Chapter 13

Limits and Derivations

"Limits and Derivatives." chapter lays the groundwork for the fundamental concepts of calculus, introducing students to the idea of limits and derivatives. The concept of a limit is essential in understanding the behavior of a function as its input approaches a particular value. Students delve into the formal definition of a limit and explore techniques for evaluating limits algebraically and graphically. The chapter then progresses to the concept of derivatives, which represents the rate at which a function changes with respect to its independent variable. The derivative is introduced as a limit, emphasizing its geometric interpretation as the slope of the tangent to the curve at a given point.

By the end of this chapter, students gain a solid foundation in the fundamental principles of calculus, setting the stage for more advanced topics in differential and integral calculus in subsequent classes.

Exercise 13.1

1. Evaluate the given limit: $\lim_{x \to 3} x + 3$

Solution:

Given,

$\lim_{x \to 3} x + 3$

Substituting x = 3, we get

= 3 + 3

= 6

2. Evaluate the given limit: $\lim_{x \to \pi}\left(x - \frac{22}{7}\right)$

Solution:

Given limit,

$\lim_{x \to \pi}\left(x - \frac{22}{7}\right)$

Substituting x = π, we get

$\lim_{x \to \pi}\left(x - \frac{22}{7}\right)$ = (π – 22 / 7)

3. Evaluate the given limit: $\lim_{r \to 1} \pi r^2$

Solution:

Given limit, $\lim_{r \to 1} \pi r^2$

Substituting r = 1, we get

$\lim_{r \to 1} \pi r^2$ = $\pi(1)^2$

= π

4. Evaluate the given limit: $\lim_{x \to 4} \frac{4x+3}{x-2}$

Solution:

Given limit,

$$\lim_{x \to 4} \frac{4x+3}{x-2}$$

Substituting x = 4, we get

$\lim_{x \to 4} \frac{4x+3}{x-2}$ = [4(4) + 3] / (4 – 2)

= (16 + 3) / 2

= 19 / 2

5. Evaluate the given limit: $\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$

Solution:

Given limit,

$$\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$$

Substituting x = -1, we get

$$\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$$

= $[(-1)^{10} + (-1)^5 + 1] / (-1 - 1)$

= (1 – 1 + 1) / – 2

= – 1 / 2

6. Evaluate the given limit: $\lim_{x\to 0} \frac{(x+1)^5 - 1}{x}$

Solution:

Given limit,

$$\lim_{x\to 0} \frac{(x+1)^5 - 1}{x}$$

$= [(0 + 1)^5 – 1] / 0$

$=0$

Since this limit is undefined,

Substitute x + 1 = y, then x = y – 1

$$\lim_{y\to 1} \frac{(y)^5 - 1}{y - 1}$$

$$= \lim_{y\to 1} \frac{(y)^5 - 1^5}{y - 1}$$

We know that,

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Hence,

$$\lim_{y\to 1} \frac{(y)^5 - 1^5}{y - 1}$$

$=5(1)^{5-1}$

$= 5(1)^4$

$= 5$

7. Evaluate the given limit: $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4}$

Solution:

By evaluating the limit at x = 2, we get

$\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4}$ = $[3(2)^2 - x - 10] / 4 - 4$

= 0

Now, by factorising numerator, we get

$\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4} = \lim_{x \to 2} \frac{3x^2 - 6x + 5x - 10}{x^2 - 2^2}$

We know that,

$a^2 - b^2 = (a - b)(a + b)$

$= \lim_{x \to 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)}$

$= \lim_{x \to 2} \frac{(3x+5)}{(x+2)}$

By substituting x = 2, we get,

= [3(2) + 5] / (2 + 2)

= 11 / 4

8. Evaluate the given limit: $\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

Solution:

First substitute x = 3 in the given limit, we get

$$= \lim_{x \to 3} \frac{(3)^4 - 81}{2(3)^2 - 5 \times 3 - 3}$$

= (81 – 81) / (18 – 18)

= 0 / 0

Since the limit is of the form 0 / 0, we need to factorise the numerator and denominator

$$\lim_{x \to 3} \frac{(x^2 - 9)(x^2 + 9)}{2x^2 - 6x + x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)(x^2 + 9)}{(2x + 1)(x - 3)}$$

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{(x + 3)(x^2 + 9)}{(2x + 1)}$$

Now substituting x = 3, we get

$$= \frac{(3 + 3)(3^2 + 9)}{(2 \times 3 + 1)}$$

= 108 / 7

Hence,

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = 108 / 7$$

9. Evaluate the given limit: $\lim_{x \to 0} \frac{ax + b}{cx + 1}$

Solution:

$$\lim_{x \to 0} \frac{ax + b}{cx + 1}$$

= [a (0) + b] / c (0) + 1

= b / 1

= b

10. Evaluate the given limit: $\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Solution:

$\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$ $= (1 - 1) / (1 - 1)$

$= 0$

Let the value of $z^{1/6}$ be x

$(z^{1/6})^2 = x^2$

$z^{1/3} = x^2$

Now, substituting $z^{1/3} = x^2$ we get

$\lim_{x\to 1}\frac{x^2-1}{x-1} = \frac{x^2-1^2}{x-1}$

We know that,

$\lim_{x\to a}\frac{x^n-a^n}{x-a}$ $= na^{n-1}$

$\lim_{x\to 1}\frac{x^2-1^2}{x-1}$ $= 2\ (1)^{2-1}$

$= 2$

11. Evaluate the given limit: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$

Solution:

Given limit,

$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$

Substituting x = 1,

$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}$

$= [a\ (1)^2 + b\ (1) + c] / [c\ (1)^2 + b\ (1) + a]$

$= (a + b + c) / (a + b + c)$

Given,

$[a+b+c \neq 0]$

= 1

12. Evaluate the given limit: $\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Solution:

By substituting x = – 2, we get

$\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2} = 0 / 0$

Now,

$\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2} = \frac{\frac{2+x}{2x}}{x+2}$

= 1 / 2x

= 1 / 2(-2)

= - 1 / 4

13. Evaluate the given limit: $\lim_{x\to 0} \frac{\sin ax}{bx}$

Solution:

Given $\lim_{x\to 0} \frac{\sin ax}{bx}$

Formula used here

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

By applying the limits in the given expression

$$\lim_{x\to 0} \frac{\sin ax}{bx} = \frac{0}{0}$$

By multiplying and dividing by ‘a’ in the given expression, we get

$$\lim_{x \to 0} \frac{\sin ax}{bx} \times \frac{a}{a}$$

We get,

$$\lim_{x \to 0} \frac{\sin ax}{ax} \times \frac{a}{b}$$

We know that,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \frac{a}{b} \lim_{ax\to 0} \frac{\sin ax}{ax} = \frac{a}{b} \times 1$$

= a / b

14. Evaluate the given limit: $\lim\limits_{x \to 0} \dfrac{\sin ax}{\sin bx}, a, b \neq 0$

Solution:

$\lim\limits_{x \to 0} \dfrac{\sin ax}{\sin bx} = 0 / 0$

By multiplying ax and bx in numerator and denominator, we get

$$\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \lim_{x \to 0} \frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx}$$

Now, we get

$$\frac{a}{b} \frac{\lim\limits_{ax \to 0} \frac{\sin ax}{ax}}{\lim\limits_{bx \to 0} \frac{\sin bx}{bx}}$$

We know that,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Hence, a / b × 1

= a / b

15. Evaluate the given limit:

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

Solution:

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)} = \lim_{\pi-x\to0}\frac{\sin(\pi-x)}{(\pi-x)}\times\frac{1}{\pi}$$

$$= \frac{1}{\pi}\lim_{\pi-x\to0}\frac{\sin(\pi-x)}{(\pi-x)}$$

We know that

$$\lim_{x\to0}\frac{\sin x}{x} = 1$$

$$\frac{1}{\pi}\lim_{\pi-x\to0}\frac{\sin(\pi-x)}{(\pi-x)} = \frac{1}{\pi}\times 1$$

$= 1 / \pi$

16. Evaluate the given limit:

$$\lim_{x\to0}\frac{\cos x}{\pi-x}$$

Solution:

$$\lim_{x\to0}\frac{\cos x}{\pi-x} = \frac{\cos 0}{\pi-0}$$

$= 1 / \pi$

17. Evaluate the given limit:

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$$

Solution:

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1} = \frac{0}{0}$$

Hence,

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x \to 0} \frac{1 - 2\sin^2 x - 1}{1 - 2\sin^2 \frac{x}{2} - 1}$$

($\cos 2x = 1 - 2\sin^2 x$)

$$\lim_{x \to 0} \frac{\sin^2 x}{\sin^2 \frac{x}{2}} = \lim_{x \to 0} \frac{\frac{\sin^2 x \times x^2}{x^2}}{\frac{\sin^2 \frac{x}{2} \times \frac{x^2}{4}}{(\frac{x}{2})^2}}$$

$$= 4 \frac{\lim_{x \to 0} \frac{\sin^2 x}{x^2}}{\lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}}$$

$$= 4 \frac{\lim_{x \to 0} \left(\frac{\sin^2 x}{x^2}\right)^2}{\lim_{x \to 0} \left(\frac{\sin^2 \frac{x}{2}}{(\frac{x}{2})^2}\right)^2}$$

We know that,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$= 4 \times 1^2 / 1^2$

$= 4$

18. Evaluate the given limit:

$$\lim_{x\to 0}\frac{ax + x\cos x}{b\sin x}$$

Solution:

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x} = \frac{0}{0}$$

Hence,

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x} = \frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$$

$$= \frac{1}{b}\lim_{x\to 0}\times \lim_{x\to 0}(a+\cos x)$$

$$= \frac{1}{b}\times\frac{1}{\lim_{x\to 0}\frac{\sin x}{x}}\times \lim_{x\to 0}(a+\cos x)$$

We know that,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$= \frac{1}{b}\times(a+\cos 0)$$

= (a + 1) / b

19. Evaluate the given limit:

$$\lim_{x\to 0} x \sec x$$

Solution:

$$\lim_{x\to 0} x\sec x = \lim_{x\to 0} \frac{x}{\cos x}$$

$$= \lim_{x\to 0} \frac{0}{\cos 0} = \frac{0}{1}$$

$$= 0$$

20. Evaluate the given limit:

$$\lim_{x\to 0} \frac{\sin ax + bx}{ax + \sin bx} a, b, a + b \neq 0$$

Solution:

$$\lim_{x\to 0} \frac{\sin ax+bx}{ax+\sin bx} = \frac{0}{0}$$

Hence,

$$\lim_{x\to 0} \frac{\sin ax+bx}{ax+\sin bx} = \lim_{x\to 0} \frac{(\sin\frac{ax}{ax})ax+bx}{ax+(\sin\frac{bx}{bx})}$$

$$= \frac{\left(\lim_{ax\to 0} \sin\frac{ax}{ax}\right)\times\lim_{x\to 0} ax+\lim_{x\to 0} bx}{\lim_{x\to 0} ax+\lim_{x\to 0} bx\times\left(\lim_{bx\to 0} \sin\frac{bx}{bx}\right)}$$

We know that,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \frac{\lim_{x\to 0} ax+\lim_{x\to 0} bx}{\lim_{x\to 0} ax+\lim_{x\to 0} bx}$$

We get,

$$= \frac{\lim_{x\to 0}(ax+bx)}{\lim_{x\to 0}(ax+bx)}$$

= 1

21. Evaluate the given limit:

$$\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$$

Solution:

$$\lim_{x \to 0}(\text{cosec } x - \cot x)$$

Applying the formulas for cosec x and cot x, we get

$$\text{cosec } x = \frac{1}{\sin x} \; and \; \cot x = \frac{\cos x}{\sin x}$$

$$\lim_{x \to 0}(\text{cosec } x - \cot x) = \lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$\lim_{x \to 0}(cosec\, x - cot\, x) = \lim_{x \to 0}\frac{1 - \cos x}{\sin x}$$

Now, by applying the formula we get,

$$1 - \cos x = 2 \sin^2\frac{x}{2} \; and \; \sin x = 2 \sin\frac{x}{2}\cos\frac{x}{2}$$

$$\lim_{x \to 0}(\text{cosec } x - \cot x) = \lim_{x \to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$\lim_{x \to 0}(cosec\, x - cot\, x) = \lim_{x \to 0}\tan\frac{x}{2}$$

= 0

22. Evaluate the given limit:

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}}$$

Solution:

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = \frac{0}{0}$$

Let $x - (\pi / 2) = y$

Then, $x \to (\pi/2) = y \to 0$

Now, we get

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0} \frac{\tan 2(y+\frac{\pi}{2})}{y}$$

$$= \lim_{y\to 0} \frac{\tan(2y+\pi)}{y}$$

$$= \lim_{y\to 0} \frac{\tan(2y)}{y}$$

We know that,

$\tan x = \sin x / \cos x$

$$= \lim_{y\to 0} \frac{\sin 2y}{y \cos 2y}$$

By multiplying and dividing by 2, we get

$$=\lim_{y\to 0}\frac{\sin 2y}{2y}\times\frac{2}{\cos 2y}$$

$$=\lim_{2y\to 0}\frac{\sin 2y}{2y}\times\lim_{y\to 0}\frac{2}{\cos 2y}$$

$= 1 \times 2 / \cos 0$

$= 1 \times 2 / 1$

$= 2$

23.

Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} 2x+3 & x \le 0 \\ 3(x+1) & x > 0 \end{cases}$

Solution:

Given function is $f(x) = \begin{cases} 2x+3 & x \le 0 \\ 3(x+1) & x > 0 \end{cases}$

$\lim_{x \to 0} f(x)$:

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (2x+3)$

$= 2(0) + 3$

$= 0 + 3$

$= 3$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} 3(x+1)$

$=3(0+1)$

$= 3(1)$

$= 3$

Hence, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 3$

Now, for $\lim_{x \to 1} f(x)$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} 3(x+1)$$

$= 3\ (1 + 1)$

$= 3\ (2)$

$= 6$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} 3(x+1)$$

$= 3\ (1 + 1)$

$= 3\ (2)$

$= 6$

Hence, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 6$

$\lim_{x \to 0} f(x)=3$ and $\lim_{x \to 1} f(x)=6$

24. Find

$\lim_{x \to 1} f(x)$ **, where**

$$f(x) = \begin{cases} x^2 - 1 & x \le 1 \\ -x^2 - 1 & x > 1 \end{cases}$$

Solution:

Given function is:

$$f(x) = \begin{cases} x^2 - 1 & x \le 1 \\ -x^2 - 1 & x > 1 \end{cases}$$

$\lim_{x \to 1} f(x)$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} x^2 - 1$$

$= 1^2 - 1$

$= 1 - 1$

$= 0$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (-x^2 - 1)$$

$= (-1^2 - 1)$

$= -1 - 1$

$= -2$

We find,

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

Hence, $\lim_{x \to 1} f(x)$ does not exist

25. Evaluate

$\lim_{x \to 0} f(x)$ **, where f(x) =**

$\begin{cases} \frac{|x|}{x}, x \neq 0 \\ x \\ 0, x = 0 \end{cases}$

Solution:

Given function is f(x) = $\begin{cases} \frac{|x|}{x}, x \neq 0 \\ x \\ 0, x = 0 \end{cases}$

We know that,

$\lim_{x \to a} f(x)$ exists only when $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

Now, we need to prove that: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

We know,

|x| = x, if x > = -x, if x < 0

Hence,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x}$

$= \lim_{x \to 0} \frac{-x}{x} = \lim_{x \to 0}(-1)$

= -1

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x}$

$= \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0}(1)$

= 1

We find here,

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence, $\lim_{x \to 0} f(x)$ does not exist.

26. Find

$\lim\limits_{x \to 0} f(x)$ **, where f (x) =**

$$\begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution:

Given function is:

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\lim\limits_{x \to 0} f(x)$:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{|x|}$$

$$= \lim_{x \to 0} \frac{x}{-x} = \lim_{x \to 0} \frac{1}{-1}$$

$= -1$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{|x|}$$

$$= \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} (1)$$

$= 1$

We find here,

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

Hence, $\lim\limits_{x \to 0} f(x)$ does not exist.

27. Find

$\lim\limits_{x \to 5} f(x)$ **, where**

$f(x) = |x| - 5$

Solution:

Given function is:

$f(x) = |x| - 5$

$\lim\limits_{x \to 5} f(x)$:

$\lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^-} |x| - 5$

$= \lim\limits_{x \to 5} (x - 5) = 5 - 5$

$= 0$

$\lim\limits_{x \to 5^+} f(x) = \lim\limits_{x \to 5^+} |x| - 5$

$= \lim\limits_{x \to 5} (x - 5)$

$= 5 - 5$

$= 0$

Hence, $\lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^+} f(x) = \lim\limits_{x \to 5} f(x) = 0$

28. Suppose

$$f(x)=\begin{cases} a+bx, x<1 \\ 4, \quad x=1 \\ b-ax\ x>1 \end{cases}$$ **and if**

$\lim\limits_{x\to 1} f(x) = f(1)$ **what are the possible values of a and b?**

Solution:

Given function is:

$$f(x)=\begin{cases} a+bx, x<1 \\ 4, x=1 \\ b-ax, x>1 \end{cases}$$ and

$$\lim_{x\to 1} f(x) = f(1)$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} a + bx$$

$$= a + b\,(1)$$

$$= a + b$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} b - ax$$

$$= b - a\,(1)$$

$$= b - a$$

Here,

$f(1) = 4$

Hence, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1} f(x) = f(1)$

Then, $a + b = 4$ and $b - a = 4$

By solving the above two equations, we get,

$a = 0$ and $b = 4$

Therefore, the possible values of a and b is 0 and 4 respectively

29. Let $a_1, a_2, \ldots\ldots a_n$ be fixed real numbers and define a function

$f(x) = (x - a_1)(x - a_2) \ldots\ldots (x - a_n)$.

What is $\lim\limits_{x \to a_1} f(x)$? **For some $a \neq a_1, a_2, \ldots\ldots a_n$, compute** $\lim\limits_{x \to a} f(x)$.

Solution:

Given function is:

$f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$

$\lim\limits_{x \to a_1} f(x)$:

$$\lim_{x \to a_1} f(x) = \lim_{x \to a_1} [(x - a_1)(x - a_2) \ldots (x - a_n)]$$

$$= [\lim_{x \to a_1}(x - a_1)] \left[\lim_{x \to a_1}(x - a_2)\right] \ldots \left[\lim_{x \to a_1}(x - a_n)\right]$$

We get,

$$= (a_1 - a_1)(a_1 - a_2) \ldots (a_1 - a_n) = 0$$

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Hence, $\lim\limits_{x \to a_1} f(x) = 0$

$\lim\limits_{x \to a} f(x)$:

$$\lim_{x \to a} f(x) = \lim_{x \to a}[(x - a_1)(x - a_2) \ldots (x - a_n)]$$

$$= [\lim_{x \to a}(x - a_1)] \left[[\lim_{x \to a}(x - a_2)\right] \ldots \left[[\lim_{x \to a}(x - a_n)\right]$$

We get,

$= (a - a_1)\ (a - a_2)\ \ldots..\ (a - a_n)$

Hence, $\lim\limits_{x \to a} f(x) = (a - a_1)\ (a - a_2)\ \ldots\ (a - a_n)$

Therefore, $\lim\limits_{x \to a_1} f(x) = 0$ and $\lim\limits_{x \to a} f(x) = (a - a_1)\ (a - a_2)\ \ldots\ (a - a_n)$

30. If $f(x)=\begin{cases}|x|+1, x<0\\ 0, \quad x=0\\ |x|-1, x>0\end{cases}$ **For what value (s) of a does** $\lim\limits_{x\to a} f(x)$ **exist?**

Solution:

Given function is:

$$f(x)=\begin{cases}|x|+1, x<0\\ 0, \quad x=0\\ |x|-1, x>0\end{cases}$$

There are three cases.

Case 1:

When $a=0$

$\lim\limits_{x\to 0} f(x)$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(|x|+1)$$

$$= \lim_{x\to 0}(-x+1) = -0+1$$

$$= 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(|x| - 1)$$

$$= \lim_{x\to 0}(x - 1) = 0 - 1$$

= -1

Here, we find

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Hence, $\lim_{x\to 0} f(x)$ does not exit.

Case 2:

When a < 0

$\lim_{x\to a} f(x)$:

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-}(|x| + 1)$$

$$= \lim_{x\to a}(-x + 1) = -a + 1$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+}(|x| + 1)$$

$$= \lim_{x\to a}(-x + 1) = -a + 1$$

Hence, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = -a + 1$

Therefore, lim (f(x)) exists at x = a and a < 0

Case 3:

When a > 0

$\lim_{x\to a} f(x)$:

$\lim_{x\to a^-} f(x) = \lim_{x\to a^-}(|x| - 1)$

$= \lim_{x\to a}(x - 1) = a - 1$

$\lim_{x\to a^+} f(x) = \lim_{x\to a^+}(|x| - 1)$

$= \lim_{x\to a}(x - 1) = a - 1$

Hence, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = a - 1$

Therefore, lim (f (x)) exists at x = a when a > 0

31. If the function f(x) satisfies $\lim_{x\to1}\frac{f(x)-2}{x^2-1}=\pi$, evaluate $\lim_{x\to1} f(x)$

Solution:

Given function that f (x) satisfies $\lim_{x\to1}\frac{f(x)-2}{x^2-1}=\pi$

$$\frac{\lim_{x\to1} f(x)-2}{\lim_{x\to1} x^2-1}=\pi$$

$$\lim_{x\to1}(f(x)-2)=\pi(\lim_{x\to1}(x^2-1))$$

Substituting x = 1, we get,

$$\lim_{x\to1}(f(x)-2)=\pi(1^2-1)$$

$$\lim_{x\to1}(f(x)-2)=\pi(1-1)$$

$$\lim_{x\to1}(f(x)-2)=0$$

$$\lim_{x\to1} f(x)-\lim_{x\to1} 2=0$$

$$\lim_{x\to1} f(x)-2=0$$

$$=2$$

32. If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m. & x>1 \end{cases}$ **For what integers m and n does both** $\lim\limits_{x\to 0} f(x)$ **and** $\lim\limits_{x\to 1} f(x)$ **exist?**

Solution:

Given function is $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m. & x>1 \end{cases}$

$\lim\limits_{x\to 0} f(x)$:

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}(mx^2+n)$$

$$=m\,(0)+n$$

$$=0+n$$

$$=n$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}(nx+m)$$

$$=n\,(0)+m$$

$$=0+m$$

$$=m$$

Hence,

$\lim\limits_{x \to 0} f(x)$ exists if n = m.

Now,

$\lim\limits_{x \to 1} f(x)$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1}(nx + m)$$

$= n(1) + m$

$= n + m$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1}(nx^3 + m)$$

$= n(1)^3 + m$

$= n(1) + m$

$= n + m$

Therefore $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1} f(x)$

Hence, for any integral value of m and n $\lim\limits_{x \to 1} f(x)$ exists.

Exercise 13.2

1. Find the derivative of $x^2 - 2$ at $x = 10$.

Solution:

Let $f(x) = x^2 - 2$

From first principle

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Put $x = 10$, we get

$$f'(10) = \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$

$$= \lim_{h\to 0}\frac{[(10+h)^2-2]-(10^2-2)}{h}$$

$$= \lim_{h\to 0}\frac{10^2+2\times 10\times h+h^2-2-10^2+2}{h}$$

$$= \lim_{h\to 0}\frac{20h+h^2}{h}$$

$$= \lim_{h\to 0}(20+h)$$

$= 20 + 0$

$= 20$

2. Find the derivative of x at x = 1.

Solution:

Let f (x) = x

Then,

From first principle

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Let f (x) = x

From first principle

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(10)}{h}$$

Put x = 1, we get

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h) - 1}{h}$$

$$= \lim_{h\to 0} \frac{1+h-1}{h}$$

$$= \lim_{h\to 0} \frac{h}{h}$$

$$= \lim_{h\to 0} 1$$

$$= 1$$

3. Find the derivative of 99x at x = l00.

Solution:

Let f (x) = 99x,

From the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Put x = 100, we get

$$f'(100) = \lim_{h \to 0} \frac{f(100+h) - f(100)}{h}$$

$$= \lim_{h \to 0} \frac{99(100+h) - 99 \times 100}{h}$$

$$= \lim_{h \to 0} \frac{99 \times 100 + 99h - 99 \times 100}{h}$$

$$= \lim_{h \to 0} \frac{99 \times h}{h}$$

$$= \lim_{h \to 0} 99$$

= 99

4. Find the derivative of the following functions from the first principle.

(i) $x^3 - 27$

(ii) $(x - 1)(x - 2)$

(iii) $1 / x^2$

(iv) $x + 1 / x - 1$

Solution:

(i) Let $f(x) = x^3 - 27$

From the first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$$= \lim_{h\to 0}(h^2 + 3x^2 + 3xh)$$

$$= 0 + 3x^2$$

$$= 3x^2$$

(ii) Let $f(x) = (x - 1)(x - 2)$

From the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}$$

$$= \lim_{h\to 0}\frac{(x^2+hx-2x+hx+h^2-2h-x-h+2)-(x^2-2x-x+2)}{h}$$

$$= \lim_{h\to 0}\frac{hx+hx+h^2-2h-h}{h}$$

$$= \lim_{h\to 0}(h+2x-3)$$

$$= 0+2x-3$$

$$= 2x-3$$

(iii) Let f (x) = 1 / x^2

From the first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0}\frac{x^2-(x+h)^2}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-h^2-2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \left[\frac{-h-2x}{x^2(x+h)^2}\right]$$

$= (0 – 2x) / [x^2 (x + 0)^2]$

$= (- 2 / x^3)$

(iv) Let f (x) = x + 1 / x – 1

From the first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{h(x-1)(x+h-1)}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1) - (x^2+hx+x-x+h-1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h \to 0} \frac{-2h}{h(x-1)(x+h-1)}$$

$$= \lim_{h \to 0} \frac{-2}{(x-1)(x+h-1)}$$

$$= -\frac{2}{(x-1)(x-1)}$$

$$= -\frac{2}{(x-1)^2}$$

5. For the function $f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots\frac{x^{2}}{2}+x+1$ **, prove that f' (1) =100 f' (0).**

Solution:

Given function is:

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots\frac{x^{2}}{2}+x+1$$

By differentiating both sides, we get

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\cdots+\frac{x^{2}}{2}+x+1\right]$$

$$=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+\cdots+\frac{d}{dx}\left(\frac{x^{2}}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

We know that,

$$\frac{d}{dx}(x^{n})=nx^{n-1}$$

$$\therefore \frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+\cdots+\frac{2x}{2}+1+0$$

$$f'(x) = x^{99} + x^{98} + \cdots + x + 1$$

At $x = 0$, we get

$$f'(0) = 0 + 0 + \ldots + 0 + 1$$

$$f'(0) = 1$$

At $x = 1$, we get

$$f'(1) = 1^{99} + 1^{98} + \ldots + 1 + 1 = [1 + 1 \ldots. + 1]\ 100 \text{ times} = 1 \times 100 = 100$$

Hence, $f'(1) = 100\, f'(0)$

6. Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$ for some fixed real number a.

Solution:

Given function is:

$$f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}\left(x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n\right)$$

$$= \frac{d}{dx}(x^n) + a\frac{d}{dx}(x^{n-1}) + a^2\frac{d}{dx}(x^{n-2}) + \cdots + a^{n-1}\frac{d}{dx}(x) + a^n\frac{d}{dx}(1)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n - 2)x^{n-3} + \ldots + a^{n-1} + a^n(0)$$

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n - 2)x^{n-3} + \ldots + a^{n-1}$$

7. For some constants a and b, find the derivative of

(i) (x − a) (x − b)

(ii) $(ax^2 + b)^2$

(iii) x – a / x – b

Solution:

(i) (x – a) (x – b)

Let f (x) = (x – a) (x – b)

$f(x) = x^2 – (a + b) x + ab$

Now, by differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(x^2 - (a+b)x + ab)$$

$$= \frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$f'(x) = 2x - (a + b) + 0$

$= 2x – a – b$

(ii) $(ax^2 + b)^2$

Let f (x) = $(ax^2 + b)^2$

f (x) = $a^2x^4 + 2abx^2 + b^2$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2)$$

$$f'(x) = \frac{d}{dx}(x^4) + (2ab)\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$f'(x) = a^2 \times 4x^3 + 2ab \times 2x + 0$$

$$= 4a^2x^3 + 4abx$$

$$= 4ax\,(ax^2 + b)$$

(iii) x – a / x – b

$$\text{Let } f(x) = \frac{(x-a)}{(x-b)}$$

By differentiating both sides and using quotient rule, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

By further calculation, we get

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

8. Find the derivative of $\frac{x^n - a^n}{x-a}$ for some constant a.

Solution:

Let $f(x) = \frac{x^n - a^n}{x-a}$

By differentiating both sides and using quotient rule, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x^n - a^n}{x-a}\right)$$

$$f'(x) = \frac{(x-a)\frac{d}{dx}(x^n - a^n) - (x^n - a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

By further calculation, we get

$$= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

9. Find the derivative of

(i) 2x – 3 / 4

(ii) $(5x^3 + 3x – 1) (x – 1)$

(iii) $x^{-3} (5 + 3x)$

(iv) $x^5 (3 – 6x^{-9})$

(v) $x^{-4} (3 – 4x^{-5})$

(vi) $(2 / x + 1) – x^2 / 3x – 1$

Solution:

(i)

Let f (x) = 2x – 3 / 4

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}\left(2x - \frac{3}{4}\right)$$

$$= 2\frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right)$$

= 2 – 0

= 2

(ii)

Let $f(x) = (5x^3 + 3x - 1)(x - 1)$

By differentiating both sides and using the product rule, we get

$$f'(x) = (5x^3 + 3x - 1)\frac{d}{dx}(x - 1) + (x - 1)\frac{d}{dx}(5x^3 + 3x + 1)$$

$$= (5x^3 + 3x - 1) \times 1 + (x - 1) \times (15x^2 + 3)$$

$$= (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3)$$

$$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$$

$$= 20x^3 - 15x^2 + 6x - 4$$

(iii)

Let $f(x) = x^{-3}(5 + 3x)$

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^{-3}\frac{d}{dx}(5+3x) + (5+3x)\frac{d}{dx}(x^{-3})$$

$$= x^{-3}(0+3) + (5+3x)(-3x^{-3-1})$$

By further calculation, we get

$$= x^{-3}(3) + (5+3x)(-3x^{-4})$$

$$= 3x^{-3} - 15x^{-4} - 9x^{-3}$$

$$= -6x^{-3} - 15x^{-4}$$

$$= -3x^{-3}\left(2 + \frac{5}{x}\right)$$

$$= \frac{-3x^{-3}}{x}(2x+5)$$

$$= \frac{-3}{x^4}(5+2x)$$

(iv)

Let $f(x) = x^5 (3 - 6x^{-9})$

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^5 \frac{d}{dx}(3-6x^{-9}) + (3-6x^{-9})\frac{d}{dx}(x^5)$$

$$= x^5 \{0 - 6(-9)x^{-9-1}\} + (3-6x^{-9})(5x^4)$$

By further calculation, we get

$$= x^5 (54x^{-10}) + 15x^4 - 30x^{-5}$$

$$= 54x^{-5} + 15x^4 - 30x^{-5}$$

$$= 24x^{-5} + 15x^4$$

$$= 15x^4 + \frac{24}{x^5}$$

(v)

Let $f(x) = x^{-4}(3 - 4x^{-5})$

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^{-4}\frac{d}{dx}\left(3-4x^{-5}\right)+\left(3-4x^{-5}\right)\frac{d}{dx}\left(x^{-4}\right)$$

$$= x^{-4}\left\{0-4(-5)x^{-5-1}\right\}+\left(3-4x^{-5}\right)(-4)x^{-4-1}$$

By further calculation, we get

$$= x^{-4}\left(20x^{-6}\right)+\left(3-4x^{-5}\right)\left(-4x^{-5}\right)$$

$$= 20x^{-10} - 12x^{-5} + 16x^{-10}$$

$$= 36x^{-10} - 12x^{-5}$$

$$= -\frac{12}{x^{5}}+\frac{36}{x^{10}}$$

(vi)

Let $f(x) = \dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

By differentiating both sides we get,

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1} - \frac{x^2}{3x-1}\right)$$

Using quotient rule we get,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - (x^2)\times 3}{(3x-1)^2}\right]$$

$$= -\frac{2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right]$$

$$= -\frac{2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

10. Find the derivative of cos x from the first principle.

Solution:

Let f (x) = cos x

Accordingly, f (x + h) = cos (x + h)

By first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

So, we get

$$= \lim_{h\to 0}\frac{1}{h}[\cos(x+h) - \cos(x)]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]$$

By further calculation, we get

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0} -\sin\left(\frac{2x+h}{2}\right) \times \lim_{h\to 0}\frac{\sin(\frac{h}{2})}{\frac{h}{2}}$$

$$= -\sin\left(\frac{2x+0}{2}\right) \times 1$$

= - sin (2x / 2)

= - sin (x)

11. Find the derivative of the following functions.

(i) sin x cos x

(ii) sec x

(iii) 5 sec x + 4 cos x

(iv) cosec x

(v) 3 cot x + 5 cosec x

(vi) 5 sin x – 6 cos x + 7

(vii) 2 tan x – 7 sec x

Solution:

(i) sin x cos x

Let f (x) = sin x cos x

Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)\cos(x+h)-\sin x\cos x}{h}$$

$$= \lim_{h\to 0}\frac{1}{2h}\left[2\sin(x+h)\cos(x+h)-2\sin x\cos x\right]$$

$$= \lim_{h\to 0}\frac{1}{2h}\left[\sin 2(x+h)-\sin 2x\right]$$

$$= \lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2}\cdot\sin\frac{2x+2h-2x}{2}\right]$$

By further calculation, we get

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos(2x+h)\sin h\right]$$

$$= \lim_{h \to 0} \cos(2x+h).\lim_{h \to 0}\frac{\sin h}{h}$$

$$= \cos(2x+0).1$$

$$= \cos 2x$$

(ii) sec x

Let f (x) = sec x

= 1 / cos x

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

Using quotient rule, we get

$$f'(x) = \frac{\cos x \frac{d}{dx}(1) - 1\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \times 0 - (-\sin x)}{\cos^2 x}$$

We get

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$$

= tan x sec x

(iii) 5 sec x + 4 cos x

Let f (x) = 5 sec x + 4 cos x

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(5\sec x + 4\cos x)$$

By further calculation, we get

$$= 5\frac{d}{dx}(\sec x) + 4\frac{d}{dx}(\cos x)$$

$$= 5\sec x\tan x + 4\times(-\sin x)$$

= 5 sec x tan x – 4 sin x

(iv) cosec x

Let f (x) = cosec x

Accordingly f (x + h) = cosec (x + h)

By first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\operatorname{cosec}(x+h)-\operatorname{cosec} x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

By further calculation, we get

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right)\sin(x+h)}\right]$$

$$=-\frac{1}{\sin x}\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$=-\frac{1}{\sin x}\times 1\times\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$=-\frac{1}{\sin x}\times\frac{\cos x}{\sin x}$$

= -cosec x cot x

(v) 3 cot x + 5 cosec x

Let f (x) = 3 cot x + 5 cosec x

f′(x) = 3 (cot x)′ + 5 (cosec x)′

Let f_1 (x) = cot x,

Accordingly f_1 (x + h) = cot (x + h)

By using first principle, we get

$$f_1'(x)=\lim_{x\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right)$$

By further calculation, we get

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\ \sin(x+h)}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right)$$

$$=1/\sin x\ \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$=-\frac{1}{\sin x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)$$

$$=-\frac{1}{\sin x}\times 1\times\frac{1}{\sin(x+0)}$$

$$=-\frac{1}{\sin^2 x}$$

$$=-\operatorname{cosec}^2 x$$

Let $f_2(x) = \operatorname{cosec} x$,

Accordingly $f_2(x+h) = \operatorname{cosec}(x+h)$

By using first principle, we get

$$f_2'(x)=\lim_{h\to 0}\frac{f_2(x+h)-f_2(x)}{h}$$

$$=\lim_{h\to 0}\frac{\operatorname{cosec}(x+h)-\operatorname{cosec} x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

By further calculation, we get

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right)\sin(x+h)}\right]$$

$$=-\frac{1}{\sin x}\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$=-\frac{1}{\sin x}\times 1\times\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$=-\frac{1}{\sin x}\times\frac{\cos x}{\sin x}$$

= -cosec x cot x

Now, substitute the value of (cot x)’ and (cosec x)’ in f’(x), we get

$f'(x) = 3\,(\cot x)' + 5\,(\operatorname{cosec} x)'$

$f'(x) = 3 \times (-\operatorname{cosec}^2 x) + 5 \times (-\operatorname{cosec} x \cot x)$

$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$

(vi)5 sin x – 6 cos x + 7

Let f (x) = 5 sin x – 6 cos x + 7

Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[5\sin(x+h) - 6\cos(x+h) + 7 - 5\sin x + 6\cos x - 7\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[5\{\sin(x+h) - \sin x\} - 6\{\cos(x+h) - \cos x\}\right]$$

$$= 5\lim_{h\to 0} \frac{1}{h}\left[\sin(x+h) - \sin x\right] - 6\lim_{h\to 0} \frac{1}{h}\left[\cos(x+h) - \cos x\right]$$

By further calculation, we get

$$= 5\lim_{h\to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right] - 6\lim_{h\to 0} \frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= 5\lim_{h\to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right] - 6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h) - \sin x\sin h}{h}\right]$$

Now, we get

$$= 5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) - 6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h} - \frac{\sin x\sin h}{h}\right]$$

$$= 5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right] - 6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right) - \sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$

$$= 5\cos x.1 - 6\left[(-\cos x).(0) - \sin x.1\right]$$
$$= 5\cos x + 6\sin x$$

(vii) 2 tan x – 7 sec x

Let f (x) = 2 tan x – 7 sec x

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\left\{\tan(x+h)-\tan x\right\}-7\left\{\sec(x+h)-\sec x\right\}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right]-7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right]$$

By further calculation, we get

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

Now, we get

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$=2.1.\frac{1}{\cos x\cos x}-7.1\left(\frac{\sin x}{\cos x\cos x}\right)$$
$$=2\sec^2 x-7\sec x\tan x$$

2Marks Questions & Answers

1. What is the limit's value $\lim\limits_{x \to 3} \left[\frac{x^2-9}{x-3}\right]$

Ans:

Here, we can see that the limit $\boldsymbol{\lim\limits_{x \to 3} \frac{x^2-9}{x-3}}$ is in the form $\frac{0}{0}$

By representing the numerator as the product of two terms, we get,

$$\lim_{x \to 3} \frac{x^2-9}{x-3}$$

$$= \frac{(x+3)(x-3)}{(x-3)}$$

$$=3+3$$

$$=6$$

2. What is the limit's value $\lim\limits_{x \to 0} \left[\frac{sin5x}{3x}\right]$ **?**

Ans:

Multiply and divide the numerator and denominator of the given limit with 5, then we get,

$$= \lim_{x \to 3} \frac{sin5x}{3x}$$

$$= \lim_{x \to 3} \frac{sin5x}{3x} \times \frac{5}{5}$$

$$= 1 \times \frac{5}{3}$$

$$= \frac{5}{3} \qquad \left[\because \lim_{x \to 0} \frac{sinax}{ax} = 1\right]$$

3. What is the result of the derivative of 2^x **with respect to** x.

Ans:

Let us assume the given expression as,

$y = 2^x$

Now, differentiating on both sides with respect to x then we get,

$$\frac{dy}{dx}$$

$= \frac{d}{dx}(2^x)$

$=2^x \log 2$

4. The result when the expression $\sqrt{sin2x}$ when it is differentiated with respect to x is.

Ans:

By using the chain rule of differentiation the derivative of given expression is given as,

$\frac{d}{dx}\sqrt{sin2x}$

$= \frac{1}{2\sqrt{sin2x}}\frac{d}{dx}\sin 2x$

$= \frac{1}{2\sqrt{sin2x}} \times 2\cos 2x$

$= \frac{\cos 2x}{\sqrt{cos2x}}$

5. What is the derivative of $\frac{2^x}{x}$ with respect to x?

Ans:

By using the $\frac{u}{v}$ formula of differentiating for the given expression, we get,

$\frac{d}{dx}(\frac{2^x}{x})$

$= \frac{\frac{d}{dx}\left(\frac{2^x}{x}\right) - 2^x\frac{d}{dx}(x)}{x^2}$

$= \frac{(x \times 2^x \ln 2) - (2^x \times 1)}{x^2}$

$= 2^x \frac{[x\ In\ 2-1]}{x^2}$

6. If the expression is $y = e^{sinx}$, then find the value of $\frac{dy}{dx}$.

Ans:

We are given the expression as,

$$y = e^{sinx}$$

Now, by differentiating on both sides with respect to x then we get,

$$\frac{dy}{dx}$$

$$=\frac{d}{dx}(e^{sinx})$$

$$= e^{sinx} \times cosx$$

$$= cosxe^{sinx}$$

7. What is the limit's value $\lim_{x \to 1} \frac{x^{15}-1}{x^{10}-1}$

Ans:

By using the L' Hospital rule that is differentiating the numerator and denominator with respect to *x* we get,

$$\lim_{x \to 1} \frac{x^{15}-1}{x^{10}-1}$$

$$= \lim_{x \to 1} \frac{15x^{14}}{10x^9}$$

$$= \lim_{x \to 1} \frac{3}{2} x^5$$

$$= \frac{3}{2}$$

8. Differentiate the expression *xsinx* **with respect to** *x*.

Ans:

By using the chain rule of differentiation, we get the derivative of the given

expression as, $\frac{dy}{dx}(xsinx)$

$$= x\left(\frac{d}{dx}(sinx)\right) + sinx \left(\frac{d}{dx}(x)\right)$$

$= x(cosx) + sinx(1)$

$= xcosx + sinx$

9. What is the limit's value $\lim_{x \to \infty} [cosecx - cotx]$?

Ans:

Rewriting $cosecx$ and $cotx$ in terms of $sinx$ and $cosx$ we get,

$$\lim_{x \to \infty} [cosecx - cotx]$$

$$= \lim_{x \to \infty} \left[\frac{1}{sinx} - \frac{cosx}{sinx}\right]$$

Now, by using the half angle formula of $sinx$ and $cosx$ we get,

$$= \lim_{x \to \infty} \frac{2\, sin^2\frac{x}{2}}{2sin\frac{x}{2}cos\frac{x}{2}}$$

$$= \lim_{x \to \infty} \tan\frac{x}{2}$$

=0

10. Find derivative of the expression $1 + x + x^2 + x^3$ +…+x^{50} at *x=1* .

Ans:

By using the power rule of differentiation, we get the derivative of given function as,

f^1 (x)=1+2x+3x^2+…+50x^{49}

Now, by substituting x=100 in the above derivative, we get,
f′(1)
=1+2+3+…+50

$= \frac{50(50+1)}{2}$

=25×51

=1275

11. Find the derivative of expression x^{-3}(5+3x) with respect to x.

Ans:

First multiply x^{-3} to each term in brackets and then differentiate the expression we get,

$$\frac{d}{dx} x^{-3}(5+3x)$$

$$= \frac{d}{dx}[5x^{-3} + 3x^{-2}]$$

$$=-15x^{-4} - 6x^{-3}$$

$$= -\frac{15}{x^4} - \frac{6}{x^3}.$$

Multiple Choice Questions

1. The derivative of x^2 cos x is

(a) 2x sin x – x^2 sin x

(b) 2x cos x – x^2 sin x

(c) 2x sin x – x^2 cos x

(d) cosx – x^2 sin x cos x

Correct option: (b) 2x cos x – x^2 sin x

Solution:

d/dx(x^2 cos x)

Using the formula d/dx [f(x) g(x)] = f(x) [d/dx g(x)] + g(x) [d/dx f(x)]

d/dx(x^2 cos x) = x^2 [d/dx (cos x)] + cos x [d/dx x^2]

= x^2(-sin x) + cos x (2x)

= 2x cos x – x^2 sin x

2. $\lim_{x\to0}$ | sin x|/x is equal to

(a) 1

(b) –1

(c) 0

(d) does not exist

Correct option: (d) does not exist

Solution:

Right hand side limit, R.H.S = $\lim_{x\to 0+} |\sin x|/ x = \lim_{x\to 0+} \sin x/x = 1$

Left hand side limit, L.H.S = $\lim_{x\to 0-} |\sin x|/ x = \lim_{x\to 0-} -\sin x/x = -1$

R.H.S ≠ L.H.S

Therefore, the solution does not exist.

3. If f(x) = x sin x, then f′(π/2) is equal to

(a) 0

(b) 1

(c) –1

(d) 1/2

Correct option: (b) 1

Solution:

Given,

f(x) = x sin x

f'(x) = x[d/dx sin x] + sin x [d/dx (x)]

= x cos x + sin x

Now,

f′(π/2) = (π/2) cos π/2 + sin π/2

$= (\pi/2)(0) + 1$

$= 1$

4. $\lim_{x\to 0}$ (cosec x – cot x)/x is

(a) -1/2

(b) 1

(c) 1/2

(d) 1

Correct option: (c) ½

5. If $f(x) = x^{100} + x^{99} + \ldots + x + 1$, then $f'(1)$ is equal to

(a) 5050

(b) 5049

(c) 5051

(d) 50051

Correct option: (a) 5050

Solution:

$f(x) = x^{100} + x^{99} + \ldots + x + 1$

$f'(x) = 100x^{99} + 99x^{98} + \ldots + 1 + 0$

$f'(1) = 100(1)^{99} + 99(1)^{98} + \ldots + 1$

$= 100 + 99 + \ldots + 1$

This is an AP with common difference -1, a = 100, n = 100 and l = 1.

So, the sum of this AP = (100/2)[100 + 1]

= 50(101)

= 5050

Therefore, $f'(1) = 5050$

6. $\lim_{x\to 0} x \sin(1/x)$ is equal to

(a) 0

(b) 1

(c) ½

(d) does not exist

Correct option: (a) 0

Solution:

We know that,

$\lim_{x\to 0} x = 0$

And

$-1 \le \sin 1/x \le 1$

By Sandwich theorem,

$\lim_{x\to 0} x \sin(1/x) = 0$

7. $\lim_{x\to \pi} (\sin x)/(x - \pi)$ is equal to

(a) 1

(b) 2

(c) -1

(d) -2

Correct option: (c) -1

Solution:

$\lim_{x\to \pi} (\sin x)/(x - \pi) = \lim_{x\to \pi} [\sin(\pi - x)])/(x - \pi)$

We know that, $\lim_{x\to 0} (\sin x)/x = 1$

When $\pi - x \to 0$

$x \to \pi$

Therefore,

$\lim_{x\to\pi} [\sin(\pi - x)]/(x - \pi) = \lim_{x\to\pi} -[\sin(\pi - x)]/(\pi - x) = -1$

8. Let f(x) = x – [x]; ∈ R, then f′(1/2) is

(a) 3/2

(b) 1

(c) 0

(d) -1

Correct option: (b) 1

Solution:

Given,

$f(x) = x - [x]$

$f'(x) = 1 - 0$ {[x] = integer less than or equal to x}

$f'(1/2) = 1$

9. If y = (sin x + cos x)/(sin x – cos x), dy/dx at x = 0 is

(a) -2

(b) 0

(c) ½

(d) does not exist

Correct option: (a) -2

Solution:

Given,

$y = (\sin x + \cos x)/(\sin x - \cos x)$

Dividing the numerator and denominator by cos x,

$y = (\tan x + 1)/(\tan x - 1)$

$y = (1 + \tan x)/ [-(1 - \tan x)]$

We know that $\tan \pi/4 = 1$,

$y = -(\tan \pi/4 + \tan x)/(1 - \tan \pi/4 \tan x)$

$y = -\tan(\pi/4 + x)$

$dy/dx = -d/dx \tan(\pi/4 + x)$

$= -\sec^2(\pi/4 + x)$ {since $d/dx \tan x = \sec^2 x$}

$(dy/dx)x = 0 = -\sec^2(\pi/4 + 0)$

$= -\sec^2(\pi/4)$

$= -(\sqrt{2})^2$

$= -2$

10. The positive integer n so that $\lim_{x\to 3} (x^n - 3^n)/(x - 3) = 108$ is

(a) 3

(b) 4

(c) -2

(d) 1

Correct option: (b) 4

Solution:

We know that,

$\lim_{x\to 3} (x^n - 3^n)/(x - 3) = n(3)^{n-1}$

Thus, n(3)n-1 = 108 {from the given}

$n(3)^{n-1} = 4(27) = 4(3^3) = 4(3)^{4-1}$

Therefore, n = 4

Summary

- The expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point. Similarly the right hand limit.
- Limit of a function at a point is the common value of the left and right hand limits, if they coincide.
- For a function f and a real number a, $\lim_{n \to \infty} f(x)$ and f (a) may not be same (In fact, one may be defined and not the other one).

- For functions f and g the following holds:

$$\lim_{x \to a}[f(x) \pm \lim_{x \to a} g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$\lim_{x \to a}[f(x). \lim_{x \to a} g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [\frac{f(x)}{g(x)}] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

- Following are some of the standard limits :

$$\lim_{x \to a} \frac{xn - an}{x - a} = na^{n-1}$$

$$\lim_{x \to a} \frac{sinx}{x} = 1$$

$$\lim_{x \to a} \frac{1 - cosx}{x} = 0$$

- The derivative of a function f at a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Derivative of a function f at any point x is defined by

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- For functions u and v the following holds:

$$(u \pm v)' = u' \pm v'$$

$$(uv)' = u'v + uv'$$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v2}$ Provided all are defined.

- Following are some of the standard derivatives.

$\frac{d}{dx}(x)^n = nx^{n-1}$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x.$