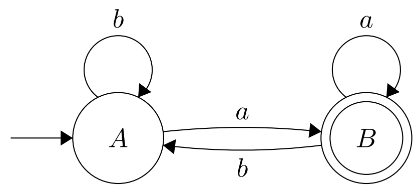
Mutual Induction (Red)

Let M be given by the following DFA:



Theorem 2. $\mathcal{L}(M) = \{w \in \{a,b\}^* \mid \exists u \in \{a,b\}^*.\, w = ua\}$

Proof. This proposition is proven by first mutually inducting with the following predicates:

$$Q(n) := \forall w \in \{a,b\}^*.\, |w| = n \rightarrow (\hat{\delta}(A,w) = A \leftrightarrow (w = \epsilon \vee \exists u \in \{a,b\}^*.\, w = ub))$$

$$R(n) := \forall w \in \{a,b\}^*.\, |w| = n \rightarrow (\hat{\delta}(A,w) = B \leftrightarrow \exists u \in \{a,b\}^*.\, w = ua)$$

Base Case $n = 0$ Let w be an arbitrary string over the alphabet $\{a,b\}$ that is of length 0. There is only one such string, the empty string, so $w = \epsilon$ and $\hat{\delta}(A,w) = \hat{\delta}(A,\epsilon)$. Further by the definition of $\hat{\delta}$, we find that $\hat{\delta}(A,\epsilon) = A$.

Thus, both sides of $Q(0)$'s biconditional are satisfied, rendering $Q(0)$ true. At the same time, since $A \neq B$ and ϵ does not end in b, both sides of $R(0)$'s biconditional are false, rendering $R(0)$ true.

Inductive Case Suppose for the inductive hypothesis that $Q(n)$ and $R(n)$ hold for some natural n. We want to now show $Q(n+1)$ and $R(n+1)$.

Let w be an arbitrary string of length $n+1$. Since $|w| > 0$, we know $w = vc$ for some $v \in \{a,b\}^*$ and $c \in \{a,b\}$.

We now proceed by cases on the result of $\hat{\delta}(A,v)$ and the identity of c.

Subcase $\hat{\delta}(A,v) = A$ **and** $c = a$ In this case the following identities hold:

$$\begin{aligned}
\hat{\delta}(A,w) &= \hat{\delta}(A,va) && [w = vc, c = a] \\
&= \delta(\hat{\delta}(A,v),a) && [\hat{\delta}\ def] \\
&= \delta(A,a) && [\hat{\delta}(A,v) = A] \\
&= B && [\delta\ def]
\end{aligned}$$

At the same time, since $\hat{\delta}(A,v) = A$, the inductive hypothesis tells us that $v = \epsilon$ or $v = ub$ for some string $u \in \{a,b\}^*$. Then $w = a$ or $w = uba$, which in either case means that the righthand side of $R(n+1)$'s biconditional is true. This leaves both sides of $R(n+1)$'s biconditional true, rendering $R(n+1)$ true.

Similarly, since $\hat{\delta}(A,w) \neq A$ and w is a non-empty string that does not end in b, both sides of $Q(n+1)$'s biconditional are false, rendering $Q(n+1)$ true.

Subcase $\hat{\delta}(A, v) = B$ and $c = a$ In this case the following identities hold:

$$\begin{aligned}
\hat{\delta}(A, w) =& \hat{\delta}(A, va) && [w = vc, c = a] \\
=& \delta(\hat{\delta}(A, v), a) && [\hat{\delta}\ def] \\
=& \delta(B, a) && [\hat{\delta}(A, v) = B] \\
=& B && [\delta\ def]
\end{aligned}$$

At the same time, since $\hat{\delta}(A, v) = B$, the inductive hypothesis tells us that $v = ua$ for some string $u \in \{a, b\}^*$. Then $w = uaa$, so the string ua witnesses that the righthand side of $R(n+1)$'s biconditional is true. This leaves both sides of $R(n+1)$'s biconditional true, rendering $R(n+1)$ true.

Similarly, since $\hat{\delta}(A, w) \neq A$ and w is a non-empty string that does not end in b, both sides of $Q(n+1)$'s biconditional are false, rendering $Q(n+1)$ true.

Subcase $\hat{\delta}(A, v) = A$ and $c = b$ In this case the following identities hold:

$$\begin{aligned}
\hat{\delta}(A, w) =& \hat{\delta}(A, vb) && [w = vc, c = b] \\
=& \delta(\hat{\delta}(A, v), b) && [\hat{\delta}\ def] \\
=& \delta(A, b) && [\hat{\delta}(A, v) = A] \\
=& A && [\delta\ def]
\end{aligned}$$

At the same time, since $\hat{\delta}(A, v) = A$, the inductive hypothesis tells us that $v = \epsilon$ or $v = ub$ for some string $u \in \{a, b\}^*$. Then $w = b$ or $w = ubb$, either of which witnesses that the righthand disjunct of the righthand side of $Q(n+1)$'s biconditional is true. This leaves both sides of $Q(n+1)$'s biconditional true, rendering $Q(n+1)$ true.

Similarly, since $\hat{\delta}(A, w) \neq B$ and w is a non-empty string ending in a, both sides of $R(n+1)$'s biconditional are false, rendering $R(n+1)$ true.

Subcase $\hat{\delta}(A, v) = B$ and $c = b$ In this case the following identities hold:

$$\begin{aligned}
\hat{\delta}(A, w) =& \hat{\delta}(A, vb) && [w = vc, c = b] \\
=& \delta(\hat{\delta}(A, v), b) && [\hat{\delta}\ def] \\
=& \delta(B, b) && [\hat{\delta}(A, v) = B] \\
=& A && [\delta\ def]
\end{aligned}$$

At the same time, since $\hat{\delta}(A, v) = A$, the inductive hypothesis tells us that $v = ua$ for some string $u \in \{a, b\}^*$. Then $w = uba$, so the string ua witnesses that the righthand side of $Q(n+1)$'s biconditional is true. This leaves both sides of $Q(n+1)$'s biconditional true, rendering $Q(n+1)$ true.

Similarly, since $\hat{\delta}(A, w) \neq A$ and w is a non-empty string ending in b, both sides of $R(n+1)$'s biconditional are false, rendering $R(n+1)$ true.

Conclusion Thus, by mutual induction, $\forall n \in \mathbb{N}.\, Q(n) \wedge R(n)$. In particular, the fact that $\forall n \in \mathbb{N}.\, R(n)$ allows us to conclude what the language of the automaton M is as follows:

$$\begin{aligned}
\mathcal{L}(M) =& \{w \in \{a,b\}^* \mid \hat{\delta}(A,w) \in \{B\}\} && [\mathcal{L}\ def]\\
=& \{w \in \{a,b\}^* \mid \hat{\delta}(A,w) = B\} && [logic]\\
=& \{w \in \{a,b\}^* \mid \exists u \in \{a,b\}^*.\, w = ub\} && [R(|w|)]
\end{aligned}$$

$\square$

Funnily enough, I actually made this DFA too easy. No induction is needed to prove the language, and this can be noticed in the proof by how the inductive hypothesis did not really play a role—it never mattered what v was, only that $w = va$ or $w = vb$.

If we never actually need the inductive hypothesis, then this is a sign that the statement can be proven directly instead. But double check this sign! Failing to use the inductive hypothesis could also just mean that your reasoning has gone wrong. In this case, the key property could actually have been proven directly by making the same casing as the above proof, just without any reference to the inductive hypothesis.