Prep Work 10 - Recurrences

CS 234

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1. Recurrences

1. The above description of the tree method finds three quantities to combine into the final result. What properties do each of those three quantities measure?

Number of nodes in a specific row: Measures the number of recursive calls that are made at each step.

How tall the tree is: Maximum row muber m and the input n can be expressed as m = log_2(n) and since the row number starts with 0, there are log_2(n) + 1 rows, which is the height of the tree.

Locally accomplished work per node in a specific row: This corresponds to the sum of the local part of the recurrence in a specific row, which is cn(1/2)^r.

Suppose for the next tasks that you have the following recurrence:

$$T(n) = \begin{cases} 0 & n \leq 1 \\ 4 \cdot T(\frac{n}{3}) + cn^2 & n > 1 \end{cases}$$

2. How many nodes will be in row r (starting from r = 0) of the call tree?

The coefficient of T(n/3) is 4, so it will be 4^r nodes.

3. How tall will the call tree be?

The input size n is divided by 3 at each rows, thus n * (1/3)^m = 1 ↔ log_3(n) = m, where m is the maximum number of rows.
Since the row numbering starts at 0, it should be log_3(n) + 1.

4. How much work is done locally to each node of row r?

By the definition of T(n), the local part is cn^2 and the argument n changes at every row by n/3^r. Thus, it should be c(n/3^r)^2.

5. How should the above quantities be combined to get the work represented by the whole tree?

$$\sum_{r=0}^{\log_3 n} 4^r \cdot \frac{cn^2}{9^r}$$

To compute these sums, it is necessary to know how to properly manipulate geometric series (sums of terms which differ by a common factor that is not equal to 0 or 1). The formula for such series is the following, where the common factor is x.

$$\sum_{i=0}^{m} a \cdot x^i = a \cdot \frac{x^{m+1} - 1}{x - 1}$$

6. How this formula is used in section G.6?

The formula is used to convert the work represented by the whole tree that is expressed in a summation form to a fractional form.

7. While the formula does not work when x = 1 because you would divide by 0, a different formula can still be derived that works with x = 1. What is this new formula? (Hint: Look back at the original summation, replace x with 1, and then simplify.)

$$\sum_{i=0}^{m} a \cdot 1^i = a \cdot (m+1)$$

To compute these sums, it is also necessary to know various logarithm identities. The textbook lists some identities, but one identity that is not listed is that $a^{log_b(c)} = c^{log_b(a)}$.

8. How this identity is used in section G.6?

The identity is used for converting 3^(log_2(m)) to m^(log_2(3)) and 2^(log_2(3)) to 3^(log_2(2)) to simplify the fraction.

Aside from using the tree method to directly derive a solution to a recurrence, one can also use strong induction to verify a guessed solution. This verification of guesses is an equally valid way of providing a solution to a recurrence.

9. Why is strong induction typically the correct choice for verifying a guessed solution to these divide-and-conquer recurrences?

The reason why strong induction is typically the correct choice is because divide-and-conquer recurrences involve the process assuming for all values of smaller input sizes that were derived from dividing the input size n holds, while normal induction would just assume that only n-1 holds.