, the two statements are insufficient to answer the question. The correct answer is **(E)**.

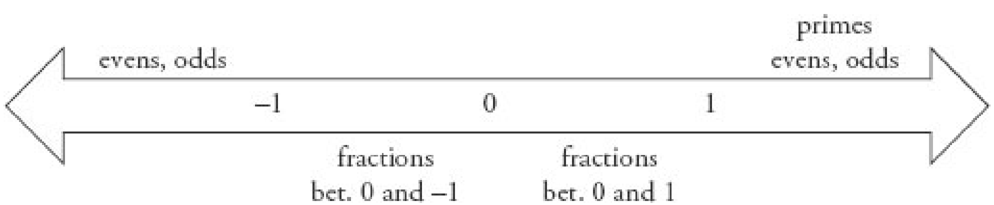
Often, you'll test cases using numbers all the way through. At times, though, you'll be able to use the real numbers you're testing to uncover the underlying theory tested by the problem. You can use this theory to shortcut the process a bit and arrive at your answer more quickly—just be sure that you really do understand the theoretical approach. If you're not 100% sure, don't use the shortcut. Test real numbers instead.

When to Test Cases

You can test cases whenever a problem allows multiple possible scenarios rather than just one numerical outcome. In that case, try some of the different possibilities allowed in order to see whether different scenarios, or cases, result in different outcomes or in the same outcome.

When testing cases, your initial starting point is every possible number on the number line. However, many problems give you restrictions that narrow the possible values, such as specifying that a number has to be an integer, or less than 0, or even. Write down your restrictions before you begin testing cases.

Think about different classes of numbers that are commonly tested on the GMAT. For example:



Problems will include clues that can help you decide which numbers to test. If a problem mentions a specific characteristic, such as odd, then of course try odds and evens. If an absolute value symbol appears, try negatives. If exponents come into play, try 0, 1, and fractions between 0 and 1 (these numbers do funny things when squared!). Picture the number line (or draw it out) and try any categories allowed by the givens that you think might make the difference in proving a DS statement insufficient or eliminating a PS answer.

How to Get Better at Testing Cases

First, practice the problems at the end of this chapter and in your OG problem sets online. Try each problem using the three-step process for testing cases. If you mess up any part of the process, try the problem again, making sure to write out all of your work.

Afterwards, review the problem. In particular, see whether you can articulate the reason why certain statements are sufficient (as the solutions to the earlier problems did). Could you explain those statements to a fellow student who is confused? If so, then you are starting to learn both the process by which you test cases and the underlying principles that these kinds of problems test.

If not, then look up the solution in this book or in GMAT Navigator™, search online, or ask an instructor or fellow student for help.

Problem Set

It's time to test out your testing cases skills. Both of these problems can be answered by testing cases.

1. If x is a positive integer, is $x^2 + 6x + 10$ odd?

(1) $x^2 + 4x + 5$ is odd.

(2) $x^2 + 3x + 4$ is even.

2. If p, q, and r are integers, is $pq + r$ even?

(1) $p + r$ is even.

(2) $q + r$ is odd.

Solutions

1. **(A):** The question can first be simplified by noting that if x is even, $x^2 + 6x + 10$ will be even, and if x is odd, $x^2 + 6x + 10$ will be odd.

Thus, you can simplify this question: “Is x odd or even?”

(A couple of shortcuts to save time in reaching that conclusion: the exponent on the first term can be ignored, since an even squared is still even and an odd squared is still odd. You know $6x$ will be even no matter what, since 6 is even, and obviously 10 is even no matter what. So, an even plus two evens is even, and an odd plus two evens is odd.)

(1) SUFFICIENT: You can test odd and even cases or simply use number theory. If x is even, you get even + even + odd = odd, and if x is odd, you get odd + even + odd = even. Thus, since $x^2 + 4x + 5$ is odd, x is even.

(2) INSUFFICIENT: $x^2 + 3x + 4$ is actually even regardless of what integer is plugged in for x. If x is even, you get even + even + even = even, and if x is odd, you get odd + odd + even = even. Thus, x could be odd or even. Plugging in numbers will yield the same conclusion—x could be any integer.

Note that you should *not* factor any of the expressions above. If you wasted time factoring, remember: factoring is meaningless if you don't have an equation set equal to 0! This problem was about number theory (or number testing), not factoring.

The correct answer is **(A)**.

2. **(E):** The yes/no question asks whether $pq + r$ is even. What would need to be true in order for the answer to be yes? Either both pq and r need to be even or both pq and r need to be odd.

(1) INSUFFICIENT: You are told that $p + r$ is even. To stay organized, test all the cases that make the statement true. Both p and r are even, or both p and r are odd. For each of those scenarios, q could be odd or even. Set up a table to keep track of all of these possibilities:

Scenario	p	q	r	$pq + r$
1	Odd	Odd	Odd	$O \times O + O = E$
2	Odd	Even	Odd	$O \times E + O = O$
3	Even	Odd	Even	$E \times O + E = E$
4	Even	Even	Even	$E \times E + E = E$

Since $pq + r$ could be odd or even, statement (1) is not sufficient. Note that you can stop as soon as you have found contradictory cases (one odd and one even); above, for example, you could have stopped after Scenario 2.

(2) INSUFFICIENT: As in statement (1), you can organize the information from statement (2) with a table. Either q is even and r is odd or q is odd and r is even, and p can be odd or even:

Scenario	p	q	r	$pq + r$
5	Odd	Even	Odd	$O \times E + O = O$
6	Even	Even	Odd	$E \times E + O = O$
7	Odd	Odd	Even	$O \times O + E = O$
8	Even	Odd	Even	$E \times O + E = E$

(1) AND (2) INSUFFICIENT: Notice that Scenarios 2 and 5 are identical, as are Scenarios 3 and 8. Therefore, both sets of scenarios meet the criteria laid forth in statements (1) and (2), but they yield opposite answers to the question:

Scenario	p	q	r	$pq + r$
2 & 5	Odd	Even	Odd	$O \times E + O = O$
3 & 8	Even	Odd	Even	$E \times O + E = E$

The correct answer is **(E)**.

Chapter 4
of
Number Properties

Combinatorics

In This Chapter…

Chapter 4

Combinatorics

The Words “OR” and “AND”

Suppose you are at a restaurant that offers a free side dish of soup or salad with any main dish. How many possible side dishes can you order?

You have two options: the soup OR the salad. This is a straightforward example, but it demonstrates an important principle: the word *or* means *add*. You will see this word show up again and again in both combinatorics and probability problems.

Now let's complicate the situation a bit. The same restaurant has three main dishes: steak, chicken, and salmon. How many possible combinations of main dish and side dish are there?

Now there are two decisions that need to be made. A diner must select a main dish AND a side dish. You can list out all the possible combinations:

Steak – Soup	Chicken – Soup	Salmon – Soup
Steak – Salad	Chicken – Salad	Salmon – Salad

There are six possible combinations. Fortunately, there is a way to avoid listing out every single combination. This brings us to the second important principle of combinatorics: the word *and* means *multiply*.

When you make two decisions, you make decision 1 AND decision 2. This is true whether the decisions are simultaneous (e.g., choosing a main dish and a side dish) or sequential (e.g., choosing among routes between successive towns on a road trip).

In this example, you have 3 options for main dishes AND 2 options for side dishes, so you have 3 × 2 = 6 options.

Believe it or not, the principles of combinatorics are derived from these two simple principles:

1. OR means *add*.
2. AND means *multiply*.

For instance, one way of interpreting the previous example is:

(steak OR chicken OR salmon)	AND	(soup OR salad)		
(1 + 1 + 1)	×	(1 + 1)	=	6
3	×	2	=	6

Unfortunately, questions will not always use the words *and* and *or* directly. Try the following example:

> An office manager must choose a five-digit lock code for the office door. The first and last digits of the code must be odd, and no repetition of digits is allowed. How many different lock codes are possible?

The question "How many…?" usually signals a combinatorics problem. If the manager has to pick a five-digit lock code, he has to make five decisions. To keep track, make a slot for each digit:

____	×	____	×	____	×	____	×	____
Digit 1	AND	Digit 2	AND	Digit 3	AND	Digit 4	AND	Digit 5

Next, fill in the number of options for each slot. This is known as the **slot method**.

How many options are there for each digit? Be careful; there are restrictions on the first and last number. Start with the most constrained decisions first.

The first digit can be 1 OR 3 OR 5 OR 7 OR 9. There are 5 options for the first digit. Remember, there can be no repeated numbers. Now that you have chosen the first digit (even though you don't know which one it is) there are only 4 odd numbers remaining for the last digit.

5	×		×		×		×	4
Digit 1	AND	Digit 2	AND	Digit 3	AND	Digit 4	AND	Digit 5

Now, fill in the rest of the slots. Make sure to account for the lack of repetition. Ten digits exist in total (0 through 9), but two have already been used, so there are 8 options for the 2nd digit, 7 options for the 3rd digit, and 6 options for the 4th digit:

5	×	8	×	7	×	6	×	4	= 6,720
Digit 1	AND	Digit 2	AND	Digit 3	AND	Digit 4	AND	Digit 5	

Any time a question involves making decisions, there are two cases:

1. Decision 1 OR Decision 2 (add possibilities)
2. Decision 1 AND Decision 2 (multiply possibilities)

Arranging Groups

Another very common type of combinatorics problem asks how many different ways there are to arrange a group.

The number of ways of arranging *n* distinct objects, if there are no restrictions, is *n*! (*n* factorial).

The term "*n* factorial" (*n*!) refers to the product of all the integers from 1 to *n*, inclusive. If you would like an 80th percentile or higher score on the Quant section of the GMAT, memorize the first six factorials:

$1! = 1$	$4! = 4 \times 3 \times 2 \times 1 = 24$
$2! = 2 \times 1 = 2$	$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
$3! = 3 \times 2 \times 1 = 6$	$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

For example, how many ways are there to arrange 4 people in 4 chairs in a row? Using the **slot method**, there is one slot for each position in the row. If you place any one of 4 people in the first chair, then you can place any one of the remaining 3 people in the second chair. For the third and fourth chairs you have

2 choices and then 1 choice.

$\underline{\quad 4 \quad} \times \underline{\quad 3 \quad} \times \underline{\quad 2 \quad} \times \underline{\quad 1 \quad} = 24$ arrangements

Once you understand what the formula means, you can just say “the number of ways to arrange 4 people equals 4 factorial, which equals 24.”

Arranging Groups Using the Anagram Grid

More complicated problems can be solved using the **anagram grid**.

For example, how many arrangements are there of the letters in the word “EEL”?

There are 3 letters, so according to the factorial formula, there should be 3! = 6 arrangements. But because two of the letters are the same, there are only 3 arrangements:

EEL	ELE	LEE

If you put subscripts on the two “E”s, you can see where the other arrangements went:

E_1E_2L	E_1LE_2	LE_1E_2
E_2E_1L	E_2LE_1	LE_2E_1

The two arrangements in each column are considered identical. Each pair of arrangements counts as one.

Try the following problem:

> Seven people enter a race. There are 4 types of medals given as prizes for completing the race. The winner gets a platinum medal, the runner-up gets a gold medal, the next two racers each get a silver medal, and the last 3 racers all get bronze medals. What is the number of different ways the medals can be awarded?

In order to keep track of all the different categories, create an **anagram grid**. Anagram grids can be used whenever you are arranging members of a group.

The number of columns in the grid will always be equal to the number of members of the group. There are 7 runners in the race, so there should be 7 columns. Next, categorize each member of the group. There are 1 platinum medal, 1 gold medal, 2 silver medals, and 3 bronze medals. Note: use only letters for the bottom row, never numbers (you'll see why in a minute).

1	2	3	4	5	6	7
P	G	S	S	B	B	B

Just as the two E's in "EEL" were indistinguishable, the 2 silver medals and the 3 bronze medals are indistinguishable, so the answer is not 7!. Use the top and bottom rows to create a fraction:

$$\frac{7!}{1!1!2!3!}$$

The numerator of the fraction is the factorial of the largest number in the top row, in this case 7!. The denominator is the product of the factorials of each *different* kind of letter in the bottom row. In this case, there are one P, one G, two S's, and three B's. (Use only letters, not numbers, in the bottom row to avoid mixing up the number of repeats with the numbers themselves.)

The above graphic shows the 1! terms for both P and G, but in practice, you don't have to write out any 1! terms, since they don't make a difference to the calculation. As you simplify the fraction, look for ways to cancel out numbers in the denominator with numbers in the numerator:

$$\frac{7!}{3!2!}=\frac{7\times6\times5\times\cancel{4}^{2}\times\cancel{3!}}{{}_{1}\cancel{2}\times1\times\cancel{3!}}=7\times6\times5\times2=420$$

Try another problem:

> A local card club will send 3 representatives to the national conference. If the local club has 8 members, how many different groups of representatives could the club send?

The problem talks about 8 members, so draw 8 columns for the anagram grid.

There are 3 representatives chosen; represent them with Y. Use N to represent the 5 members of the group who are not chosen.

1	2	3	4	5	6	7	8
Y	Y	Y	N	N	N	N	N

Set up your fraction:

$$\frac{8!}{3!5!} = \frac{8 \times 7 \times \cancel{6}^{\cancel{3}} \times \cancel{5!}}{({}_1\cancel{3} \times {}_1\cancel{2} \times 1)\cancel{5!}} = 8 \times 7 = 56$$

Don't write out all of the numbers on the top of the fraction; only write out the numbers down to the largest factorial on the bottom of the fraction. In the above case, you can cancel out the two 5! terms without having to write them out.

Multiple Groups

So far, the discussion has revolved around two main themes: (1) making decisions, and (2) arranging groups. Difficult combinatorics questions will actually combine the two topics. In other words, you may have to make multiple decisions, each of which will involve arranging different groups.

Try the following problem:

> The I Eta Pi fraternity must choose a delegation of 3 senior members and 2 junior members for an annual interfraternity conference. If I Eta Pi has 6 senior members and 5 junior members, how many different delegations are possible?

First, note that you are choosing senior members AND junior members. These are different decisions, so you need to determine each separately and then multiply the possible arrangements.

You have to pick 3 seniors out of a group of 6. That means that 3 are chosen (and identical) and the remaining 3 are not chosen (and also identical):

$$\frac{6!}{3!3!}=\frac{{}_{3}\cancel{6}\times 5\times 4\times \cancel{3!}}{({}_{1}\cancel{3}\times {}_{1}\cancel{2}\times 1)\cancel{3!}}=5\times 4=20$$

Similarly, you need to pick 2 juniors out of a group of 5; 2 members are chosen (and identical) and the remaining 3 members are not chosen (and also identical):

$$\frac{5!}{2!3!}=\frac{5\times \cancel{4}^{2}\times \cancel{3!}}{({}_{1}\cancel{2}\times 1)\cancel{3!}}=5\times 2=10$$

There are 20 possible senior delegations AND 10 possible junior delegations. Remember, AND means multiply. Together there are 20 × 10 = 200 possible delegations.

Questions will not always make it clear that you are dealing with multiple decisions. Try the following problem:

> The yearbook committee has to pick a color scheme for this year's yearbook. There are 7 colors to choose from (red, orange, yellow, green, blue, indigo, and violet). How many different color schemes are possible if the committee can select at most 2 colors?

Although this question concerns only one group (colors), it also involves multiple decisions. Notice the question states there can be *at most* 2 colors chosen. That means the color scheme can contain 1 color OR 2 colors.

Figure out how many combinations are possible if 1 color is chosen and if 2 colors are chosen, and then add them together:

$$\text{1 color chosen and 6 colors not chosen} = \frac{7!}{1!6!}=7$$

$$\text{2 colors chosen and 5 colors not chosen} = \frac{7!}{2!5!}=21$$

Together there are 7 plus 21, or 28, possible color schemes.

Problem Set

1. In how many different ways can the letters in the word “LEVEL” be arranged?
2. Amy and Adam are making boxes of truffles to give out as wedding favors. They have an unlimited supply of 5 different types of truffles. If each box holds 2 truffles of different types, how many different boxes can they make?
3. A pod of 6 dolphins always swims single file, with 3 females at the front and 3 males in the rear. In how many different arrangements can the dolphins swim?

Save the below problems for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

4. Mario's Pizza has 2 choices of crust: deep dish and thin-and-crispy. The restaurant also has a choice of 5 toppings: tomatoes, sausage, peppers, onions, and pepperoni. Finally, Mario's offers every pizza in extra cheese as well as regular. If Linda's volleyball team decides to order a pizza with 4 toppings, how many different choices do the teammates have at Mario's Pizza?
5. What is the sum of all the possible three-digit numbers that can be constructed using the digits 3, 4, and 5 if each digit can be used only once in each number?

Solutions

1. **30 ways:** There are two repeated E's and two repeated L's in the word “LEVEL.” To find the anagrams for this word, set up a fraction in which the numerator is the factorial of the number of letters and the denominator is the factorial of the number of each repeated letter:

$$\frac{5!}{2!2!} = \frac{5\times4\times3\times2\times1}{2\times1\times2\times1} = 30$$

Alternatively, you can solve this problem using the slot method, as long as you correct for over-counting (since you have some indistinguishable elements). There are 5 choices for the first letter, 4 for the second, and so on, making the product 5 × 4 × 3 × 2 × 1 = 120. However, there are two sets of 2 indistinguishable elements each, so you must divide by 2! to account for each of these. Thus, the total number of combinations is $\frac{5\times4\times3\times2\times1}{2!\times2!} = 30$.

2. **10 boxes:** In every combination, 2 types of truffles will be in the box, and 3 types of truffles will not. Therefore, this problem is a question about the number of anagrams that can be made from the “word” YYNNN:

$$\frac{5!}{2!3!}=\frac{5\times4\times3\times2\times1}{3\times2\times1\times2\times1}=5\times2=10$$

A	B	C	D	E
Y	Y	N	N	N

3. **36 arrangements:** There are 3! ways in which the 3 females can swim. There are 3! ways in which the 3 males can swim. Therefore, there are 3! × 3! ways in which the entire pod can swim:

$$3! \times 3! = 6 \times 6 = 36$$

This is a multiple arrangements problem, in which you have 2 separate pools (females and males).

4. **20 choices:** Consider the toppings first. Model the toppings with the “word” YYYYN, in which four of the toppings are on the pizza and one is not. The number of anagrams for this “word” is:

$$\frac{5!}{4!}=5$$

A	B	C	D	E
Y	Y	Y	Y	N

If each of these pizzas can also be offered in 2 choices of crust, there are 5 × 2 = 10 pizzas. The same logic applies for extra cheese and regular: 10 × 2 = 20.

5. **2,664:** There are 6 ways in which to arrange these digits: 345, 354, 435, 453, 534, and 543. Notice that each digit appears twice in the hundreds column, twice in the tens column, and twice in the ones column. Therefore, you can use your knowledge of place value to find the sum quickly. Because each digit appears twice in the hundreds column, you have 3 + 3 + 4 + 4 + 5 + 5 = 24 in the hundreds column. If you multiply 24 by 100, you get the value of all of the numbers in that column. Repeat this reasoning for the tens column and the ones column:

$$100(24) + 10(24) + (24) = 2{,}400 + 240 + 24 = 2{,}664$$

Chapter 5
of
Number Properties

Probability

In This Chapter…

Chapter 5

Probability

Probability is a quantity that expresses the chance, or likelihood, of an event.

Think of probability as a fraction:

$$\text{Probability} = \frac{\text{Number of } \textit{desired} \text{ or } \textit{successful} \text{ outcomes}}{\text{Total number of } \textit{possible} \text{ outcomes}}$$

For instance, if you flip a coin, the probability that heads turns up is $\frac{1}{2}$. There were two possible outcomes (heads or tails), but only one of them is considered desirable (heads).

Notice that the numerator of the fraction is *always* a subset of the denominator. If there are *n* possible outcomes, then the number of desirable outcomes must be between 0 and *n* (the number of outcomes cannot be negative). Simply put, *any probability will be between 0 and 1*. An impossible event has a probability of 0; a certain event has a probability of 1.

Additionally, you may be required to express probability as a fraction, a decimal, or a percent. For instance, $\frac{3}{4}$ = 0.75 = 75%. Although a question may ask for a probability in any one of these forms, you will first need to think of it as a fraction in order to make the necessary calculations.

Calculate the Numerator and

Denominator Separately

Numerators and denominators of probabilities are related, but they must be calculated separately. Often, it will be easier to begin by calculating the denominator.

There are two ways to calculate a number of outcomes for either the numerator or the denominator:

1. Use an appropriate combinatorics formula.
2. Manually count the number of outcomes.

Try the following problem:

> Two number cubes with faces numbered 1 to 6 are rolled. What is the probability that the sum of the rolls is 8?

Start with the total number of possible outcomes (the denominator). For this calculation you can use combinatorics. Notice that rolling two number cubes is like rolling cube 1 AND rolling cube 2. For each of these rolls, there are 6 possible outcomes (the numbers 1 to 6). Since AND equals multiply, there are $6 \times 6 = 36$ possible outcomes. This is the denominator of your fraction.

Next, figure out how many of those 36 possible rolls represent the desired outcome (a sum of 8). Can you think of an appropriate combinatorics formula? Probably not. Truth be told, the writers of this guide can't think of a formula either.

Fortunately, you don't need a formula. Only a limited number of combinations would work. Count them up! When you *have to* count, you never have to count too much. If the first die turns up a 1, the other die would need to roll a 7. This isn't possible, so eliminate that possibility. Keep counting; here are the rolls that work:

> 2 − 6
> 3 − 5
> 4 − 4
> 5 − 3
> 6 − 2

That's it; there are 5 combinations that work. Therefore, the probability of a sum of 8 is $\frac{5}{36}$.

More Than One Event: "AND" vs. "OR"

Combinatorics and probability have another connection: the meaning of the words AND and OR. In probability, as well as in combinatorics, the word AND means multiply and the word OR means add.

> There is a $\frac{1}{2}$ probability that a certain coin will turn up heads on any given toss.
>
> What is the probability that two tosses of the coin will yield heads both times?

To answer this question, calculate the probability that the coin lands on heads on the first flip AND heads on the second flip. The probability of heads on the first flip is $\frac{1}{2}$. The probability of heads on the second flip is also $\frac{1}{2}$. Since AND means multiply, the probability is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Try another example:

> The weather report for today states that there is a 40% chance of sun, a 25% chance of rain, and a 35% chance of hail. Assuming only one of the three outcomes can happen, what is the probability that it rains or hails today?

The question is asking for the probability of rain OR hail. Therefore, the probability is 25% + 35% = 60%. The calculation would change if *both* rain and hail can happen, but don't worry about that for now.

P(A) + P(Not A) = 1

P(A) + P(Not A) = 1 is really just a fancy way of saying that the probability of something happening plus the probability of that thing *not* happening must sum to 1. For example, the probability that it rains or doesn't rain equals 1: if there's a 25% chance of rain, then there must be a 75% chance that it will *not* rain. Try an example:

> A person has a 40% chance of winning a game every time he or she plays it. If there are no ties, what is the probability that Asha loses the game the first time she plays and wins the second time she plays?

If the probability of winning the game is 40%, then the odds of *not* winning the game (losing) are 100% − 40% = 60%. Calculate the odds that Asha loses the game the first time AND wins the game the second time:

$$(60\%) \times (40\%) = 0.6 \times 0.4 = 0.24$$

The probability is 0.24, or 24%.

The 1 − *x* Probability Trick

Suppose that a salesperson makes 5 sales calls, and you want to find the likelihood that he or she makes *at least 1* sale. If you try to calculate this probability directly, you will have to confront 5 separate possibilities that constitute “success”: exactly 1 sale, exactly 2 sales, exactly 3 sales, exactly 4 sales, or exactly 5 sales. This would almost certainly be more work than you can reasonably do in two minutes.

There is, however, another option. Instead of calculating the probability that the salesperson makes at least one sale, you can calculate the probability that the salesperson does *not* make at least one sale. Then, you could subtract that probability from 1. This shortcut works because the thing that does *not* happen represents a smaller number of the possible outcomes: that is, *not* getting at least 1 sale is the same thing as getting 0 sales, which is just one of the total possible outcomes. By contrast, making at least 1 sale represents 5 separate possible outcomes. When this occurs, it is much easier to calculate the probability for that one possible outcome (0 sales) and then subtract from 1.

For complicated probability problems, decide whether it is easier to calculate the probability you want or the probability you do *not* want. Try an example:

> A bag contains equal numbers of red, green, and yellow marbles. If Geeta pulls three marbles out of the bag, replacing each marble after she picks it, what is the probability that at least one will be red?

The quick way to answer this question is to calculate the probability that *none* of the marbles are red. For each of the three picks, there is a $\frac{2}{3}$ probability that the marble will not be red. The probability that all three marbles will not be red is $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$.

If the probability that *none* of the marbles is red is $\frac{8}{27}$, then the probability that at least one marble is red is $1 - \frac{8}{27} = \frac{19}{27}$.

If you need to calculate the probability of an event (P(A)), there are two ways to calculate the probability:

P(A) or 1 − P(Not A)

When the question includes *at least* or *at most* language, the 1 − P(Not A) method is usually faster.

Problem Set

For problems #1 and #2, assume that each number cube has 6 sides with faces numbered 1 to 6.

1. What is the probability that the sum of two number cubes will yield a 10 or lower?
2. What is the probability that the sum of two number cubes will yield a 7, and then when both are thrown again, their sum will again yield a 7?
3. There is a 30% chance of rain and a 70% chance of shine. If it rains, there is a 50% chance that Bob will cancel his picnic, but if the sun is shining, he will

definitely have his picnic. What is the chance that Bob will have his picnic?

Save the below problem set for review, either after you finish this book or after you finish all of the Quant books that you plan to study.

4. In a diving competition, each diver has a 20% chance of a perfect dive. The first perfect dive of the competition, but no subsequent dives, will receive a perfect score. If Janet is the third diver to dive, what is her chance of receiving a perfect score? (Assume that each diver can perform only one dive per turn.)
5. A magician has five animals in his magic hat: 3 doves and 2 rabbits. If he pulls two animals out of the hat at random, what is the chance that he will have a matched pair?

Solutions

1. $\mathbf{\frac{11}{12}}$**:** Solve this problem by calculating the probability that the sum will be higher than 10 and subtracting the result from 1. There are 3 combinations of 2 number cubes that yield a sum higher than 10: 5 + 6, 6 + 5, and 6 + 6. Therefore, the probability that the sum will be higher than 10 is $\frac{3}{36}$, or $\frac{1}{12}$. The probability that the sum will be 10 or lower is $1 - \frac{1}{12} = \frac{11}{12}$.

2. $\mathbf{\frac{1}{36}}$**:** There are 36 ways in which 2 number cubes can be thrown (6 × 6 = 36). The combinations that yield a sum of 7 are 1 + 6, 2 + 5, 3 + 4, 4 + 3, 5 + 2, and 6 + 1: 6 different combinations. Therefore, the probability of rolling a 7 is $\frac{6}{36}$, or $\frac{1}{6}$. To find the probability that this will happen twice in a row, you need to multiply: $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

3. **85%:** There are two possible chains of events in which Bob will have the picnic:

One: The sun shines: P = 70% OR

Two: It rains AND Bob chooses to have the picnic anyway:

$$P = 30\%\left(\frac{1}{2}\right) = 15\%$$

Add the probabilities together to find the total probability that Bob will have the picnic:

70% + 15% = 85%

4. $\mathbf{\frac{16}{125}}$**:** In order for Janet to receive a perfect score, neither of the previous two divers can receive one. Therefore, you are finding the probability of a chain of three events: that diver one will *not* get a perfect score AND diver two will *not* get a perfect score AND Janet *will* get a perfect score. Multiply the probabilities: $\frac{4}{5}\times\frac{4}{5}\times\frac{1}{5}=\frac{16}{125}$.

The probability is $\frac{16}{125}$ that Janet will receive a perfect score.

5. **40%:** Use an anagram model to find out the total number of different pairs the magician can pull out of his hat. Since two animals will be in the pair and the other three will not, use the "word" YYNNN.

A	B	C	D	E
Y	Y	N	N	N

$$\frac{5!}{2!3!}=\frac{5\times4}{2\times1}=10$$

Thus, there are 10 possible pairs.

Then, list the pairs in which the animals will match. Represent the rabbits with the letters *a* and *b*, and the doves with the letters *x*, *y*, and *z*.

Matched Pairs: R_aR_b D_xD_y D_xD_z D_yD_z There are four pairs in which the animals will be a matched set.

Therefore, the probability that the magician will randomly draw a matched set is $\frac{4}{10}=40\%$.

Chapter 6

of

Number Properties

Extra Divisibility & Primes

In This Chapter…

Chapter 6

Extra Divisibility & Primes

This chapter covers harder material within the topic of divisibility and primes. Before you read this chapter, read the first four chapters of this book and complete some *Official Guide for GMAT Review* problems that involve divisibility and primes. If your goal is not 80th percentile (or higher) on the Quant section of the GMAT, you may want to skip this section.

Primes

You should become very comfortable with small prime numbers—at least the first 10. Even better, know (or be able to derive quickly) all the primes up to 100: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97. Here are some additional facts about primes that may be helpful on the GMAT:

- **There are an infinite number of prime numbers.** There is no upper limit to the size of prime numbers.
- **There is no simple pattern in the prime numbers.** Since 2 is the only even prime number, all other primes are odd. However, there is no easy pattern to determine which odd numbers will be prime. Each number needs to be tested directly to determine whether it is prime.
- **Positive integers with exactly two factors must be prime, and positive integers with more than two factors are never prime.** Any integer greater than or equal to 2 has at least two factors: 1 and itself. Thus, if there are only two factors of x (with x equal to an integer > 2), then the factors of x *must* be 1 and x. Therefore, x must be prime. Also, do not

forget that the number 1 is *not* prime. The number 1 has only one factor (itself), so it is defined as a non-prime number.

These facts can be used to disguise the topic of prime numbers on the GMAT. Take a look at the following Data Sufficiency (DS) examples:

What is the value of integer x?

(1) x has exactly 2 factors.

(2) When x is divided by 2, the remainder is 0.

Statement (1) indicates that x is prime, because it has only 2 factors. This statement is insufficient by itself, since there are infinitely many prime numbers. Statement (2) indicates that 2 divides evenly into x, meaning that x is even; that is also insufficient by itself. Taken together, however, the two statements reveal that x must be an even prime—and the only even prime number is 2. The answer is (C).

If x is a prime number, what is the value of x?

(1) There are a total of 50 prime numbers between 2 and x, inclusive.

(2) There is no integer n such that x is divisible by n and $1 < n < x$.

At first, this problem seems outlandishly difficult. How are you to list out the first 50 prime numbers in under two minutes? Remember, however, that this is a Data Sufficiency problem. You do not need to list the first 50 primes. Instead, all you need to do is determine *whether* you can do so.

For statement (1), you know that certain numbers are prime and others are not. You also know that x is prime. Therefore, if you were to list all the primes from 2 on up, you eventually would find the 50th prime number. That number must equal x: because x is prime, it *must* be the 50th item on that list of primes. This inforzvmation is sufficient.

For statement (2), you are told that x is not divisible by any integer greater than 1 and less than x. There fore, x can only have 1 and x as factors. In other words, x is prime. You already know this result, in fact: it was given to you in the question stem. So statement (2) does not help you determine what x is. This statement is insufficient.

The correct answer is **(A)**. (Incidentally, for those who are curious, the 50th prime number is 229.)

Divisibility and Addition/Subtraction

Part 1 of Divisibility & Primes showed that if you add or subtract multiples of an integer, you get another multiple of that integer. This rule can be generalized to other situations. For the two following rules, assume that N is an integer:

1. If you add a multiple of N to a non-multiple of N, the result is a non-multiple of N. (The same holds true for subtraction.) For example:

   $18 - 10 = 8$ (Multiple of 3) − (Non-multiple of 3) = (Non-multiple of 3)

2. If you add two non-multiples of N, the result could be either a multiple of N or a non-multiple of N. For example:

   $19 + 13 = 32$ (Non-multiple of 3) + (Non-multiple of 3) = (Non-multiple of 3)

   $19 + 14 = 33$ (Non-multiple of 3) + (Non-multiple of 3) = (Multiple of 3)

   The exception to this rule is when $N = 2$. Two odds always sum to an even.

Try the following Data Sufficiency example:

Is N divisible by 7?

(1) $N = x - y$, where x and y are integers.

(2) x is divisible by 7, and y is not divisible by 7.

Statement (1) indicates that N is the difference between two integers (x and y), but it does not tell you anything about whether x or y is divisible by 7. This statement is insufficient.

Statement (2) tells you nothing about N. This statement is insufficient.

Statements (1) and (2) combined indicate that x is a multiple of 7, but y is not a multiple of 7. The difference between x and y can *never* be divisible by 7 if x is divisible by 7 but y is not. (If you are not convinced, try testing out some real numbers.) Therefore, N cannot be a multiple of 7.

The correct answer is **(C)**.

Greatest Common Factor and Least Common Multiple

On the GMAT, you may have to find the greatest common factor (GCF) or least common multiple (LCM) of a set of two or more numbers.

> **The greatest common *factor* (GCF)** is the largest divisor of two or more integers; this *factor* will be smaller than or equal to the starting integers.
>
> **The least common *multiple* (LCM)** is the the smallest multiple of two or more integers; this *multiple* will be larger than or equal to the starting integers.

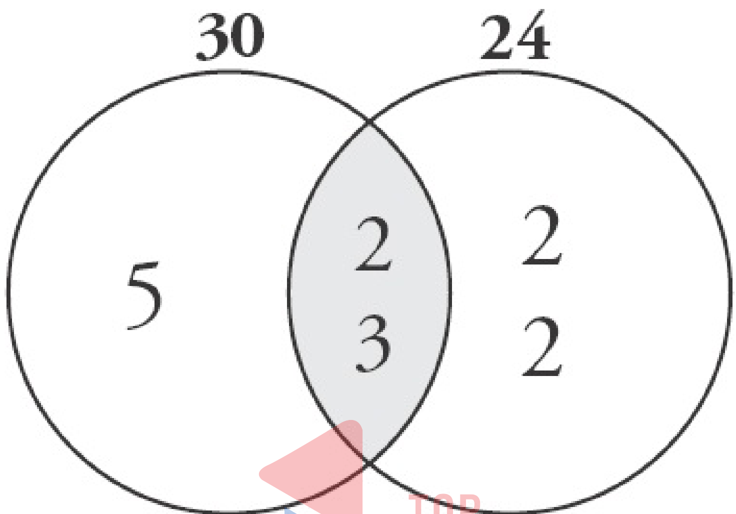
You may already know how to find the LCM. When you add together the fractions $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}$, you convert the fractions to thirtieths: $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}=\frac{15}{30}+\frac{10}{30}+\frac{6}{30}=\frac{31}{30}$. Why thirtieths? Because 30 is the LCM of the denominators 2, 3, and 5.

Finding GCF and LCM Using Venn Diagrams

You can find the GCF and LCM of two numbers by placing prime factors into a **Venn diagram**—a diagram of circles showing the overlapping and non-overlapping elements of two sets. To find the GCF and LCM of two numbers using a Venn diagram, perform the following steps:

1. Factor the numbers into primes. For example, $30 = 2 \times 3 \times 5$ and $24 = 2 \times 2 \times 2 \times 3$.
2. Create a Venn diagram.

3. Place each shared factor into the shared area of the diagram (the shaded region to the right). In this example, 30 and 24 share one 2 and one 3.
4. Place the remaining (non-shared) factors into the non-shared areas.
5. The GCF is the product of the primes in the shared region: $2 \times 3 = 6$.
6. The LCM is the product of all primes in the diagram: $5 \times 2 \times 3 \times 2 \times 2 = 120$.

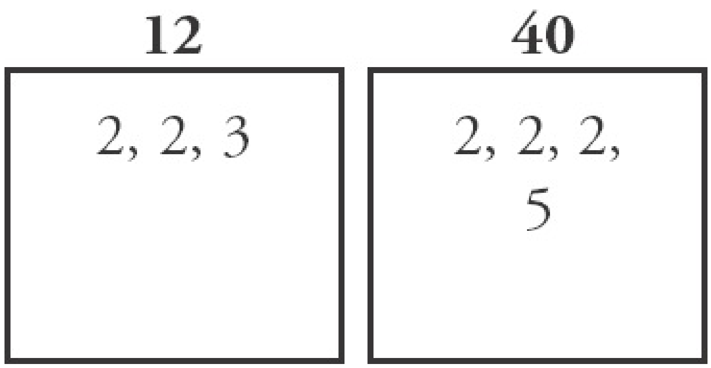


Try it an example:

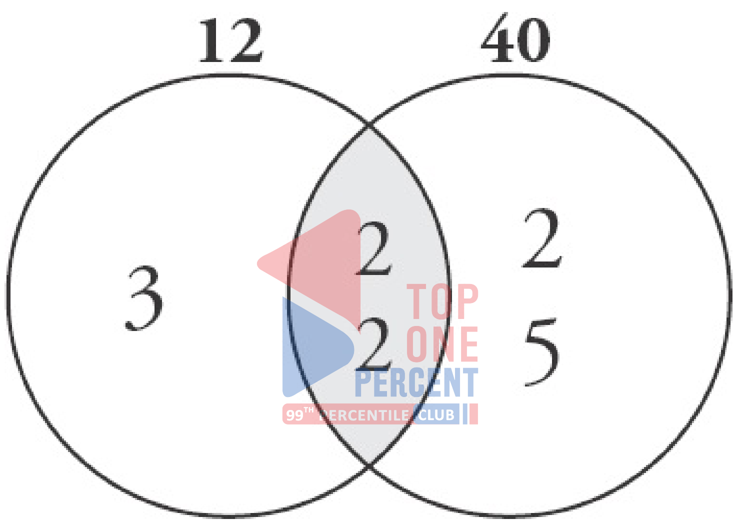
> Compute the GCF and LCM of 12 and 40 using the Venn diagram approach.

The prime factorizations of 12 and 40 are $2 \times 2 \times 3$ and $2 \times 2 \times 2 \times 5$, respectively:

The only common factors of 12 and 40 are two 2’s. Therefore, place two 2’s in the shared area of the Venn diagram and remove them from *both* prime factorizations. Then, place the remaining factors in the zones belonging exclusively to 12 and 40. These two outer regions must have *no* primes in common!

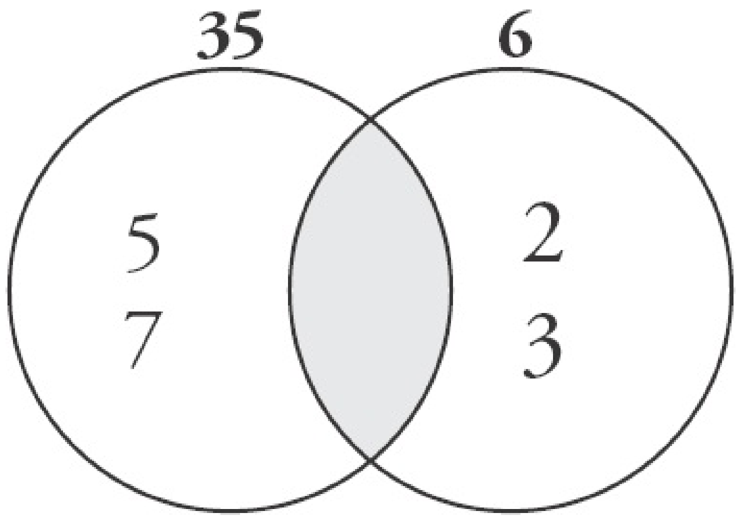


The GCF of 12 and 40 is therefore 2 × 2 = 4, the product of the primes in the *shared area*. (An easy way to remember this is that the common factors are in the common area.)



The LCM is 3 × 2 × 2 × 2 × 5 = 120, the product of *all* the primes in the diagram.

Note that if two numbers have *no* primes in common, then their GCF is 1 and their LCM is their product. For example, 35 (= 5 × 7) and 6 (= 2 × 3) have no prime numbers in common. Therefore, their GCF is 1 (the common factor of *all* positive integers) and their LCM is 35 × 6 = 210. Be careful: even though you have no primes in the common area, the GCF is not 0 but 1.



Advanced GCF and LCM Techniques

While Venn diagrams are helpful for visualizing the steps needed to compute the GCF and LCM, they can be cumbersome if you want to find the GCF or LCM of large numbers or of three or more numbers.

Finding GCF and LCM Using Prime Columns

Here are the steps:

Step 1: Calculate the prime factors of each integer.

Step 2: Create a column for each prime factor found within any of the integers.

Step 3: Create a row for each integer.

Step 4: In each cell of the table, place the prime factor raised to a *power*. This power counts how many copies of the column's prime factor appear in the prime box of the row's integer.

To calculate the GCF, take the *lowest* count of each prime factor found across *all* the integers. This counts the shared primes. To calculate the LCM, take the *highest* count of each prime factor found across *all* the integers. This counts all the primes less the shared primes.

Try this problem:

Find the GCF and LCM of 100, 140, and 250.

Step 1: Calculate the prime factors of each integer.

$100 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$

$140 = 2 \times 2 \times 5 \times 7 = 2^2 \times 5 \times 7$

$250 = 2 \times 5 \times 5 \times 5 = 2 \times 5^3$

Step 2: Create a column for each prime factor base and a row for each integer. The different prime factors are 2, 5, and 7, so you need three columns. There are three integers (100, 140, and 250), so you also need three rows.

Step 3: Fill in the table with each prime factor raised to the appropriate power:

Number	2		5		7
100	2^2	×	5^2	×	7^0
140	2^2	×	5^1	×	7^1
250	2^1	×	5^3	×	7^0

To calculate the GCF, take the *smallest* count (the lowest power) in any column, because the GCF is formed only out of the *shared* primes. The smallest count of the factor 2 is one, in 250 ($= 2^1 \times 5^3$). The smallest count of the factor 5 is one, in 140 ($= 2^2 \times 5^1 \times 7^1$). The smallest count of the factor 7 is zero, since 7 does not appear in 100 or in 250. Therefore, the GCF is $2^1 \times 5^1 = 10$.

To calculate the LCM, take the *largest* count (the highest power) in any column, because the LCM is formed out of *all* the primes less the shared primes. The largest count of the factor 2 is two, in 140 and 100. The largest count of the factor 5 is three, in 250. The largest count of the factor 7 is one, in 140. Therefore, the LCM is $2^2 \times 5^3 \times 7^1 = 3{,}500$.

Number	2		5		7		
100	2^2	×	5^2	×	7^0		
140	2^2	×	5^1	×	7^1		
250	2^1	×	5^3	×	7^0		
GCF	2^1	×	5^1	×	7^0	=	$2^1 \times 5^1 = 10$
LCM	2^2	×	5^3	×	7^1	=	$2^2 \times 5^3 \times 7^1 = 3{,}500$

Finding GCF and LCM Using Prime Boxes or Factorizations

Also, you can use a shortcut directly from the prime boxes or the prime factorizations to find the GCF and LCM. Once you become familiar with the prime columns method, you can just scan the boxes or the factorizations and take all the lowest powers to find the GCF and the highest powers to find the LCM.

Try an example:

What are the GCF and LCM of 30 and 24?

30	24
2, 3, 5	2, 2, 2, 3

The prime factorization of 30 is 2 × 3 × 5.

The prime factorization of 24 is 2 × 2 × 2 × 3, or $2^3 \times 3$.

The GCF is 2 × 3 = 6.

The LCM is $2^3 \times 3 \times 5 = 120$.

Finally, you may be asked to determine what combinations of numbers could lead to a specific GCF or LCM. This is a difficult task. Consider the following DS problem :

Is the integer *z* divisible by 6?

(1) The greatest common factor of *z* and 12 is 3.

(2) The greatest common factor of z and 15 is 15.

When calculating the GCF for a set of numbers, determine the prime factors of each number and then take each prime factor to the *lowest* power it appears in any factorization. In this problem, you are given the GCF, so you can work backwards to determine what additional information can be determined.

Statement (1) indicates that z and 12 ($2 \times 2 \times 3$) have a GCF of 3. Set up that information in a prime columns table (like the one on the right) to figure out what you can deduce about the prime factors of z.

Number	**2**	**3**
z	–	3^1
12	2^2	3^1

Notice that the GCF of 12 and z contains a 3. Since the GCF contains each prime factor to the power it appears the *least*, you know that z must also contain at least one 3. Therefore, z is divisible by 3.

Notice also that the GCF contains NO 2's. Since 12 contains two 2's, z must not contain any 2's. Therefore, z is *not* divisible by 2. Since z is not divisible by 2, it cannot be divisible by 6. This statement is sufficient.

Statement (2) indicates that z and 15 (3×5) have a GCF of 15. Set that up in a prime columns table (see below) to figure out what you can deduce about the prime factors of z.

The GCF of 15 and z contains a 3. Since the GCF contains each prime factor to the power it appears the *least*, you know that z must also contain at least one 3. Therefore, z is divisible by 3.

Number	**3**	**5**
z	3^1	5^1
15	3^1	5^1

Likewise, z must also contain a 5. Therefore, z is divisible by 5.

However, this does *not* indicate whether z contains any 2's. In order to be divisible by 6, z has to contain at least one 2 and at least one 3. Thus, it's impossible to tell whether z is divisible by 6; this statement is not sufficient. The correct answer is **(A)**.

Now consider this example:

> If the LCM of *a* and 12 is 36, what are the possible values of *a*?

As in the earlier example, you can use the prime columns technique to draw conclusions about the prime factors of *a*.

First, notice that *a* cannot be larger than 36. The LCM of two or more integers is always *at least* as large as any of the integers. Therefore, the maximum value of *a* is 36.

Next, the prime factorization of 36 is 2 × 2 × 3 × 3. Notice that the LCM of 12 and *a* contains two 2's. Since the LCM contains each prime factor to the power it appears the *most*, and since 12 also contains two 2's, *a* cannot contain more than two 2's. It does not necessarily need to contain any 2's, so *a* can contain zero, one, or two 2's.

Number	**2**	**3**
a	$\leq 2^2$	3^2
12	2^2	3^1

Finally, observe that the LCM of 12 and *a* contains two 3's. But 12 only contains *one* 3. The 3^2 factor in the LCM must have come from the prime factorization of *a*. Therefore, *a* contains exactly two 3's.

Since *a* must contain exactly two 3's, and can contain no 2's, one 2, or two 2's, *a* could be 3 × 3 = 9, 3 × 3 × 2 = 18, or 3 × 3 × 2 × 2 = 36. Thus, 9, 18, and 36 are the possible values of *a*.

Other Applications of Primes & Divisibility

Counting Factors and Primes

The GMAT can ask you to count factors of some number in several different ways. For example, consider the number 1,400. The prime factorization of this number is $2 \times 2 \times 2 \times 5 \times 5 \times 7$, or $2^3 \times 5^2 \times 7$. Here are three different questions that the GMAT could ask you about this integer:

Factors of 1,400 ($2^3 \times 5^2 \times 7$):

1. How many different prime factors?

- May be phrased as "different prime factors," "unique prime factors," or "distinct prime factors."
- Count each repeated prime factor only ONCE.
- In this example, 2, 5, and 7 are distinct, so there are three different prime factors.

2. How many total prime factors (length)?

- Length is defined as the number of primes (not necessarily distinct) whose product is *x* (in this case, whose product is 1,400).
- Add the exponents of the prime factors. If there is no exponent, count it as 1.
- In this example, the length is 3 + 2 + 1 = 6.

3. How many total factors?

- Includes all factors, not necessarily just prime factors.
- Can be determined using "factor pairs" approach, but this is cumbersome for larger numbers.
- Advanced technique discussed later in this chapter.
- Do not forget to include 1 as a factor!

Consider the number 252.

(a) How many unique prime factors of 252 are there?

(b) What is the length of 252 (as defined above)?

(c) How many total factors of 252 are there?

For (a), determine the number of unique prime factors by looking at the prime factorization of 252: $2 \times 2 \times 3 \times 3 \times 7$. There are three different prime factors in

252: 2, 3, and 7. Do *not* count repeated primes to answer this particular question.

For (b), the "length" of an integer is defined as the total number of primes that, when multiplied together, equal that integer. (Note: on the GMAT, any question that asks about the length of an integer will provide this definition of length, so you do not need to memorize it.) The prime factorization of 252 is $2 \times 2 \times 3 \times 3 \times 7$. There are five total prime factors in 252: 2, 2, 3, 3, and 7. In other words, the length of an integer is just the total number of primes in the prime box of that integer. *Do* count repeated primes to answer this particular question.

You can also answer this question by looking at the prime factorization in exponential form: $252 = 2^2 \times 3^2 \times 7$. Add the exponents: $2 + 2 + 1 = 5$. Notice that a number written in this form without an exponent has an implicit exponent of 1.

For (c), one way to determine the total number of factors is to determine the factor pairs of 252, using the process described in Chapter 1 of this book and shown in the chart to the right. Simply start at 1 and "walk up" through all the integers, determining whether each is a factor, as shown in the table. You can stop once the small column "meets" the large column. For example, since the last entry in the large column is 18, you can stop searching once you have evaluated 17 as a possible factor. Count up the number of factors in the table: there are 18.

Small	Large
1	252
2	126
3	84
4	63
6	42
7	36
9	28
12	21
14	18

This method will be too cumbersome for larger numbers, so a more advanced method is introduced in the next section.

Counting Total Factors

When a problem has a large number of factors, the factor pair method can be too slow. Therefore, you need a general method to apply to more difficult problems of this type. For example:

> How many different factors does 2,000 have?

It would take a very long time to list all of the factors of 2,000. However, prime factorization can shorten the process considerably. First, factor 2,000 into primes: $2{,}000 = 2^4 \times 5^3$. The key to this method is to consider each distinct prime factor separately.

Consider the prime factor 2 first. Because the prime factorization of 2,000 contains four 2's, there are *five* possibilities for the number of 2's in any factor of 2,000: none, one, two, three, or four. (Do not forget the possibility of *no* occurrences! For example, 5 is a factor of 2,000, and 5 does not have *any* 2's in its prime box.)

Next, consider the prime factor 5. There are three 5's, so there are *four* possibilities for the number of 5's in a factor of 2,000: none, one, two, or three. (Again, do not forget the possibility of *no* occurrences of 5.)

In general, if a prime factor appears to the Nth power, then there are $(N + 1)$ possibilities for the occurrences of that prime factor. This is true for each of the individual prime factors of any number.

You can borrow a principle from combinatorics to simplify the calculation of the number of prime factors in 2,000: when you are making a number of separate decisions, then multiply the number of ways to make each *individual* decision to find the number of ways to make *all* the decisions. Because there are five possible decisions for the 2 factor and four possible decisions for the 5 factor, there are $5 \times 4 = 20$ different factors.

The logic behind this process can also be represented in the following table of factors. (Note that there is no reason to make this table, unless you are interested in the specific factors themselves. It simply illustrates the reasoning behind multiplying the possibilities.)

	2^0	2^1	2^2	2^3	2^4
5^0	1	2	4	8	16
5^1	5	10	20	40	80
5^2	25	50	100	200	400
5^3	125	250	500	1,000	2,000

Each entry in the table is the unique product of a power of 2 (the columns) and a power of 5 (the rows). For instance, $50 = 2^1 \times 5^2$. Notice that the factor in the top left corner contains no 5's and no 2's. That factor is 1 (which equals $2^0 \times 5^0$).

The table has five columns (representing the possible powers of 2 in the factor) and four rows (representing the possible powers of 5 in the factor). The total number of factors is given by 5 columns multiplied by 4 rows, so there are 20 different factors.

Although a table like the one above cannot be easily set up for more than two prime factors, the process can be generalized to numbers with more than two prime factors. If a number has prime factorization $a^x \times b^y \times c^z$ (where a, b, and c are all prime), then the number has $(x + 1)(y + 1)(z + 1)$ different factors.

For instance, $9{,}450 = 2^1 \times 3^3 \times 5^2 \times 7^1$, so 9,450 has $(1 + 1)(3 + 1)(2 + 1)(1 + 1) = 48$ different factors.

Perfect Squares, Cubes, etc.

The GMAT occasionally tests properties of perfect squares, which are squares of other integers. The numbers 4 ($= 2^2$) and 25 ($= 5^2$) are examples of perfect squares. One special property of perfect squares is that **all perfect squares have an odd number of total factors**. Similarly, any integer that has an odd number of total factors *must* be a perfect square. All other non-square integers have an even number of factors. Why is this the case?

Think back to the factor pair exercises you have done so far. Factors come in pairs. If a and b are integers and $a \times b = c$, then a and b are a factor pair of c. However, if c is a perfect square, then in *one* of its factor pairs, a equals b. That is, in this particular pair you have $a \times a = c$, or $a^2 = c$. This “pair” does not consist of two different factors. Rather, you have a single unpaired factor: the square root.

Consider the perfect square 36. It has 5 factor pairs that yield 36, as shown to the right. Notice that the *final* pair is 6 and 6, so instead of $5 \times 2 = 10$ total factors, there are only 9 different factors of 36.

Small	Large
1	36
2	18
3	12
4	9
6	**6**

Notice also that any number that is not a perfect square will *never* have an odd number of factors. That is because the only way to arrive at an odd number of factors is to have a factor pair in which the two factors are equal.

For larger numbers, it would be much more difficult to use the factor pair technique to prove that a number is a perfect square or that it has an odd number of factors. Thankfully, you can use a different approach. Notice that perfect squares are formed from the product of two copies of the same prime factors. For instance, $90^2 = (2 \times 3^2 \times 5)(2 \times 3^2 \times 5) = 2^2 \times 3^4 \times 5^2$. Therefore, **the prime factorization of a perfect square contains only even powers of primes**. It is also true that any number whose prime factorization contains only even powers of primes must be a perfect square.

Here are some examples:

$144 = 2^4 \times 3^2$ $\quad\quad$ $9 = 3^2$

$36 = 2^2 \times 3^2$ $\quad\quad$ $40{,}000 = 2^6 \times 5^4$

All of these integers are perfect squares.

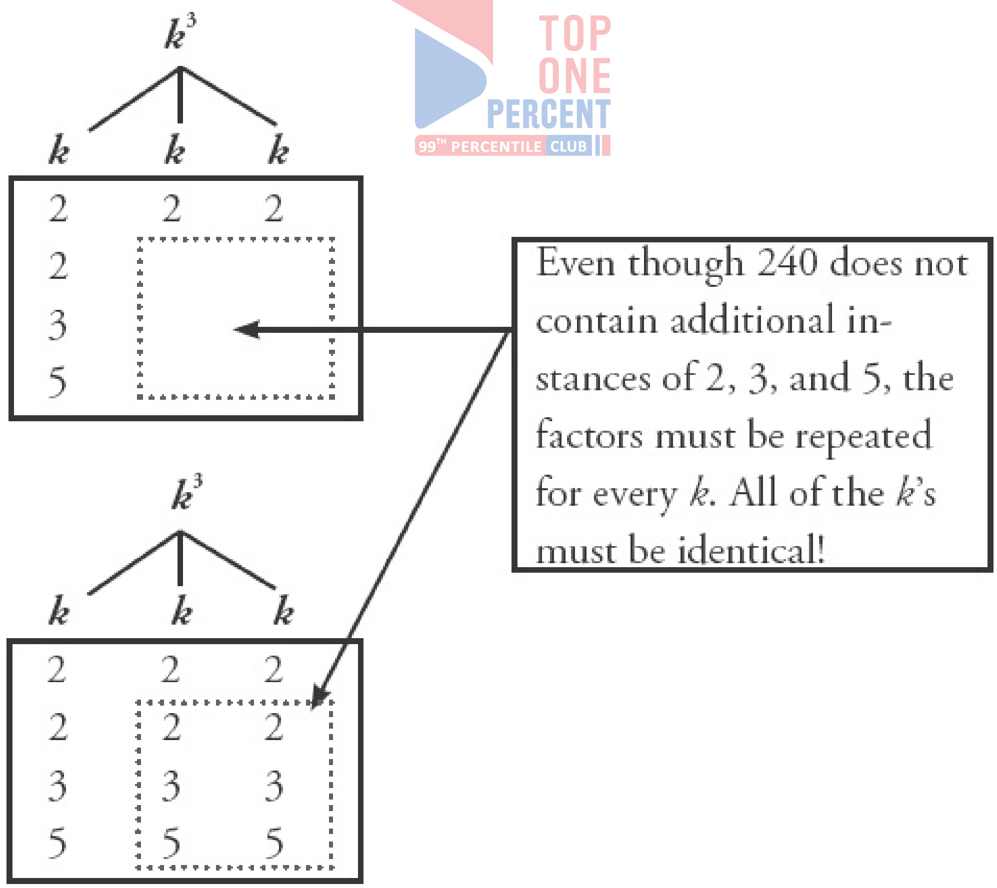
By contrast, if a number's prime factorization contains any odd powers of primes, then the number is not a perfect square. For instance, $132{,}300 = 2^2 \times 3^3 \times 5^2 \times 7^2$ is not a perfect square, because the 3 is raised to an odd power.

The same logic used for perfect squares extends to perfect cubes and to other "perfect" powers. If a number is a perfect cube, then it is formed from three identical sets of primes, so all the powers of primes are multiples of 3 in the factorization of a perfect cube: for instance: $90^3 = (2 \times 3^2 \times 5)(2 \times 3^2 \times 5)(2 \times 3^2 \times 5) = 2^3 \times 3^6 \times 5^3$.

Try an example:

> If k^3 is divisible by 240, what is the least possible value of integer k?
>
> (A) 12
>
> (B) 30
>
> (C) 60
>
> (D) 90
>
> (E) 120

The prime box of k^3 contains the prime factors of 240, which are 2, 2, 2, 2, 3, and 5. The prime factors of k^3 should be the prime factors of k appearing in sets of three, so distribute the prime factors of k^3 into three columns to represent the prime factors of k, as shown to the right.



There is a complete set of three 2's in the prime box of k^3, so k must have a factor of 2. However, there is a fourth 2 left over. That additional factor of 2 must be from k as well, so assign it to one of the component k columns. There is an incomplete set of 3's in the prime box of k^3, but you can still infer that k has a factor of 3; otherwise k^3 would not have any. Similarly, k^3 has a single 5 in its prime box, but that factor must be one of the factors of k as well. Thus, k has 2, 2, 3, and 5 in its prime box, so k must be a multiple of 60.

The correct answer is **(C)**.

Factorials and Divisibility

Because $N!$ is the product of all the integers from 1 to N, $N!$ must be divisible by all integers from 1 to N. Another way of saying this is that $N!$ is a multiple of all the integers from 1 to N.

This fact works in concert with other properties of divisibility and multiples. For instance, the quantity 10! + 7 must be a multiple of 7, because both 10! and 7 are multiples of 7.

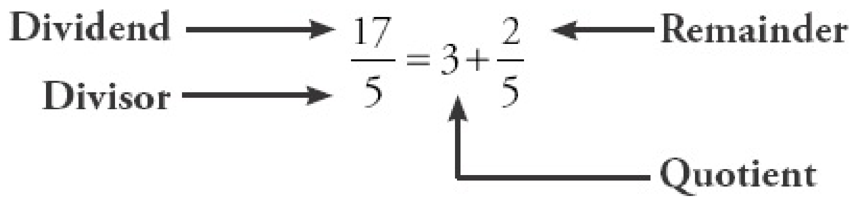
Therefore, 10! + 15 must be a multiple of 15, because 10! is divisible by 5 and 3, and 15 is divisible by 5 and 3. Thus, both numbers are divisible by 15, and the sum is divisible by 15. Finally, 10! + 11! is a multiple of any integer from 1 to 10, because every integer between 1 and 10 inclusive is a factor of both 10! and 11!, separately.

Advanced Remainders

Remainders are often expressed as integers; however, they can also be expressed as fractions or as decimals. It is important to understand the connection between these three different ways of expressing remainders.

How can you express 17 divided by 5 using remainders? It can be expressed as $17 = 3 \times 5 + 2$. Divide both sides of the equation by 5 (because the initial problem was to divide 17 by 5):

$$\frac{17}{5} = 3 + \frac{2}{5}$$



The fraction form of a remainder is the integer form divided by the divisor. And, as you may have guessed, the decimal form of a remainder is just the decimal equivalent of the fraction: $\frac{2}{5} = 0.4$.

The quotient can be 0. For instance, when 3 is divided by 5, the solution is 0 remainder 3, because 5 goes into 3 zero times with 3 left over.

In sum, you can express the division of 17 by 5 in three ways:

$\frac{17}{5} = 3$ with a remainder of 2 $\qquad \frac{17}{5} = 3\frac{2}{5} \qquad \frac{17}{5} = 3.4$

Try the following example:

> When positive integer *A* is divided by positive integer *B*, the result is 4.35. Which of the following could be the remainder when *A* is divided by *B*?
>
> (A) 13
>
> (B) 14
>
> (C) 15
>
> (D) 16
>
> (E) 17

It may seem as if this question has not given you a whole lot to go on. First, notice the language in the question. When a GMAT question refers to a remainder, it is referring to the integer form of the remainder. The key to this problem will be to connect the integer form of the remainder with the decimal form of the remainder provided in the question.

You know that $\frac{A}{B} = 4.35$. That means that 4 is the quotient and 0.35 is the remainder (expressed as a decimal). If you let *R* equal the remainder, then you can set up the following relationship:

$$0.35 = \frac{\text{Remainder}}{\text{Divisor}} = \frac{R}{B}$$

This relationship may not appear particularly useful. The value, however, comes from a hidden constraint of this relationship: *R* and *B* must both be integers. You know *B* is an integer because of information given in the question. And you know that *R* is an integer because this relationship is the connection between the integer form of remainders and the decimal (or fraction) form.

The next step is to change the decimal to a fraction:

$$0.35 = \frac{R}{B}$$

$$\frac{35}{100} = \frac{R}{B}$$

$$\frac{7}{20} = \frac{R}{B}$$

Now the decimal remainder is expressed as a division involving two integers (7 and 20).

Finally, cross multiply the fractions:

$$7B = 20R$$

In order for this equation to involve only integers, the prime factors on the left side of the equation must equal the prime factors on the right side of the equation. You know that the divisor (*B*) *must* be a multiple of 20 and, more importantly, the remainder (*R*) *must* be a multiple of 7.

Take a look at the answer choices. The only choice that is a multiple of 7 is answer (B) 14. Therefore, the correct answer is **(B)**.

You could even use this information to calculate the values of *A* and *B*.

First, go back to the original relationship you wrote down:

$$0.35 = \frac{\text{Remainder}}{\text{Divisor}} = \frac{R}{B}$$

You now know a possible remainder, so replace *R* with (14):

$$0.35 = \frac{R}{B}$$

$$\frac{7}{20} = \frac{14}{B}$$

$$\left(\frac{7}{20}\right)B = 14$$

$$B = 14 \times \frac{20}{7} = 2 \times 20 = 40$$

You can then use the value of *B* to solve for *A*:

$$\frac{A}{B} = 4.35$$

$$\frac{A}{(40)} = 4.35$$

$$A = 4.35 \times 40 = 174$$

Creating Numbers with a Certain Remainder

Occasionally, a GMAT question will give you the following type of information:

"When positive integer *n* is divided by 7, there is a remainder of 2."

To answer this question, you will need to be able to list different possible values of *n*.

So, what are the possible values of *n*? Since the remainder is given to you as an integer, set up the integer remainder relationship:

Dividend = Quotient × Divisor + Remainder

n = (integer) × 7 + 2

Perform two calculations to generate possible values of *n*: first, multiply 7 by an integer and, second, add 2 to that product, as in the chart below:

Integer (Quotient)	×	7 (Divisor)	+	2 (Remainder)	=	n (Dividend)
0	×	7	+	2	=	2
1	×	7	+	2	=	9
2	×	7	+	2	=	16
3	×	7	+	2	=	23

Another way to think of all these values of n is that they are 2 more than a multiple of 7. Remember that 0 is a multiple of *any* number. In this example, 2 divided by 7 has a quotient of 0 and a remainder of 2.

Notice also that the possible values of n follow a pattern: the successive value of n is 7 more than the previous value. You can keep adding 7 to generate more possible values for n.

On the GMAT, it will be fairly straightforward to calculate possible values of n. Focus on the *important* or relevant values. Try the following example:

> When positive integer x is divided by 5, the remainder is 2. When positive integer y is divided by 4, the remainder is 1. Which of the following values CANNOT be the sum of x and y?

(A) 12

(B) 13

(C) 14

(D) 16

(E) 21

To answer this question efficiently, you will need to list out possible values of x and y. Notice that the answer choices are not very large. Listing out a few of the smallest possibilities for x and y should be sufficient:

$x = 2, 7, 12, 17$

$y = 1, 5, 9, 13$

The answer choices represent potential values for $x + y$. Which answers can you create from your list of possible values for x and y?

(A) $12 = 7 + 5$

(B) $13 = 12 + 1$

(C) $14 = ???$

(D) $16 = 7 + 9$

(E) $21 = 12 + 9$

The correct answer is **(C)**. There is no way for $x + y$ to equal 14.

Problem Set

1. If $y = 30p$, and p is prime, what is the greatest common factor of y and $14p$, in terms of p?
2. What is the greatest common factor of 420 and 660?
3. What is the least common multiple of 18 and 24?
4. Is p divisible by 168?

   (1) p is divisible by 14.

   (2) p is divisible by 12.

5. Is pq divisible by 168?

   (1) p is divisible by 14.

   (2) q is divisible by 12.

6. What is the greatest common factor of x and y?

   (1) x and y are both divisible by 4.

   (2) $x - y = 4$

7. What is the value of integer x?

   (1) The least common multiple of x and 45 is 225.

   (2) The least common multiple of x and 20 is 300.

8. If x^2 is divisible by 216, what is the smallest possible value for positive integer x?

9. If x and y are positive integers and $x \div y$ has a remainder of 5, what is the smallest possible value of xy?

10. All of the following have the same set of unique prime factors EXCEPT:

(A) 420
(B) 490
(C) 560
(D) 700
(E) 980

11. Which of the following numbers is NOT prime? (Hint: avoid actually computing these numbers.)

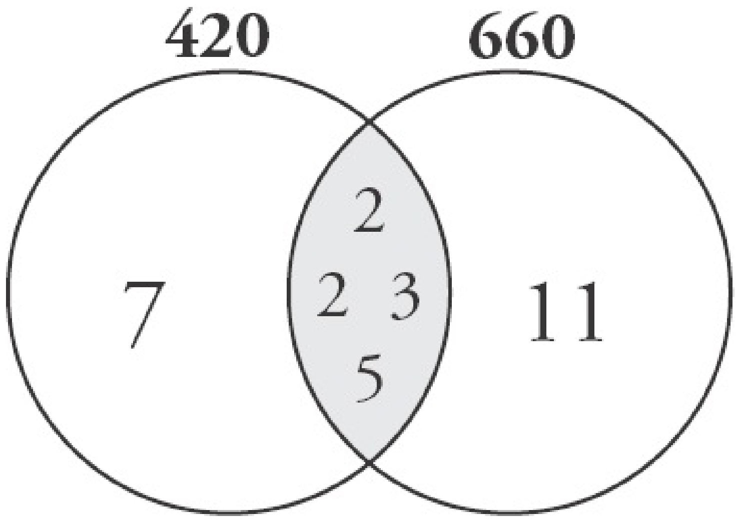
(A) $6! - 1$
(B) $6! + 21$
(C) $6! + 41$
(D) $7! - 1$
(E) $7! + 11$

Solutions

1. **2*p*:** The greatest common factor of y (= 30p) and 14p is the product of all the common prime factors, using the lower power of repeated factors. The only repeated factors are 2 and p: $2^1 \times p^1 = 2 \times p = 2p$. Again, you would get the same answer if p were any positive integer.

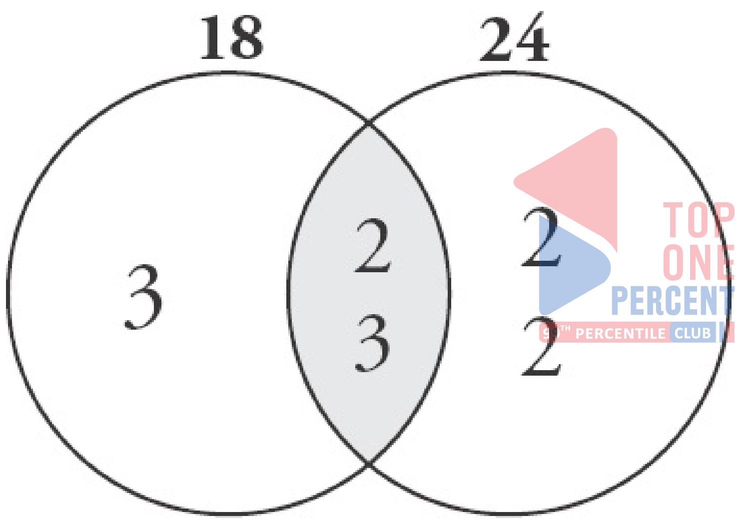
Number	2		3		5		7		p
$30p$	2^1	×	3^1	×	5^1		–	×	p^1
$14p$	2^1		–		–	×	7^1	×	p^1
GCF	$\mathbf{2^1}$								$\mathbf{p^1}$

2. **60:**



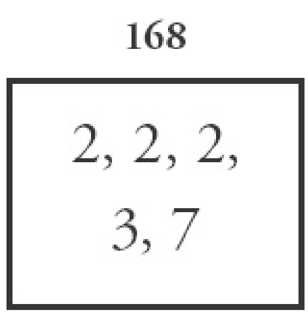
420 = 2 × 2 × 3 × 5 × 7
660 = 2 × 2 × 3 × 5 × 11
The greatest common factor is the product of the primes in the shared factors *only*: 2 × 2 × 3 × 5 = 60.

3. **72:**



18 = 2 × 3 × 3
24 = 2 × 2 × 2 × 3
The least common multiple is the product of all the primes in the diagram: 3 × 2 × 3 × 2 × 2 = 72.

4. **(E):** The first step in this kind of problem is to determine what prime factors *p* needs in order to be divisible by 168. The prime factorization of 168 is 2 × 2 × 2 × 3 × 7, so the question can be restated as follows:



Are there at least three 2's, one 3, and one 7 in the prime box of *p*?

(1) INSUFFICIENT: Statement (1) tells you that *p* is divisible by 14, which is 2 × 7. Therefore, you know that *p* has at least a 2 and a 7 in its prime box. However, you do not know anything else about the possible prime factors in *p*, so you cannot determine whether *p* is divisible by 168. For example, *p* could equal 2 × 2 × 2 × 3 × 7 = 168, in which case the answer to the question would be, "Yes, *p* is divisible by 168." Alternatively, *p* could equal 2 × 7 = 14, in which case the answer to the question would be, "No, *p* is *not* divisible by 168."

Statement (1): ***p***

2, 7
. . . ?

(2) INSUFFICIENT: Statement (2) tells you that *p* is divisible by 12, which is 2 × 2 × 3. Therefore, you know that *p* has at least two 2's and one 3 in its prime box. However, you do not know anything else about *p*, so you cannot determine whether *p* is divisible by 168. For example, *p* could equal 168, in which case the answer to the question would be, "Yes, *p* is divisible by 168." Alternatively, *p* could equal 12, in which case the answer to the question would be "No, *p* is *not* divisible by 168."

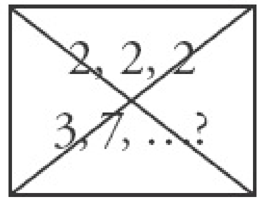
Statement (2): ***p***

2, 2, 3
. . . ?

(1) AND (2) INSUFFICIENT: Combining the primes from statements (1) and (2), you seem to have three 2's, one 3, and one 7. That should be sufficient to prove that *p* is divisible by 168.

However, you cannot do this. Consider the number 84: 84 is divisible by 14 and it is also divisible by 12. Therefore, following from statements (1) and (2), *p* could be 84. However, 84 is not divisible by 168: $84 = 2 \times 2 \times 3 \times 7$, so you are missing a needed 2.

INCORRECT:
Statement's (1) & (2): *p*



Both statements mention that *p* contains at least one 2 in its prime factorization. It is possible that these statements are referring to the *same* 2. Therefore, one of the 2's in statement (2) *overlaps* with the 2 from statement (1). You have been given *redundant* information. The two boxes you made for statements (1) and (2) are not truly different boxes. Rather, they are two different views of the same box (the prime box of *p*).

Thus, you have to eliminate the redundant 2 when you combine the two views of *p*'s prime box from statements (1) and (2). Given both statements, you only know that *p* has two 2's, one 3, and one 7 in its prime box. The correct answer is **(E)**.

5. **(C):** How is this problem different from problem 4? A new variable, *q*, has been introduced, and you're now told that *q* is divisible by 12 (rather than *p*). Because of this change, the information in the two statements is no longer redundant. There is no overlap between the prime boxes, because the prime boxes belong to different variables (*p* and *q*). Statement (1) tells you that *p* has at least one 2 and one 7 in its prime box. Statement (2) tells you that *q* has at least two 2's and one 3 in its prime box. As with question 4, the two statements individually are not sufficient to answer the question, so you can eliminate answers (A), (B), and (D). When you combine the two statements, you combine the prime boxes without removing any overlap, because there is no such overlap. As a result, you know that the product *pq* contains *at least* three 2's, one 3, and one 7 in its combined prime box. You can now answer the question "Is *pq* divisible by 168?" with a definitive "Yes," since the question is really asking whether *pq* contains *at least* three 2's, one 3, and one 7 in its prime box.

The correct answer to this problem is **(C)**.

6. **(C):** (1) INSUFFICIENT: Statement (1) tells you that *x* and *y* are both

divisible by 4, but that does not tell you the GCF of x and y. For example, if $x = 16$ and $y = 20$, then the GCF is 4. However, if $x = 16$ and $y = 32$, then the GCF is 16.

(2) INSUFFICIENT: Statement (2) tells you that $x - y = 4$, but that does not tell you the GCF of x and y. For example, if $x = 1$ and $y = 5$, then the GCF is 1. However, if $x = 16$ and $y = 20$, then the GCF is 4.

(1) AND (2) SUFFICIENT: Combined, statements (1) and (2) tell you that x and y are multiples of 4 and that they are 4 apart on the number line. Therefore, ***x* and *y* are consecutive multiples of 4**. Since x and y are consecutive multiples of 4, their GCF must equal 4 (test some consecutive multiples of 4 to see that this is true). The correct answer is **(C)**.

7. **(C):** Try to determine the value of x using the LCM of x and certain other integers.

(1) INSUFFICIENT: Statement (1) tells you that x and 45 ($3 \times 3 \times 5$) have an LCM of 225 ($= 3 \times 3 \times 5 \times 5 = 3^2 \times 5^2$).

Notice on the chart to the right that the LCM of x and 45 contains two 3's. Because 45 contains two 3's, x can contain zero, one, or two 3's. The LCM of x and 45 contains two 5's. Because 45 contains only ONE 5, x must contain **exactly** two 5's. (If x contained more 5's, the LCM would contain more 5's. If x contained fewer 5's, the LCM would contain fewer 5's.)

Number	3		5
x	?	$\times$	?
45	3^2	$\times$	5^1
LCM	3^2	$\times$	5^2

Therefore, x can be any of the following numbers:

$x = 5 \times 5 = 25$

$x = 3 \times 5 \times 5 = 75$

$x = 3 \times 3 \times 5 \times 5 = 225$

(2) INSUFFICIENT: Statement (2) tells you that x and 20 ($2 \times 2 \times 5$) have an LCM of 300 ($= 2 \times 2 \times 3 \times 5 \times 5 = 2^2 \times 3^1 \times 5^2$).

The LCM of x and 20 contains two 2's. Because 20 contains two 2's, x can contain zero, one, or two 2's. The LCM of x and 20 contains one 3. Because 20 contains no 3's, x must contain **exactly** one 3.

Number	2		3		5
x	?	×	?	×	?
20	2^2		–	×	5^1
LCM	2^2	×	3^1	×	5^2

Furthermore, the LCM of x and 20 contains two 5's. Because 20 contains one 5, x must contain **exactly** two 5's.

Therefore, x can be any of the following numbers:

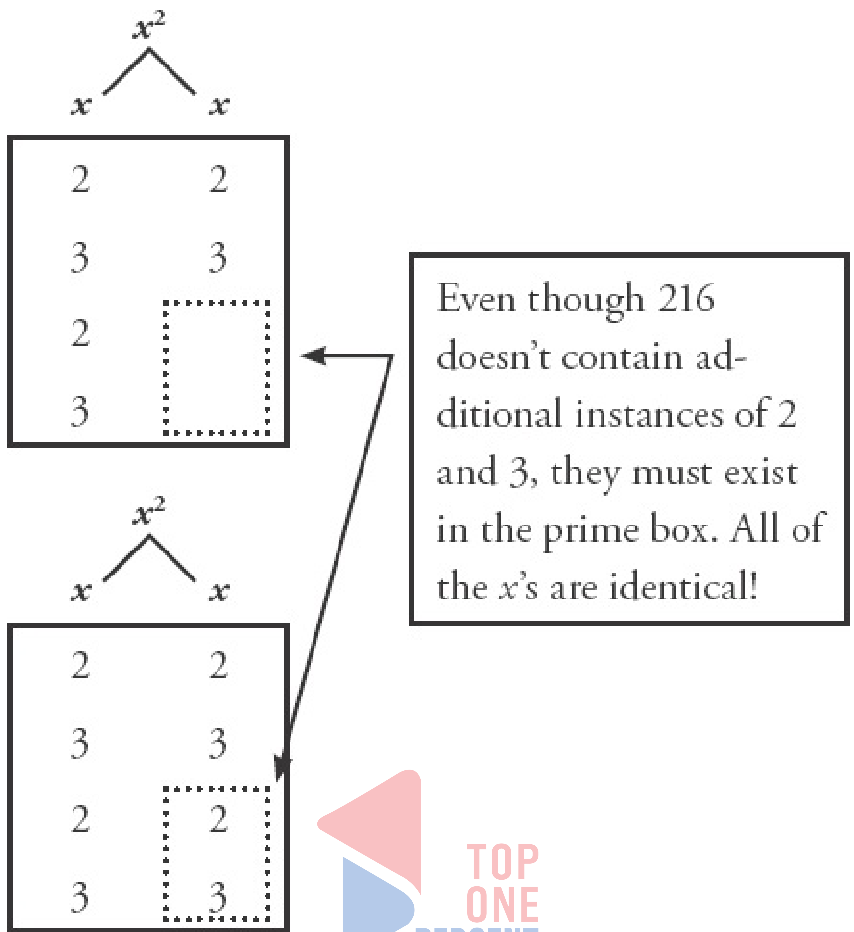
$x = 3 \times 5 \times 5 = 75.$

$x = 2 \times 3 \times 5 \times 5 = 150.$

$x = 2 \times 2 \times 3 \times 5 \times 5 = 300.$

(1) AND (2) SUFFICIENT: Statement (1) tells you that x could be 25, 75, or 225. Statement (2) tells you that x could be 75, 150, or 300. The only number that satisfies both of these conditions is $x = 75$. Therefore, you know that x must be 75. The correct answer is **(C)**.

8. **36:** The prime box of x^2 contains the prime factors of 216, which are 2, 2, 2, 3, 3, and 3. You know that the prime factors of x^2 should be the prime factors of x appearing in sets of two, or pairs. Therefore, you should distribute the prime factors of x^2 into two columns to represent the prime factors of x, as shown to the right.



There is a complete pair of two 2's in the prime box of x^2, so x must have a factor of 2. However, there is a third 2 left over. That additional factor of 2 must be from x as well, so assign it to one of the component x columns. Also, there is a complete pair of two 3's in the prime box of x^2, so x must have a factor of 3. However, there is a third 3 left over. That additional factor of 3 must be from x as well, so assign it to one of the component x columns. Thus, x has 2, 3, 2, and 3 in its prime box, so x must be a positive multiple of 36.

9. **30:** The remainder must always be smaller than the divisor. In this problem, 5 must be smaller than y. Additionally, y must be an integer, so y must be at least 6. If y is 6, then the smallest possible value of x is 5. (Other values of x that leave a remainder of 5 when divided by 6 would be 11, 17, 23, etc.) If y is chosen to be larger than 6, then the smallest possible value of x is still 5. Thus, you will get the smallest possible value of the product xy by choosing the smallest x together with the smallest y. The smallest possible value of xy is $5 \times 6 = 30$.

10. **(A):** To solve this problem, take the prime factorization of each answer

choice and note the unique prime factors. One of the answer choices will have a different set of unique prime factors than the other answer choices:

(A) 420 = 42 × 10 = 21 × 2 × 2 × 5 = 3 × 7 × 2 × 2 × 5. (Unique primes: 2, 3, 5, and 7)

(B) 490 = 49 × 10 = 7 × 7 × 2 × 5 (Unique primes: 2, 5, and 7)

(C) 560 = 56 × 10 = 7 × 8 × 2 × 5 = 7 × 2 × 2 × 2 × 2 × 5. (Unique primes: 2, 5, and 7)

(D) 700 = 70 × 10 = 7 × 2 × 5 × 2 × 5 (Unique primes: 2, 5, and 7)

(E) 980 = 98 × 10 = 49 × 2 × 2 × 5 = 7 × 7 × 2 × 2 × 5. (Unique primes: 2, 5, and 7)

The correct answer is **(A)**, because it is the only answer choice with a prime factor of 3.

11. **(B):** You could solve this problem by computing each answer choice and testing each one to see whether it is divisible by any smaller integer. However, some of the numbers in the answer choices will be very large (e.g., 7! is equal to 5,040), so testing to see whether these numbers are prime will be extremely time consuming.

A different approach can be taken: try to find an answer choice that *cannot* be prime based on the properties of divisibility. Earlier in this chapter, you learned the following property of factorials and divisibility: *N*! is a multiple of all integers from 1 to *N*. In Chapter 1, you also learned that if two numbers share a factor, their sum or difference also shares the same factor. You can apply these concepts directly to the answer choices:

(A)6! − 1: 6! is not prime, but 6! − 1 might be prime, because 6! and 1 do not share any prime factors.

(B)6! + 21: 6! is not prime, and 6! + 21 CANNOT be prime, because 6! and 21 are both multiples of 3. Therefore, 6! + 21 is divisible by 3.

(C)6! + 41: 6! is not prime, but 6! + 41 might be prime, because 6! and 41 do not share any prime factors.

(D)7! − 1: 7! is not prime, but 7! − 1 might be prime, because 7! and 1 do not share any prime factors.

(E)7! + 11: 7! is not prime, but 7! + 11 might be prime, because 7! and 11 do not share any prime factors.

By the way, because answer (B) cannot be prime, you can infer that all the other answer choices *must* be prime, without having to actually check them. There *cannot* be more than one correct answer choice.

Chapter 7
of
Number Properties

Extra Combinatorics & Probability

In This Chapter…

Chapter 7

Extra Combinatorics & Probability

Disguised Combinatorics

Some combinatorics problems are written disguise, with problem statements that seem to bear little resemblance to the typical examples shown earlier in the general combinatorics chapter. If you are not aiming for a 90th percentile or higher score on the GMAT, consider skipping this section.

Many word problems involving the words “how many” are combinatorics problems. Also, many combinatorics problems masquerade as probability problems. The difficult part of the problem draws on combinatorics to count desired or total possibilities, whereas creating the probability fraction is trivial. If you think creatively enough, looking for *analogies* to known problem types, you may be able to find a viable combinatorics solution.

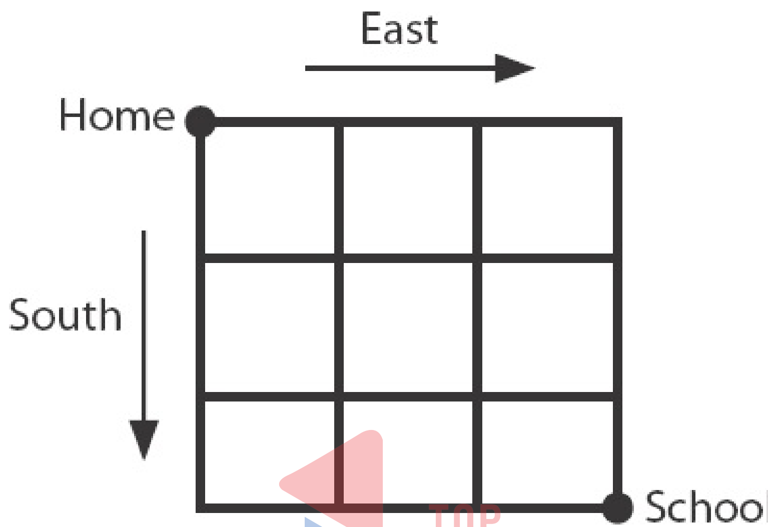
Here are some examples of combinatorics problems that at first may appear to have little to do with combinatorics:

- How many four-digit integers have digits with some specified properties?
- How many paths exist from point A to point B in a given diagram?
- How many diagonals, triangles, lines, etc., exist in a given geometrical figure?
- How many *pairings* (handshakes, games between two teams, nonstop flights between cities, etc.) exist in a given situation? (Pairings are groups of two.)

Try this example:

> Alicia lives in a town whose streets are on a grid system, with all streets running east–west or north–south without breaks. Her school, located on a corner, lies three blocks south and three blocks east of her home, also located on a corner. If Alicia only walks south or east on her way to school, how many possible routes can she take to school?

First, draw a diagram of the situation:



It may be tempting to draw all the different routes, but this method will be time consuming. Fortunately, you can answer this question using combinatorics.

Alicia will have to walk south 3 times and east 3 times on her way to school. That's 6 blocks in total that she'll walk. She'll have to make 6 decisions total: to walk south or east at each corner. For instance, she could walk south–east–east–south–south–east to get to her goal.

You're arranging 6 decisions. But you also have repeats: 3 blocks south and 3 blocks east. Divide 6! by 3! and 3!:

$$\frac{6!}{3!3!} = \frac{\cancel{6} \times 5 \times 4 \times \cancel{3!}}{\cancel{3 \times 2} \times 1 \times \cancel{3!}} = 5 \times 4 = 20$$

There are 20 routes to school.

You could also have made an anagram grid of the situation:

1	2	3	4	5	6
S	S	S	E	E	E

How many anagrams can you make of the “word” SSSEEE? You can make 6! divided by the product of 3! and 3!.

Arrangements with Constraints

The most complex combinatorics problems include unusual constraints: one person refuses to sit next to another, for example. Try the following:

> Greg, Marcia, Peter, Jan, Bobby, and Cindy go to a movie and sit next to each other in six adjacent seats in the front row of the theater. If Marcia and Jan will not sit next to each other, in how many different arrangements can the six people sit?

This is a simple arrangement with one unusual constraint: Marcia and Jan will not sit next to each other. To solve the problem, ignore the constraint for now. Just find the number of ways in which 6 people can sit in 6 chairs:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

Because of the constraint on Jan and Marcia, though, not all of those 720 seating arrangements are viable. Count the arrangements in which Jan *is* sitting next to Marcia (the *undesirable* seating arrangements), and subtract them from the total of 720.

To count the ways in which Jan *must* sit next to Marcia, use the **glue method**:

> For problems in which items or people must be next to each other, pretend that the items are glued together into one larger item.

Imagine that Jan and Marcia are stuck together into one person. There are now effectively 5 people: JM (stuck together), G, P, B, and C. These 5 people can be arranged in 5! = 120 different ways.

Each of those 120 different ways, though, represents *two* different possibilities, because the “stuck together” moviegoers could be in order either as J–M or as

M–J. Therefore, the total number of seating arrangements with Jan next to Marcia is 2 × 120 = 240.

Finally, do not forget that those 240 possibilities are the ones to be *excluded* from consideration. The number of allowed seating arrangements is therefore 720 − 240, or 480.

The Domino Effect

Sometimes the outcome of the first event will affect the probability of a subsequent event. For example:

> A box contains 10 blocks, 3 of which are red. If you pick two blocks out of the box, what is the probability that they are both red? Assume that you do NOT replace the first block after you have picked it.

You're asked to find the probability of a red AND a red, so calculate the probability of each event and multiply the numbers together.

The probability of selecting a red block on the first try is $\frac{3}{10}$. But, for the second try, you now have only 9 blocks to choose from, and only 2 red blocks, so the probability of selecting a red block on the second try is $\frac{2}{9}$. Now multiply.

$$\frac{3}{10}\times\frac{2}{9}=\frac{\cancel{3}^{1}}{\cancel{10}^{5}}\times\frac{\cancel{2}^{1}}{\cancel{9}^{3}}=\frac{1}{15}$$

Thus, the probability of picking two reds is $\frac{1}{15}$.

Do not forget to consider whether one event affects subsequent events. The first roll of a die or flip of a coin has no effect on any subsequent rolls or flips. However, the first pick of an object out of a box does affect subsequent picks if you do not replace that object. Check whether the object is or is not placed back into the container before the second and subsequent picks.

Combinatorics and the Domino Effect

The **domino-effect rule** states that you multiply the probabilities of events in a sequence, taking earlier events into account. Some domino-effect problems are difficult because of the sheer number of possibilities involved. When all possibilities are equivalent, though, combinatorics can save the day. Consider the following:

> A miniature gumball machine contains 7 blue, 5 green, and 4 red gumballs, which are identical except for their colors. If the machine dispenses three gumballs at random, what is the probability that it dispenses one gumball of each color?

Consider one specific case: blue first, then green, then red. By the domino-effect rule, the probability of this case is
$$\frac{7\,\text{blue}}{16\,\text{total}} \times \frac{5\,\text{green}}{15\,\text{total}} \times \frac{4\,\text{red}}{14\,\text{total}} = \frac{\cancel{7}}{\cancel{16}\ 4} \times \frac{\cancel{5}}{\cancel{15}\ 3} \times \frac{\cancel{4}}{\cancel{14}\ 2} = \frac{1}{24}.$$

Now consider another case: green first, then red, then blue. The probability of this case is $\frac{5\,\text{green}}{16\,\text{total}} \times \frac{4\,\text{red}}{15\,\text{total}} \times \frac{7\,\text{blue}}{14\,\text{total}} = \frac{\cancel{5}}{\cancel{16}\ 4} \times \frac{\cancel{4}}{\cancel{15}\ 3} \times \frac{\cancel{7}}{\cancel{14}\ 2} = \frac{1}{24}$. Notice that the probability is the same! This is no accident; the order in which the balls come out does not matter. The three numerators will always be 7, 5, and 4, and the three denominators will always be 16, 15, and 14.

Because the three desired gumballs can come out in any order, there are 3!, or 6, different cases. *All of these cases have the same probability*. Therefore, the overall probability is $6 \times \frac{1}{24} = \frac{1}{4}$.

In general, when you have a symmetrical problem (a problem with multiple equivalent cases), calculate the probability of one case (often by using the domino-effect rule). Then multiply by the number of cases. Use combinatorics to calculate the number of cases, if necessary.

When you apply a symmetry argument, the situation must truly be symmetrical. In the case above, if you swapped the order of “red” and “green” emerging from the gumball machine, nothing would change about the problem. As a result, you can use symmetry to simplify the computation.

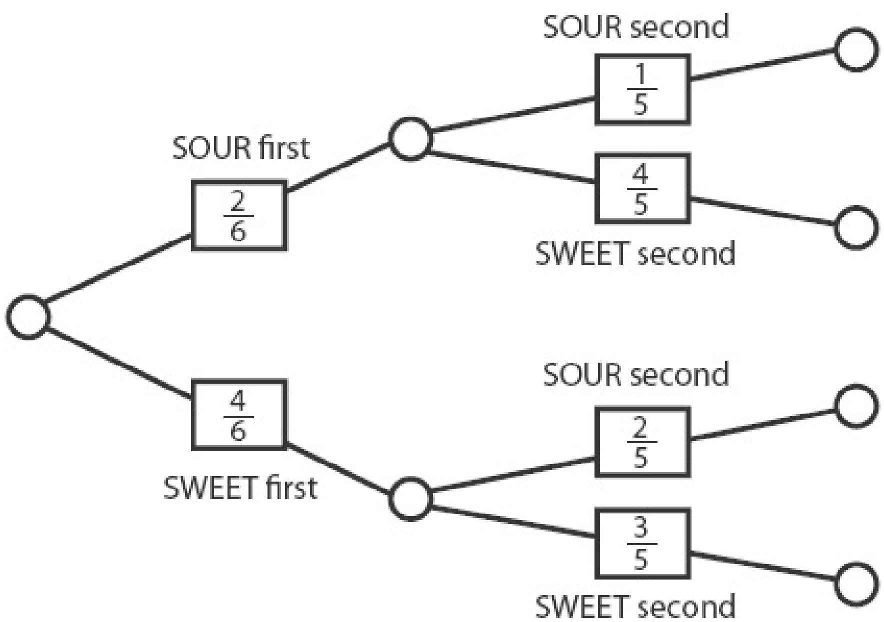
Probability Trees

Trees can be a useful tool to keep track of branching possibilities and winning scenarios. Consider the following problem:

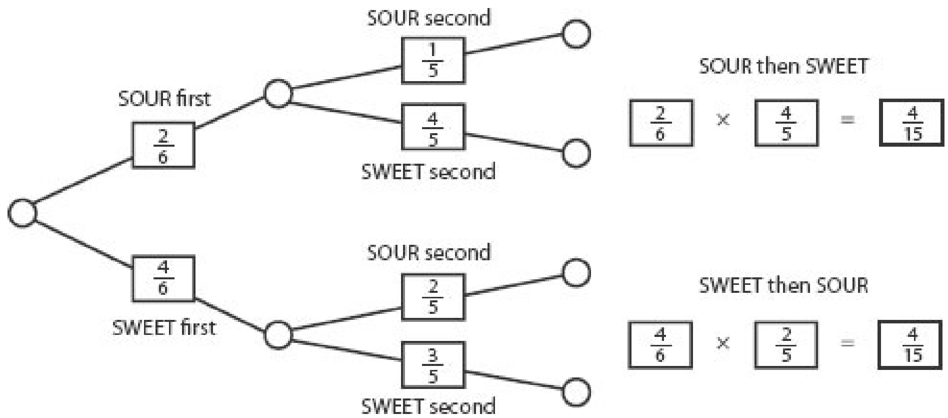
> Renee has a bag of 6 candies, 4 of which are sweet and 2 of which are sour. Jack picks two candies simultaneously and at random. What is the chance that exactly one of the candies he has picked is sour?

Even though Jack picks the two candies simultaneously, you can pretend that he picks them in a sequence. This trick allows you to set up a tree reflecting Jack's picks at each stage.

The tree is shown below. Label each branch and put in probabilities. Jack has a $\frac{2}{6}$ chance of picking a sour candy first and a $\frac{4}{6}$ chance of picking a sweet candy first. Note that these probabilities add to 1. On the second set of branches, put the probabilities *as if* Jack has already made his first pick. Remember the domino effect! When making the second pick, don't forget to remove one of the type of candy chosen on the first pick.



Now compute the probabilities of the desired scenarios. One scenario is *sour* *first* AND *sweet second*; the other is *sweet first* AND *sour second*. Since each scenario is one event AND another event occurring together, multiply the basic probabilities. In other words, you multiply the branches:



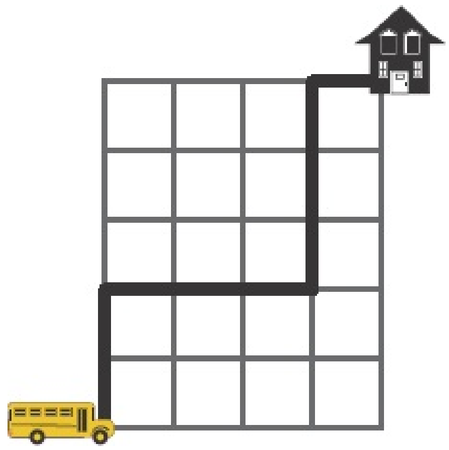
Finally, EITHER one scenario OR the other scenario works: in either case, Jack picks exactly one sour candy. Add these probabilities: $\frac{4}{15}+\frac{4}{15}=\frac{8}{15}$.

Problem Set

1. Three gnomes and three elves sit down in a row of six chairs. If no gnome will sit next to another gnome and no elf will sit next to another elf, in how many different ways can the elves and gnomes sit?

2. Gordon buys 5 dolls for his 5 nieces. The gifts include 2 identical Sun-and-Fun beach dolls, 1 Elegant Eddie dress-up doll, 1 G.I. Josie army doll, and 1 Tulip Troll doll. If the youngest niece does not want the G.I. Josie doll, in how many different ways can he give the gifts?

3. Every morning, Grishma walks from her house to the bus stop; the placement of the house and bus stop are shown in the diagram to the right. She always travels exactly nine blocks from her house to the bus, but she varies the route she takes every day. (One sample route is shown.) How many days can Casey walk from her house to the bus stop without repeating the same route?



4. In a bag of marbles, there are 3 red, 2 white, and 5 blue marbles. If Kia takes 2 marbles out of the bag, what is the probability that he will have 1 white and 1 blue marble? (Assume that Kia does not replace the marbles in the bag.)

5. A florist has 2 azaleas, 3 buttercups, and 4 petunias. She puts two flowers together at random in a bouquet. However, the customer calls and says that she does not want two of the same flower. What is the probability that the florist does not have to change the bouquet?

6. Five A-list actresses are vying for the three leading roles in the new film “Catfight” in Denmark. The actresses are Julia Robards, Meryl Strep, Sally Fieldstone, Nicole Kadman, and Hallie Strawberry. Assuming that no actress has any advantage in getting any role, what is the probability that Julia and Hallie will star in the film together?

7. For one roll of a certain number cube with six faces, numbered 1 through 6, the probability of rolling a two is $\frac{1}{6}$. If this number cube is rolled 4 times, which of the following is the probability that the outcome will be a two *at least* 3 times?

(A) $\left(\frac{1}{6}\right)^4$

(B) $2\left(\frac{1}{6}\right)^3+\left(\frac{1}{6}\right)^4$

(C) $3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)+\left(\frac{1}{6}\right)^4$

(D) $4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)+\left(\frac{1}{6}\right)^4$

(E) $6\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)+\left(\frac{1}{6}\right)^4$

Solutions

1. **72:** The only way to ensure that no two gnomes and no two elves sit next to each other is to have the gnomes and elves alternate seats (GEGEGE or EGEGEG). Use the slot method to assign seats to gnomes or elves. Begin by seating the first gnome. As he is the first to be seated, he can sit anywhere. He has 6 choices. If the first gnome sits in an odd-numbered chair, the second gnome can sit in either of the two remaining odd-numbered chairs. (Likewise, if the first gnome sits in an even-numbered chair, the second gnome can sit in either of the two remaining even-numbered chairs.) Either way, the second gnome has two choices. The last gnome has only 1 chair option, since she is not to be seated next to another gnome.

Person	Choices	Seat Assigned
Gnome A	6 choices (1, 2, 3, 4, 5, 6)	#1
Gnome B	2 choices (3, 5)	#3
Gnome C	1 choice (5)	#5
Elf A	3 choices (2, 4, 6)	#2
Elf B	2 choices (4, 6)	#4
Elf C	1 choice (6)	#6

Then, seat the elves. The first elf can sit in any of the three empty chairs, the second in any of the other two, and the last in the final remaining chair. Therefore, the first elf has three choices, the second elf has two choices, and the third elf has one choice.

Finally, find the product of the number of choices for each “person”:

$$6 \times 2 \times 1 \times 3 \times 2 \times 1 = 72$$

You can also think of this problem as a succession of three choices: 1) choosing whether to arrange the little guys as GEGEGE or EGEGEG, 2) choosing the order of the gnomes, and then 3) choosing the order of the elves.

The first choice has only two options: EGEGEG and GEGEGE. Each of the subsequent choices has 3!, or 6 options, because those choices involve unrestricted rearrangements (simple factorials). Therefore, the total number of seating arrangements is: $2 \times 3! \times 3! = 2 \times 6 \times 6 = 72$.

2. **48:** First, solve the problem without considering the fact that the youngest girl does not want the G.I. Josie doll.

Gordon's nieces could get either one of the Sun-and-Fun dolls, which we'll call S, or they could get the Elegant Eddie doll (E), the Tulip Troll doll (T), or the G.I. Josie doll (G). This problem can be modeled with anagrams for the “word” SSETG:

$$\frac{5!}{2!} = 5 \times 4 \times 3 = 60$$

A	B	C	D	E
S	S	E	T	G

Note that you should divide by 2! because of the two identical Sun-and-Fun dolls.

Thus, there are 60 ways in which Gordon can give the gifts to his nieces.

However, you know that the youngest girl (niece E) does not want the G.I. Josie doll. So, calculate the number of arrangements in which the youngest girl *does* get the G.I. Josie doll. If niece E gets doll G, then you still have 2 S dolls, 1 E doll, and 1 T doll to give out to nieces A, B, C, and D. Model this situation with the anagrams of the “word” SSET:

$$\frac{4!}{2!} = 12$$

A	B	C	D
S	S	E	T

There are 12 ways in which the youngest niece *will* get the G.I. Josie doll.

Therefore, there are $60 - 12$, or 48, ways in which Gordon can give the dolls to his nieces.

3. **126 days:** No matter which route Grishma walks, she will travel 4 blocks to the left and 5 blocks down. This can be modeled with the “word” LLLLDDDDD. Find the number of anagrams for this “word”:

$$\frac{9!}{5!4!} = \frac{9\times8\times7\times6}{4\times3\times2\times1} = 126$$

This problem can also be solved with the combinations formula. Grishma is going to walk 9 blocks in a row, no matter what. Imagine that those blocks are already marked 1, 2, 3, and 4 (the first block she walks, the second block she walks, and so on), up to 9. Now, to create a route, four of those blocks will be dubbed “Left” and the other five will be “Down.” The question is, in how many ways can she assign those labels to the numbered blocks?

The answer is given by the fact that she is choosing a combination of either 4 blocks out of 9 (“Left”) or 5 blocks out of 9 (“Down”). (Either method gives the same answer.) At first it may seem as though “order matters” here, because Grishma is choosing routes, but “order” does not matter in the combinatorial sense. That is, designating blocks 1, 2, 3, and 4 as “Left” blocks is the same as designating blocks 3, 2, 4, and 1 as “Left” blocks (or any other order of those same four blocks). Therefore, use combinations, not permutations, to derive the expression: $\frac{9!}{5!\times4!} = 126$.

4. $\mathbf{\frac{2}{9}}$**:** You can solve this problem by listing the winning scenarios or by using combinatorics counting methods. Both solutions are presented below:

1. List the winning scenarios

First Pick	Second Pick	Probability
(1) Blue $\left(\frac{1}{2}\right)$	White $\left(\frac{2}{9}\right)$	$\frac{1}{2}\times\frac{2}{9}=\frac{1}{9}$
(2) White $\left(\frac{1}{5}\right)$	Blue $\left(\frac{5}{9}\right)$	$\frac{1}{5}\times\frac{5}{9}=\frac{1}{9}$

To find the probability, add the probabilities of the winning scenarios: $\frac{1}{9}+\frac{1}{9}=\frac{2}{9}$.

2. Use the counting method

A	B	C	D	E	F	G	H	I	J
Y	Y	N	N	N	N	N	N	N	N

There are $\frac{10!}{2!8!} = 45$ different combinations of marbles.

Since there are 2 white marbles and 5 blue marbles, there are 2 × 5 = 10 different white–blue combinations. Therefore, the probability of selecting a blue and white combination is $\frac{10}{45}$, or $\frac{2}{9}$.

5. $\mathbf{\frac{13}{18}}$**:** Solve this problem by finding the probability that the two flowers in the bouquet *will* be the same, and then subtract the result from 1. The table to the right indicates that there are 10 different bouquets in which both flowers are the same. Then, find the number of different 2-flower bouquets that can be made in total, using an anagram model. In how many different ways can you arrange the letters in the "word" YYNNNNNNN?

$$\frac{9!}{7!2!} = \frac{9 \times 8}{2 \times 1} = 36$$

The probability of randomly putting together a bouquet that contains two of the same type of flower is $\frac{10}{36}$, or $\frac{5}{18}$. Therefore, the probability of randomly putting together a bouquet that contains two different flowers and that therefore will *not* need to be changed is $1 - \frac{5}{18}$, or $\frac{13}{18}$.

Flower #1	**Flower #2**
A_1	A_2
B_1	B_2
B_1	B_3
B_2	B_3
P_1	P_2

P_1	P_3
P_1	P_4
P_2	P_3
P_2	P_4
P_3	P_4

6. $\mathbf{\frac{3}{10}}$**:** The probability of Julia being cast first is $\frac{1}{5}$. If Julia is cast, the probability of Hallie being cast second is $\frac{1}{4}$. The probability of any of the remaining 3 actresses being cast is $\frac{3}{3}$, or 1. Therefore, the probability of this chain of events is:

$$\frac{1}{5} \times \frac{1}{4} \times 1 = \frac{1}{20}$$

There are six event chains that yield this outcome, shown in the chart to the right. Therefore, the total probability that Julia and Hallie will be among the 3 leading actresses is:

$$\frac{1}{20} \times 6 = \frac{6}{20} = \frac{3}{10}$$

Actress (1)	**Actress (2)**	**Actress (3)**
Julia	Hallie	X
Julia	X	Hallie
Hallie	Julia	X
Hallie	X	Julia
X	Julia	Hallie
X	Hallie	Julia

Alternatively, you can solve this problem with counting methods.

The number of different combinations in which the actresses can be cast in the roles, assuming you are not concerned with which actress is given which role, is $\frac{5!}{3!2!} = 5 \times 2 = 10.$

A	B	C	D	E
Y	Y	N	N	N

There are 3 possible combinations that feature both Julia and Hallie:

(1) Julia, Hallie, Sally
(2) Julia, Hallie, Meryl
(3) Julia, Hallie, Nicole

Therefore, the probability that Julia and Hallie will star together is $\frac{3}{10}$.

7. **(D):** $4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)+\left(\frac{1}{6}\right)^4$

Unfortunately, you cannot easily use the $1 - x$ trick here, so you must express the probability directly. You must regard the desired outcome in two separate parts: first, rolling a two *exactly* 4 times, and second, rolling a two *exactly* 3 times out of 4 attempts. First, the probability of rolling a two *exactly* 4 times is $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)=\left(\frac{1}{6}\right)^4$.

Next, if you roll a two exactly 3 times out of 4 attempts, then on exactly one of those attempts, you do *not* roll a two. Hence, the probability of rolling a two exactly 3 times out of 4 attempts is the sum of the following four probabilities:

Outcome		Probability
(Two)(Two)(Two)(Not a Two)	→	$\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)=\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)$
(Two)(Two)(Not a Two)(Two)	→	$\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)=\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)$
(Two)(Not a Two)(Two)(Two)	→	$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)=\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)$
(Not a Two)(Two)(Two)(Two)	→	$\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)=\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)$
		$4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)$

Notice that there are 4 rearrangements of 3 “Twos” and 1 “Not a two.” In other words, you have to count as separate outcomes the 4 different positions in which the “Not a two” roll occurs: first, second, third, or fourth.

There is no way to roll a two exactly 4 times AND exactly 3 times, so you can now just add up these probabilities. Thus, the desired probability is

$$4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)+\left(\frac{1}{6}\right)^4.$$