Contents

1 Question 1

Consider the second-stage problem defined by

$$\begin{aligned}
&\min_y\ 2y_1 + y_1 \\
&\text{s.t. } y_1 + y_2 \geq 1 - x_1, \\
&\qquad y_1 \geq \boldsymbol{\xi} - x_1 - x_2, \\
&\qquad y_1, y_2 \geq 0
\end{aligned}$$

Show that this program has a finite recourse if $\mathbb{E}[\boldsymbol{\xi}]$ *is finite.*

Let's write the second-stage program in the standard form.

$$\begin{aligned}
&\min_y\ 2y_1 + y_1 \\
&\text{s.t. } y_1 + y_2 - s_1 \qquad = 1 - x_1 \qquad, \\
&\qquad y_1 \qquad\quad - s_2 = \boldsymbol{\xi} - x_1 - x_2, \\
&\qquad y_1, y_2, s_1, s_2 \geq 0
\end{aligned}$$

We can determine q, y, W, $h(\xi)$ and $T(\xi)$ to identify the problem as follows,

$$\begin{aligned}
&\min_y\ q^T y \\
&\text{s.t. } Wy = h(\xi) - T(\xi)\,x \\
&\qquad y \geq 0
\end{aligned}$$

$$q^T = \begin{pmatrix} 2 & 1 & 0 & 0 \end{pmatrix} \qquad W = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad h(\xi) = \begin{pmatrix} 1 \\ \xi \end{pmatrix}$$

$$y^T = \begin{pmatrix} y_1 & y_2 & s_1 & s_2 \end{pmatrix} \qquad T(\xi) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

1.1 Complete recourse

1.1.1 Using positive hull

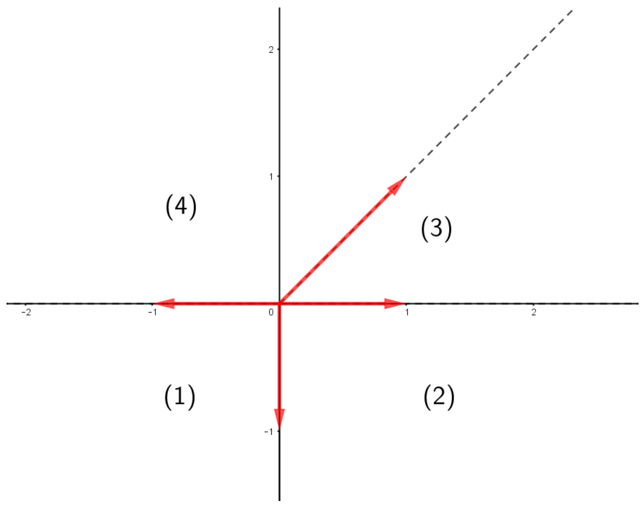
Because $\mathbb{E}[\boldsymbol{\xi}]$ is finite, we have that $\boldsymbol{\xi} < +\infty$ almost surely, i.e. $\mathbb{P}(\boldsymbol{\xi} < +\infty) = 1$. Therefore, all realizations ξ of the random variable are finite, i.e $\xi < +\infty$, and for all $\xi \in \tilde{\Xi}$ and $x \in \mathbb{R}^2$,

$$z = \begin{pmatrix} 1 - x_1 \\ \xi - x_1 - x_2 \end{pmatrix} < +\infty.$$

The problem has complete recourse *if*

$$\operatorname{pos} W = \{z \in \mathbb{R}^2 \mid \exists y \in \mathbb{R}^4_+, Wy = z\} = \mathbb{R}^2 \tag{1}$$

Consider $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2$, and let's distinguish the four following cases, where the vectors are the column vectors of the matrix W,



(1) If $z_1 \leq 0$ and $z_2 \leq 0$, then take $y = \begin{pmatrix} 0 \\ 0 \\ -z_1 \\ -z_2 \end{pmatrix} \geq 0$, and $Wy = z$.

(2) If $z_1 \geq 0$ and $z_2 \leq 0$, then take $y = \begin{pmatrix} 0 \\ z_1 \\ 0 \\ -z_2 \end{pmatrix} \geq 0$, and $Wy = z$.

(3) If $z_1 - z_2 \geq 0$ and $z_2 \geq 0$, then take $y = \begin{pmatrix} z_2 \\ z_1 - z_2 \\ 0 \\ 0 \end{pmatrix} \geq 0$, and $Wy = z$.

(4) If $z_1 - z_2 \leq 0$ and $z_2 \geq 0$, then take $y = \begin{pmatrix} z_2 \\ 0 \\ z_2 - z_1 \\ 0 \end{pmatrix} \geq 0$, and $Wy = z$.

Conclusion: for all $z \in \mathbb{R}^2$, there exists $y \in \mathbb{R}^4_+$, such that $Wy = z$, so

$$\operatorname{pos} W = \mathbb{R}^2$$

and the recourse is *complete*.

1.1.2 What about the use of ξ?? Computation of K_2.

By considering the definitions from Birge and Louveaux [1], the second-stage feasibility set is defined as,

$$K_2 = \{x \mid \mathcal{Q}(x) < \infty\}. \tag{2}$$

Let's now show that the expected recourse $\mathcal{Q}(x)$ is finite for all $x \in \mathbb{R}^2$, i.e. $K_2 = \mathbb{R}^2$ *if* $\mathbb{E}[\boldsymbol{\xi}]$ is finite.

What is $Q(x, \xi)$? Let's take $\xi \in \tilde{\Xi}$,

1. If $\xi - x_1 - x_2 \geq 0$, then $y_1^* = \xi - x_1 - x_2$ and $s_2^* = 0$. So, $y_2 - s_1 = 1 - \xi + x_2$
   (a) If $1 - \xi + x_2 \geq 0$, then $y_2^* = 1 - \xi + x_2$ and $s_1^* = 0$. So,
   $$Q(x, \xi) = 1 + \xi - 2x_1 - x_2. \tag{3}$$
   (b) If $1 - \xi + x_2 < 0$, then $y_2^* = 0$ and $s_1^* = -1 + \xi - x_2$. So,
   $$Q(x, \xi) = 2\xi - 2x_1 - 2x_2. \tag{4}$$
2. If $\xi - x_1 - x_2 < 0$, then $y_1^* = 0$ and $s_2^* = -\xi + x_1 + x_2$. So, $y_2 - s_1 = 1 - x_1$
   (a) If $1 - x_1 \geq 0$, then $y_2^* = 1 - x_1$ and $s_1^* = 0$. So,
   $$Q(x, \xi) = 1 - x_1. \tag{5}$$
   (b) If $1 - x_1 < 0$, then $y_2^* = 0$ and $s_2^* = -1 + x_1$. So,
   $$Q(x, \xi) = 0. \tag{6}$$

What is $\mathcal{Q}(x)$?

$$\begin{aligned} \text{For all } x \in \mathbb{R}^2, \quad \mathcal{Q}(x) &= \int_{\xi \in \Xi} Q(x, \xi)\ \mathbb{P}(\mathrm{d}\xi) \\ &= \int_{[-\infty, x_1+x_2]} Q(x, \xi)\ \mathbb{P}(\mathrm{d}\xi) + \int_{[x_1+x_2, +\infty]} Q(x, \xi)\ \mathbb{P}(\mathrm{d}\xi) \end{aligned}$$

From (5) and (6),

$$\int_{[-\infty, x_1+x_2]} Q(x, \xi)\ \mathbb{P}(\mathrm{d}\xi) = \max\{0, 1 - x_1\} < \infty. \tag{7}$$

Then, we have

- If $1 \leq x_1$ then $1 + x_2 \leq x_1 + x_2$ and from (4),
  $$\begin{aligned} \int_{[x_1+x_2, +\infty]} Q(x, \xi)\ \mathbb{P}(\mathrm{d}\xi) &= \int_{[x_1+x_2, +\infty]} 2\xi - 2x_1 - 2x_2\ \mathbb{P}(\mathrm{d}\xi) \\ &= 2\int_{[x_1+x_2, +\infty]} \xi\ \mathbb{P}(\mathrm{d}\xi) - 2(x_1 + x_2)\,\mathbb{P}(\xi \geq x_1 + x_2) \end{aligned}$$

  Therefore,
  $$\mathcal{Q}(x) = 2\int_{[x_1+x_2, +\infty]} \xi\ \mathbb{P}(\mathrm{d}\xi) - 2(x_1 + x_2)\,\mathbb{P}(\xi \geq x_1 + x_2)$$

- If $1 > x_1$ then $1 + x_2 > x_1 + x_2$, and from (3) and (4), we have

$$\begin{aligned}\int_{[x_1+x_2,+\infty]} Q(x,\,\xi)\;\mathbb{P}(\mathrm{d}\xi) &= \int_{[x_1+x_2,1+x_2]} 1+\xi-2x_1-x_2\;\mathbb{P}(\mathrm{d}\xi) + \int_{[1+x_2,+\infty]} 2\xi-2x_1-2x_2\;\mathbb{P}(\mathrm{d}\xi)\\ &= (1-2x_1-x_2)\,\mathbb{P}(1+x_2\le\xi\le x_1+x_2) + \int_{[x_1+x_2,1+x_2]} \xi\;\mathbb{P}(\mathrm{d}\xi)\\ &\quad - 2(x_1+2x_2)\,\mathbb{P}(\xi\ge 1+x_2) + 2\int_{[1+x_2,+\infty]} \xi\;\mathbb{P}(\mathrm{d}\xi)\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{Q}(x) = 1 - x_1 &+ (1-2x_1-x_2)\,\mathbb{P}(1+x_2\le\xi\le x_1+x_2) - 2(x_1+2x_2)\,\mathbb{P}(\xi\ge 1+x_2)\\ &+ \int_{x_1+x_2}^{1+x_2} \xi\;\mathbb{P}(\mathrm{d}\xi) + 2\int_{1+x_2}^{+\infty} \xi\;\mathbb{P}(\mathrm{d}\xi)\end{aligned}$$

Conclusion: if $\int_{[0,+\infty]} \xi\;\mathbb{P}(\mathrm{d}\xi) = \mathbb{E}[\boldsymbol{\xi}\,\mathbb{1}\{\boldsymbol{\xi}\ge 0\}] < +\infty$, then for all $x\in\mathbb{R}^2$, $\mathcal{Q}(x) < +\infty$ and $K_2 = \mathbb{R}^2$.

Indeed, in this situation, for all $0 \le a < b \le +\infty$,

$$\mathbb{E}[\boldsymbol{\xi}\,\mathbb{1}\{a\le\boldsymbol{\xi}\le b\}] = \int_{[a,b]} \xi\;\mathbb{P}(\mathrm{d}\xi) \le \int_{[0,+\infty]} \xi\;\mathbb{P}(\mathrm{d}\xi) = \mathbb{E}[\boldsymbol{\xi}\,\mathbb{1}\{\boldsymbol{\xi}\ge 0\}] < +\infty \tag{8}$$

Remark 1. In exercise 4 from Birge and Louveaux [1, p.123], the support of the probability distribution is supposed to be non-negative, i.e. $\Xi\subset\mathbb{R}_+$, and consequently, $\int_{[0,+\infty]} \xi\;\mathbb{P}(\mathrm{d}\xi) = \mathbb{E}[\boldsymbol{\xi}] < \infty$, when $\mathbb{E}[\boldsymbol{\xi}] < \infty$.
And from the above reasoning, $K_2 = \mathbb{R}^2$.

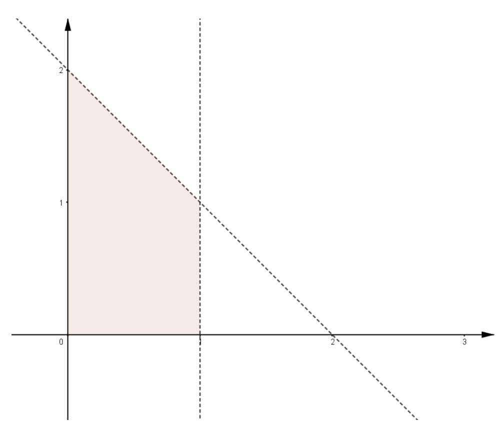
1.2 Interpretation with the dual problem

The dual of the problem is the following,

$$\begin{aligned}\max_{\lambda}\;& (1-x_1)\lambda_1 + (\xi-x_1-x_2)\lambda_2\\ \text{s.t. }& \lambda_1+\lambda_2\le 2,\\ & \lambda_1\le 1,\\ & \lambda_1,\lambda_2\ge 0\end{aligned}$$

The feasible set for λ is closed and bounded, and the function $(\lambda_1,\lambda_2)\mapsto(1-x_1)\lambda_1+(\xi-x_1-x_2)\lambda_2$ is continuous. So there exists λ_1^*,λ_2^* in the dual feasible set, such that $(1-x_1)\lambda_1^*+(\xi-x_1-x_2)\lambda_2^* = \max_\lambda\,(1-x_1)\lambda_1+(\xi-x_1-x_2)\lambda_2$. By the strong duality theorem, there exists y_1^*,y_2^* in the primal feasible set such that

$$2y_1^*+y_2^* = (1-x_1)\lambda_1^* + (\xi-x_1-x_2)\lambda_2^*.$$



We can recover the 4 possible situations for $Q(x,\xi)$:

- $\lambda_1^*=1,\lambda_2^*=1$, and $Q(x,\xi) = 1+\xi-2x_1-x_2$
- $\lambda_1^*=1,\lambda_2^*=0$, and $Q(x,\xi) = 1-x_1$
- $\lambda_1^*=0,\lambda_2^*=2$, and $Q(x,\xi) = 2(\xi-x_1-x_2)$
- $\lambda_1^*=0,\lambda_2^*=0$, and $Q(x,\xi) = 0$

Remark 2. The λ_1^* and λ_2^* depend on ξ.

1.3 Discussions

By considering the definition

$$K_2 = \{x \mid \mathcal{Q}(x) < \infty\}$$

some inconsistencies can occur.

1.3.1 Complete recourse but not relatively complete

Let's consider again the problem

$$\begin{aligned}\min_{y}\ & 2y_1 + y_1 \\ \text{s.t.}\ & y_1 + y_2 \geq 1 - x_1, \\ & y_1 \geq \boldsymbol{\xi} - x_1 - x_2, \\ & y_1, y_2 \geq 0\end{aligned}$$

with the random variable $\boldsymbol{\xi}$ defined as,

$$\mathbb{P}(\boldsymbol{\xi} = 2^n) = \frac{1}{2^{n+1}} \quad \forall\, n \in \mathbb{N}^* \tag{9}$$

where $\mathbb{N}^* = \{1, 2, 3, \dots\}$ are the strictly positive integers, and $\Xi = \{2, \dots, 2^n, \dots\}$.

Remark 3. $\boldsymbol{\xi}$ is a random variable because,

- for all $\xi \in \Xi$, $\mathbb{P}(\boldsymbol{\xi} = \xi) \in [0\,;\,1]$,
- $\sum_{\xi\in\Xi} \mathbb{P}(\boldsymbol{\xi} = \xi) = \sum_{n\in\mathbb{N}^*} \mathbb{P}(\boldsymbol{\xi} = 2^n) = \sum_{n\in\mathbb{N}^*} 2^{-n} = 1$

Complete recourse
We have seen previously that the problem has *complete* recourse because $\operatorname{pos} W = \mathbb{R}^2$ when considering the standard form of the second-stage problem.

Not relatively complete
However, we can show that,

$$K_2 = \{x \mid \mathcal{Q}(x) < \infty\} = \emptyset.$$

Consequently, $K_1 \not\subseteq K_2$, and the problem doesn't have *relatively complete* recourse.

1. $\boldsymbol{\xi}$ has not finite expectation,

$$\mathbb{E}[\boldsymbol{\xi}] = \sum_{\xi\in\Xi} \xi\, \mathbb{P}(\boldsymbol{\xi} = \xi) = \sum_{n\in\mathbb{N}^*} 2^n \frac{1}{2^{n+1}} = \sum_{n\in\mathbb{N}^*} 1 = +\infty$$

2. Let's consider $x \in \mathbb{R}^2$,

$$\begin{aligned}
\mathcal{Q}(x) &= \mathbb{E}_{\boldsymbol{\xi}}\left[\min_{y_1,y_2\in\mathbb{R}^+}\ 2y_1 + y_2 \,\middle|\, y_1 + y_2 \geq 1 - x_1 \text{ and } y_1 \geq \boldsymbol{\xi} - x_1 - x_2\right] \\
&\geq \mathbb{E}_{\boldsymbol{\xi}}\left[\min_{y_1,y_2\in\mathbb{R}^+}\ 2y_1 \,\middle|\, y_1 + y_2 \geq 1 - x_1 \text{ and } y_1 \geq \boldsymbol{\xi} - x_1 - x_2\right] && \text{because } y_2 \geq 0 \text{ and same feasible set} \\
&\geq \mathbb{E}_{\boldsymbol{\xi}}\left[\min_{y_1,y_2\in\mathbb{R}^+}\ 2y_1 \,\middle|\, y_1 \geq \boldsymbol{\xi} - x_1 - x_2\right] && \text{by the relaxation of a constraint the feasible set is larger} \\
&= \mathbb{E}_{\boldsymbol{\xi}}\left[2\max\{0,\, \boldsymbol{\xi} - x_1 - x_2\}\right] = \sum_{\xi\in\Xi} 2\max\{0,\, \boldsymbol{\xi} - x_1 - x_2\}\, \mathbb{P}(\boldsymbol{\xi} = \xi) \\
&= \sum_{n\in\mathbb{N}^*} 2\max\{0,\, 2^n - x_1 - x_2\} \frac{1}{2^n} \\
&= \sum_{n\in\mathbb{N}^*} \max\{0,\, 2^n - x_1 - x_2\} \frac{1}{2^{n-1}}
\end{aligned}$$

There exists $n_0 \in \mathbb{N}$ such that $2^{n_0} > x_1 - x_2$, and

$$\begin{aligned}\mathcal{Q}(x) &= \sum_{n\in\mathbb{N}^*} \max\{0,\, 2^n - x_1 - x_2\} \frac{1}{2^{n-1}} \\ &= \sum_{n\geq n_0} (2^n - x_1 - x_2) \frac{1}{2^{n-1}} \\ &= +\infty\end{aligned}$$

This is true for all $x \in \mathbb{R}^2$, so $K_2 = \emptyset$.

Conclusion: we have then a confusion, *complete recourse* is presented as a special case of *relatively completeness*. So *complete recourse* should imply a *relatively complete* recourse which is not the case here.

1.4 Clarifications on definitions

1.4.1 Statement of the problem

A stochastic program with recourse can be formulated as,

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} & z(x) = \mathbb{E}_{\boldsymbol{\xi}}\left[c^T x + Q(x, \boldsymbol{\xi})\right] \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \tag{10}$$

where,

$$Q(x,\,\xi) = \left\{ \begin{array}{ll} \min_y & q(\xi)^T y \\ \text{s.t.} & W(\xi)y = h(\xi) - T(\xi)x \\ & y \geq 0 \end{array}\right. \tag{11}$$

is referred as the *second-stage program*.

A and b are fixed matrices of size $m \times n$ and $m \times 1$ respectively and q, h, T, and W are matrices of size $\overline{n} \times 1$, $\overline{m} \times 1$, $\overline{m} \times n$, and $\overline{m} \times \overline{n}$ respectively whose elements are components of a random variable $\boldsymbol{\xi}$ defined on $\mathbb{R}^N$, $N = \overline{m}(1 + n + \overline{n}) + \overline{n}$.
Strictly speaking, $\mathbb{R}^N$ could be replaced by some Borel subset Ξ of $\mathbb{R}^N$ with probability measure 1, thus generating an abstract probability space $(\Xi, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a σ-field on Ξ including the Borel sets, μ is the probability measure defined on $\mathcal{F}$, and $\mathcal{F}$ is completed with respect to μ. As observed in [2], such a replacement has no effect on the objective function $z(x)$. By $\tilde{\Xi}$ we denote the support set of the random variable $\boldsymbol{\xi}$, i. e., the smallest closed subset of $\mathbb{R}^N$.

If (11) is infeasible or unbounded below, we set $Q(x, \xi)$ equal to $+\infty$ or $-\infty$ respectively. In Section 2 of Walkup and Wets [2] is given a precise definition of the expectation operator $\mathbb{E}$ which accommodates values $+\infty$ and $-\infty$ for $Q(x, \xi)$.

By setting $z(x, \xi) = c^T x + Q(x, \xi)$, we define

$$z(x) = \mathbb{E}\left[z(x,\,\xi)\right] = \int z(x, \xi)\,\mathrm{d}\mu \tag{12}$$

to be the sum of the four quantities,

$$\begin{aligned} A\big[z(x,\xi)\big] &= \int_{0\leq z(x,\xi)<+\infty} z(x,\xi)\,\mathrm{d}\mu \\ B\big[z(x,\xi)\big] &= \int_{-\infty<z(x,\xi)<0} z(x,\xi)\,\mathrm{d}\mu \\ C\big[z(x,\xi)\big] &= \left\{\begin{array}{ll} +\infty & \text{if } z(x,\xi) = +\infty \text{ on a set of positive measure} \\ 0 & \text{otherwise}\end{array}\right. \\ D\big[z(x,\xi)\big] &= \left\{\begin{array}{ll} -\infty & \text{if } z(x,\xi) = -\infty \text{ on a set of positive measure} \\ 0 & \text{otherwise}\end{array}\right. \end{aligned}$$

In particular, $z(x) = +\infty$ if $Q(x, \xi) = \infty$ with positive probability or if the positive contribution to the integral diverges.
In view of the two ways in which $\mathcal{Q}(x)$ may be $+\infty$, the second-stage feasibility set is defined in several ways.

1.4.2 Feasibility sets

The objective function $z(x)$ has been defined for all values of x in $\mathbb{R}^n$. Nonetheless, it seems desirable to have specific knowledge about where $z(x)$ or, equivalently, $\mathcal{Q}(x)$ is finite, and this requires in part a knowledge of where $Q(x,\xi) < +\infty$, i.e., where the second-stage program (11) is feasible. As in Walkup and Wets [3], we define the following sets.

Definition 1. We define the **weak feasibility set**,

$$\begin{aligned} K_2 &= \{x \in \mathbb{R}^n \,|\, \text{ (11) is feasible with probability one}\} && (13)\\ &= \{x \in \mathbb{R}^n \,|\, Q(x,\xi) < +\infty \text{ with probability one}\} && (14)\\ &= \{x \in \mathbb{R}^n \,|\, \big(h(\xi) - T(\xi)x\big) \in \text{pos}W \text{ with probability one}\} && (15)\end{aligned}$$

Definition 2. We define also the **strong feasibility set**,

$$K_2^s = \{x \in \mathbb{R}^n \,|\, \mathcal{Q}(x) < +\infty\} \tag{16}$$

Definition 3. For each $\xi \in \mathbb{R}^N$, the **elementary feasibility set** is defined as,

$$K_2(\xi) = \{x \in \mathbb{R}^n \,|\, Q(x,\xi) < +\infty\} \tag{17}$$

Definition 4. The set

$$K_2^P = \bigcap_{\xi \in \tilde{\Xi}} K_2(\xi)$$

is said to define the **possibility interpretation** of the second-stage feasibility set.

Remark 4.

$$K_2^s \subset K_2$$

In the light of these definitions, we can define the special cases,

- *relatively complete recourse*: $K_1 \subset K_2$
  This holds if and only if for all values of $x \in K_1$, $h - Tx$ belongs to $\text{pos}\, W$ with probability 1.
- *complete recourse*: $\text{pos}\, W = \mathbb{R}^{\overline{m}}$ with probability 1.
  This is a special case of relatively complete recourse. This also allows to determine from the structure of W that $K_2 = \mathbb{R}^n$.
- *fixed recourse*: W is fixed, i.e. , for all values of $\xi \in \tilde{\Xi}$, W is constant.
- *simple recourse*: W is fixed and equal to $\begin{pmatrix} I & | & -I \end{pmatrix}$.

As explained in Walkup and Wets [2] theorem 3.5,

Prop. 1. *The weak feasibility set K_2 is closed and convex.*

As seen in Walkup and Wets [2] proposition 3.16, we have

Prop. 2. *If* $\boldsymbol{W}$ *is fixed,* $\boldsymbol{p}$ *and* $\boldsymbol{T}$ *are independent, and* $\tilde{\Xi}_T$ *(or the closure of its positive hull) is polyhedral, then* K_2 *is polyhedral.*

Furthermore, we have from proposition 3 in Birge and Louveaux [1, p.110], we have

Prop. 3. *If* $\boldsymbol{\xi}$ *has finite second moments, then*

$$\mathbb{P}(\xi \,|\, Q(x,\xi) < +\infty) \Longrightarrow \mathcal{Q}(x) < +\infty$$

And consequently,

$$K_2 = K_2^s.$$

Finally, from Theorem 4 in Birge and Louveaux [1, p.111],

Prop. 4. *For a stochastic program with fixed recourse where* $\boldsymbol{\xi}$ *has finite second moments, the sets* K_2, K_2^{μ} *and* K_2^P *coincide, i.e.*

$$K_2 = K_2^s = K_2^P = \bigcap_{\xi \in \tilde{\Xi}} K_2(\xi).$$

2 Question 2

Consider the second-stage problem defined by,

$$\begin{aligned}
\min_y\ & 2y_1 + y_2 \\
\text{s.t. } & y_1 - y_2 \le 2 - \boldsymbol{\xi} x_1, \\
& y_2 \le x_2, \\
& y_1, y_2 \ge 0
\end{aligned}$$

Let's write the problem in the standard form and determine the different elements.

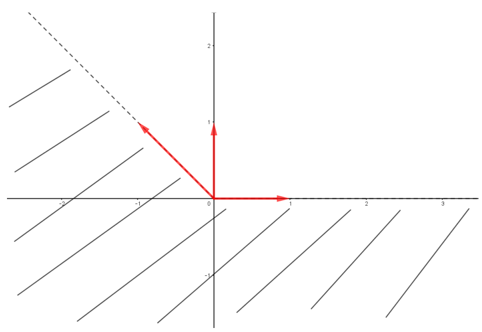
$$\begin{aligned}
\min_y\ & 2y_1 + y_2 \\
\text{s.t. } & y_1 - y_2 + s_1 = 2 - \boldsymbol{\xi} x_1 \quad, \\
& \phantom{y_1 - {}} y_2 + s_2 = \phantom{2 - \boldsymbol{\xi}} x_2, \\
& y_1, y_2, s_1, s_2 \ge 0
\end{aligned}$$

So,

$$q^T = \begin{pmatrix} 2 & 1 & 0 & 0 \end{pmatrix} \qquad W = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad h(\xi) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$y^T = \begin{pmatrix} y_1 & y_2 & s_1 & s_2 \end{pmatrix} \qquad T(\xi) = \begin{pmatrix} \xi & 0 \\ 0 & -1 \end{pmatrix}$$

Remark 5. The problem is this time not complete, because the points in the shaded area cannot be reached as a non-negative linear combination of the column vectors of the matrix W.



Computation $K_2(\xi)$. Let's take $\xi \in \Xi$,

$$\begin{aligned}
K_2(\xi) &= \left\{ x \,\middle|\, \text{second-stage is feasible for the realization } \xi \right\} \\
&= \left\{ x \,\middle|\, Q(x, \xi) < +\infty \right\} \\
&= \left\{ x \,\middle|\, \exists\, y \ge 0, \text{ such that } W(\xi) y = h(\xi) - T(\xi) x \right\}
\end{aligned}$$

We have,

$$x \in K_2(\xi) \Longleftrightarrow \exists\, y_1, y_2 \ge 0, \text{ such that } \left\{ \begin{array}{l} y_1 + \xi x_1 \le 2 + y_2 \\ y_2 \le x_2 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} \xi x_1 \le 2 + x_2 \\ x_2 \ge 0 \end{array} \right.$$

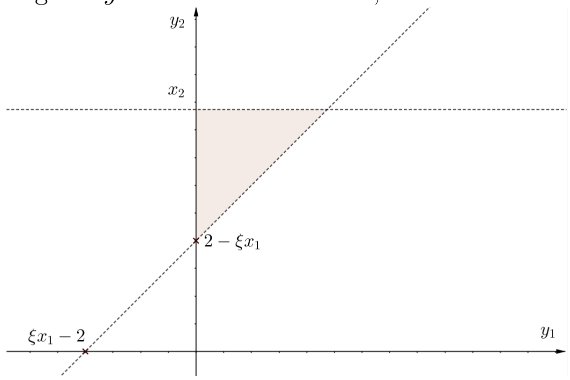
Indeed,

- Suppose $\exists\, y_1, y_2 \ge 0$, such that $\left\{ \begin{array}{l} y_1 + \xi x_1 \le 2 + y_2 \\ y_2 \le x_2 \end{array} \right.$. Then, $x_2 \ge y_2 \ge 0$, so $x_2 \ge 0$.
  And, $\xi x_1 \le 2 + y_2 \le 2 + x_2$.

- Suppose $\begin{cases} \xi x_1 \leq 2 + x_2 \\ x_2 \geq 0 \end{cases}$ then by taking, $y_1 = 2 + x_2 - \xi x_1 \geq 0$ and $y_2 = x_2 \geq 0$, we have $y_1 + \xi x_1 = 2 + x_2 \leq 2 + y_2$ and $y_2 \leq x_2$.

Visually, it can be illustrated by the following figure, by rewriting the y constraints as follows,

$$\begin{cases} y_1 - y_2 \leq 2 - \xi x_1 \\ y_2 \leq x_2 \\ y_1, y_2 \geq 0 \end{cases}$$



Therefore,

$$K_2(\xi) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x_2 \geq 0 \quad \text{and} \quad \xi x_1 - x_2 \leq 2 \right\}$$

Computation K_2. The matrix W doesn't depend on ξ, so the recourse is fixed, and for both cases (a) and (b), the distributions have finite second order moments,

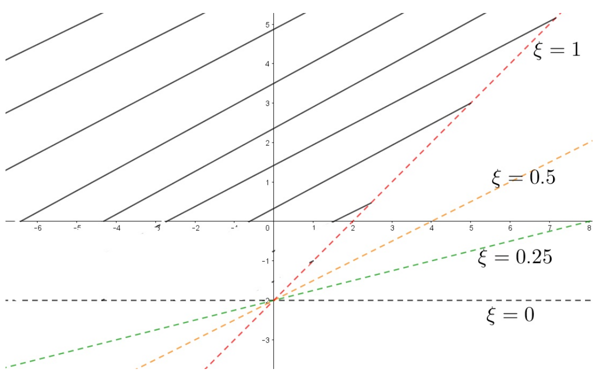
(a) when $\boldsymbol{\xi} \sim \mathcal{U}[0;1]$, $\mathbb{E}[\boldsymbol{\xi}] = \dfrac{1}{2}$ and $\mathbb{E}[\boldsymbol{\xi}^2] = \mathbb{V}\text{ar}[\boldsymbol{\xi}] + \mathbb{E}[\boldsymbol{\xi}]^2 = \dfrac{1}{12} + \dfrac{1}{4} = \dfrac{1}{3} < \infty$.

(b) when $\boldsymbol{\xi} \sim \mathcal{P}(\lambda)$, $\mathbb{E}[\boldsymbol{\xi}] = \lambda$ and $\mathbb{E}[\boldsymbol{\xi}^2] = \mathbb{V}\text{ar}[\boldsymbol{\xi}] + \mathbb{E}[\boldsymbol{\xi}]^2 = \lambda + \lambda^2 < \infty$.

Therefore, for (a) and (b), from Theorem 4 in Birge and Louveaux [1, p.111],

$$K_2 = K_2^P = \bigcap_{\xi \in \Xi} K_2(\xi). \tag{18}$$

(a) $\boldsymbol{\xi} \sim U[0,\, 1]$.



As seen with the figure above

- when $x_1 \geq 0$, for $0 \leq \xi_1 \leq \xi_2 \leq 1$, $\xi_1 x_1 - x_2 \leq 2 \Longrightarrow \xi_2 x_1 - x_2 \leq 2$,
- when $x_1 < 0$, for all $\xi \in [0,1]$, $\xi x_1 \leq 0$, so $x_2 \geq 0 \Longrightarrow \xi x_1 - x_2 \leq 2$,

therefore, we have $K_2(\xi_2) \subset K_2(\xi_1)$, and

$$K_2 = \bigcap_{\xi \in \Xi} K_2(\xi) = K_2(1) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x_2 \geq 0 \text{ and } x_1 - x_2 \leq 2 \right\} \tag{19}$$

(b) $\boldsymbol{\xi} \sim \text{Poisson}(\lambda)$, $\lambda > 0$.

In this situation, $\xi \in \{0,1,2,\ldots\}$. Therefore, we must distinguish two cases,

- $x_1 \leq 0$

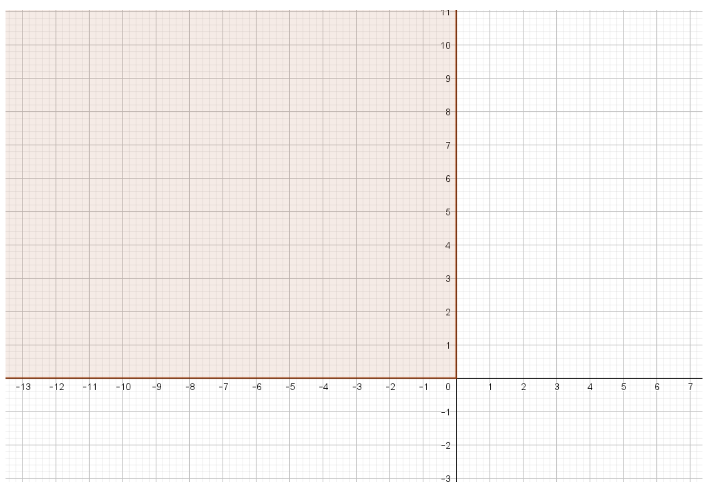
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K_2 \,,\; x_1 \leq 0 \iff \forall \xi,\; x_2 \geq 0 \leq 0 \text{ and } \xi x_1 - x_2 \leq 2 \,,\; x_1 \leq 0$$
$$\iff \forall \xi,\; x_2 \geq 0 \text{ and } x_2 \geq \underbrace{-2 + \xi x_1}_{\leq 0} \,,\; x_1 \leq 0$$
$$\iff x_2 \geq 0 \,,\; x_1 \leq 0$$

- $x_1 > 0$
  Let's consider some $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ with $x_1 > 0$.
  For ξ_0 sufficiently large, $\xi_0 x_1 > x_2 + 2$, for example, take $\xi_0 = \left\lfloor \frac{x_2 + 2}{x_1} \right\rfloor + 1$. So, $x \notin K_2(\xi_0)$ and,

$$x \notin \bigcap_{\xi \in \{0\,,1,\,2,\,\ldots\}} K_2(\xi) = K_2$$

Conclusion,

$$K_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x_1 \leq 0 \text{ and } x_2 \geq 0 \right\}$$



What properties can we reasonably expect for K_2 ?

The recourse matrix W is fixed and in both cases (a) and (b) the random variable $\boldsymbol{\xi}$ has finite second order moments. We can deduce the following properties,

- from Theorem 5 in Birge and Louveaux [1, p.111], $K_2 = \bigcap_{\xi \in \tilde{\Xi}} K_2(\xi)$,
- K_2 should be closed and convex,
- also, $\boldsymbol{h}$ and $\boldsymbol{T}$ are independent because $\boldsymbol{h}$ is constant, and the positive hull of $\tilde{\Xi}_T$ is polyhedral, so K_2 is polyhedral.

We can observe such properties in both examples.

3 Question 3

We consider a baker who decides the morning how many breads he will cook, and can sell his/her production during the day with profit. In the evening, the unsold items can be sold at a reduced price as the breads are less fresh. Two types of bread can be produced : white bread and whole wheat bread.
A unit of white bread cost 1.5$ to produce, while the whole wheat bread costs 1.8$. The are sold at 3$ and 4$ per unit respectively. The unsold breads are sold the evening at 1$ and 1.2$.
200g of flour are required to produce one unit of whole wheat bread, while for one unit of white bread, 150g are sufficient. The baker has a total of 12kg of flour.
We assume that the demand for white bread follows a normal distribution with mean 50 and standard deviation 5, and the demand for whole wheat bread follows a normal distribution with mean 30 and standard deviation 2. The two demands are correlated, with a covariance between of 0.4.

1. *Describe the problem as a two-stage stochastic program. Explain the choice of first- and second-stage decision variables, and form the mathematical program.*

   First-stage variables:

   - x_{wi} : units of white bread produced for the day
   - x_{we} : units of whole wheat bread produced for the day

   Second-stage variables:

   - y_{wi} : units of white bread sold during the day at the high price
   - y_{we} : units of whole wheat bread sold during the day at the high price
   - w_{wi} : units of white bread sold at the end of the day at the low price
   - w_{we} : units of whole wheat bread sold at the end of the day at the low price

   Similarly, the random variable realizations will be divided in $\xi = \begin{pmatrix} \xi_{wi} \\ \xi_{we} \end{pmatrix}$,

   - ξ_{wi} : white bread demand
   - ξ_{we} : whole wheat bread demand

   The problem can be written as,

$$\begin{aligned} \min_{x=(x_{wi},x_{we})\in\mathbb{R}^2_+} \quad & 1.5x_{wi} + 1.8x_{we} + \mathcal{Q}(x) \\ \text{s.t.} \quad & 0.15x_{wi} + 0.2x_{we} \leq 12 \\ & x_{wi}, x_{we} \geq 0 \end{aligned}$$

$$\text{where } \mathcal{Q}(x) = \mathbb{E}_{\boldsymbol{\xi}}[Q(x, \boldsymbol{\xi})] \quad \text{and} \quad Q(x,\boldsymbol{\xi}) = \begin{cases} \min_{y,w} & -3y_{wi} - 4y_{we} - w_{wi} - 1.2w_{we} \\ \text{s.t.} & y_{wi} \leq \boldsymbol{\xi}_{wi} \\ & y_{wi} + w_{wi} \leq x_{wi} \\ & y_{we} \leq \boldsymbol{\xi}_{we} \\ & y_{we} + w_{we} \leq x_{we} \\ & y_{wi}, y_{we}, w_{wi}, w_{we} \geq 0 \end{cases}$$

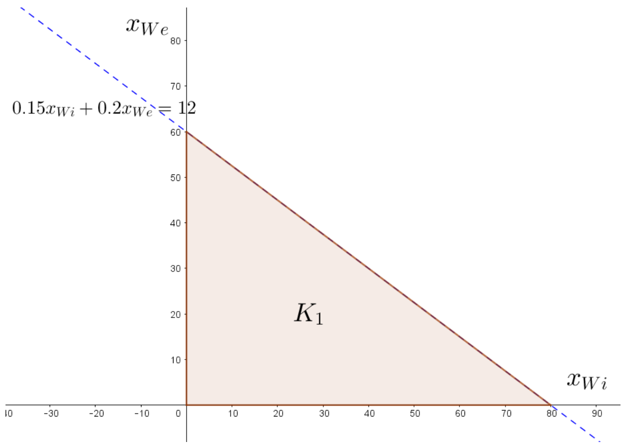
2. *Give the expression of K_1, K_2, and K_2^P . Is the recourse complete, relatively complete, simple ?*

   (a) $K_1 = \left\{ \begin{pmatrix} x_{wi} \\ x_{we} \end{pmatrix} \geq 0 \,\middle|\, 0.15x_{wi} + 0.2x_{we} \leq 12 \right\}$.

   (b)

$$\xi = \begin{pmatrix} \xi_{wi} \\ \xi_{we} \end{pmatrix} \sim \mathcal{N}\left(m = \begin{pmatrix} 50 \\ 30 \end{pmatrix}, \Sigma = \begin{pmatrix} 25 & 0.4 \\ 0.4 & 4 \end{pmatrix} \right)$$

   where m is the mean vector and Σ is the covariance matrix.



The second stage problem can be written in the standard form as,

$$
\begin{aligned}
\min_{y,w}\ & -3y_{wi} - 4y_{we} - w_{wi} - 1.2w_{we} \\
\text{s.t. } & y_{wi} + s_1 = \boldsymbol{\xi}_{wi} \\
& y_{wi} + w_{wi} + s_2 = x_{wi} \\
& y_{we} + s_3 = \boldsymbol{\xi}_{we} \\
& y_{we} + w_{we} + s_4 = x_{we} \\
& y_{wi}, y_{we}, w_{wi}, w_{we}, s_{1,2,3,4} \geq 0
\end{aligned}
$$

$$q^T = \begin{pmatrix} -4 & -1 & -1 & -1.2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$y^T = \begin{pmatrix} y_{wi} & y_{we} & w_{wi} & w_{we} & s_1 & s_2 & s_3 & s_4 \end{pmatrix}$$

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$h(\xi) = \begin{pmatrix} \xi_{wi} \\ 0 \\ \xi_{we} \\ 0 \end{pmatrix}$$

$$T(\xi) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

We can see that the recourse matrix W is independent of ξ and the random variable $\boldsymbol{\xi}$ have finite second order moment. Therefore,

$$K_2 = K_2^P = \bigcap_{\xi \in \Xi} K_2(\xi)$$

For the computation of K_2, let's discuss two situations, when considering the true distribution for $\boldsymbol{\xi}$ and when considering the modeling distribution,

(i) the true distribution of $\boldsymbol{\xi}$ has support $\Xi \subset \mathbb{R}_+^2$, so

$$\forall \xi \in \Xi,\ K_2(\xi) = \left\{ \begin{pmatrix} x_{wi} \\ x_{we} \end{pmatrix} \geq 0 \right\} = \mathbb{R}_+^2.$$

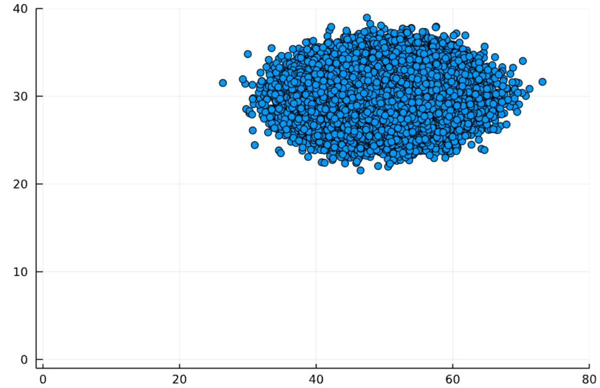
And,

$$K_2 = K_2^P = \mathbb{R}_+^2$$

(ii) the modeling distribution for $\boldsymbol{\xi}$ we choose has support $\Xi = \mathbb{R}^2$. Therefore, for $\xi = \begin{pmatrix} \xi_{wi} \\ \xi_{we} \end{pmatrix}$ such that $\xi_{wi} \times \xi_{we} < 0$, we have

$$K_2(\xi) = \emptyset \quad \text{and} \quad K_2 = \bigcap_{\xi \in \Xi} K_2(\xi) = \emptyset$$

But as illustrated in the figure below, $\mathbb{P}(\boldsymbol{\xi} \notin \mathbb{R}_+^2) \approx 0$, so we can easily consider that $\Xi \subset \mathbb{R}_+^2$ and $K_2 = \mathbb{R}_+^2$.



Plot of 100 000 realizations of $\boldsymbol{\xi} \sim \mathcal{N}(m\,;\,\Sigma)$.

Conclusion, the second-stage problem,

- is *relatively complete* because $K_1 \subset K_2$.

- is **not** *complete* because $pos\, W \neq \mathbb{R}^4$.
  By considering $b = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, there exists $y \in \mathbb{R}^4$ such that: $y^T W \leq 0$ and $y^T b > 0$, take $y = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.
  Therefore, by Farkas lemma, there is no $x \in \mathbb{R}^4_+$ such that $Wx = b$, so $pos\, W \neq \mathbb{R}^4$.

- is **not** *simple* because W is not in the form $\begin{pmatrix} I \mid -I \end{pmatrix}$

3. *Draw* 100 *demand scenarios and implement the problem using Julia and JuMP.*

See notebook.

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\xi} wi \\ \boldsymbol{\xi} we \end{pmatrix} \sim \mathcal{N}\left(m = \begin{pmatrix} 50 \\ 30 \end{pmatrix}, \Sigma = \begin{pmatrix} 25 & 0.4 \\ 0.4 & 4 \end{pmatrix} \right)$$

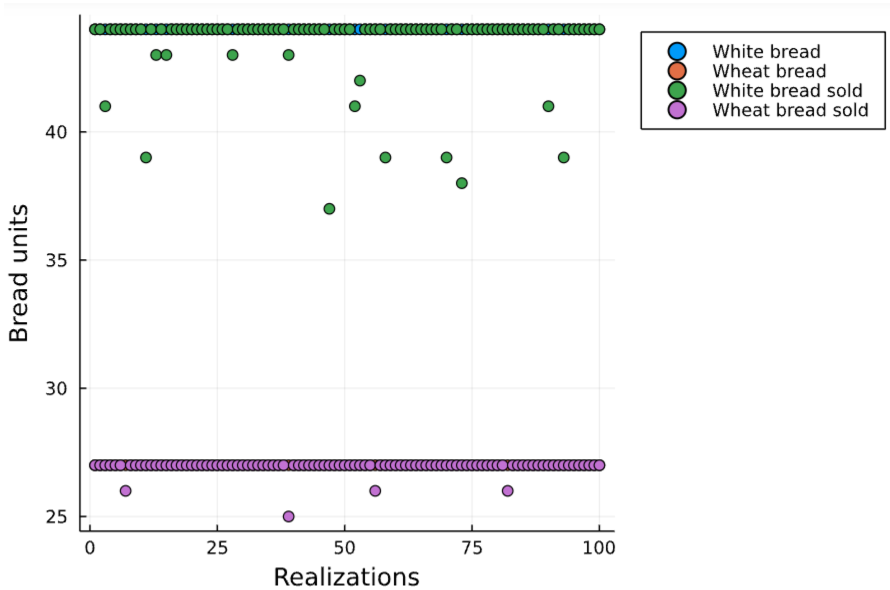
where m is the mean vector and Σ is the covariance matrix.

To solve this problem, we consider a Monte-Carlo approximation by generating a random vector $\tilde{\boldsymbol{\xi}}$ of S independent and identically distributed copies of the random variable $\boldsymbol{\xi}$. We have $\tilde{\boldsymbol{\xi}} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_S)^T$ where for all $s \in \{1, \ldots, S\}$, $\boldsymbol{\xi}_s = (\boldsymbol{\xi}_{wi,s}, \boldsymbol{\xi}_{we,s})^T$. Then, we can write the extensive form of the problem as,

$$\begin{aligned}
\min_{x=(x_{wi},x_{we})\in\mathbb{R}^2_+} \quad & 1.5x_{wi} + 1.8x_{we} + \frac{1}{S}\sum_{s=1}^{S}(-3y_{wi,s} - 4y_{we,s} - w_{wi,s} - 1.2w_{we,s}) \\
\text{s.t. } & 0.15x_{wi} + 0.2x_{we} \leq 12 \\
& y_{wi,s} \leq \boldsymbol{\xi}_{wi,s} \\
& y_{wi,s} + w_{wi} \leq x_{wi,s} \\
& y_{we,s} \leq \boldsymbol{\xi}_{we,s} \\
& y_{we,s} + w_{we,s} \leq x_{we,s} \\
& y_{wi,s}, y_{we,s}, w_{wi,s}, w_{we,s} \geq 0 \quad \text{for all } s \in \{1, \ldots, S\} \\
& x_{wi}, x_{we} \geq 0
\end{aligned}$$

4. *Interpret the solution.*

For a benefit of about 125\$, the baker produced ≈ 42.04 white breads and ≈ 28.5 wheat breads. We can observe that the baker produced a number of wheat bread close to the mean value, but slightly less to make sure that in most situations all wheat breads are sold at the high price. This can be understood by observing that the largest benefit is given by the wheat bread, 2.2\$ against 1.5\$ for white bread. Then, the rest of flower is used to produce the white bread. This is illustrated by the figure below, where for every realization, we plot the selling strategy compared to the number of bread produced. The solutions have been computed with the integer constraint.



5. *Adapt the Julia code in order to obtain integer solutions.*

   See notebook. When solving using integer programming, we obtain a production of 44 white breads and 27 wheat breads, for a total benefit of 124.74\$. Concerning the production strategy the same reasoning can be applied than in the previous question.

   Other possible modelling with simple recourse. We suppose that every bread is sold, either during the day or in the evening. The strategy is to count the profit done for every bread, but then to penalize in the second-stage if there are some bread left.

   $$\text{Let's denote, } \begin{cases} y_{wi}^+ : & \text{rest of white bread} \\ y_{wi}^- : & \text{missing white bread} \\ y_{we}^+ : & \text{rest of wheat bread} \\ y_{we}^- : & \text{rest of wheat bread} \end{cases} \quad \text{and} \quad \begin{cases} \text{Profit for white bread : } 1.5\$ - 3\$ = -1.5\$ \\ \text{Profit for whet bread : } 1.8\$ - 4\$ = -2.2\$ \\ \text{Penalization for left white bread : } 3\$ - 1\$ = 2\$ \\ \text{Profit for white bread : } 4\$ - 1.2\$ = 2.8\$ \end{cases}$$

   Therefore, the problem becomes

   $$\begin{aligned} \min_{x=(x_{wi},x_{we})\in\mathbb{R}_+^2} \quad & -1.5x_{wi} - 2.2x_{we} + \mathcal{Q}(x) \\ \text{s.t. } & 0.15x_{wi} + 0.2x_{we} \leq 12 \\ & x_{wi}, x_{we} \geq 0 \end{aligned}$$

   $$\text{where } \mathcal{Q}(x) = \mathbb{E}_{\boldsymbol{\xi}}[Q(x,\boldsymbol{\xi})] \quad \text{and} \quad Q(x,\boldsymbol{\xi}) = \begin{cases} \min_{y^+,y^-,} & 2y_{wi}^+ + 2.8y_{we}^+ \\ \text{s.t.} & y_{wi}^+ - y_{wi}^- = x_{wi} - \boldsymbol{\xi}_{wi} \\ & y_{we}^+ - y_{we}^- = x_{we} - \boldsymbol{\xi}_{we} \\ & y_{wi}^+, y_{we}^+, y_{wi}^-, y_{we}^-, \geq 0 \end{cases}$$

   Then, we have that the recourse is *simple* because,

   $$q^T = \begin{pmatrix} 2 & 2.8 & 0 & 0 \end{pmatrix} \qquad y^T = \begin{pmatrix} y_{wi}^+ & y_{we}^+ & y_{wi}^- & y_{we}^- \end{pmatrix} \qquad W = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

4 Some common mistakes

1. $\mathbb{E}[\xi]$ not finite does not necessarily imply that $\mathcal{Q}(x) = +\infty$. Consider,

$$\begin{aligned} \min_y \; & y_1 - y_2 \\ \text{s.t. } & y_1 \geq \xi \\ & y_2 \leq \xi \end{aligned}$$

   so, we have for all ξ, $Q(x, \xi) = 0$ even if $\mathbb{E}[\xi] = \infty$.

2. Considering, $z(x) = \min_x c^T x + \mathbb{E}[Q(x, \xi)]$. There is a big difference between:

   - minimizing z for a fixed ξ and then doing the expectation of the x found for each realization
   - minimizing z, i.e. finding minimal x such that the second-stage is minimised in expectation.

   This is because, in general,

$$\min_x \{c^T x + \mathbb{E}[Q(x, \xi)]\} \neq \mathbb{E}[\min_x \{c^T x + Q(x, \xi)\}]$$

3. **Integer minimization:** It is very different :

   - to minimize a problem considering real values, then rounding the solutions
   - minimizing a problem using integer constraint methods.

   See notebook *Example_Integer_Programming*.

5 Appendices

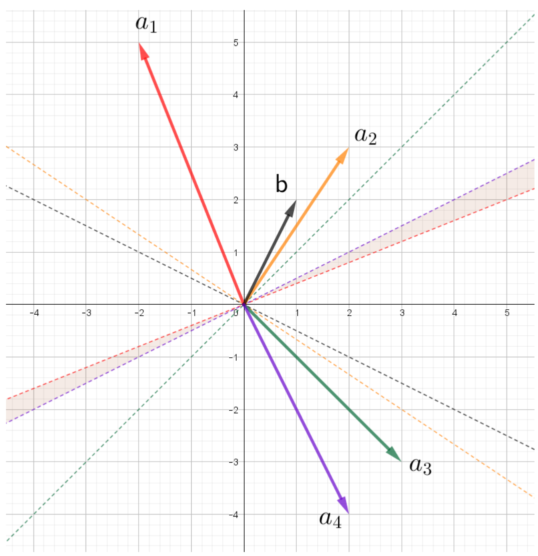
5.1 Farkas Lemma

Lemma 5. *For a matrix $A = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$, only one of the following situation can be true,*

1. *there exists $x \in \mathbb{R}^n$, such that $x \geq 0$ and $Ax = b$,*
2. *there exists $y \in \mathbb{R}^m$ such that $y^T A \geq 0$ and $b^T y < 0$, or conversely by considering $-y$.*

Example 1. Let's consider the matrix $A = \begin{pmatrix} -2 & 2 & 3 & 2 \\ 5 & 3 & -3 & -4 \end{pmatrix}$.

1. If $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *There exists $x \in \mathbb{R}^n$, such that $x \geq 0$ and $Ax = b$,* for example take $x = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.



As you can see, all vectors y in the shaded region, $y^T b$ and $y^T A$ will have the same sign.

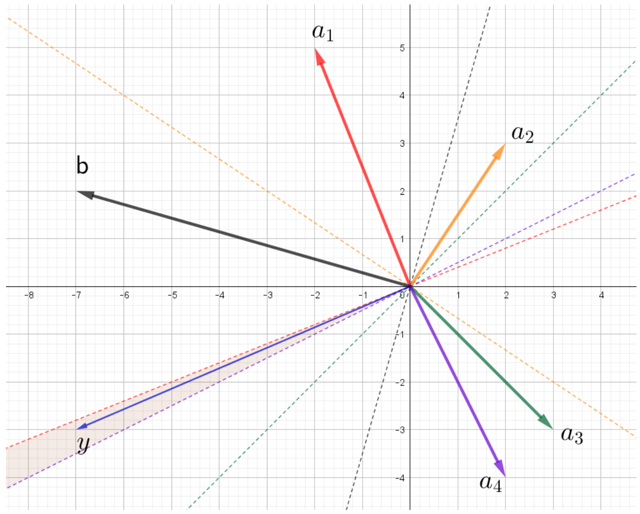
2. If $b = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$. *There exists $y \in \mathbb{R}^m$ such that $y^T A < 0$ and $b^T y \geq 0$.*

For example by taking $y = \begin{pmatrix} -7 \\ -3 \end{pmatrix}$ we can see that,

$$y^T A = \begin{pmatrix} -1 & -23 & -12 & -2 \end{pmatrix} < 0$$

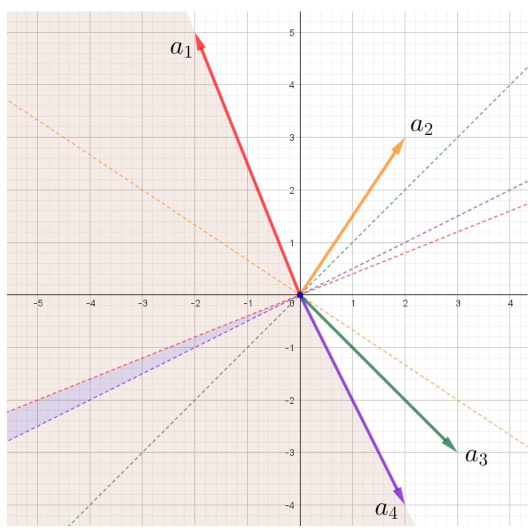
and

$$y^T b = 44 > 0.$$



Conclusion. For all vectors b,

- in the shaded region, the condition (2) is true, because there will be a vector in the blue shaded region satisfying (2),
- otherwise, condition (1) is valid.



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[2] Walkup, D. W. and Wets, R. J.-B. Stochastic programs with recourse. *SIAM Journal on Applied Mathematics*, 15(5):1299–1314, 1967.

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