Chapter 4 –

Principle of Mathematical Induction

Mathematical induction is widely applied in various branches of mathematics, including algebra, number theory, and combinatorics. It provides a rigorous and systematic way to prove statements that involve an infinite number of cases, offering a valuable tool for mathematicians in establishing the validity of conjectures and theorems.

Mathematical Induction is commonly used to prove statements about natural numbers, such as properties of sequences, series, and divisibility. It's a powerful and widely applicable method in mathematical reasoning.

Exercise 4.1

Prove the following by using the principle of mathematical induction for all n ∈ N:

1.

$$1+3+3^2+\ldots+3^{n-1}=\frac{(3^n-1)}{2}$$

Solution:

We can write the given statement as

$$P(n): 1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{(3^n-1)}{2}$$

If n = 1 we get

$$P(1): 1 = \frac{(3^1-1)}{2}=\frac{3-1}{2}=\frac{2}{2}=1$$

Which is true.

Consider P (k) be true for some positive integer k

$$1+3+3^2+\ldots+3^{k-1}=\frac{(3^k-1)}{2} \qquad \ldots(i)$$

Now let us prove that P (k + 1) is true.

Here

$$1 + 3 + 3^2 + \ldots + 3^{k-1} + 3^{(k+1)-1} = (1 + 3 + 3^2 + \ldots + 3^{k-1}) + 3^k$$

By using equation (i)

$$=\frac{(3^k-1)}{2}+3^k$$

Taking LCM

$$=\frac{(3^k-1)+2.3^k}{2}$$

Taking the common terms out

$$=\frac{(1+2)3^k-1}{2}$$

We get

$$=\frac{3.3^k-1}{2}$$

$$=\frac{3^{k+1}-1}{2}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

2.

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Solution:

We can write the given statement as

$$\text{P}(n)\text{: } 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

If n = 1 we get

$$\text{P}(1)\text{: } 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{1.2}{2}\right)^2 = 1^2 = 1$$

Which is true.

Consider P (k) be true for some positive integer k

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 \qquad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = (1^3 + 2^3 + 3^3 + \ldots\ldots + k^3) + (k + 1)^3$

By using equation (i)

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

So we get

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

Taking LCM

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

Taking the common terms out

$$= \frac{(k+1)^2\{k^2+4(k+1)\}}{4}$$

We get

$$= \frac{(k+1)^2\{k^2+4k+4\}}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

By expanding using formula

$$= \frac{(k+1)^2(k+1+1)^2}{4}$$

$$= \left(\frac{(k+1)(k+1+1)}{2}\right)^2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

3.

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots n)}=\frac{2n}{(n+1)}$$

Solution:

We can write the given statement as

$$P(n):\ 1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots n}=\frac{2n}{n+1}$$

If n = 1 we get

$$P(1):\ 1=\frac{2.1}{1+1}=\frac{2}{2}=1$$

Which is true.

Consider P (k) be true for some positive integer k

$$1+\frac{1}{1+2}+\ldots+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}=\frac{2k}{k+1} \quad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

$$=\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots k}\right)+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

By using equation (i)

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

We know that

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

So we get

$$=\frac{2k}{k+1}+\frac{1}{\left(\frac{(k+1)(k+1+1)}{2}\right)}$$

It can be written as

$$=\frac{2k}{(k+1)}+\frac{2}{(k+1)(k+2)}$$

Taking the common terms out

$$=\frac{2}{(k+1)}\left(k+\frac{1}{k+2}\right)$$

By taking LCM

$$=\frac{2}{k+1}\left(\frac{k(k+2)+1}{k+2}\right)$$

We get

$$=\frac{2}{(k+1)}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$=\frac{2\cdot(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{2(k+1)}{(k+2)}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

4.

$$1.2.3 + 2.3.4 + \ldots + n(n + 1)(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

Solution:

We can write the given statement as

$$P(n): 1.2.3 + 2.3.4 + \ldots + n(n + 1)(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

If n = 1 we get

$$P(1): 1.2.3 = 6 = \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1.2.3.4}{4} = 6$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.2.3 + 2.3.4 + \ldots + k(k + 1)(k + 2) = \frac{k(k+1)(k+2)(k+3)}{4} \quad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$1.2.3 + 2.3.4 + \ldots + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3) = \{1.2.3 + 2.3.4 + \ldots + k(k + 1)(k + 2)\} + (k + 1)(k + 2)(k + 3)$

By using equation (i)

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

So we get

$$= (k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$

It can be written as

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

By further simplification

$$= \frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

5.

$$1.3+2.3^2+3.3^3+...+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

Solution:

We can write the given statement as

$$P(n): 1.3+2.3^2+3.3^3+...+n3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

If n = 1 we get

$$P(1): 1.3 = 3=\frac{(2.1-1)3^{1+1}+3}{4}=\frac{3^2+3}{4}=\frac{12}{4}=3$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.3+2.3^2+3.3^3+...+k3^k=\frac{(2k-1)3^{k+1}+3}{4} \quad ... (i)$$

Now let us prove that P (k + 1) is true.

Here

$1.3 + 2.3^2 + 3.3^3 + \dots + k3^k + (k + 1)\ 3^{k+1} = = (1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k) + (k + 1)\ 3^{k+1}$

By using equation (i)

$$=\frac{(2k-1)3^{k+1}+3}{4}+(k+1)3^{k+1}$$

By taking LCM

$$=\frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

Taking the common terms out

$$= \frac{3^{k+1}\{2k-1+4(k+1)\}+3}{4}$$

By further simplification

$$= \frac{3^{k+1}\{6k+3\}+3}{4}$$

Taking 3 as common

$$= \frac{3^{k+1}.3\{2k+1\}+3}{4}$$

$$= \frac{3^{(k+1)+1}\{2k+1\}+3}{4}$$

$$= \frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

6.

$$1.2+2.3+3.4+\ldots+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

Solution:

We can write the given statement as

$$P(n):\ 1.2+2.3+3.4+\ldots+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

If n = 1 we get

$$P(1):\ 1.2=2=\frac{1(1+1)(1+2)}{3}=\frac{1.2.3}{3}=2$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.2+2.3+3.4+\ldots\ldots+k.(k+1)=\left[\frac{k(k+1)(k+2)}{3}\right] \qquad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$1.2 + 2.3 + 3.4 + \ldots + k.\ (k + 1) + (k + 1).\ (k + 2) = [1.2 + 2.3 + 3.4 + \ldots + k.\ (k + 1)] + (k + 1).\ (k + 2)$

By using equation (i)

$$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)$$

We can write it as

$$=(k+1)(k+2)\left(\frac{k}{3}+1\right)$$

We get

$$=\frac{(k+1)(k+2)(k+3)}{3}$$

By further simplification

$$=\frac{(k+1)(k+1+1)(k+1+2)}{3}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

7.

$$1.3+3.5+5.7+...+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

Solution:

We can write the given statement as

$$P(n): 1.3+3.5+5.7+...+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

If n = 1 we get

$$P(1): 1.3=3=\frac{1(4.1^2+6.1-1)}{3}=\frac{4+6-1}{3}=\frac{9}{3}=3$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.3+3.5+5.7+......+(2k-1)(2k+1)=\frac{k(4k^2+6k-1)}{3} \quad ... (i)$$

Now let us prove that P (k + 1) is true.

Here

$(1.3 + 3.5 + 5.7 + \dots + (2k-1)(2k+1) + \{2(k+1)-1\}\{2(k+1)+1\}$

By using equation (i)

$$=\frac{k(4k^2+6k-1)}{3}+(2k+2-1)(2k+2+1)$$

$$= \frac{k\left(4k^2+6k-1\right)}{3}+\left(2k+2-1\right)\left(2k+2+1\right)$$

On further calculation

$$= \frac{k\left(4k^2+6k-1\right)}{3}+\left(2k+1\right)\left(2k+3\right)$$

By multiplying the terms

$$= \frac{k\left(4k^2+6k-1\right)}{3}+\left(4k^2+8k+3\right)$$

Taking LCM

$$= \frac{k\left(4k^2+6k-1\right)+3\left(4k^2+8k+3\right)}{3}$$

By further simplification

$$= \frac{4k^3+6k^2-k+12k^2+24k+9}{3}$$

So we get

$$= \frac{4k^3+18k^2+23k+9}{3}$$

It can be written as

$$= \frac{4k^3+14k^2+9k+4k^2+14k+9}{3}$$

$$= \frac{k\left(4k^2+14k+9\right)+1\left(4k^2+14k+9\right)}{3}$$

Separating the terms

$$= \frac{\left(k+1\right)\left\{4k^2+8k+4+6k+6-1\right\}}{3}$$

Taking the common terms out

$$= \frac{\left(k+1\right)\left\{4\left(k^2+2k+1\right)+6\left(k+1\right)-1\right\}}{3}$$

Using the formula

$$= \frac{\left(k+1\right)\left\{4\left(k+1\right)^2+6\left(k+1\right)-1\right\}}{3}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

8. $1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\, 2^{n+1} + 2$

Solution:

We can write the given statement as

P (n): $1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\, 2^{n+1} + 2$

If n = 1 we get

P (1): $1.2 = 2 = (1 - 1)\, 2^{1+1} + 2 = 0 + 2 = 2$

Which is true.

Consider P (k) be true for some positive integer k

$1.2 + 2.2^2 + 3.2^2 + \ldots + k.2^k = (k - 1)\, 2^{k+1} + 2 \ldots$ (i)

Now let us prove that P (k + 1) is true.

Here

$$\left\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k\right\} + (k+1) \cdot 2^{k+1}$$

By using equation (i)

$$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$

Taking the common terms out

$$= 2^{k+1}\left\{(k-1) + (k+1)\right\} + 2$$

So we get

$$= 2^{k+1}.2k + 2$$

It can be written as

$$= k.2^{(k+1)+1} + 2$$

$$= \left\{(k+1) - 1\right\} 2^{(k+1)+1} + 2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

9.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

If n = 1 we get

$$P(1): \frac{1}{2}=1-\frac{1}{2^1}=\frac{1}{2}$$

Which is true.

Consider P (k) be true for some positive integer k

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots.+\frac{1}{2^k}=1-\frac{1}{2^k} \qquad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots.+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

By using equation (i)

$$=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

We can write it as

$$=1-\frac{1}{2^k}+\frac{1}{2.2^k}$$

Taking the common terms out

$$= 1 - \frac{1}{2^k}\left(1 - \frac{1}{2}\right)$$

So we get

$$= 1 - \frac{1}{2^k}\left(\frac{1}{2}\right)$$

It can be written as

$$= 1 - \frac{1}{2^{k+1}}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

10.

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

If n = 1 we get

$$P(1)=\frac{1}{2.5}=\frac{1}{10}=\frac{1}{6.1+4}=\frac{1}{10}$$

Which is true.

Consider P (k) be true for some positive integer k

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4} \qquad ... \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.......+\frac{1}{(3k-1)(3k+2)}+\frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

By using equation (i)

$$=\frac{k}{6k+4}+\frac{1}{(3k+3-1)(3k+3+2)}$$

By simplification of terms

$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$

Taking 2 as common

$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

Taking the common terms out

$$=\frac{1}{(3k+2)}\left(\frac{k}{2}+\frac{1}{3k+5}\right)$$

Taking LCM

$$= \frac{1}{(3k+2)}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$$

By multiplication

$$= \frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$$

Separating the terms

$$= \frac{1}{(3k+2)}\left(\frac{(3k+2)(k+1)}{2(3k+5)}\right)$$

By further calculation

$$= \frac{(k+1)}{6k+10}$$

So we get

$$= \frac{(k+1)}{6(k+1)+4}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

11.

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

If n = 1 we get

$$P(1): \frac{1}{1\cdot 2\cdot 3}=\frac{1\cdot(1+3)}{4(1+1)(1+2)}=\frac{1\cdot 4}{4\cdot 2\cdot 3}=\frac{1}{1\cdot 2\cdot 3}$$

Which is true.

Consider P (k) be true for some positive integer k

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\ldots+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)} \quad \ldots (i)$$

Now let us prove that P (k + 1) is true.

Here

$$\left[\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+.....+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

By using equation (i)

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

Taking out the common terms

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4}+\frac{1}{k+3}\right\}$$

Taking LCM

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2+4}{4(k+3)}\right\}$$

Expanding using formula

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k^2+6k+9)+4}{4(k+3)}\right\}$$

By further calculation

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+6k^2+9k+4}{4(k+3)}\right\}$$

We can write it as

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^2+k+4k^2+8k+4}{4(k+3)}\right\}$$

Taking the common terms

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k^2+2k+1)+4(k^2+2k+1)}{4(k+3)}\right\}$$

We get

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+1)^2+4(k+1)^2}{4(k+3)}\right\}$$

Here

$$= \frac{(k+1)^2 (k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

12.

$$a+ar+ar^2+\ldots+ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

Solution:

We can write the given statement as

$$\mathrm{P}(n): a+ar+ar^2+\ldots+ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

If n = 1 we get

$$\mathrm{P}(1): a=\frac{a\left(r^1-1\right)}{(r-1)}=a$$

Which is true.

Consider P (k) be true for some positive integer k

$$a+ar+ar^2+\ldots\ldots+ar^{k-1}=\frac{a\left(r^k-1\right)}{r-1} \qquad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$$\left\{a+ar+ar^2+\ldots\ldots+ar^{k-1}\right\}+ar^{(k+1)-1}$$

By using equation (i)

$$=\frac{a\left(r^k-1\right)}{r-1}+ar^k$$

Taking LCM

$$=\frac{a\left(r^k-1\right)+ar^k(r-1)}{r-1}$$

Multiplying the terms

$$= \frac{a\left(r^k - 1\right) + ar^{k+1} - ar^k}{r-1}$$

So we get

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r-1}$$

By further simplification

$$= \frac{ar^{k+1} - a}{r-1}$$

Taking the common terms out

$$= \frac{a\left(r^{k+1} - 1\right)}{r-1}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

13.

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2$$

Solution:

We can write the given statement as

$$\mathrm{P}(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2$$

If n = 1 we get

$$\mathrm{P}(1):\left(1+\frac{3}{1}\right) = 4 = (1+1)^2 = 2^2 = 4,$$

Which is true.

Consider P (k) be true for some positive integer k

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right) = (k+1)^2 \qquad \ldots (1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{\{2(k+1)+1\}}{(k+1)^2}\right\}$$

By using equation (i)

$$=(k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right)$$

Taking LCM

$$=(k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$$

So we get

$$=(k+1)^2+2(k+1)+1$$

By further simplification

$$=\{(k+1)+1\}^2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

14.

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

Solution:

We can write the given statement as

$$P(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

If n = 1 we get

$$P(1):\left(1+\frac{1}{1}\right)=2=(1+1)$$

Which is true.

Consider P (k) be true for some positive integer k

$$P(k):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)=(k+1) \qquad \ldots (1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

By using equation (i)

$$=(k+1)\left(1+\frac{1}{k+1}\right)$$

Taking LCM

$$=(k+1)\left(\frac{(k+1)+1}{(k+1)}\right)$$

By further simplification

= (k + 1) + 1

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

15.

$$1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Solution:

We can write the given statement as

$$P(n)=1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

If n = 1 we get

$$P(1)=1^2=1=\frac{1(2.1-1)(2.1+1)}{3}=\frac{1.1.3}{3}=1,$$

Which is true.

Consider P (k) be true for some positive integer k

$$P(k)=1^2+3^2+5^2+...+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3} \qquad ...(1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left\{1^2+3^2+5^2+...+(2k-1)^2\right\}+\left\{2(k+1)-1\right\}^2$$

By using equation (i)

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+2-1)^2$$

So we get

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

Taking LCM

$$= \frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

Taking the common terms out

$$= \frac{(2k+1)\{k(2k-1)+3(2k+1)\}}{3}$$

By further simplification

$$= \frac{(2k+1)\{2k^2-k+6k+3\}}{3}$$

So we get

$$= \frac{(2k+1)\{2k^2+5k+3\}}{3}$$

We can write it as

$$= \frac{(2k+1)\{2k^2+2k+3k+3\}}{3}$$

Splitting the terms

$$= \frac{(2k+1)\{2k(k+1)+3(k+1)\}}{3}$$

We get

$$= \frac{(2k+1)(k+1)(2k+3)}{3}$$

$$= \frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

16.

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

Solution:

We can write the given statement as

$$P(n):\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

If n = 1 we get

$$P(1)=\frac{1}{1.4}=\frac{1}{3.1+1}=\frac{1}{4}=\frac{1}{1.4}$$

Which is true.

Consider P (k) be true for some positive integer k

$$P(k)=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \quad ...(1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

By using equation (i)

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

So we get

$$=\frac{1}{(3k+1)}\left\{k+\frac{1}{(3k+4)}\right\}$$

Taking LCM

$$=\frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$$

Multiplying the terms

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$$

It can be written as

$$= \frac{1}{(3k+1)}\left\{\frac{3k^2+3k+k+1}{(3k+4)}\right\}$$

Separating the terms

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

By further calculation

$$= \frac{(k+1)}{3(k+1)+1}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

17.

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Solution:

We can write the given statement as

$$\mathrm{P}(n):\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

If n = 1 we get

$$\mathrm{P}(1):\frac{1}{3.5}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}$$

Which is true.

Consider P (k) be true for some positive integer k

$$\mathrm{P}(k):\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)} \qquad ... (1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

By using equation (i)

$$= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)}$$

So we get

$$= \frac{1}{(2k+3)}\left[\frac{k}{3} + \frac{1}{(2k+5)}\right]$$

Taking LCM

$$= \frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{3(2k+5)}\right]$$

Multiplying the terms

$$= \frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$$

It can be written as

$$= \frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{3(2k+5)}\right]$$

Separating the terms

$$= \frac{1}{(2k+3)}\left[\frac{2k(k+1)+3(k+1)}{3(2k+5)}\right]$$

By further calculation

$$= \frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$= \frac{(k+1)}{3\{2(k+1)+3\}}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

18.

$$1+2+3+\ldots+n<\frac{1}{8}(2n+1)^2$$

Solution:

We can write the given statement as

$$P(n):1+2+3+\ldots+n<\frac{1}{8}(2n+1)^2$$

If n = 1 we get

$$1<\frac{1}{8}(2.1+1)^2=\frac{9}{8}$$

Which is true.

Consider P (k) be true for some positive integer k

$$1+2+\ldots+k<\frac{1}{8}(2k+1)^2 \qquad \ldots (1)$$

Now let us prove that P (k + 1) is true.

Here

$$(1+2+\ldots+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$$

By using equation (i)

$$<\frac{1}{8}\{(2k+1)^2+8(k+1)\}$$

Expanding terms using formula

$$<\frac{1}{8}\{4k^2+4k+1+8k+8\}$$

By further calculation

$$<\frac{1}{8}\{4k^2+12k+9\}$$

So we get

$$<\frac{1}{8}(2k+3)^2$$

$$<\frac{1}{8}\{2(k+1)+1\}^2$$

$$(1+2+3+\ldots+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

19. *n* (*n* + 1) (*n* + 5) is a multiple of 3

Solution:

We can write the given statement as

P (n): n (n + 1) (n + 5), which is a multiple of 3

If n = 1 we get

1 (1 + 1) (1 + 5) = 12, which is a multiple of 3

Which is true.

Consider P (k) be true for some positive integer k

k (k + 1) (k + 5) is a multiple of 3

k (k + 1) (k + 5) = 3m, where m ∈ N …… (1)

Now let us prove that P (k + 1) is true.

Here

(k + 1) {(k + 1) + 1} {(k + 1) + 5}

We can write it as

= (k + 1) (k + 2) {(k + 5) + 1}

By multiplying the terms

= (k + 1) (k + 2) (k + 5) + (k + 1) (k + 2)

So we get

= {k (k + 1) (k + 5) + 2 (k + 1) (k + 5)} + (k + 1) (k + 2)

Substituting equation (1)

= 3m + (k + 1) {2 (k + 5) + (k + 2)}

By multiplication

$= 3m + (k + 1) \{2k + 10 + k + 2\}$

On further calculation

$= 3m + (k + 1) (3k + 12)$

Taking 3 as common

$= 3m + 3 (k + 1) (k + 4)$

We get

$= 3 \{m + (k + 1) (k + 4)\}$

$= 3 \times q$ where $q = \{m + (k + 1) (k + 4)\}$ is some natural number

$(k + 1) \{(k + 1) + 1\} \{(k + 1) + 5\}$ is a multiple of 3

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

20. $10^{2n-1} + 1$ is divisible by 11

Solution:

We can write the given statement as

P (n): $10^{2n-1} + 1$ is divisible by 11

If n = 1 we get

P (1) = $10^{2.1-1} + 1 = 11$, which is divisible by 11

Which is true.

Consider P (k) be true for some positive integer k

$10^{2k-1} + 1$ is divisible by 11

$10^{2k-1} + 1 = 11m$, where $m \in N$ …… (1)

Now let us prove that P (k + 1) is true.

Here

$10^{2(k+1)-1} + 1$

We can write it as

$= 10^{2k+2-1} + 1$

$= 10^{2k+1} + 1$

By addition and subtraction of 1

$= 10^2 (10^{2k-1} + 1 - 1) + 1$

We get

$= 10^2 (10^{2k-1} + 1) - 10^2 + 1$

Using equation 1 we get

$= 10^2 . 11m - 100 + 1$

$= 100 \times 11m - 99$

Taking out the common terms

$= 11 (100m - 9)$

$= 11$ r, where r $= (100m - 9)$ is some natural number

$10^{2(k+1)-1} + 1$ is divisible by 11

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

21. $x^{2n} - y^{2n}$ is divisible by $x + y$

Solution:

We can write the given statement as

P (n): $x^{2n} - y^{2n}$ is divisible by $x + y$

If n = 1 we get

$P(1) = x^{2 \times 1} - y^{2 \times 1} = x^2 - y^2 = (x + y)(x - y)$, which is divisible by (x + y)

Which is true.

Consider P (k) be true for some positive integer k

$x^{2k} - y^{2k}$ is divisible by $x + y$

$x^{2k} - y^{2k} = m(x + y)$, where $m \in N$ …… (1)

Now let us prove that P (k + 1) is true.

Here

$x^{2(k+1)} - y^{2(k+1)}$

We can write it as

$= x^{2k} . x^2 - y^{2k} . y^2$

By adding and subtracting y^{2k} we get

$= x^2 (x^{2k} - y^{2k} + y^{2k}) - y^{2k}. y^2$

From equation (1) we get

$= x^2 \{m(x + y) + y^{2k}\} - y^{2k}. y^2$

By multiplying the terms

$= m(x + y) x^2 + y^{2k}. x^2 - y^{2k}. y^2$

Taking out the common terms

$= m(x + y) x^2 + y^{2k} (x^2 - y^2)$

Expanding using formula

$= m(x + y) x^2 + y^{2k} (x + y)(x - y)$

So we get

$= (x + y)\{mx^2 + y^{2k} (x - y)\}$, which is a factor of (x + y)

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

22. $3^{2n+2} - 8n - 9$ is divisible by 8

Solution:

We can write the given statement as

P (n): $3^{2n+2} - 8n - 9$ is divisible by 8

If n = 1 we get

P (1) = $3^{2 \times 1 + 2} - 8 \times 1 - 9 = 64$, which is divisible by 8

Which is true.

Consider P (k) be true for some positive integer k

$3^{2k+2} - 8k - 9$ is divisible by 8

$3^{2k+2} - 8k - 9 = 8m$, where $m \in$ N …… (1)

Now let us prove that P (k + 1) is true.

Here

$3^{2(k+1)+2} - 8(k+1) - 9$

We can write it as

$= 3^{2k+2} . 3^2 - 8k - 8 - 9$

By adding and subtracting 8k and 9 we get

$= 3^2 (3^{2k+2} - 8k - 9 + 8k + 9) - 8k - 17$

On further simplification

$= 3^2 (3^{2k+2} - 8k - 9) + 3^2 (8k + 9) - 8k - 17$

From equation (1) we get

$= 9. 8m + 9 (8k + 9) - 8k - 17$

By multiplying the terms

$= 9.\ 8m + 72k + 81 – 8k – 17$

So we get

$= 9.\ 8m + 64k + 64$

By taking out the common terms

$= 8\ (9m + 8k + 8)$

= 8r, where r = (9m + 8k + 8) is a natural number

So $3^{2(k + 1) + 2} – 8\ (k + 1) – 9$ is divisible by 8

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

23. $41^n – 14^n$ is a multiple of 27

Solution:

We can write the given statement as

P (n):$41^n – 14^n$is a multiple of 27

If n = 1 we get

P (1) = $41^1 – 14^1 = 27$, which is a multiple by 27

Which is true.

Consider P (k) be true for some positive integer k

$41^k – 14^k$is a multiple of 27

$41^k – 14^k = 27m$, where $m \in N$ …… (1)

Now let us prove that P (k + 1) is true.

Here

$41^{k + 1} – 14^{k + 1}$

We can write it as

$= 41^k. 41 – 14^k. 14$

By adding and subtracting 14^k we get

$= 41 (41^k – 14^k + 14^k) – 14^k. 14$

On further simplification

$= 41 (41^k – 14^k) + 41. 14^k – 14^k. 14$

From equation (1) we get

$= 41. 27m + 14^k (41 – 14)$

By multiplying the terms

$= 41. 27m + 27. 14^k$

By taking out the common terms

$= 27 (41m – 14^k)$

$= 27r$, where $r = (41m – 14^k)$ is a natural number

So $41^{k + 1} – 14^{k + 1}$ is a multiple of 27

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

24. $(2n +7) < (n + 3)^2$

Solution:

We can write the given statement as

P(n): $(2n +7) < (n + 3)2$

If n = 1 we get

$2.1 + 7 = 9 < (1 + 3)^2 = 16$

Which is true.

Consider P (k) be true for some positive integer k

$(2k + 7) < (k + 3)^2$ … (1)

Now let us prove that P (k + 1) is true.

Here

$\{2 (k + 1) + 7\} = (2k + 7) + 2$

We can write it as

$= \{2 (k + 1) + 7\}$

From equation (1) we get

$(2k + 7) + 2 < (k + 3)^2 + 2$

By expanding the terms

$2 (k + 1) + 7 < k^2 + 6k + 9 + 2$

On further calculation

$2 (k + 1) + 7 < k^2 + 6k + 11$

Here $k^2 + 6k + 11 < k^2 + 8k + 16$

We can write it as

$2 (k + 1) + 7 < (k + 4)^2$

$2 (k + 1) + 7 < \{(k + 1) + 3\}^2$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

2 Marks Questions & Answers

1. Prove by PMI$(ab)^2 = a^n b^n$.

Ans: In this Question we have to prove that $(ab)^2 = a^n b^n$ by using the method of PMI.

Given,

P (n): $(ab)^2 = a^n b^n$

Checking if the statement is true or not for n=1.

So, For n=1

P (1): ab=ab

Which is true.

Thus, P (1) is true.

Let, P (n): $(ab)^2 = a^n b^n$ is true for n=k.

That is, P (k): $(ab)^k = a^k b^k$….. (i)

Now, we have to show that the given statement P(n): $(ab)^2 = a^n b^n$ Is true for n=k+1

So, P (k+1): $(ab)^{k+1} = a^{k+1} b^{k+1}$

Now, L.H.S =$(ab)^{k+1}$

= $(ab)^k (ab)$

= $a^k b^k$ (ab) .... {From equation 1}

$= a^{k+1} b^{k+1}$ =R.H.S

Which is true.

Hence, P (k+1): $(ab)^{k+1}$= $a^{k+1} b^{k+1}$ **is true.**

Thus, P (k+1) is true when P (k) is true.

Therefore by P.M.I. the statement $((ab)^n = a^n b^n$ is true.

2. Show that the sum of the first n odd natural number is n^2.

Ans: In this Question we have to prove that 1+3+5+.....+ (2n−1)= n^2 by using the method of PMI.

Given,

$P(n): 1+3+5+.....+(2n-1) = \boldsymbol{n^2}$

Checking if the statement is true or not for $n=1$.

So, For $n=1$

$P(1): 1=1$

Which is true.

Thus, $P(1)$ is true.

Let, $P(n): 1+3+5+.....+(2n-1)= \boldsymbol{n^2}$ is true for $n=k$.

That is, $P(k): 1+3+5+.....+(2k-1)= \boldsymbol{k^2}$..... (i)

Now, we have to show that the given statement

$P(n): 1+3+5+.....+(2n-1) = \boldsymbol{n^2}$ is true for $n=k+1$

So, $P(k+1): 1+3+5+.....+(2k+1) = (\boldsymbol{k+1})^2$

Now, L.H.S $=1+3+5+.....+(2k-1)+(2k+1)$

$=\boldsymbol{k^2}+(2k+1)$........{ from equation 1 }

$(\boldsymbol{k+1})^2$ =R.H.S

Which is true.

Hence, $P(k+1): 1+3+5+.....+(2k+1) = (\boldsymbol{k+1})^2$ is true.

Thus $P(k+1)$ is true when $P(k)$ is true.

Therefore, by P.M.I. the statement $1+3+5+.....+(2n-1) = \boldsymbol{n^2}$ is true.

3. Prove by PMI $\boldsymbol{x^n - y^n}$ is divisible by $(x-y)$ whenever $x\text{-}y \neq 0$.

Ans: In this Question we have to prove that $\boldsymbol{x^n - y^n}$ is divisible by $x-y$ by using the method of PMI.
Given,

$P(n): \boldsymbol{x^n - y^n}$.

Checking if the statement is true or not for $n=1$

So, For $n=1$
$P(1): x-y$

Which is divisible by x−y

Thus, P (1) is true.

Let, P (n): $x^n - y^n$ is true for n=k

That is, P (k): $x^k - y^k$ = (x−y)λ, where λ∈N.......(i)

Now, we have to show that the given statement P (n): $x^n - y^n$ is true for n=k+1.

P (k+1):$x^{k+1} - y^{k+1}$

= $x^{k+1} - y^{k+1}$

= $x^k.x - y^k.y$

= ((x−y) λ+y^k) x−y^k·y from equation (i)

=λ (x−y) x + x·y^k−y^k·y

=λ (x−y) x+y^k (x−y)

= (x−y) (y^k+λx)

Hence, P (k+1): x^{k+1}− y^{k+1} is divisible by x−y

Thus, P (k+1) is true when P (k) is true.

Therefore by P.M.I. the given statement is true for every positive integer *n*.

4. The sum of the cubes of three consecutive natural number is divisible by 9.

Ans: In this Question we have to prove that $[n^3+(n+1)^3+(n+2)^3]$ is divisible by 9 by using the method of PMI.

Given, P (n): $[n^3+(n+1)^3+(n+2)^3]$

Checking if the statement is true or not for n=1

So, For n=1

P (1): $1^3 + 2^3 + 3^3 = 243$

Which is divisible by 9

Thus, P (1) is true.

Let, P (n): $[n^3+(n+1)^3+(n+2)^3]$ is true for n= k.

That is, P (k): $[k^3+(k+1)^3+(k+2)^3]$ = 9λ, where λ∈N.......(i)

Now, we have to show that the given statement

P (n): $[n^3+(n+1)^3+(n+2)^3]$ is true for n=k+1

P (k+1): $[(k+1)^3+(k+2)^3+(k+3)^3]$

= $[(k+1)^3+(k+2)^3+k^3+9k^2 27k+27]$

= $[9\lambda+9k^2+27k+27]$ from equation (i)

=$9[\lambda+k^2+3k+3]$

Which is divisible by 9

.Hence, P (k+1): $[k^3+(k+1)^3+(k+2)^3]$ is divisible by 9

.

Thus, P (k+1) is true when P (k) is true.

Therefore by P.M.I. the given statement is true for every positive integer n

.

5. Prove that 12^n+25^{n-1} is divisible by 13 .

Ans: In this Question we have to prove that 12^n+25^{n-1} is divisible by 13 by using the method of PMI.

Given,

P (n): 12^n+25^{n-1}

Checking if the statement is true or not for n=1

So, For n=1

P (1):12+(25)°=13

Which is divisible by 13

Thus, P (1)is true.

Let, P (n): 12^n+25^{n-1} is true for n=k

That is, P (k): 12^k+25^{k-1}=13λ, where λ∈N.......(i)

Now, we have to show that the given statement

P (n): $\mathbf{12^n+25^{n-1}}$is true for n=k+1

P (k+1): $\mathbf{12^{k+1}+25^k}$

= $\mathbf{12^k.12+25^k}$

= (13λ−$\mathbf{25^{k-1}}$) 12+$\mathbf{25^k}$ from equation (i)

=13·12λ−12·$\mathbf{25^{k-1}}$+$\mathbf{25^k}$

=13·12λ+$\mathbf{25^{k-1}}$ (−12+25)

=13·12λ+13·$\mathbf{25^{k-1}}$

=13(12λ+$\mathbf{25^{k-1}}$)

Which is divisible by 13.

Hence, P (k+1):$\mathbf{12^{k+1}+25^k}$ is divisible by 13.

Thus P (k+1) is true when P (k) is true.

Therefore by P.M.I. the given statement is true for every positive integer n
.

6. Prove that 1+3+$\mathbf{3^2}$+...+$\mathbf{3^{n-1}}$= $\mathbf{\frac{3^n-1}{2}}$

Ans: In this Question we have to prove that 1+3+$\mathbf{3^2}$+...+$\mathbf{3^{n-1}}$= $\mathbf{\frac{3^n-1}{2}}$ by using the method of PMI.
Given,

P (n): 1+3+$\mathbf{3^2}$+...+$\mathbf{3^{n-1}}$= $\mathbf{\frac{3^n-1}{2}}$

Checking if the statement is true or not for n=1

So, For n=1

P (1):1= $\mathbf{\frac{3-1}{2}}$

1=1

Which is true.

Thus, P (1) is true.

Let, P (n):1+3+$\mathbf{3^2}$+...+$\mathbf{3^{n-1}}$= $\mathbf{\frac{3^n-1}{2}}$ is true for n=k

That is, P (k):1+3+3^2+...+3^{n-1}= $\frac{3^n-1}{2}$

Now, we have to show that the given statement

P (n): 1+3+3^2+...+3^{n-1}= $\frac{3^n-1}{2}$ is true for n=k+1

So, P (k+1): 1+3+3^2+...+3^{k-1}+3^k= $\frac{3^{k+1}-1}{2}$

Now, L.H.S =1+3+3^2+...+3^{k-1}+3^k

=$\frac{3^k-1}{2}+3^k$ .... { from equation 1 }

= $\frac{3^k-1+2\times3^k}{2}$

= $\frac{3^k(1+2)-1}{2}$

=$\frac{3^k\times3-1}{2}$

$\frac{3^{k+1}-1}{2}$=R.H.S

Which is true.

Hence, P (k+1): 1+3+3^2+...+3^{k-1}+3^k = $\frac{3^{k+1}-1}{2}$ is true.

Thus, P (k+1) is true when P (k) is true.

Therefore by P.M.I. the statement 1+3+3^2+...+3^{n-1}= $\frac{3^n-1}{2}$ is true.

Multiple Choice Questions

1. If x^n – 1 is divisible by x – k, then the least positive integral value of k is

(a) 1

(b) 2

© 3

(d) 4

Correct option: (a) 1

Solution:

Given,

P(n): xn – 1 is divisible by x – k

Let us substitute n = 1, 2, 3,..

⇒ P(1) : $x – 1$

⇒ P(2) : $x^2 – 1 = (x−1)(x+1)$

⇒ P(3) : $x^3 – 1 = (x – 1)(x^2 + x + 1)$

⇒ P(4) : $x^4 – 1 = (x^2 – 1)(x^2 + 1) = (x – 1)(x + 1)(x^2 + 1)$

Therefore, the least positive integral value of k is 1.

2. For any natural number n, $2^{2n} – 1$ is divisible by

(a) 2

(b) 3

© 4

(d) 5

Correct option: (b) 3

Solution:

Let $P(n) = 2^{2n} – 1$

Substituting n = 1, 2, 3,….

$P(1) = 2^{2(1)} – 1 = 4 – 1 = 3$

This is divisible by 3.

$P(2) = 2^{2(2)} – 1 = 16 – 1 = 15$

This is divisible by 3.

$P(3) = 2^{2(3)} – 1 = 256 – 1 = 255$

This is also divisible by 3.

Assume that P(n) is true for some natural number k, i.e., P(k): $2^{2k} – 1$ is divisible by 3, i.e., $2^{2k} – 1 = 3q$, where $q \in N$

Now,

P(k + 1) : $2^{2(k+1)} – 1$

$= 2^{2k + 2} – 1$

$= 2^{2k} . 2^2 – 1$

$= 2^{2k} . 4 – 1$

$= 3.2^{2k} + (2^{2k} – 1)$

$= 3.2^{2k} + 3q$

$= 3 (2^{2k} + q) = 3m$, where $m \in N$

Thus P(k + 1) is true, whenever P(k) is true.

Therefore, for any natural number n, $2^{2n} – 1$ is divisible by 3.

3. A student was asked to prove a statement P(n) by induction. He proved that P(k + 1) is true whenever P(k) is true for all k > 5 ∈ N and also that P(5) is true. Based on this, he could conclude that P(n) is true

(a) for all n ∈ N

(b) for all n > 5

© for all n ≥ 5

(d) for all n < 5

Correct option: © for all n ≥ 5

Solution:

The student could be able to conclude that P(n) is true for all n ≥ 5 since P(5) is true for all k > 5 ∈ N as well as true for P(5) and P(k + 1) is true, whenever P(k) is true.

4. If P (n): “$49^n + 16^n + k$ is divisible by 64 for n ∈ N” is true, then the least negative integral value of k is

(a) 1

(b) -2

© -1

(d) -3

Correct option: © -1

Solution:

Given that P(n) : $49^n + 16^n + k$ is divisible by 64 for $n \in N$

For n = 1,

P(1) : 49 + 16 + k = 65 + k is divisible by 64.

Thus k, should be -1 since, 65 – 1 = 64 is divisible by 64.

5. For all $n \in N$, $3.5^{2n+1} + 2^{3n+1}$ is divisible by

(a) 19

(b) 17

© 23

(d) 25

Correct option: (b) 17

Solution:

Let P(n) be the statement that $3.5^{2n+1} + 2^{3n+1}$ is divisible by 17

If n = 1, then given expression = $3 * 5^3 + 2^4 + 375 + 16 = 391 = 17 * 23$, divisible by 17.

P(1) is true

Assume that P(k) is true.

$3.5^{2k+1} + 2^{3k+1}$ is divisible by 17.

$3.5^{2k} = 1 + 2^{3k+1} = 17m$ where $m \in N$

$3.5^{2(k+1)+1} + 23^{(k+1)+1}$

$= 3.5^{2k+1} * 5^2 + 2^{3k+1} * 2^3$

$= 25^{(17m - 23k+1)} + 8.2^{3k+1}$

$= 425m - 25.2^{3k+1} + 8.2^{3k+1}$

$= 425m - 17.2^{3k+1}$

$= 17(25m - 2^{3k+1})$, divisible by 17

P(k + 1) is true by Principle of Mathematical Induction

P(n) is true for all n ∈ N. $3.5^{2n+1} + 2^{3n+1}$ is divisible by 17 for all n ∈ N

6. n(n + 1) (n + 5) is a multiple of

(a) 3

(b) 8

© 5

(d) 7

Correct option: (a) 3

Solution:

Let P(n) = n(n + 1)(n + 5)

Substituting n = 1, 2, 3,….

P(1) = 1(1 + 1)(1 + 5) = 2(6) = 12; multiple of 2, 3, 4, 6

P(2) = 2(2 + 1)(2 + 5) = 2(3)(7) = 42; multiple of 2, 3, 6, 7

P(3) = 3(3 + 1)(3 + 5) = 3(4)(8) = 96; multiple of 2, 3, 4, 6, 8, 12..

Thus, from the above statements and verifying the options, we can say that n(n + 1)(n + 5) is a multiple of 3.

7. $n^2 < 2^n$ for all natural numbers

(a) $n \geq 5$

(b) n < 5

© n > 1

(d) $n \leq 3$

Correct option: (a) $n \geq 5$

Solution:

Consider, P(n) : $n^2 < 2^n$

Substituting n = 1, 2, 3,…

P(1): $1^2 < 2^1$

1 < 2 (not true)

P(2): $2^2 < 2^2$

4 < 4 (not true)

P(3): $3^2 < 2^3$

9 < 8 (not true)

P(4): $4^2 < 2^4$

16 < 16 (not true)

P(5): $5^2 < 2^5$

25 < 32 (true)

P(6): $6^2 < 2^6$

26 < 64 (true)

Thus, $n^2 < 2^n$ for all natural numbers $n \geq 5$.

8. If $10^n + 3.4^{n+2} + k$ is divisible by 9 for all $n \in N$, then the least positive integral value of k is

(a) 5

(b) 3

© 7

(d) 1

Correct option: (a) 5

Solution:

Given that $10^n + 3.4^{n+2} + k$ is exactly divisible by 9.

Consider: $P(n) = 10^n + 3.4^{n+2} + k$

Substituting n = 1,

$P(1) = 10^1 + 3.4^{1+2} + k$

$= 10 + 3(64) + k$

$= 10 + 192 + k$

$= 202 + k$ is exactly divisible by 9, the value of k will be 5.

9. Let P(n) : "$2^n < (1 \times 2 \times 3 \times \ldots \times n)$". Then the smallest positive integer for which P(n) is true is

(a) 1

(b) 2

© 3

(d) 4

Correct option: (d) 4

Solution:

$P(1) : 2^1 < 1$

$2 < 1$ is false

$P(2) : 2^2 < 1 \times 2$

$4 < 2$ is false

$P(3) : 2^3 < 1 \times 2 \times 3$

8 < 6 is false

P(4) : $2^4 < 1 \times 2 \times 3 \times 4$

16 < 24 is true

10. For every positive integer n, $7^n - 3^n$ is divisible by

(a) 3

(b) 4

© 7

(d) 5

Correct option: (b) 4

Solution:

Let $P(n) = 7^n - 3^n$

Substituting n = 1, 2, 3,…

$P(1) = 7^1 - 3^1 = 7 - 3 = 4$

$P(2) = 7^2 - 3^2 = 49 - 9 = 40$

$P(3) = 7^3 - 3^3 = 343 - 27 = 316$

Thus, for every positive integer n, $7^n - 3^n$ is divisible by 4.

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