Chapter 14

Mathematical Reasoning

"Mathematical Reasoning." chapter serves as a valuable introduction to the principles of logical reasoning and its application in mathematical contexts. Students explore the basics of mathematical reasoning, including the notion of statements, logical connectives (like 'and,' 'or,' 'not'), and the importance of forming clear and precise mathematical statements.

Additionally, the chapter introduces students to mathematical arguments, emphasizing the importance of deductive reasoning in proving mathematical statements. Inductive reasoning, counterexamples, and different methods of mathematical proofs are discussed to provide students with a well-rounded understanding of mathematical reasoning.

Propositional logic, which deals with compound statements formed from simpler statements, is a key focus. Students learn to identify tautologies and contradictions and explore the concept of implications.

Exercise 14.1

1. Which of the following sentences are statements? Give reasons for your answer.

(i) There are 35 days in a month.

(ii) Mathematics is difficult.

(iii) The sum of 5 and 7 is greater than 10.

(iv) The square of a number is an even number.

(v) The sides of a quadrilateral have equal lengths.

(vi) Answer this question.

(vii) The product of (–1) and 8 is 8.

(viii) The sum of all interior angles of a triangle is 180°.

(ix) Today is a windy day.

(x) All real numbers are complex numbers.

Solution:

(i) The maximum number of days in a month is 31, so this sentence is incorrect. Therefore, it is a statement.

(ii) This sentence is subjective. For some people, Mathematics can be easy, and for others, it can be difficult. Therefore, it is not a statement.

(iii) The sum of 5 and 7 is 12, and it is greater than 10. Therefore, this sentence is always correct. Hence, it is a statement.

(iv) This sentence can be sometimes correct and sometimes incorrect. For example, the square of 2 is an even number, but the square of 3 is an odd number. Hence, it is not a statement.

(v) This sentence can be sometimes correct and sometimes incorrect. For example, squares and rhombi have sides of equal lengths, whereas trapezia and rectangles have sides of unequal lengths. Therefore it is not a statement.

(vi) It is an order. Hence, it is not a statement.

(vii) The given sentence is incorrect because the product of (-1) and 8 is -8. Hence, it is a statement.

(viii) The given sentence is correct, and therefore, it is a statement.

(ix) The given sentence is not a statement because the day that is being referred to is not evident from the sentence.

(x) The given sentence is always correct because all real numbers can be written as $a \times 1 + 0 \times i$. Hence, it is a statement.

2. Give three examples of sentences which are not statements. Give reasons for the answers.

Solution:

The three examples of sentences which are not statements are given below:

(i) He is a doctor.

In the given sentence, it is not evident to whom 'he' is referred to. Hence, it is not a statement.

(ii) Geometry is difficult.

For some people, geometry can be easy, and for others, it can be difficult. Hence, this is not a statement.

(iii) Where is she going?

In this question, it is not evident to whom 'she' is referred to. Hence, it is not a statement.

Exercise 14.2

1. Write the negation of the following statements:

(i) Chennai is the capital of Tamil Nadu.

(ii) $\sqrt{2}$ is not a complex number.

(iii) All triangles are not equilateral triangles.

(iv) The number 2 is greater than 7.

(v) Every natural number is an integer.

Solution:

(i) Chennai is not the capital of Tamil Nadu.

(ii) $\sqrt{2}$ is a complex number.

(iii) All triangles are equilateral triangles.

(iv) The number 2 is not greater than 7.

(v) Every natural number is not an integer.

2. Are the following pairs of statements negations of each other?

(i) The number *x* is not a rational number.

The number *x* is not an irrational number.

(ii) The number *x* is a rational number.

The number *x* is an irrational number.

Solution:

(i) The negation of the first statement is ‘the number x is a rational number’.

This is the same as the second statement because if a number is not an irrational number, then the number is a rational number.

Hence, the given statements are negations of each other.

(ii) The negation of the first statement is ‘the number x is not a rational number. This means that the number x is an irrational number which is the same as the second statement.

Hence, the given statements are negations of each other.

3. Find the component statements of the following compound statements and check whether they are true or false.

(i) Number 3 is prime, or it is odd.

(ii) All integers are positive or negative.

(iii) 100 is divisible by 3, 11 and 5.

Solution:

(i) The component statements are

(a) Number 3 is prime

(b) Number 3 is odd

Here, both statements are true.

(ii) The component statements are as follows:

(a) All integers are positive

(b) All integers are negative

Here, both statements are false.

(iii) The component statements are as follows:

(a) 100 is divisible by 3

(b) 100 is divisible by 11

(c) 100 is divisible by 5

Here, the statements (a) and (b) are false, and (c) is true.

Exercise 14.3

1. For each of the following compound statements, first identify the connecting words and then break them into component statements.

(i) All rational numbers are real and all real numbers are not complex.

(ii) Square of an integer is positive or negative.

(iii) The sand heats up quickly in the Sun and does not cool down fast at night.

(iv) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

Solution:

(i) In this sentence, ‘and’ is the connecting word

The component statements are as follows:

(a) All rational numbers are real

(b) All real numbers are not complex

(ii) In this sentence, ‘or’ is the connecting word

The component statements are as follows:

(a) Square of an integer is positive

(b) Square of an integer is negative

(iii) In this sentence, ‘and’ is the connecting word

The component statements are as follows:

(a) The sand heats up quickly in the Sun

(b) The sand does not cool down fast at night

(iv) In this sentence, ‘and’ is the connecting word

The component statements are as follows:

(a) $x = 2$ is the root of the equation $3x^2 - x - 10 = 0$

(b) $x = 3$ is the root of the equation $3x^2 - x - 10 = 0$

2. Identify the quantifier in the following statements and write the negation of the statements.

(i) There exists a number which is equal to its square.

(ii) For every real number x, x is less than $x + 1$.

(iii) There exists a capital for every state in India.

Solution:

(i) Here, the quantifier is 'there exists'.

The negation of this statement is as follows:

There does not exist a number which is equal to its square.

(ii) Here, the quantifier is 'for every'.

The negation of this statement is as follows:

There exist a real number x, such that x is not less than x + 1.

(iii) Here, the quantifier is 'there exists'.

The negation of this statement is as follows:

In India, there exists a state which does not have a capital.

3. Check whether the following pair of statements is a negation of each other. Give reasons for the answer.

(i) $x + y = y + x$ is true for every real number x and y.

(ii) There exists real numbers x and y for which $x + y = y + x$.

Solution:

The negation of statement (i) is as given below:

There exist real numbers x and y for which $x + y \neq y + x$

Now, this statement is not the same as statement (ii).

Hence, the given statements are not a negation of each other.

4. State whether the "Or" used in the following statements is "exclusive "or" inclusive. Give reasons for your answer.

(i) Sun rises or Moon sets.

(ii) To apply for a driving licence, you should have a ration card or a passport.

(iii) All integers are positive or negative.

Solution:

(i) It is not possible for the Sun to rise and the Moon to set together. Hence, the 'or' in the given statement is exclusive.

(ii) Since a person can have both a ration card and a passport to apply for a driving license. Hence, the 'or' in the given statement is inclusive.

(iii) Since all integers cannot be both positive and negative. Hence, the 'or' in the given statement is exclusive.

Exercise 14.4

1. Rewrite the following statement with "if-then" in five different ways conveying the same meaning.

If a natural number is odd, then its square is also odd.

Solution:

The five different ways of the given statement can be written as follows:

(i) A natural number is odd, indicating that its square is odd.

(ii) A natural number is odd only if its square is odd.

(iii) For a natural number to be odd, it is necessary that its square is odd.

(iv) It is sufficient that the number is odd for the square of a natural number to be odd.

(v) If the square of a natural number is not odd, then the natural number is not odd.

2. Write the contrapositive and converse of the following statements.

(i) If *x* is a prime number, then *x* is odd.

(ii) If the two lines are parallel, then they do not intersect in the same plane.

(iii) Something that is cold implies that it has a low temperature.

(iv) You cannot comprehend geometry if you do not know how to reason deductively.

(v) *x* is an even number implies that *x* is divisible by 4

Solution:

(i) The contra positive of the given statement is as follows:

If a number x is not odd, then x is not a prime number.

The converse of the given statement is as follows:

If a number x is odd, then it is a prime number.

(ii) The contra positive of the given statement is as follows:

If two lines intersect in the same plane, then the two lines are not parallel.

The converse of the given statement is as follows:

If two lines do not intersect in the same plane, then they are parallel.

(iii) The contra positive of the given statement is as follows:

If something does not have a low temperature, then it is not cold.

The converse of the given statement is as follows:

If something is at a low temperature, then it is cold.

(iv) The contra positive of the given statement is as follows:

If you know how to reason deductively, then you can comprehend geometry.

The converse of the given statement is as follows:

If you do not know how to reason deductively, then you cannot comprehend geometry.

(v) The given statement can be written as 'if x is an even number, then x is divisible by 4'.

The contra positive of the given statement is as follows:

If x is not divisible by 4, then x is not an even number.

The converse of the given statement is as follows:

If x is divisible by 4, then x is an even number.

3. Write each of the following statements in the form "if-then".

(i) You get a job implies that your credentials are good.

(ii) The Banana trees will bloom if it stays warm for a month.

(iii) A quadrilateral is a parallelogram if its diagonals bisect each other.

(iv) To get A^+ in the class, it is necessary that you do the exercises in the book.

Solution:

(i) If you get a job, then your credentials are good.

(ii) If the Banana trees stay warm for a month, then the trees will bloom.

(iii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

(iv) If you want to score an A^+ in the class, then you do all the exercises in the book.

4. Given statements in (a) and (b). Identify the statements given below as contra positive or converse of each other.

(a) If you live in Delhi, then you have winter clothes.

(i) If you do not have winter clothes, then you do not live in Delhi.

(ii) If you have winter clothes, then you live in Delhi.

(b) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(i) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

(ii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Solution:

(a) If you live in Delhi, then you have winter clothes.

(i) If you do not have winter clothes, then you do not live in Delhi

[Contra positive of statement (a)].

(ii) If you have winter clothes, then you live in Delhi [Converse of statement (a)]

(b) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(i) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram [Contra positive of statement (b)].

(ii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram [Converse of statement (b)].

Exercise 14.5

1. Show that the statement

***p*: "If *x* is a real number such that $x^3 + 4x = 0$, then *x* is 0" is true by**

(i) direct method

(ii) method of contradiction

(iii) method of contrapositive

Solution:

Let p: 'If x is a real number such that $x^3 + 4x = 0$, then x is 0'

q: x is a real number such that $x^3 + 4x = 0$

r: x is 0

(i) We assume that q is true to show that statement p is true and then show that r is true.

Therefore, let statement q be true

Hence, $x^3 + 4x = 0$

$x (x^2 + 4) = 0$

$x = 0$ or $x^2 + 4 = 0$

Since x is real, it is 0.

So, statement r is true.

Hence, the given statement is true.

(ii) By contradiction, to show statement p to be true, we assume that p is not true.

Let x be a real number such that $x^3 + 4x = 0$ and let $x \neq 0$

Hence, $x^3 + 4x = 0$

$x (x^2 + 4) = 0$

$x = 0$ or $x^2 + 4 = 0$

$x = 0$ or $x^2 = -4$

However, x is real. Hence, $x = 0$, which is a contradiction since we have assumed that $x \neq 0$.

Therefore, the given statement p is true.

(iii) By the contra positive method, to prove statement p to be true, we assume that r is false and prove that q must be false.

$\sim r: x \neq 0$

Clearly, it can be seen that

$(x^2 + 4)$ will always be positive

$x \neq 0$ implies that the product of any positive real number with x is not zero.

Now, consider the product of x with $(x^2 + 4)$

$\therefore x(x^2 + 4) \neq 0$

$x^3 + 4x \neq 0$

This shows that statement q is not true.

Hence, it proved that

$\sim r \Rightarrow \sim q$

Hence, the given statement p is true.

2. Show that the statement "For any real numbers *a* and *b*, $a^2 = b^2$ implies that $a = b$" is not true by giving a counter-example.

Solution:

The given statement can be written in the form of 'if then' is given below:

If a and b are real numbers such that $a^2 = b^2$, then $a = b$

Let p: a and b are real numbers such that $a^2 = b^2$

q: $a = b$

The given statement has to be proved false. To show this, two real numbers, a and b, with $a^2 = b^2$, are required such that $a \neq b$.

Let us consider $a = 1$ and $b = -1$

$a^2 = (1)^2$

$= 1$ and

$b^2 = (-1)^2$

$= 1$

Hence, $a^2 = b^2$

However, $a \neq b$

Therefore, it can be concluded that the given statement is false.

3. Show that the following statement is true by the method of contra positive.

p*: *If x is an integer and x^2 is even, then x is also even.

Solution:

Let p: If x is an integer and x^2 is even, then x is also even

Let q: x be an integer and x^2 be even

r: x is even

By the contra positive method, to prove that p is true, we assume that r is false and prove that q is also false

Let x is not even

To prove that q is false, it has to be proved that x is not an integer or x^2 is not even.

x is not even indicates that x^2 is also not even.

Hence, statement q is false.

Therefore, the given statement p is true.

4. By giving a counter example, show that the following statements are not true.

(i) *p*: If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.

(ii) *q*: The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.

Solution:

(i) Let q: All the angles of a triangle are equal

r: The triangle is an obtuse angled triangle

The given statement p has to be proved false.

To show this, the required angles of a triangle should not be an obtuse angle.

We know that sum of all the angles of a triangle is 180^0. Therefore, if all three angles are equal, then each angle measures 60^0, which is not obtuse.

In an equilateral triangle, all angles are equal. However, the triangle is not an obtuse-angled triangle.

Hence, it can be concluded that the given statement p is false.

(ii) The given statement is

q: The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.

This statement has to be proven false

To show this, let us consider

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

One root of the equation $x^2 - 1 = 0$, i.e. the root $x = 1$, lies between 0 and 2

Therefore, the given statement is false.

5. Which of the following statements are true and which are false? In each case, give a valid reason for saying so.

(i) *p*: Each radius of a circle is a chord of the circle.

(ii) *q*: The centre of a circle bisects each chord of the circle.

(iii) *r*: Circle is a particular case of an ellipse.

(iv) *s*: If x and y are integers such that $x > y$, then $-x < -y$.

(v) *t*: √11 is a rational number.

Solution:

(i) The given statement p is false.

As per the definition of a chord, it should intersect the circle at two distinct points.

(ii) The given statement q is false.

The centre will not bisect that chord which is not the diameter of the circle.

In other words, the centre of a circle only bisects the diameter, which is the chord of the circle.

(iii) The equation of an ellipse is,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

If we put a = b = 1, then, we get

$x^2 + y^2 = 1$, which is an equation of a circle

Hence, a circle is a particular case of an ellipse.

Therefore, statement r is true.

(iv) $x > y$

By the rule of inequality

$-x < -y$

Hence, the given statement s is true.

(v) 11 is a prime number

We know that the square root of any prime number is an irrational number.

Therefore $\sqrt{11}$ is an irrational number.

Hence, the given statement t is false.

2Marks Questions & Answers

1. Find the component statements of the following compound statements and check whether they are true or false.

(i) The number 3 is prime or it is odd.

Ans:

The given compound statements can be written as component statements as:

p: "number 3 is prime"

q: "number 3 is odd".

The given statement is connected by connective "or"

Since, both the statements are true, hence, the given statement is true.

2. Check whether the following pair of statements are negative of each other. Give reasons for your answer.

(i) X + y = y + x is true for every real numbers x and y.

(ii) There exists real number x and y for which x + y = y + x.

Ans: The statements are:

p: "$x + y = y + x$ is true for every real number x and y"

q: "There exists real number x and y for which $x + y = y + x$"

The negation of p is

~p: "There are real numbers x and y for which $x + y \neq y + x$"
Therefore, the given statements are not negations of each other and they can be true at the same time.

3. Write the contra-positive and converse of the following statements.

(i) If x is a prime number, then x is odd.

Ans: We know that, if a statement $p \Rightarrow q$, then its contra-positive will be $\sim q \Rightarrow \sim p \sim q$ and its converse will be $q \Rightarrow p$

(i) **Contra-positive:** "If x is not odd, then it is not a prime number."

Converse: "If x is odd, then it is a prime number."

(ii) If the two lines are parallel, then they do not intersect in the same plane.

Ans: Contra-positive: "If two lines intersect in the same plane, then they are not parallel."

Converse: "If two lines do not intersect in the same plane, then they are parallel."

4. Given the below two statements

p: 25 is a multiple of 5.

q: 25 is a multiple of 8.

Write the compound statement connecting these two statements with "and" and "or". In both the cases check the validity of the compound statement.

Ans: Case 1: By using "and", we get the compound statement "p and q"

i.e., "25 is multiple of 5 and 8."

It is not a true statement as q is always false. ($\because$ 25 is not a multiple of 8)($\because$ 25 is not a multiple of 8)

Case 2: By using "or", we get the compound statement "p or q"

i.e., "25 is multiple of 5 or 8."

It is a true statement as p is always true. ($\because$ 25 is a multiple of 5)

5. Write the following statement in five different ways, conveying the same meaning.

p: If a triangle is equiangular, then it is an obtuse angled triangle.

Ans: Given: "If a triangle is equiangular, then it is an obtuse angled triangle".

The given statement can be written as follows:

1. "A triangle is equiangular only if it is an obtuse angled triangle".
2. "If a triangle is not obtuse angled triangle then it is not an equiangular triangle".
3. "Equiangularity is a sufficient condition for the triangle to be obtuse angled".
4. "A triangle being obtuse angled, is a necessary condition for it to be equiangular".
5. "A triangle is obtuse angled if it is equiangular".

Multiple Choice Questions

1. Which of the following is a statement?

(a) Roses are black.

(b) Mind your own business.

(c) Be punctual.

(d) Do not tell lies.

Correct option: (a) Roses are black.

Solution:

The sentences in (b), (c), and (d) are neither true nor false. All these sentences are pieces of advice.

Sentence (a) is a definite statement.

2. Which of the following is not a statement?

(a) Smoking is injurious to health.

(b) $2 + 2 = 4$

(c) 2 is the only even prime number.

(d) Come here.

Correct option: (d) Come here.

Solution:

Smoking is injurious to health. – It is a statement.

2 + 2 = 4; It is a mathematical statement.

2 is the only even prime number. – Mathematical statement.

Come here. – It is not a statement but it is an order.

3. The negation of the statement "7 is greater than 8" is

(a) 7 is equal to 8.

(b) 7 is not greater than 8.

(c) 8 is less than 7.

(d) None of these.

Correct option: (b) 7 is not greater than 8.

Solution:

Statement: 7 is greater than 8

Negation: 7 is not greater than 8

4. The contra positive of the statement "If p, then q", is

(a) If q, then p.

(b) If p, then ~ q.

(c) If ~ q, then ~ p.

(d) If ~ p, then ~ q.

Correct option: (c) If ~ q, then ~ p.

Solution:

The contra positive of the statement "If p, then q", is If ~ q, then ~ p.

5. The connective in the statement "Earth revolves around the Sun and Moon is a satellite of earth" is

(a) or

(b) Earth

(c) Sun

(d) And

Correct option: (d) and

Solution:

Given statement: Earth revolves around the Sun and Moon is a satellite of earth

Here, the connective word is "and."

6. The converse of the statement "If $x > y$, then $x + a > y + a$" is

(a) If $x < y$, then $x + a < y + a$.

(b) If $x + a > y + a$, then $x > y$.

(c) If $x < y$, then $x + a > y + a$.

(d) If $x > y$, then $x + a < y + a$.

Correct option: (b) If $x + a > y + a$, then $x > y$.

Solution:

As we know, the converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.

So, the converse of the statement "If $x > y$, then $x + a > y + a$" is If $x + a > y + a$, then $x > y$.

7. Which of the following statements is a conjunction?

(a) Ram and Shyam are friends.

(b) Both Ram and Shyam are tall.

(c) Both Ram and Shyam are enemies.

(d) None of the above

Correct option: (d) None of the above

Solution:

When the word connects two statements “and,” we call the combined statement conjunction.

None of the statements is connected by “and” from the given.

8. The contra positive of the statement ‘If Chandigarh is the capital of Punjab, then Chandigarh is in India’ is

(a) If Chandigarh is not in India, then Chandigarh is not the capital of Punjab.

(b) If Chandigarh is in India, then Chandigarh is the capital of Punjab.

(c) If Chandigarh is not the capital of Punjab, then Chandigarh is not the capital of India.

(d) If Chandigarh is the capital of Punjab, then Chandigarh is not in India.

Correct option: (a) If Chandigarh is not in India, then Chandigarh is not the capital of Punjab.

Solution:

The contra positive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$.

Therefore, the contra positive of the statement ‘If Chandigarh is the capital of Punjab, then Chandigarh is in India’ is “If Chandigarh is not in India, then Chandigarh is not the capital of Punjab.”

9. The method(s) that are used to check the validity of statements is/are

(a) contra positive method

(b) method of contradiction

(c) using a counter example

(d) All the above

Correct option: (d) All the above

Solution:

The methods that are used to check the validity of statements include the following.

(i) direct method

(ii) contra positive method

(iii) method of contradiction

(iv) using a counter example

10. The negation of the statement "Akash or Ankitha lived in Goa" is

(a) Akash did not live in Goa or Ankitha lives in Goa.

(b) Akash lives in Goa and Ankitha did not live in Goa.

(c) Akash did not live in Goa and Ankitha did not live in Goa.

(d) Akash did not live in Goa or Ankitha did not live in Goa.

Correct option: (c) Akash did not live in Goa and Ankitha did not live in Goa.

Solution:

Given,

Statement: Akash or Ankitha lived in Goa

Negation of the above statement is:

Akash did not live in Goa and Ankitha did not live in Goa.