Assignment 5 - Asymptotics

CS 234

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1 Asymptotic Proofs

Prove the following by giving constants and then showing that the inequalities hold:

D.4 $2n^2 + 4n = O\left(n^2\right)$

Proof. Let n be an arbitrary real number that is at least 4.
Then the following inequality holds:

$$\begin{aligned} 2n^2 + 4n &\leq 2n^2 + n^2 \quad [n \geq 4] \\ &= 3n^2. \end{aligned}$$

Thus $2n^2 + 4n \leq 3n^2$, so picking $n_0 = 4$ and $c = 3$ witnesses
$\exists n_0, c > 0. \quad \forall n \geq n_0. \quad 2n^2 + 4n \leq c \cdot n^2$.
Thus, by the definition of big-O, $2n^2 + 4n \in O\left(n^2\right)$.

$\square$

D.5 $3n^2 - 4n + 5 = O\left(n^2\right)$

Proof. Let n be an arbitrary real number that is at least 1.
Then the following inequality holds:

$$\begin{aligned} 3n^2 - 4n + 5 &\leq 3n^2 + n^2 + 5n^2 \quad \left[\text{ as } -4n \leq n^2 \text{ and } 5 \leq 5n^2 \text{ for all } n \geq 1\right] \\ &= 9n^2. \end{aligned}$$

Thus $3n^2 - 4n + 5 \leq 9n^2$, so picking $n_0 = 1$ and $c = 9$ witnesses
$\exists n_0, c > 0. \quad \forall n \geq n_0. \quad 3n^2 - 4n + 5 \leq c \cdot n^2$.
Thus, by the definition of big-O, $3n^2 - 4n + 5 \in O\left(n^2\right)$.

$\square$

D.9 $4n^2 - 3n = \Omega\left(n^2\right)$

Proof. Let n be an arbitrary real number that is at least 1 and let c be 1. We want to show that there exists $n_0, c > 0$ such that for all $n \geq n_0$ we have that $4n^2 - 3n \geq c \cdot n^2$.
Observe the following inequality:

$$4n^2 - 3n = n(4n-3) \geq c \cdot n \cdot n = n^2 \quad [c = 1].$$

Since we initially assumed that n is an arbitrary real number that is at least 1, we know by arithmetic that

$$4n - 3 \geq n \quad [\text{ math }].$$

Therefore,

$$3(n-1) \geq 0 \quad [\text{ math }].$$

Thus picking $n_0 = 1$ and $c = 1$ witnesses
$\exists n_0, c > 0. \quad \forall n \geq n_0. \quad 4n^2 - 3n \geq c \cdot n^2.$
Thus, by the definition of big-Ω, $4n^2 - 3n \in \Omega\left(n^2\right)$.

□

D.10 $n^2 - 2n + 3 = \Omega\left(n^2\right)$

Proof. Let n be an arbitrary real number that is at least 1 and let c be $\frac{1}{2}$. We want to show that there exists $n_0, c > 0$ such that for all $n \geq n_0$ we have that $n^2 - 2n + 3 \geq c \cdot n^2$.
Observe the following inequality:

$$n^2 - 2n + 3 \geq c \cdot n \cdot n = \frac{1}{2} \cdot n^2 \quad \left[c = \frac{1}{2}\right].$$

Therefore, by arithmetics, we know that

$$\frac{1}{2}n^2 - 2n + 3 \geq 0 \quad [\text{ math }].$$

We can rewrite the above mathematical expression as

$$\frac{1}{2}\left(n^2 - 4n\right) + 3 \geq 0 \quad [\text{ math }].$$

Therefore, by arithmetics, we know that

$$\frac{1}{2}(n-2)^2 - 2 + 3 = \frac{1}{2}(n-2)^2 + 1 \geq 0 \quad [\text{ math }].$$

Thus picking $n_0 = 1$ and $c = \frac{1}{2}$ witnesses
$\exists n_0, c > 0. \quad \forall n \geq n_0. \quad n^2 - 2n + 3 \geq c \cdot n^2.$
Thus, by the definition of big-Ω, $n^2 - 2n + 3 \in \Omega\left(n^2\right)$.

□

D.13 $n+8=\Theta(n)$

Proof. To show this, by definition of big-Θ, it suffices to show $n+8 \in O(n)$ and $n+8 \in \Omega(n)$.
Let n be an arbitrary real number that is at least 1.
Then the following inequality holds:

$$\begin{aligned} n+8 &\leq n+8n \quad [n \geq 1] \\ &= 9n. \end{aligned}$$

Thus $n+8 \leq 9n$, so picking $n_0 = 1$ and $c_1 = 9$ witnesses
$\exists n_0, c_1 > 0. \quad \forall n \geq n_0. \quad n+8 \leq c_1 \cdot n.$
Thus, by the definition of big-O, $n+8 \in O(n)$.
Also, let n be an arbitrary real number that is at least 1 and let c_2 be 1.
Observe the following inequality:

$$n+8 \geq n \quad [n \geq 1].$$

Thus picking $n_0 = 1$ and $c_2 = 1$ witnesses
$\exists n_0, c_2 > 0. \quad \forall n \geq n_0. \quad n+8 \geq c_2 \cdot n.$
Thus, by the definition of big-Ω, $n+8 \in \Omega(n)$.
Finally, because both $n+8 \in O(n)$ and $n+8 \in \Omega(n)$, it follows that $n+8 \in \Theta(n)$ by definition.

$\square$

D.14 $n^2+2n=\Theta\left(n^2\right)$

Proof. To show this, by definition of big-Θ, it suffices to show $n^2+2n \in O\left(n^2\right)$ and $n^2+2n \in \Omega\left(n^2\right)$.
Let n be an arbitrary real number that is at least 1.
Then the following inequality holds:

$$\begin{aligned} n^2+2n &\leq n^2+2n^2 \quad [n \geq 1] \\ &= 3n^2. \end{aligned}$$

Thus $n^2+2n \leq 3n^2$, so picking $n_0 = 1$ and $c_1 = 3$ witnesses
$\exists n_0, c_1 > 0. \quad \forall n \geq n_0. \quad n^2+2n \leq c_1 \cdot n^2.$
Thus, by the definition of big-O, $n^2+2n \in O\left(n^2\right)$.
Also, let n be an arbitrary real number that is at least 1 and let c_2 be 1.
Observe the following inequalities:

$$n^2+2n \geq n^2 \quad [n \geq 1].$$

By arithmetic, we know that

$$2n \geq 0 \quad [\text{ math }].$$

Thus picking $n_0 = 1$ and $c_2 = 1$ witnesses
$\exists n_0, c_2 > 0. \quad \forall n \geq n_0. \quad n^2 + 2n \geq c_2 \cdot n^2.$
Thus, by the definition of big-Ω, $n^2 + 2n \in \Omega\left(n^2\right)$.
Finally, because both $n^2 + 2n \in O\left(n^2\right)$ and $n^2 + 2n \in \Omega\left(n^2\right)$, it follows that $n^2 + 2n \in \Theta(n^2)$ by definition.

$\square$

D.16 $n^2 = o\left(n^3\right)$

Proof. Let c be an arbitrary positive real number.
Then let n be an arbitrary real number that is strictly greater than $\frac{1}{c}$.
Observe the following inequalities that they hold.

$$\begin{aligned} c \cdot n^3 &> c \cdot \frac{1}{c} \cdot n^2 && \left[n > \frac{1}{c}\right] \\ &= n^2 && [\text{ math }]. \end{aligned}$$

Thus picking an arbitrary positive real number n_0 to be strictly greater than $\frac{1}{c}$ witnesses that $\forall c > 0. \quad \exists n_0 > 0. \quad \forall n \geq n_0. \quad n^2 < c \cdot n^3$. By the definition of little-o, this means that $n^2 \in o\left(n^3\right)$.

□

D.17 $100n^3 = o\left(n^4\right)$

Proof. Let c be an arbitrary positive real number.
Then let n be an arbitrary real number that is strictly greater than $\frac{100}{c}$.
Observe the following inequalities that they hold.

$$\begin{aligned} c \cdot n^4 &> c \cdot \frac{100}{c} \cdot n^3 && \left[n > \frac{100}{c}\right] \\ &= 100n^3 && [\text{ math }]. \end{aligned}$$

Thus picking an arbitrary positive real number n_0 to be strictly greater than $\frac{100}{c}$ witnesses that $\forall c > 0. \quad \exists n_0 > 0. \quad \forall n \geq n_0. \quad 100n^3 < c \cdot n^4$. By the definition of little-o, this means that $100n^3 \in o\left(n^4\right)$.

□

D.18 $n^5 = \omega\left(n^4\right)$

Proof. Let c be an arbitrary positive real number. Then let n be an arbitrary real number that is strictly greater than c. We want to show that for all $c > 0$, there exists a constant $n_0 > 0$ such that for all $n \geq n_0$, $n^5 > c \cdot n^4$. Observe the following inequalities that they hold.

$$\begin{aligned} n^5 &> c \cdot n^4 && [n > c > 0] \\ \longleftrightarrow n &> c && [\text{ math }]. \end{aligned}$$

We earlier assumed that $n > c$ so we can observe that for all $n \geq n_0$, $n \geq n_0 > c$. Thus picking n_0 to be strictly greater than c witnesses that $\forall c > 0.\ \exists n_0 > 0.\ \forall n \geq n_0.\ n^5 > c \cdot n^4$.
By the definition of little-ω, it follows that $n^5 \in \omega\left(n^4\right)$.

□

D.19 $10n^3 = \omega\left(n^2\right)$

Proof. Let c be an arbitrary positive real number. Then let n be an arbitrary real number that is strictly greater than $\frac{c}{10}$. We want to show that for all $c > 0$, there exists a constant $n_0 > 0$ such that for all $n \geq n_0$, $10n^3 > c \cdot n^2$. Observe the following inequalities that they hold.

$$\begin{aligned} 10n^3 &> c \cdot n^2 \quad \left[n > \frac{c}{10} > 0\right] \\ \longleftrightarrow n &> \frac{c}{10} \quad [\text{ math }]. \end{aligned}$$

We earlier assumed that $n > \frac{c}{10}$ so we can observe that for all $n \geq n_0$, $n \geq n_0 > \frac{c}{10}$. Thus picking n_0 to be strictly greater than $\frac{c}{10}$ witnesses that $\forall c > 0.\ \exists n_0 > 0.\ \forall n \geq n_0.\ 10n^3 > c \cdot n^2$.
By the definition of little-ω, it follows that $10n^3 \in \omega\left(n^2\right)$.

$\square$