Chapter 5

Complex Numbers and Quadratic Equations

Complex Numbers: This chapter introduces complex numbers, which are numbers in the real numbers and the imaginary unit. It covers the representation of complex numbers in the form of a complex plane, operations on complex numbers (addition, subtraction, multiplication, and division), and the modulus and argument of a complex number.

Quadratic Equations: The chapter also explores quadratic equations, which are equations of the for where a, b, and c are constants and the variable. It covers methods for solving quadratic equations, including factorization, completing the square, and using the quadratic formula. The discriminate of a quadratic equation is discussed, and its role in determining the nature of roots (real or complex) is explained.

Complex Roots of Quadratic Equations:

The connection between complex numbers and quadratic equations is established by discussing the nature of roots. If the discriminate is negative, the roots are complex conjugates. The chapter illustrates how complex numbers are a natural extension of the real number system when dealing with quadratic equations.

Quadratic Equations in Real life:

The practical applications of quadratic equations in various real-life situations are also highlighted in this chapter, demonstrating the relevance of these mathematical concepts in solving problems from different fields.

Exercise 5.1

Express each of the complex numbers given in Exercises 1 to 10 in the form a + ib.

1. (5i) (-3/5i)

Solution:

$(5i)(-3/5i) = 5 \times (-3/5) \times i^2$

$= -3 \times -1 \; [i^2 = -1]$

$= 3$

Hence,

$(5i)(-3/5i) = 3 + i0$

2. $i^9 + i^{19}$

Solution:

$i^9 + i^{19} = (i^2)^4 . i + (i^2)^9 . i$

$= (-1)^4 . i + (-1)^9 .i$

$= 1 \times i + -1 \times i$

$= i - i$

$= 0$

Hence,

$i^9 + i^{19} = 0 + i0$

3. i^{-39}

Solution:

$i^{-39} = 1/ i^{39} = 1/ i^{4 \times 9 + 3} = 1/ (1^9 \times i^3) = 1/ i^3 = 1/ (-i) \; [i^4 = 1, i^3 = -I \text{ and } i^2 = -1]$

Now, multiplying the numerator and denominator by i we get

$i^{-39} = 1 \times i / (-i \times i)$

$= i/ 1 = i$

Hence,

$i^{-39} = 0 + i$

4. $3(7 + i7) + i(7 + i7)$

Solution:

$3(7 + i7) + i (7 + i7) = 21 + i21 + i7 + i^2 7$

$= 21 + i28 – 7$ [$i^2 = -1$]

$= 14 + i28$

Hence,

$3(7 + i7) + i (7 + i7) = 14 + i28$

5. $(1 – i) – (–1 + i6)$

Solution:

$(1 – i) – (–1 + i6) = 1 – i + 1 – i6$

$= 2 – i7$

Hence,

$(1 – i) – (–1 + i6) = 2 – i7$

6.

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$$

Solution:

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$$
$$=\frac{1}{5}+\frac{2}{5}i-4-\frac{5}{2}i$$
$$=\left(\frac{1}{5}-4\right)+i\left(\frac{2}{5}-\frac{5}{2}\right)$$
$$=\frac{-19}{5}+i\left(\frac{-21}{10}\right)$$
$$=\frac{-19}{5}-\frac{21}{10}i$$

Hence,

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)=\frac{-19}{5}-\frac{21}{10}i$$

7. $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

Solution:

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(\frac{-4}{3}+i\right)$$
$$=\frac{1}{3}+\frac{7}{3}i+4+\frac{1}{3}i+\frac{4}{3}-i$$
$$=\left(\frac{1}{3}+4+\frac{4}{3}\right)+i\left(\frac{7}{3}+\frac{1}{3}-1\right)$$
$$=\frac{17}{3}+i\frac{5}{3}$$

Hence,

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)=\frac{17}{3}+i\frac{5}{3}$$

8. $(1 - i)^4$

Solution:

$(1 - i)^4 = [(1 - i)^2]^2$

$= [1 + i^2 - 2i]^2$

$= [1 - 1 - 2i]^2$ $[i^2 = -1]$

$= (-2i)^2$

$= 4(-1)$

$= -4$

Hence, $(1 - i)^4 = -4 + 0i$

9. $(1/3 + 3i)^3$

Solution:

$$\left(\frac{1}{3}+3i\right)^3 = \left(\frac{1}{3}\right)^3 + (3i)^3 + 3\left(\frac{1}{3}\right)(3i)\left(\frac{1}{3}+3i\right)$$

$$= \frac{1}{27} + 27i^3 + 3i\left(\frac{1}{3}+3i\right)$$

$$= \frac{1}{27} + 27(-i) + i + 9i^2 \qquad [i^3 = -i]$$

$$= \frac{1}{27} - 27i + i - 9 \qquad [i^2 = -1]$$

$$= \left(\frac{1}{27} - 9\right) + i(-27+1)$$

$$= \frac{-242}{27} - 26i$$

Hence, $(1/3 + 3i)^3 = -242/27 - 26i$

10. $(-2 - 1/3i)^3$

Solution:

$$\left(-2-\frac{1}{3}i\right)^3 = (-1)^3\left(2+\frac{1}{3}i\right)^3$$

$$= -\left[2^3 + \left(\frac{i}{3}\right)^3 + 3(2)\left(\frac{i}{3}\right)\left(2+\frac{i}{3}\right)\right]$$

$$= -\left[8 + \frac{i^3}{27} + 2i\left(2+\frac{i}{3}\right)\right]$$

$$= -\left[8 - \frac{i}{27} + 4i + \frac{2i^2}{3}\right] \qquad \left[i^3 = -i\right]$$

$$= -\left[8 - \frac{i}{27} + 4i - \frac{2}{3}\right] \qquad \left[i^2 = -1\right]$$

$$= -\left[\frac{22}{3} + \frac{107i}{27}\right]$$

$$= -\frac{22}{3} - \frac{107}{27}i$$

Hence,

$(-2 - 1/3i)^3 = -22/3 - 107/27i$

Find the multiplicative inverse of each of the complex numbers given in Exercises 11 to 13.

11. 4 – 3i

Solution:

Let's consider $z = 4 - 3i$

Then,

$= 4 + 3i$ and

$|z|^2 = 4^2 + (-3)^2 = 16 + 9 = 25$

Thus, the multiplicative inverse of $4 - 3i$ is given by z^{-1}

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3}{25}i$$

12. $\sqrt{5} + 3i$

Solution:

Let's consider $z = \sqrt{5} + 3i$

$$\text{Then, } \bar{z} = \sqrt{5} - 3i \text{ and } |z|^2 = \left(\sqrt{5}\right)^2 + 3^2 = 5 + 9 = 14$$

$|z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14$

Thus, the multiplicative inverse of $\sqrt{5} + 3i$ is given by z^{-1}

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3i}{14}$$

13. – i

Solution:

Let's consider $z = -i$

$$\text{Then, } \bar{z} = i \text{ and } |z|^2 = 1^2 = 1$$

Thus, the multiplicative inverse of $-i$ is given by z^{-1}

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{i}{1} = i$$

14. Express the following expression in the form of a + ib:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

Solution:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

$$=\frac{(3)^2-\left(i\sqrt{5}\right)^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i} \qquad \left[(a+b)(a-b)=a^2-b^2\right]$$

$$=\frac{9-5i^2}{2\sqrt{2}i}$$

$$=\frac{9-5(-1)}{2\sqrt{2}i} \qquad \left[i^2=-1\right]$$

$$=\frac{9+5}{2\sqrt{2}i}\times\frac{i}{i}$$

$$=\frac{14i}{2\sqrt{2}i^2}$$

$$=\frac{14i}{2\sqrt{2}(-1)}$$

$$=\frac{-7i}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{-7\sqrt{2}i}{2}$$

Hence,

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-i\sqrt{2}\right)} = 0+\frac{-7\sqrt{2}i}{2}$$

Given,

$z = -1 - i\sqrt{3}$

Let $r\cos\theta = -1$ and $r\sin\theta = -\sqrt{3}$

On squaring and adding, we get

$(r\cos\theta)^2 + (r\sin\theta)^2 = (-1)^2 + \left(-\sqrt{3}\right)^2$

$r^2\left(\cos^2\theta + \sin^2\theta\right) = 1 + 3$

$r^2 = 4$ $\quad\left[\cos^2\theta + \sin^2\theta = 1\right]$

$r = \sqrt{4} = 2$ $\quad$ [Conventionally, $r > 0$]

Thus, modulus = 2

So, we have

$2\cos\theta = -1$ and $2\sin\theta = -\sqrt{3}$

$\cos\theta = \frac{-1}{2}$ and $\sin\theta = \frac{-\sqrt{3}}{2}$

As the values of both sin θ and cos θ are negative, θ lies in III Quadrant,

$\text{Argument} = -\left(\pi - \frac{\pi}{3}\right) = \frac{-2\pi}{3}$

Therefore, the modulus and argument of the complex number $-1-\sqrt{3}\,i$ are 2 and $\frac{-2\pi}{3}$ respectively.

2Marks Questions & Answers

1. Evaluate the value of i^{-39}

Ans:

Let us solve the given expression further –

$i^{-39} = i^{-38-1}$

$\Rightarrow i^{-39} = (\, i^2\,)^{-19} \cdot \frac{1}{i}$

$\Rightarrow i^{-39} = (\, i\,)^{-19} \cdot \frac{1}{i}$

$\Rightarrow i^{-39} = -\frac{1}{i}\,\frac{i}{i}$

Therefore,

$i^{-39} = i$

2. If $(\frac{1+i}{1-i})^m = 1$, then find the least positive integral value of m.

Ans:

Let us simplify the given expression –

$(\frac{1+i}{1-i})^m = (\frac{1+i}{1-i} \times \frac{1+i}{1+i})^m$

$\Rightarrow (\frac{1+2i-1}{1+1})^m$

$\Rightarrow (\frac{2i}{2})^m$

$\Rightarrow i^m = 1$

Now, we know that $i^{4n} = 1$

Therefore, $i^m = i^{4n}$

$\Rightarrow m = 4n$.

Where n is an integer.

Hence, the least positive integral value of m is 4.

3. Evaluate the expression $(1+i)^4$

Ans:

Let us simplify the given expression –

$$(1+i)^4 = [(1+i)^2]^2$$

$$\Rightarrow (1+i2+i^2)^2$$

$$\Rightarrow (1+i2-1)^2$$

$$\Rightarrow 4i^2$$

$$\Rightarrow -4$$

Hence, $(1+i)^4 = -4+0i$

4. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$

Ans:

Let us simplify the given expression –

$$Z=\frac{1+i}{1-i}-\frac{1-i}{1+i}=\frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}$$

$$=\frac{(1+2i-1)-(1-2i-1)}{1+1}$$

$$=\frac{2i+2i}{2}$$

$$\Rightarrow z=2i$$

Hence, the modulus will be –

$$|z|=\sqrt{0^2+2^2}$$

$$\Rightarrow|z|=2$$

5. Express in the form of a+ib: $(1+3i)^{-1}$.

Ans:

Let us simplify the given expression –

$$(1+3i)^{-1}=\frac{1}{(1+3i)}$$

$$=\frac{1}{(1+3i)}\times\frac{1-3i)}{(1+3i)}$$

$$=\frac{1-3i}{1-9i^2}$$

$$=\frac{1-3i}{10}$$

Therefore,

$$(1+3i)^{-1}=\frac{1}{10}-i\frac{3}{10}$$

.

6. Find the conjugate of $\frac{1}{2-3i}$.

Ans:

Let us further solve the expression as –

$$\frac{1}{2-3i}=\frac{1}{2-3i}\times\frac{2+3i}{2+3i}$$

$$=\frac{2+3i}{4-9i^2}$$

$$=\frac{2+3i}{4-9}$$

$$= \frac{2+3i}{13}$$

Therefore, the conjugate will be

$$= \frac{2-3i}{13}$$

7. Express in the form of a+ib: (3i−7) + (7−4i) − (6+3i) + i^{23}

Ans:

Let us simplify the given expression –

(3i−7) + (7−4i) − (6+3i) + i^{23}=3i−7+7−4i−6−3i+i^{23}

$$\Rightarrow -4i-6+i^{22+1}$$

$$\Rightarrow -4i-6+(i^2)^{11}.i$$

$$\Rightarrow -4i-6+(-1)^{11}.i$$

$$\Rightarrow -4i-6-i$$

Therefore, (3i−7) + (7−4i) − (6+3i) + i^{23} = 6 – 5i.

8. Solve for x **and** y**,** 3x+(2x−y)i=6−3i**.**

Ans:

We will equate the real part of the right-hand side with the real part of the left-hand side. Similarly, we will equate their imaginary parts as well.

Therefore,

$$3x=6 \text{ and}$$

$$2x-y=-3$$

$$\Rightarrow x=2 \text{ and}$$

$$\Rightarrow 2\,(2)-y=-3$$

$$\Rightarrow 4-y=-3$$

$$\Rightarrow y=7$$

.

Hence,

$$x=2 \text{ and}$$

$$y=7.$$

9. Multiply 3−2i **by its conjugate.**

Ans:

Let there be a complex number

$$z=3-2i$$

Hence, its conjugate will be

$$\bar{z}=3+2i$$

.

Therefore, the product of the complex number with its conjugate will be –

$$z\bar{z} = (3-2i)(3+2i)$$

$$\Rightarrow 9+6i-6i-4i^2$$

$$\Rightarrow 9+4$$

Hence,

$$(3-2i)(3+2i)=13.$$

10. Find the multiplicative inverse of 4−3i.

Ans:

Let us assume a complex number z=4−3i

.

To find the multiplicative inverse, we need $\bar{z}$ and $|z|$

Therefore,

$$\bar{z} = 4+3i$$

.

$$|z| = \sqrt{x^2+y^2}$$

$$\Rightarrow \sqrt{4^2+-3^2}$$

$$\Rightarrow \sqrt{16+9}$$

$$\Rightarrow |z| = 5$$

Hence, multiplicative inverse $\Rightarrow z^{-1}=\frac{\bar{z}}{|z|^2}$ will be –

$$\Rightarrow z^{-1} = \frac{4+3i}{25}$$

.

$$\Rightarrow z^{-1} = \frac{4}{25} + i\frac{3}{25}.$$

Multiple Choice Questions

1. The value of $1 + i^2 + i^4 + i^6 + \ldots + i^{2n}$ is

(a) positive

(b) Negative

(c) 0

(d) Cannot be evaluated

Correct option: (d) cannot be evaluated

Solution:

$1 + i^2 + i^4 + i^6 + \ldots + i^{2n} = 1 - 1 + 1 - 1 + \ldots (-1)^n$

This cannot be evaluated unless the value of n is known.

2. If a + ib = c + id, then

(a) $a^2 + c^2 = 0$

(b) $b^2 + c^2 = 0$

(c) $b^2 + d^2 = 0$

(d) $a^2 + b^2 = c^2 + d^2$

Correct option: (d) $a^2 + b^2 = c^2 + d^2$

Solution:

Given,

a + ib = c + id

$\Rightarrow |a + ib| = |c + id|$

$\Rightarrow \sqrt{(a^2 + b^2)} = \sqrt{(c^2 + d^2)}$

Squaring on both sides, we get;

$a^2 + b^2 = c^2 + d^2$

3. If a complex number z lies in the interior or on the boundary of a circle of radius 3 units and centre (– 4, 0), the greatest value of |z +1| is

(a) 4

(b) 6

(c) 3

(d) 10

Correct option: (b) 6

Solution:

The distance of the point representing z from the centre of the circle is $|z – (-4 + i0)| = |z + 4|$

According to the given,

$|z + 4| \leq 3$

Now,

$|z + 1| = |z + 4 – 3| \leq |z + 4| + |-3| \leq 3 + 3 \leq 6$

Hence, the greatest value of $|z + 1|$ is 6.

4. The value of arg (x) when x < 0 is

(a) 0

(b) $\pi/2$

(c) π

(d) none of these

Correct option: (c) π

Solution:

Let $z = x + 0i$ and $x < 0$

Since the point (-x, 0) lies on the negative side of the real axis,

$|z| = |x + oi| = \sqrt{[(-1)^2 + 0)]} = 1$

$\therefore$ Principal argument (z) = π

Alternative method:

Let $x = \cos\theta + i \sin\theta$

For $\theta = \pi$, x should be negative.

Thus, $x < 0$ for $\theta = \pi$.

5. If 1 – i, is a root of the equation $x^2 + ax + b = 0$, where a, b ∈ R, then the value of a – b is

(a) -4

(b) 0

(c) 2

(d) 1

Correct option: (a) -4

Solution:

Given that 1 – i is the root of $x^2 + ax + b = 0$.

Thus, 1 + i is also the root of the given equation since non-real complex roots occur in conjugate pairs.

Sum of roots = $-a/1 = (1 - i) + (1 + i)$

$\Rightarrow a = -2$

Product of roots, $b/1 = (1 - i)(1 + i)$

$b = 1 - i^2$

$b = 1 + 1$ {since $i^2 = -1$}

$\Rightarrow b = 2$

Now, $a - b = -2 - 2 = -4$

6. Number of solutions of the equation $z^2 + |z|^2 = 0$ is

(a) 1

(b) 2

(c) 3

(d) infinitely many

Correct option: (d) infinitely many

Solution:

Given,

$z^2 + |z|^2 = 0, z \neq 0$

$\Rightarrow (x + iy)^2 + [\sqrt{(x^2 + y^2)}]^2 = 0$

$\Rightarrow x^2 - y^2 + i2xy + x^2 + y^2 = 0$

$\Rightarrow 2x^2 + i2xy = 0$

$\Rightarrow 2x (x + iy) = 0$

$\Rightarrow x = 0$ or $x + iy = 0$ (not possible)

Therefore, $x = 0$ and $z \neq 0$.

Thus, y can have any real value.

Hence, there exist infinitely many solutions.

7. If $[(1 + i)/(1 - i)]^x = 1$, then

(a) $x = 2n + 1$, where $n \in N$

(b) $x = 4n$, where $n \in N$

(c) $x = 2n$, where $n \in N$

(d) $x = 4n + 1$, where $n \in N$

Correct option: (b) $x = 4n$, where $n \in N$

Solution:

Given,

$[(1 + i)/(1 – i)]^x = 1$
By rationalising the denominator,

$[(1 + i)(1 + i)/ (1 – i)(1 + i)]^x = 1$ $[(1 + i)^2/ (1 – i + i – i^2)]^x = 1$ $[(1 + i^2 + 2i)/(1 + 1)]^x = 1$ $[(1 – 1 + 2i)/ 2]^x = 1$
$i^x = 1$

Thus, $i^x = i^{4n}$, where n is any positive integer.

8. If the complex number $z = x + iy$ satisfies the condition $|z + 1| = 1$, then z lies on

(a) x-axis

(b) circle with centre (1, 0) and radius 1

(c) circle with centre (–1, 0) and radius 1

(d) y-axis

Correct option: (c) circle with centre (–1, 0) and radius 1

Solution:

Given,

$z = x + iy$

and

$|z + 1| = 1$

$|x + iy + 1| = 1$

⇒ $|(x + 1) + iy| = 1$

⇒ $\sqrt{[(x + 1)^2 + y^2]} = 1$

Squaring on both sides,

$(x + 1)^2 + y^2 = 1$

This is the equation of a circle with centre (–1, 0) and radius 1.

9. The simplified value of $(1 - i)^3/(1 - i^3)$ is

(a) 1

(b) -2

(c) -i

(d) 2i

Correct option: (b) -2

Solution:

$(1 - i)^3/(1 - i^3)$

$= (1 - i)^3/(1^3 - i^3)$

$= (1 - i)^3/ [(1 - i)(1 + i + i^2)]$

$= (1 - i)^2/(1 + i - 1)$

$= (1 - i)^2/i$

$= (1 + i^2 - 2i)/i$

$= (1 - 1 - 2i)/i$

$= -2i/i$

$= -2$

10. sin x + i cos 2x and cos x – i sin 2x are conjugate to each other for:

(a) $x = n\pi$

(b) $x = [n + (1/2)] (\pi/2)$

(c) $x = 0$

(d) No value of x

Correct option: (d) No value of x

Solution:

Consider sin x + i cos 2x and cos x – i sin 2x are conjugate to each other.

So, sin x – i cos 2x = cos x – i sin 2x

On comparing real and imaginary parts of both sides, we get

⇒ sin x = cos x and cos 2x = sin 2x

⇒ sin x/cos x = 1 and (cos 2x/sin 2x) = 1

⇒ tan x = 1 and tan 2x = 1

Now, consider tan 2x = 1

Using the formula $\tan 2A = 2 \tan A/(1 – \tan^2 A)$,

$(2 \tan x)/(1 – \tan^2 x) = 1$

However, this is not possible for tan x = 1.

Therefore, for no value of x, sinx + i cos 2x and cos x – i sin 2x are conjugate to each other

Summary

- A number of the form a + ib, where a and b are real numbers, is called a complex number, a is called the real part and b is called the imaginary part of the complex number.
- Let $z_1 = a + ib$ and $z_2 = c + id$. Then

  (i) $z_1 + z_2 = (a + c) + i (b + d)$

(ii) z_1z_2= (ac – bd) + i (ad + bc) a^2+b^2

- For any non-zero complex number z = a + ib (a ≠ 0, b ≠ 0), there exists the complex number $\frac{a}{a2+b2} + i\,\frac{-b}{a2+b2}$, denoted by $\frac{1}{z}$ or z^{-1} ,called the multiplicative inverse of z such that (a + ib) $\frac{a}{a2+b2} + i\,\frac{-b}{a2+b2}$ = 1+ i0 =1
- For any integer k, $i^{-4k} = 1$, $i^{-4k+1} = i$, $i^{-4k+2} = -1$, $i^{-4k+3} = -i$
- For any integer k, i 4k = 1, i 4k + 1 = i, i 4k + 2 = – 1, i 4k + 3 = – i
- The conjugate of the complex number z = a + ib, denoted by $\bar{z}$, is given by $\bar{z}$ = a – ib.