Inclass Recurrence Relation Proofs

CS 234

Daniel Lee

1 Green

Green Recursion Tree Method

$$W(n) = \begin{cases} 0 & n \leq 1 \\ 8 \cdot W(n/4) + n & n > 1 \end{cases}$$

Proof. Consider unrolling the call tree for $W(n)$. This tree will have:

- $\log_4(n) + 1$ rows because the argument is divided by 4 until it is less than or equal to 1
- 8^r nodes in row r (0-indexed) because having 8 recursive call children from each node multiplies the number of nodes each row by 8.
- $\frac{n}{4^r}$ work done in each node of row r, because the original argument n will have been divded by 4 a total r times before it is an argument to a row-r recursive call, and that argument is precisely the non-recursive work done

Because 0 work is done in the last row, we can ignore the last row and consider there to only be $\log_4(n)$ rows.

Putting these values together, the work done is $\sum_{r=0}^{\log_4(n)-1} 8^r \cdot \frac{n}{4^r}$, which can be algebraically simplified as follows:

$$
\begin{aligned}
\sum_{r=0}^{\log_4(n)-1} 8^r \cdot \frac{n}{4^r} &= n \cdot \sum_{r=0}^{\log_4(n)-1} \left(\frac{8^r}{4^r}\right) \\
&= n \cdot \sum_{r=0}^{\log_4(n)-1} 2^r \\
&= n \left(\frac{2^{\log_4(n)} - 1}{2-1}\right) \\
&= n \left(n^{\log_4(2)} - 1\right) \\
&= n \left(n^{\frac{\log_2 2}{\log_2 4}} - 1\right) \\
&= n \left(n^{\frac{1}{2}} - 1\right) \\
&= n^{\frac{3}{2}} - n
\end{aligned}
$$

Since the total work $n^{\frac{3}{2}} - n$ is always less than or equal to $n^{\frac{3}{2}}$ for $n \geq 1$, it follows that $W(n) = n^{\frac{3}{2}} - n \in O\left(n^{\frac{3}{2}}\right)$.

$\square$

Green Induction

Proof. This asymptotic bound can be shown by inducting with the inductive predicate $P(n) := W(n) \leq 2n^{\frac{3}{2}} - n$.

Base case $n = 2$: If $n = 2$, then $W(n)$ satisfies the following chain of equalities:

$$\begin{aligned} W(2) &= 8 \cdot W(\tfrac{1}{2}) + 2 && W \text{ def} \\ &= 2 && W(\tfrac{1}{2}) = 0 \end{aligned}$$

Thus $W(2) \leq 2 \cdot 2^{\frac{3}{2}} - 2$, which satisfies $W(2)$.

Inductive case : Suppose that $P(j)$ holds for all $0 \leq j \leq k$ for an arbitrary k. We now want to show $P(k+1)$.

Consider $W(k+1)$. This value satisfies the following chain of inequalities:

$$\begin{aligned} W(k+1) &= 8 \cdot W\left(\frac{k+1}{4}\right) + (k+1) && T \text{ def} \\ &\leq 8 \cdot \left(2 \cdot \left(\frac{k+1}{4}\right)^{\frac{3}{2}} - \frac{k+1}{4}\right) + (k+1) && IH \\ &= 16 \cdot \frac{(k+1)^{\frac{3}{2}}}{4^{\frac{3}{2}}} - 2(k+1) + (k+1) && \text{algebra} \\ &= 16 \cdot \frac{(k+1)^{\frac{3}{2}}}{2^{2 \cdot \frac{3}{2}}} - (k+1) && \text{algebra} \\ &= 16 \cdot \frac{(k+1)^{\frac{3}{2}}}{2^{3}} - (k+1) && \text{algebra} \\ &= 2(k+1)^{\frac{3}{2}} - (k+1) && \text{algebra} \end{aligned}$$

Thus $W(k+1) \leq 2(k+1)^{\frac{3}{2}} - (k+1)$, which satisfies $P(k+1)$.

Conclusion : As a result of induction, we find that $P(n)$ holds for all $n \geq 2$. That is, $\forall n \geq 2. W(n) \leq 2n^{\frac{3}{2}} - n$. By definition, it is therefore the case that $W(n) \in O\left(n^{\frac{3}{2}}\right)$. $\square$

2 Blue

Blue Recursion Tree Method

$$S(k) = \begin{cases} 0 & k \leq 1 \\ 4 \cdot S(k/2) + k^2 & k > 1 \end{cases}$$

Proof. Consider unrolling the call tree for $S(k)$. This tree will have:

- $\log_2(k) + 1$ rows because the argument is divided by 2 until it is less than or equal to 1
- 4^r nodes in row r (0-indexed) because having 4 recursive call children from each node multiplies the number of nodes each row by 4.
- $\frac{k^2}{4^r}$ work done in each node of row r, because the original argument k will have been divded by 2 a total r times before it is an argument to a row-r recursive call, yielding $\frac{k}{2^r}$ and the square of that argument is the non-recursive work done

Because 0 work is done in the last row, we can ignore the last row and consider there to only be $\log_2(k)$ rows.

Putting these values together, the work done is $\sum_{r=0}^{\log_2(k)-1} 4^r \cdot \frac{k^2}{4^r}$, which can be algebraically simplified as follows:

$$\begin{aligned} \sum_{r=0}^{\log_2(k)-1} 4^r \cdot \frac{k^2}{4^r} &= \sum_{r=0}^{\log_2(k)-1} k^2 \\ &= k^2 \left(\log_2(k) - 1 + 1\right) \\ &= k^2 \log_2 k \end{aligned}$$

Since the total work $k^2 \log_2 k$ is always less than or equal to $k^2 \log_2 k$ for $k \geq 1$, it follows that $S(k) = k^2 \log_2 k \in O\left(k^2 \log_2 k\right)$.

□

Blue Induction

Proof. This asymptotic bound can be shown by inducting with the inductive predicate $P(k) := S(k) \leq k^2 \log_2 k$.

Base case $k = 2$: If $k = 2$, then $S(k)$ satisfies the following chain of equalities:

$$\begin{aligned} S(2) &= 4 \cdot S(1) + 4 && S \text{ def} \\ &= 4 && S(1) = 0 \end{aligned}$$

Thus $S(2) \leq 4 \log_2 2 = 4$, which satisfies $S(2)$.

Inductive case : Suppose that $P(j)$ holds for all $0 \leq j \leq q$ for an arbitrary q. We now want to show $P(q+1)$.

Consider $S(q+1)$. This value satisfies the following chain of inequalities:

$$\begin{aligned} S(q+1) &= 4S\left(\tfrac{q+1}{2}\right) + (q+1)^2 && \text{S def} \\ &\leq 4\left(\tfrac{q+1}{2}\right)^2 \log_2 \tfrac{q+1}{2} + (q+1)^2 && \text{IH} \\ &= (q+1)^2 \left(\log_2 \tfrac{q+1}{2}\right) + (q+1)^2 && \text{algebra} \\ &= (q+1)^2 \left(\log_2(q+1) - \log_2 2\right) + (q+1)^2 && \text{algebra} \\ &= (q+1)^2 \log_2(q+1) - (q+1)^2 + (q+1)^2 && \text{algebra} \\ &= (q+1)^2 \log_2(q+1) && \text{algebra} \end{aligned}$$

Thus $S(q+1) \leq (q+1)^2 \log_2(q+1)$, which satisfies $P(q+1)$.

Conclusion : As a result of induction, we find that $P(k)$ holds for all $k \geq 2$. That is, $\forall k \geq 2. S(k) \leq k^2 \log_2 k$. By definition, it is therefore the case that $S(k) \in O\left(k^2 \log_2 k\right)$. $\square$

3 Red

Red Recursion Tree Method

$$T(x) = \begin{cases} 0 & x \le 1 \\ 100 \cdot T(x/5) + x^3 & x > 1 \end{cases}$$

Proof. Consider unrolling the call tree for $S(k)$. This tree will have:

- $\log_5(x) + 1$ rows because the argument is divided by 5 until it is less than or equal to 1
- 100^r nodes in row r (0-indexed) because having 100 recursive call children from each node multiplies the number of nodes each row by 100.
- $\frac{x^3}{125^r}$ work done in each node of row r, because the original argument x will have been divded by 5 a total r times before it is an argument to a row-r recursive call, yielding $\frac{x}{5^r}$ and the cube of that argument is the non-recursive work done

Because 0 work is done in the last row, we can ignore the last row and consider there to only be $\log_5(x)$ rows.

Putting these values together, the work done is $\sum_{r=0}^{\log_5(x)-1} 100^r \cdot \frac{x^3}{125^r}$, which can be algebraically simplified as follows:

$$\begin{aligned}
\sum_{r=0}^{\log_5(x)-1} 100^r \cdot \frac{x^3}{125^r} &= x^3 \cdot \sum_{r=0}^{\log_5(x)-1} \left(\frac{100}{125}\right)^r \\
&= x^3 \cdot \sum_{r=0}^{\log_5(x)-1} \left(\frac{4}{5}\right)^r \\
&= x^3 \cdot \frac{1 - \left(\frac{4}{5}\right)^{\log_5 x}}{1 - \frac{4}{5}} \\
&= 5x^3 \left(1 - \left(\frac{4}{5}\right)^{\log_5 x}\right) \\
&= 5x^3 \left(1 - \frac{4^{\log_5 x}}{5^{\log_5 x}}\right) \\
&= 5x^3 \left(1 - \frac{x^{\log_5 4}}{x^{\log_5 5}}\right) \\
&= 5x^3 \left(1 - x^{\log_5(4)-1}\right) \\
&= 5x^3 - 5 \cdot x^{3+\log_5(4)-1} \\
&= 5x^3 - 5 \cdot x^{2+\log_5 4}
\end{aligned}$$

Since the total work $5x^3 - 5 \cdot x^{2+\log_5 4}$ is always less than or equal to $5x^3$ for $x \geq 1$ because $0 < \log_5 4 < 1$, it follows that $T(x) = 5x^3 - 5 \cdot x^{2+\log_5 4} \in O\left(x^3\right)$.

$\square$

Red Induction

Proof. This asymptotic bound can be shown by inducting with the inductive predicate $P(x) := T(x) \leq 5x^3$.

Base case $x = 2$: If $x = 2$, then $T(x)$ satisfies the following chain of equalities:

$$\begin{aligned} T(2) &= 100 \cdot T(\tfrac{2}{5}) + 2^3 && T \text{ def} \\ &= 8 && T(\tfrac{2}{5}) = 0 \end{aligned}$$

Thus $T(2) \leq 5 \cdot 2^3$, which satisfies $T(2)$.

Inductive case : Suppose that $P(j)$ holds for all $0 \leq j \leq k$ for an arbitrary k. We now want to show $P(k+1)$.

Consider $T(k+1)$. This value satisfies the following chain of inequalities:

$$\begin{aligned} T(k+1) &= 100 \cdot T\left(\frac{k+1}{5}\right) + (k+1)^3 && T \text{ def} \\ &\leq 100 \cdot 5 \cdot \left(\frac{k+1}{5}\right)^3 + (k+1)^3 && IH \\ &= 500 \cdot \frac{(k+1)^3}{125} + (k+1)^3 && \text{algebra} \\ &= 4(k+1)^3 + (k+1)^3 && \text{algebra} \\ &= 5(k+1)^3 && \text{algebra} \end{aligned}$$

Thus $T(k+1) \leq 5(k+1)^3$, which satisfies $P(k+1)$.

Conclusion : As a result of induction, we find that $P(x)$ holds for all $x \geq 2$. That is, $\forall x \geq 2. T(x) \leq 5x^3$. By definition, it is therefore the case that $T(x) \in O\left(x^3\right)$. □