Assignment 6 - Inductive Proofs

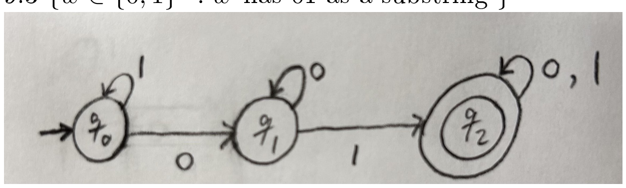
CS 234

Daniel Lee

1 Last 2 Proofs on Paper

Draw DFAs with as few states as possible for each of the following languages and then prove that they accept that language.

9.3 $\{w \in \{0,1\}^* : w \text{ has 01 as a substring }\}$



Proof. This statement is proven by mutual induction using the following 3 predicates:

$$A(n) := \forall w \in \{0,1\}^*.|w| = n \rightarrow \Big(\hat{\delta}(q_0, w) = q_0 \leftrightarrow w \text{ doesn't have 01 as a substring and doesn't end with 0}\Big)$$

$$B(n) := \forall w \in \{0,1\}^*.|w| = n \rightarrow \Big(\hat{\delta}(q_0, w) = q_1 \leftrightarrow w \text{ doesn't have 01 as a substring and ends with 0}\Big)$$

$$C(n) := \forall w \in \{0,1\}^*.|w| = n \rightarrow \Big(\hat{\delta}(q_0, w) = q_2 \leftrightarrow w \text{ has 01 as a substring }\Big)$$

Base case $n = 0$: Let w be an arbitrary string over the alphabet $\{0,1\}$ that is of length 0. Then we know that $w = \lambda$ and $\hat{\delta}(q_0, w) = \hat{\delta}(q_0, \lambda)$. By definition of $\hat{\delta}$, we find that $\hat{\delta}(q_0, \lambda) = q_0$.
Thus, both sides of $A(0)$'s biconditional are satisfied, rendering $A(0)$ true. At the same time, since $q_0 \neq q_1, q_0 \neq q_2$, λ does not end with 0, and λ does not have 01 as a substring, both sides of $B(0)$'s biconditional are false and both sides of $C(0)$'s biconditional are false, rendering $B(0)$ and $C(0)$ true.

Inductive case : Suppose for the inductive hypothesis that all of $A(n), B(n)$, and $C(n)$ hold for some natural n. We want to show each of $A(n+1), B(n+1)$, and $C(n+1)$.

Let w be an arbitrary string over the alphabet $\{0,1\}$ of length $n+1$. Because $n+1 \geq 1$, it must be that $w = vc$ for some string $v \in \{0,1\}^*$ of length n and $c \in \{0,1\}$.

The proof now proceeds by cases over the result of $\hat{\delta}(q_0, v)$ and the identity of c.

Subcase $\hat{\delta}(q_0, v) = q_0$ **and** $c = 0$: Suppose that $\hat{\delta}(q_0, v) = q_0$ and $c = 0$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v0) && [w = vc, c = 0] \\
&= \delta\left(\hat{\delta}(q_0, v), 0\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_0, 0) && \left[\hat{\delta}(q_0, v) = q_0\right] \\
&= q_1 && [\delta \text{ def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_0$, the inductive hypothesis tells us that v does not have 01 as a substring and does not end with 0. Thus, we know that $w = v0$ does not have 01 as a substring and ends with 0.

This leaves both sides of $B(n+1)$'s biconditional true, rendering $B(n+1)$ true.

At the same time, because $\hat{\delta}(q_0, w)$ is not q_0 or q_2 and w does not have 01 as a substring and ends with 0, both sides of $A(n+1)$ and $C(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $C(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_0$ **and** $c = 1$: Suppose that $\hat{\delta}(q_0, v) = q_0$ and $c = 1$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v1) && [w = vc, c = 1] \\
&= \delta\left(\hat{\delta}(q_0, v), 1\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_0, 1) && \left[\hat{\delta}(q_0, v) = q_0\right] \\
&= q_0 && [\delta \text{ def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_0$, the inductive hypothesis tells us that v does not have 01 as a substring and does not end with 0. Thus, we know that $w = v1$ does not have 01 as a substring and doesn't end with 0.

This leaves both sides of $A(n+1)$'s biconditional true, rendering $A(n+1)$ true.

At the same time, because $\hat{\delta}(q_0, w)$ is not q_1 or q_2 and w does not have 01 as a substring and doesn't end with 0, both sides of $B(n+1)$ and

$C(n+1)$'s biconditionals are false, rendering both $B(n+1)$ and $C(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_1$ **and** $c = 0$: Suppose that $\hat{\delta}(q_0, v) = q_1$ and $c = 0$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v0) && [w = vc, c = 0] \\
&= \delta\left(\hat{\delta}(q_0, v), 0\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_1, 0) && \left[\hat{\delta}(q_0, v) = q_1\right] \\
&= q_1 && [\delta \text{ def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_1$, the inductive hypothesis tells us that v does not have 01 as a substring and ends with 0. Thus, we know that $w = v0$ does not have 01 as a substring and ends with 0.
This leaves both sides of $B(n+1)$'s biconditional true, rendering $B(n+1)$ true.
At the same time, because $\hat{\delta}(q_0, w)$ is not q_0 or q_2 and w does not have 01 as a substring and ends with 0, both sides of $A(n+1)$ and $C(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $C(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_1$ **and** $c = 1$: Suppose that $\hat{\delta}(q_0, v) = q_1$ and $c = 1$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v1) && [w = vc, c = 1] \\
&= \delta\left(\hat{\delta}(q_0, v), 1\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_1, 1) && \left[\hat{\delta}(q_0, v) = q_1\right] \\
&= q_2 && [\delta \text{ def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_1$, the inductive hypothesis tells us that v does not have 01 as a substring and ends with 0. Thus, we know that $w = v1$ has 01 as a substring.
This leaves both sides of $C(n+1)$'s biconditional true, rendering $C(n+1)$ true.
At the same time, because $\hat{\delta}(q_0, w)$ is not q_0 or q_1 and w has 01 as a substring, both sides of $A(n+1)$ and $B(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $B(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_2$ **and** $c = 0$: Suppose that $\hat{\delta}(q_0, v) = q_2$ and $c = 0$.

Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v0) && [w = vc, c = 0] \\
&= \delta\left(\hat{\delta}(q_0, v), 0\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_2, 0) && \left[\hat{\delta}(q_0, v) = q_2\right] \\
&= q_2 && [\delta \,\text{def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_2$, the inductive hypothesis tells us that v has 01 as a substring. Thus, we know that $w = v0$ has 01 as a substring.
This leaves both sides of $C(n+1)$'s biconditional true, rendering $C(n+1)$ true.
At the same time, because $\hat{\delta}(q_0, w)$ is not q_0 or q_1 and w has 01 as a substring, both sides of $A(n+1)$ and $B(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $B(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_2$ **and** $c = 1$: Suppose that $\hat{\delta}(q_0, v) = q_2$ and $c = 1$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v1) && [w = vc, c = 1] \\
&= \delta\left(\hat{\delta}(q_0, v), 1\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_2, 1) && \left[\hat{\delta}(q_0, v) = q_2\right] \\
&= q_2 && [\delta \,\text{def}].
\end{aligned}$$

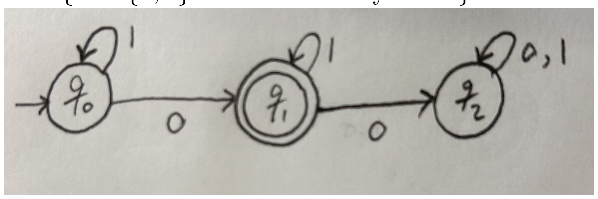
Further, because $\hat{\delta}(q_0, v) = q_2$, the inductive hypothesis tells us that v has 01 as a substring. Thus, we know that $w = v1$ has 01 as a substring.
This leaves both sides of $C(n+1)$'s biconditional true, rendering $C(n+1)$ true.
At the same time, because $\hat{\delta}(q_0, w)$ is not q_0 or q_1 and w has 01 as a substring, both sides of $A(n+1)$ and $B(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $B(n+1)$ true.
Conclusion : Thus, by mutual induction, $A(n)$, $B(n)$, and $C(n)$ hold for all naturals n.
Now observe that the following identities hold for the language of the automaton M:

$$\begin{aligned}
\mathcal{L}(M) &= \left\{w \in \{0,1\}^* \mid \hat{\delta}(q_0, w) \in \{q_2\}\right\} && [\mathcal{L} \text{ def }] \\
&= \left\{w \in \{0,1\}^* \mid \hat{\delta}(q_0, w) = q_2\right\} && [\text{ logic }] \\
&= \{w \in \{0,1\}^* \mid w \text{ has 01 as a substring }\} && [C(|w|)]
\end{aligned}$$

This confirms the desired identity for the language of M.

□

9.12 $\{w \in \{0,1\}^* : w \text{ has exactly one } 0\}$



Proof. This statement is proven by mutual induction using the following 3 predicates:

$$A(n) := \forall w \in \{0,1\}^*. |w| = n \to \left(\hat{\delta}(q_0, w) = q_0 \leftrightarrow w \text{ has no } 0\right)$$

$$B(n) := \forall w \in \{0,1\}^*. |w| = n \to \left(\hat{\delta}(q_0, w) = q_1 \leftrightarrow w \text{ has exactly one } 0\right)$$

$$C(n) := \forall w \in \{0,1\}^*. |w| = n \to \left(\hat{\delta}(q_0, w) = q_2 \leftrightarrow w \text{ has at least two 0s }\right)$$

Base case $n = 0$: Let w be an arbitrary string over the alphabet $\{0,1\}$ that is of length 0. Then we know that $w = \lambda$ and $\hat{\delta}(q_0, w) = \hat{\delta}(q_0, \lambda)$. By definition of $\hat{\delta}$, we find that $\hat{\delta}(q_0, \lambda) = q_0$.
Thus, both sides of $A(0)$'s biconditional are satisfied, rendering $A(0)$ true. At the same time, since $q_0 \neq q_1, q_0 \neq q_2, \lambda$ does not have exactly one 0, and λ does not have at least two 0s, both sides of $B(0)$'s biconditional are false and both sides of $C(0)$'s biconditional are false, rendering $B(0)$ and $C(0)$ true.

Inductive case : Suppose for the inductive hypothesis that all of $A(n), B(n)$, and $C(n)$ hold for some natural n. We want to show each of $A(n+1), B(n+1)$, and $C(n+1)$.
Let w be an arbitrary string over the alphabet $\{0,1\}$ of length $n+1$. Because $n + 1 \geq 1$, it must be that $w = vc$ for some string $v \in \{0,1\}^*$ of length n and $c \in \{0,1\}$.
The proof now proceeds by cases over the result of $\hat{\delta}(q_0, v)$ and the identity of c.

Subcase $\hat{\delta}(q_0, v) = q_0$ **and** $c = 0$: Suppose that $\hat{\delta}(q_0, v) = q_0$ and $c = 0$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v0) && [w = vc, c = 0] \\
&= \delta\left(\hat{\delta}(q_0, v), 0\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_0, 0) && \left[\hat{\delta}(q_0, v) = q_0\right] \\
&= q_1 && [\delta \text{ def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_0$, the inductive hypothesis tells us that v has no 0. Thus, we know that $w = v0$ has exactly one 0.
This leaves both sides of $B(n+1)$'s biconditional true, rendering $B(n+1)$ true.
At the same time, because $\hat{\delta}(q_0, w)$ is not q_0 or q_2 and w has exactly one 0, both sides of $A(n+1)$ and $C(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $C(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_0$ **and** $c = 1$: Suppose that $\hat{\delta}(q_0, v) = q_0$ and $c = 1$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v1) && [w = vc, c = 1] \\
&= \delta\left(\hat{\delta}(q_0, v), 1\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_0, 1) && \left[\hat{\delta}(q_0, v) = q_0\right] \\
&= q_0 && [\delta \,\text{def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_0$, the inductive hypothesis tells us that v has no 0. Thus, we know that $w = v1$ has no 0.
This leaves both sides of $A(n+1)$'s biconditional true, rendering $A(n+1)$ true.
At the same time, because $\hat{\delta}(q_0, w)$ is not q_1 or q_2 and w has no 0, both sides of $B(n+1)$ and $C(n+1)$'s biconditionals are false, rendering both $B(n+1)$ and $C(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_1$ **and** $c = 0$: Suppose that $\hat{\delta}(q_0, v) = q_1$ and $c = 0$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v0) && [w = vc, c = 0] \\
&= \delta\left(\hat{\delta}(q_0, v), 0\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_1, 0) && \left[\hat{\delta}(q_0, v) = q_1\right] \\
&= q_2 && [\delta \,\text{def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_1$, the inductive hypothesis tells us that v has exactly one 0. Thus, we know that $w = v0$ has at least two 0s.
This leaves both sides of $C(n+1)$'s biconditional true, rendering $C(n+1)$ true.
At the same time, because $\hat{\delta}(q_0, w)$ is not q_0 or q_1 and w has at least two 0s, both sides of $A(n+1)$ and $B(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $B(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_1$ **and** $c = 1$: Suppose that $\hat{\delta}(q_0, v) = q_1$ and $c = 1$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v1) && [w = vc, c = 1] \\
&= \delta\left(\hat{\delta}(q_0, v), 1\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_1, 1) && \left[\hat{\delta}(q_0, v) = q_1\right] \\
&= q_1 && [\delta \,\text{def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_1$, the inductive hypothesis tells us that v has exactly one 0. Thus, we know that $w = v1$ has exactly one 0.
This leaves both sides of $B(n+1)$'s biconditional true, rendering $B(n+1)$ true.
At the same time, because $\hat{\delta}(q_0, w)$ is not q_0 or q_2 and w has exactly one 0, both sides of $A(n+1)$ and $C(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $C(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_2$ **and** $c = 0$: Suppose that $\hat{\delta}(q_0, v) = q_2$ and $c = 0$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v0) && [w = vc, c = 0] \\
&= \delta\left(\hat{\delta}(q_0, v), 0\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_2, 0) && \left[\hat{\delta}(q_0, v) = q_2\right] \\
&= q_2 && [\delta \,\text{def}].
\end{aligned}$$

Further, because $\hat{\delta}(q_0, v) = q_2$, the inductive hypothesis tells us that v has at least two 0s. Thus, we know that $w = v0$ has at least two 0s.
This leaves both sides of $C(n+1)$'s biconditional true, rendering $C(n+1)$ true.
At the same time, because $\hat{\delta}(q_0, w)$ is not q_0 or q_1 and w has at least two 0s, both sides of $A(n+1)$ and $B(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $B(n+1)$ true.

Subcase $\hat{\delta}(q_0, v) = q_2$ **and** $c = 1$: Suppose that $\hat{\delta}(q_0, v) = q_2$ and $c = 1$. Observe then the following:

$$\begin{aligned}
\hat{\delta}(q_0, w) &= \hat{\delta}(q_0, v1) && [w = vc, c = 1] \\
&= \delta\left(\hat{\delta}(q_0, v), 1\right) && [\hat{\delta} \text{ def}] \\
&= \delta(q_2, 1) && \left[\hat{\delta}(q_0, v) = q_2\right] \\
&= q_2 && [\delta \,\text{def}].
\end{aligned}$$

Further, because $\hat{\delta}\,(q_0, v) = q_2$, the inductive hypothesis tells us that v has at least two 0s. Thus, we know that $w = v1$ has at least two 0s.
This leaves both sides of $C(n+1)$'s biconditional true, rendering $C(n+1)$ true.
At the same time, because $\hat{\delta}\,(q_0, w)$ is not q_0 or q_1 and w has at least two 0s, both sides of $A(n+1)$ and $B(n+1)$'s biconditionals are false, rendering both $A(n+1)$ and $B(n+1)$ true.

Conclusion : Thus, by mutual induction, $A(n)$, $B(n)$, and $C(n)$ hold for all naturals n.
Now observe that the following identities hold for the language of the automaton M:

$$\begin{aligned}
\mathcal{L}(M) &= \left\{ w \in \{0,1\}^* \mid \hat{\delta}(q_0, w) \in \{q_1\} \right\} && [\mathcal{L} \text{ def }] \\
&= \left\{ w \in \{0,1\}^* \mid \hat{\delta}(q_0, w) = q_1 \right\} && [\text{ logic }] \\
&= \{ w \in \{0,1\}^* \mid w \text{ has exactly one } 0 \ \} && [B(|w|)]
\end{aligned}$$

This confirms the desired identity for the language of M. $\square$