Assignment 7 - Proving a Negative

CS 234

Daniel Lee

1 Proofs on Paper

10.7 Prove that $\log_2 3$ is irrational.

Proof. Suppose for the sake of contradiction that actually $\log_2 3 \in \mathbb{Q}$. Then there must exist some integers p, q where $q \neq 0$ such that $\log_2 3 = p/q$. Moreover, since $\log_2 3 > 0$, we can let p and q both themselves be greater than 0, so p and q are natural numbers at least 1. Now observe the following:

$$\begin{aligned}
&\log_2 3 = p/q \quad \text{[above]} \\
&\Leftrightarrow q \log_2 3 = p \quad \text{[math]} \\
&\Leftrightarrow \log_2 3^q = p \quad [\text{ math }] \\
&\Leftrightarrow 2^{\log_2 3^q} = 2^p \quad [\text{ math }] \\
&\Leftrightarrow 3^{q \log_2 2} = 2^p \quad [\text{ math }] \\
&\Leftrightarrow \quad 3^q = 2^p \quad [\text{ math }].
\end{aligned}$$

The above shows that $3^q = 2^p$. However, the number that only contains the prime factor 3 cannot be same as the number that only contains the prime factor 2. This is a contradiction. Since assuming $\log_2 3 \in \mathbb{Q}$ leads to a contradiction, we can conclude $\log_2 3 \notin \mathbb{Q}$, completing the proof.

$\square$

10.15 Prove that the average of n numbers is at most as large as at least one of the numbers.

Proof. Suppose for the sake of contradiction that actually the average of n numbers is strictly larger than all of the each numbers.
We know that we can rewrite this statement as $\frac{\sum_{i=1}^{n} a_i}{n} > a_k$ for all $1 \leq k \leq n. \quad k \in \mathbb{N}$.

Now observe the following:

$$\begin{aligned} n \cdot \frac{\sum_{i=1}^{n} a_i}{n} &> \sum_{k=1}^{n} a_k \quad \text{[math]} \\ \Leftrightarrow \quad \sum_{i=1}^{n} a_i &> \sum_{k=1}^{n} a_k \quad \text{[math]} \end{aligned}$$

The above shows that $\sum_{i=1}^{n} a_i > \sum_{k=1}^{n} a_k$, which is clearly false therefore this is a contradiction. Since assuming that the average of n numbers is strictly larger than all of the each numbers leads to a contradiction, we can conclude that the average of n numbers is at most as large as at least one of the numbers, completing the proof.

□

Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

Proof. Suppose for the sake of contradiction that $\mathcal{P}(\mathbb{N})$ is countable. Then its elements can be completely listed out. Let N_i be the i^{th} set of natural numbers in such a listing.
Consider the following set S :

$$S = \{i \in \mathbb{N} : i \notin N_i\}$$

As every element of set S is a natural number, it must be that $S \subseteq \mathbb{N}$, and so $S \in \mathcal{P}(\mathbb{N})$ by the definition of powerset. This means that S must be equal to N_k for some natural k.
However, if $S = N_k$, the following also holds:

$$\begin{aligned} k \in S &\Leftrightarrow k \in \{i \in \mathbb{N} : i \notin N_i\} \quad [S\,\text{def}] \\ &\Leftrightarrow k \notin N_k \quad [\in \text{def}] \\ &\Leftrightarrow k \notin S \quad [S = N_k] \end{aligned}$$

The fact that $k \in S$ iff $k \notin S$ is a contradiction. Thus the original assumption must be false and in fact $\mathcal{P}(\mathbb{N})$ is not countable.

□

Prove that $\mathbb{R}$ is uncountable. (Hint: Maybe consider the decimal representation of those numbers between 0 and 1.)

Proof. Suppose for the sake of contradiction that $\mathbb{R}$ was in fact countable. Then its elements can be completely listed out. Consider the subset of $\mathbb{R}$ which is the set of real numbers in the range of $(0, 1)$. We know that if $\mathbb{R}$ was countable, then the set of real numbers in the range of $(0, 1)$ would also be countable.
Let us assume that the set of real numbers in the range of $(0, 1)$ is countable. Then its elements can be completely listed out. Let r_i be the i^{th} real number in this listing and note that each r_i is able to be denoted in

its unique decimal representation as follows:

$$\begin{aligned}
r_0 &= 0.d_{00}d_{01}d_{02}\dots \\
r_1 &= 0.d_{10}d_{11}d_{12}\dots \\
r_2 &= 0.d_{20}d_{21}d_{22}\dots \\
&\dots
\end{aligned}$$

Note that $i, j, d_{ij} \in \mathbb{N}$. $0 \leq d_{ij} \leq 9$. and each r_i has no trailing 9s.
Now consider the real number s constructed as follows:

$$s = 0.s_0s_1s_2\cdots$$

where the i^{th} digit s_i is definded as follows:

$$s_i = 9 - d_{ii}$$

We know that s is a real number in the range of $(0, 1)$ and this means that s must be equal to r_k for some natural k.
However, s differs from each r_k in at least the k^{th} decimal place such that $s_k \neq d_{kk}$ by the definition of s_i. Thus, $s \neq r_k$. This is a contradiction. Thus the assumption that the set of real numbers in the range of $(0, 1)$ is countable must be false, therefore $\mathbb{R}$ is uncountable. $\square$

Prove that the following languages are not regular:

11.12 $\{ww : w \in \{0, 1\}^*\}$

Proof. Let $L = \{ww : w \in \{0, 1\}^*\}$. Suppose for the sake of contradiction that L is regular. Then there exists some number $n > 0$ such that all strings in L of length at least n can be pumped.

Consider the string $s = 0^n10^n1$ and note that $s \in L$ and that $|s| = 2n+2 > n$.
We now consider all possible ways to break up s into $s = xyz$ such that $|y| > 0$ and $|xy| \leq n$. Since the first n symbols in s are all 0s, the string x and y can only have 0s in them.
Let us say that $x = 0^i$ and $y = 0^j$, where $i \geq 0$ and $j \geq 1$. This means that the remaining symbols in s go into $z = 0^{n-i-j}10^n1$
We now pick a natural number $k \geq 0$ such that xy^kz is not in L. Selecting $k = 2$, we get that $xy^kz = xyyz = 0^i0^j0^j0^{n-i-j}10^n1 = 0^{n+j}10^n1$, which is not in L since $j > 0$ so $\left|0^{n+j}1\right| \neq |0^n1|$. This is a contradiction, so L cannot be regular.

$\square$

11.18 $\{0^n : n \text{ is a power of } 2\}$

Proof. Let $L = \{0^n : n \text{ is a power of } 2\}$. Suppose for the sake of contradiction that L is regular. Then there exits some number $k > 0$ such that all strings in L of length at least k can be pumped.
Consider the string 0^{2^a} where a is some integer and that $2^a > k$. This string is in L and has length $2^a \geq k$, so it can be pumped. Thus, $0^{2^a} = xyz$ for some strings x, y, z where $|y| > 0$ and $|xy| \leq k$ such that, for all $i \in \mathbb{N}$, the string xy^iz is also in L.
Because $|xy| \leq k$ and the string 0^{2^a} is only constituted with 0s, it must be that y is made up entirely of 0s. Pumping the string once therefore yields the string $0^{2^a+|y|}$ which is guaranteed by the pumping lemma to be in L. However, $2^a + |y|$ is not a power of 2, since we know that $|y| > 0$, therefore $2^a + |y| > 2^a$ and at the sametime, $|y| \leq |xy| \leq k < 2^a$ and the subsequent power of 2 that comes after 2^a is 2^{a+1}, which means $2^a + |y| < 2^a + 2^a = 2^{a+1}$. Thus, $2^a < 2^a + |y| < 2^{a+1}$ so we now know that $2^a + |y|$ is not a power of 2 . This is a contradiction, so L cannot be regular.

□