Chapter 11

Conic Sections

This chapter explores the fascinating geometric shapes obtained by intersecting a cone with a plane at different angles. The primary conic sections discussed are the circle, ellipse, parabola, and hyperbola.

The chapter begins by introducing the concept of a conic section and elucidating how varying the inclination of the cutting plane results in distinct shapes. Students delve into the characteristics of a circle, understanding its definition in terms of locus and exploring its key properties. The discussion then transitions to ellipses, introducing their definitions, major and minor axes, and the standard equation for an ellipse.

Parabolas take center stage next, with the chapter providing insights into their definitions, focal points, and various forms of equations. Students learn to distinguish between parabolas opening upward, downward, leftward, and rightward. Hyperbolas follow, with the chapter delving into their eccentricity, asymptotes, and equations.

Exercise 11.1

In each of the following Exercise 1 to 5, find the equation of the circle with
1. Centre (0, 2) and radius 2

Solution:

Given:

Centre (0, 2) and radius 2

Let us consider the equation of a circle with centre (h, k) and

Radius r is given as $(x – h)^2 + (y – k)^2 = r^2$

So, centre (h, k) = (0, 2) and radius (r) = 2

The equation of the circle is

$(x – 0)^2 + (y – 2)^2 = 2^2$

$x^2 + y^2 + 4 – 4y = 4$

$x^2 + y^2 – 4y = 0$

∴ The equation of the circle is $x^2 + y^2 – 4y = 0$

2. Centre (–2, 3) and radius 4

Solution:

Given:

Centre (-2, 3) and radius 4

Let us consider the equation of a circle with centre (h, k).

Radius r is given as $(x – h)^2 + (y – k)^2 = r^2$

So, centre (h, k) = (-2, 3) and radius (r) = 4

The equation of the circle is

$(x + 2)^2 + (y – 3)^2 = (4)^2$

$x^2 + 4x + 4 + y^2 – 6y + 9 = 16$

$x^2 + y^2 + 4x – 6y – 3 = 0$

∴ The equation of the circle is $x^2 + y^2 + 4x – 6y – 3 = 0$

3. Centre (1/2, 1/4) and radius (1/12)

Solution:

Given:

Centre (1/2, 1/4) and radius 1/12

Let us consider the equation of a circle with centre (h, k).

Radius r is given as $(x – h)^2 + (y – k)^2 = r^2$

So, centre (h, k) = (1/2, 1/4) and radius (r) = 1/12

The equation of the circle is

$(x – 1/2)^2 + (y – 1/4)^2 = (1/12)^2$

$x^2 – x + ¼ + y^2 – y/2 + 1/16 = 1/144$

$x^2 – x + ¼ + y^2 – y/2 + 1/16 = 1/144$

$144x^2 – 144x + 36 + 144y^2 – 72y + 9 – 1 = 0$

$144x^2 – 144x + 144y^2 – 72y + 44 = 0$

$36x^2 + 36x + 36y^2 – 18y + 11 = 0$

$36x^2 + 36y^2 – 36x – 18y + 11= 0$

∴ The equation of the circle is $36x^2 + 36y^2 – 36x – 18y + 11= 0$

4. Centre (1, 1) and radius √2

Solution:

Given:

Centre (1, 1) and radius √2

Let us consider the equation of a circle with centre (h, k).

Radius r is given as $(x – h)^2 + (y – k)^2 = r^2$

So, centre (h, k) = (1, 1) and radius (r) = √2

The equation of the circle is

$(x-1)^2 + (y-1)^2 = (\sqrt{2})^2$

$x^2 – 2x + 1 + y^2 -2y + 1 = 2$

$x^2 + y^2 – 2x -2y = 0$

∴ The equation of the circle is $x^2 + y^2 – 2x -2y = 0$

5. Centre (–a, –b) and radius $\sqrt{(a^2 – b^2)}$

Solution:

Given:

Centre (-a, -b) and radius $\sqrt{(a^2 – b^2)}$

Let us consider the equation of a circle with centre (h, k) and

Radius r is given as $(x – h)^2 + (y – k)^2 = r^2$

So, centre (h, k) = (-a, -b) and radius (r) = $\sqrt{(a^2 – b^2)}$

The equation of the circle is

$(x + a)^2 + (y + b)^2 = (\sqrt{(a^2 – b^2)})^2$

$x^2 + 2ax + a^2 + y^2 + 2by + b^2 = a^2 – b^2$

$x^2 + y^2 +2ax + 2by + 2b^2 = 0$

∴ The equation of the circle is $x^2 + y^2 +2ax + 2by + 2b^2 = 0$

In each of the following Exercise 6 to 9, find the centre and radius of the circles.

6. $(x + 5)^2 + (y – 3)^2 = 36$

Solution:

Given:

The equation of the given circle is $(x + 5)^2 + (y – 3)^2 = 36$

$(x – (-5))^2 + (y – 3)^2 = 6^2$ [which is of the form $(x – h)^2 + (y – k)^2 = r^2$]

Where, h = -5, k = 3 and r = 6

∴ The centre of the given circle is (-5, 3) and its radius is 6.

7. $x^2 + y^2 – 4x – 8y – 45 = 0$

Solution:

Given:

The equation of the given circle is $x^2 + y^2 – 4x – 8y – 45 = 0$.

$x^2 + y^2 – 4x – 8y – 45 = 0$

$(x^2 – 4x) + (y^2 -8y) = 45$

$(x^2 – 2(x) (2) + 2^2) + (y^2 – 2(y) (4) + 4^2) – 4 – 16 = 45$

$(x – 2)^2 + (y – 4)^2 = 65$

$(x - 2)^2 + (y - 4)^2 = (\sqrt{65})^2$ [which is form $(x-h)^2 + (y-k)^2 = r^2$]

Where $h = 2$, $K = 4$ and $r = \sqrt{65}$

∴ The centre of the given circle is (2, 4) and its radius is $\sqrt{65}$.

8. $x^2 + y^2 - 8x + 10y - 12 = 0$

Solution:

Given:

The equation of the given circle is $x^2 + y^2 - 8x + 10y - 12 = 0$.

$x^2 + y^2 - 8x + 10y - 12 = 0$

$(x^2 - 8x) + (y^2 + 10y) = 12$

$(x^2 - 2(x)(4) + 4^2) + (y^2 - 2(y)(5) + 5^2) - 16 - 25 = 12$

$(x - 4)^2 + (y + 5)^2 = 53$

$(x - 4)^2 + (y - (-5))^2 = (\sqrt{53})^2$ [which is form $(x-h)^2 + (y-k)^2 = r^2$]

Where $h = 4$, $K = -5$ and $r = \sqrt{53}$

∴ The centre of the given circle is (4, -5) and its radius is $\sqrt{53}$.

9. $2x^2 + 2y^2 - x = 0$

Solution:

The equation of the given circle is $2x^2 + 2y^2 - x = 0$.

$2x^2 + 2y^2 - x = 0$

$(2x^2 + x) + 2y^2 = 0$

$(x^2 - 2(x)(1/4) + (1/4)^2) + y^2 - (1/4)^2 = 0$

$(x - 1/4)^2 + (y - 0)^2 = (1/4)^2$ [which is form $(x-h)^2 + (y-k)^2 = r^2$]

Where, $h = ¼$, $K = 0$, and $r = ¼$

∴ The center of the given circle is (1/4, 0) and its radius is 1/4.

10. Find the equation of the circle passing through the points (4,1) and (6,5) and whose centre is on the line 4x + y = 16.

Solution:

Let us consider the equation of the required circle to be $(x – h)^2 + (y – k)^2 = r^2$

We know that the circle passes through points (4,1) and (6,5)

So,

$(4 – h)^2 + (1 – k)^2 = r^2$ ……………..(1)

$(6– h)^2 + (5 – k)^2 = r^2$ ………………(2)

Since, the centre (h, k) of the circle lies on line 4x + y = 16,

$4h + k = 16$………………… (3)

From the equation (1) and (2), we obtain

$(4 – h)^2 + (1 – k)^2 = (6 – h)^2 + (5 – k)^2$

$16 – 8h + h^2 + 1 - 2k + k^2 = 36 - 12h + h^2 + 15 – 10k + k^2$

$16 – 8h + 1 - 2k + 12h - 25 - 10k$

$4h + 8k = 44$

$h + 2k = 11$……………. (4)

On solving equations (3) and (4), we obtain h=3 and k= 4.

On substituting the values of h and k in equation (1), we obtain

$(4 – 3)^2 + (1 – 4)^2 = r^2$

$(1)^2 + (-3)^2 = r^2$

$1+9 = r^2$

$r = \sqrt{10}$

so now, $(x – 3)^2 + (y – 4)^2 = (\sqrt{10})^2$

$x^2 – 6x + 9 + y^2 – 8y + 16 = 10$

$x^2 + y^2 – 6x – 8y + 15 = 0$

∴ The equation of the required circle is $x^2 + y^2 – 6x – 8y + 15 = 0$

11. Find the equation of the circle passing through the points (2, 3) and (–1, 1) and whose centre is on the line x – 3y – 11 = 0.

Solution:

Let us consider the equation of the required circle to be $(x – h)^2 + (y – k)^2 = r^2$

We know that the circle passes through points (2,3) and (-1,1).

$(2 – h)^2 + (3 – k)^2 = r^2$ ……………..(1)

$(-1 – h)^2 + (1 – k)^2 = r^2$ ………………(2)

Since, the centre (h, k) of the circle lies on line x – 3y – 11= 0,

$h – 3k = 11$………………… (3)

From the equation (1) and (2), we obtain

$(2 – h)^2 + (3 – k)^2 = (-1 – h)^2 + (1 – k)^2$

$4 – 4h + h^2 + 9 - 6k + k^2 = 1 + 2h + h^2 + 1 – 2k + k^2$

$4 – 4h + 9 - 6k = 1 + 2h + 1 - 2k$

$6h + 4k = 11$……………. (4)

Now let us multiply equation (3) by 6 and subtract it from equation (4) to get,

$6h + 4k – 6(h-3k) = 11 – 66$

$6h + 4k – 6h + 18k = 11 – 66$

$22 k = – 55$

$K = -5/2$

Substitute this value of K in equation (4) to get,

$6h + 4(-5/2) = 11$

$6h – 10 = 11$

$6h = 21$

$h = 21/6$

$h = 7/2$

We obtain $h = 7/2$and $k = -5/2$

On substituting the values of h and k in equation (1), we get

$(2 – 7/2)^2 + (3 + 5/2)^2 = r^2$

$[(4-7)/2]^2 + [(6+5)/2]^2 = r^2$
$(-3/2)^2 + (11/2)^2 = r^2$

$9/4 + 121/4 = r^2$

$130/4 = r^2$

The equation of the required circle is

$(x – 7/2)^2 + (y + 5/2)^2 = 130/4$

$[(2x-7)/2]^2 + [(2y+5)/2]^2 = 130/4$
$4x^2 -28x + 49 +4y^2 + 20y + 25 =130$

$4x^2 +4y^2 -28x + 20y – 56 = 0$

$4(x^2 +y^2 -7x + 5y – 14) = 0$

$x^2 + y^2 – 7x + 5y – 14 = 0$

∴ The equation of the required circle is $x^2 + y^2 – 7x + 5y – 14 = 0$

12. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2, 3).

Solution:

Let us consider the equation of the required circle to be $(x – h)^2+ (y – k)^2 = r^2$

We know that the radius of the circle is 5 and its centre lies on the x-axis, $k = 0$ and $r = 5$.

So now, the equation of the circle is $(x – h)^2 + y^2 = 25$.

It is given that the circle passes through the point (2, 3) so the point will satisfy the equation of the circle.

$(2 - h)^2 + 3^2 = 25$

$(2 - h)^2 = 25-9$

$(2 - h)^2 = 16$

$2 - h = \pm \sqrt{16} = \pm 4$

If 2-h = 4, then h = -2

If 2-h = -4, then h = 6

Then, when h = -2, the equation of the circle becomes

$(x + 2)^2 + y^2 = 25$

$x^2 + 12x + 36 + y^2 = 25$

$x^2 + y^2 + 4x - 21 = 0$

When h = 6, the equation of the circle becomes

$(x - 6)^2 + y^2 = 25$

$x^2 - 12x + 36 + y^2 = 25$

$x^2 + y^2 - 12x + 11 = 0$

∴ The equation of the required circle is $x^2 + y^2 + 4x - 21 = 0$ and $x^2 + y^2 - 12x + 11 = 0$

13. Find the equation of the circle passing through (0,0) and making intercepts a and b on the coordinate axes.

Solution:

Let us consider the equation of the required circle to be $(x - h)^2 + (y - k)^2 = r^2$

We know that the circle passes through (0, 0),

So, $(0 - h)^2 + (0 - k)^2 = r^2$

$h^2 + k^2 = r^2$

Now, The equation of the circle is $(x - h)^2 + (y - k)^2 = h^2 + k^2$.

It is given that the circle intercepts a and b on the coordinate axes.

i.e., the circle passes through points (a, 0) and (0, b).

So, $(a - h)^2 + (0 - k)^2 = h^2 + k^2$……………..(1)

$(0 - h)^2 + (b - k)^2 = h^2 + k^2$………………(2)

From equation (1), we obtain

$a^2 - 2ah + h^2 + k^2 = h^2 + k^2$

$a^2 - 2ah = 0$

$a(a - 2h) = 0$

$a = 0$ or $(a - 2h) = 0$

However, $a \neq 0$; hence, $(a - 2h) = 0$

$h = a/2$

From equation (2), we obtain

$h^2 - 2bk + k^2 + b^2 = h^2 + k^2$

$b^2 - 2bk = 0$

$b(b - 2k) = 0$

$b = 0$ or $(b - 2k) = 0$

However, $a \neq 0$; hence, $(b - 2k) = 0$

$k = b/2$

So, the equation is

$(x - a/2)^2 + (y - b/2)^2 = (a/2)^2 + (b/2)^2$

$[(2x-a)/2]^2 + [(2y-b)/2]^2 = (a^2 + b^2)/4$
$4x^2 - 4ax + a^2 + 4y^2 - 4by + b^2 = a^2 + b^2$

$4x^2 + 4y^2 -4ax – 4by = 0$

$4(x^2 +y^2 -7x + 5y – 14) = 0$

$x^2 + y^2 – ax – by = 0$

∴ The equation of the required circle is $x^2 + y^2 – ax – by = 0$

14. Find the equation of a circle with centre (2,2) and passes through the point (4,5).

Solution:

Given:

The centre of the circle is given as (h, k) = (2,2)

We know that the circle passes through point (4,5), the radius (r) of the circle is the distance between the points (2,2) and (4,5).

$r = \sqrt{[(2-4)^2 + (2-5)^2]}$

$= \sqrt{[(-2)^2 + (-3)^2]}$

$= \sqrt{[4+9]}$

$= \sqrt{13}$

The equation of the circle is given as

$(x– h)^2+ (y – k)^2 = r^2$

$(x –h)^2 + (y – k)^2 = (\sqrt{13})^2$

$(x –2)^2 + (y – 2)^2 = (\sqrt{13})^2$

$x^2 – 4x + 4 + y^2 – 4y + 4 = 13$

$x^2 + y^2 – 4x – 4y = 5$

∴ The equation of the required circle is $x^2 + y^2 – 4x – 4y = 5$

15. Does the point (–2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?

Solution:

Given:

The equation of the given circle is $x^2 + y^2 = 25$.

$x^2 + y^2 = 25$

$(x – 0)^2 + (y – 0)^2 = 5^2$ [which is of the form $(x – h)^2 + (y – k)^2 = r^2$]

Where, h = 0, k = 0 and r = 5.

So the distance between point (-2.5, 3.5) and the centre (0,0) is

$= \sqrt{[(-2.5 – 0)^2 + (-3.5 – 0)^2]}$

$= \sqrt{(6.25 + 12.25)}$

$= \sqrt{18.5}$

= 4.3 [which is < 5]

Since, the distance between point (-2.5, -3.5) and the centre (0, 0) of the circle is less than the radius of the circle, point (-2.5, -3.5) lies inside the circle.

Exercise 11.2

In each of the following Exercises 1 to 6, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum.

1. $y^2 = 12x$

Solution:

Given:

The equation is $y^2 = 12x$

Here, we know that the coefficient of x is positive.

So, the parabola opens towards the right.

On comparing this equation with $y^2 = 4ax$, we get,

$4a = 12$

$a = 3$

Thus, the co-ordinates of the focus = (a, 0) = (3, 0)

Since the given equation involves y^2, the axis of the parabola is the x-axis.

∴ The equation of directrix, x = -a, then,

$x + 3 = 0$

Length of latus rectum = 4a = 4 × 3 = 12

2. $x^2 = 6y$

Solution:

Given:

The equation is $x^2 = 6y$

Here, we know that the coefficient of y is positive.

So, the parabola opens upwards.

On comparing this equation with $x^2 = 4ay$, we get,

$4a = 6$

$a = 6/4$

$= 3/2$

Thus, the co-ordinates of the focus = (0,a) = (0, 3/2)

Since the given equation involves x^2, the axis of the parabola is the y-axis.

∴ The equation of directrix, y =-a, then,

$y = -3/2$

Length of latus rectum = 4a = 4(3/2) = 6

3. $y^2 = -8x$

Solution:

Given:

The equation is $y^2 = -8x$

Here, we know that the coefficient of x is negative.

So, the parabola open towards the left.

On comparing this equation with $y^2 = -4ax$, we get,

$-4a = -8$

$a = -8/-4 = 2$

Thus, co-ordinates of the focus = (-a,0) = (-2, 0)

Since the given equation involves y^2, the axis of the parabola is the x-axis.

∴ Equation of directrix, x =a, then,

$x = 2$

Length of latus rectum = 4a = 4 (2) = 8

4. $x^2 = -16y$

Solution:

Given:

The equation is $x^2 = -16y$

Here, we know that the coefficient of y is negative.

So, the parabola opens downwards.

On comparing this equation with $x^2 = -4ay$, we get,

$-4a = -16$

$a = -16/-4$

= 4

Thus, co-ordinates of the focus = (0,-a) = (0,-4)

Since the given equation involves x^2, the axis of the parabola is the y-axis.

∴ The equation of directrix, y =a, then,

y = 4

Length of latus rectum = 4a = 4(4) = 16

5. $y^2 = 10x$

Solution:

Given:

The equation is $y^2 = 10x$

Here, we know that the coefficient of x is positive.

So, the parabola open towards the right.

On comparing this equation with $y^2 = 4ax$, we get,

4a = 10

a = 10/4 = 5/2

Thus, co-ordinates of the focus = (a,0) = (5/2, 0)

Since the given equation involves y^2, the axis of the parabola is the x-axis.

∴ The equation of directrix, x =-a, then,

x = – 5/2

Length of latus rectum = 4a = 4(5/2) = 10

6. x^2 = – 9y

Solution:

Given:

The equation is $x^2 = -9y$

Here, we know that the coefficient of y is negative.

So, the parabola open downwards.

On comparing this equation with $x^2 = -4ay$, we get,

$-4a = -9$

$a = -9/-4 = 9/4$

Thus, co-ordinates of the focus = $(0,-a) = (0, -9/4)$

Since the given equation involves x^2, the axis of the parabola is the y-axis.

∴ The equation of directrix, $y = a$, then,

$y = 9/4$

Length of latus rectum = $4a = 4(9/4) = 9$

In each of the Exercises 7 to 12, find the equation of the parabola that satisfies the given conditions:

7. Focus (6,0); directrix x = – 6

Solution:

Given:

Focus (6,0) and directrix $x = -6$

We know that the focus lies on the x–axis is the axis of the parabola.

So, the equation of the parabola is either of the form $y^2 = 4ax$ or $y^2 = -4ax$.

It is also seen that the directrix, $x = -6$ is to the left of the y- axis,

While the focus (6, 0) is to the right of the y –axis.

Hence, the parabola is of the form $y^2 = 4ax$.

Here, a = 6

∴ The equation of the parabola is $y^2 = 24x$.

8. Focus (0,–3); directrix y = 3

Solution:

Given:

Focus (0, -3) and directrix y = 3

We know that the focus lies on the y–axis, the y-axis is the axis of the parabola.

So, the equation of the parabola is either of the form $x^2 = 4ay$ or $x^2 = -4ay$.

It is also seen that the directrix, y = 3 is above the x- axis,

While the focus (0,-3) is below the x-axis.

Hence, the parabola is of the form $x^2 = -4ay$.

Here, a = 3

∴ The equation of the parabola is $x^2 = -12y$.

9. Vertex (0, 0); focus (3, 0)

Solution:

Given:

Vertex (0, 0) and focus (3, 0)

We know that the vertex of the parabola is (0, 0) and the focus lies on the positive x-axis. [x-axis is the axis of the parabola.]

The equation of the parabola is of the form $y^2 = 4ax$.

Since, the focus is (3, 0), a = 3

∴ The equation of the parabola is $y^2 = 4 \times 3 \times x$,

$y^2 = 12x$

10. Vertex (0, 0); focus (–2, 0)

Solution:

Given:

Vertex (0, 0) and focus (-2, 0)

We know that the vertex of the parabola is (0, 0) and the focus lies on the positive x-axis. [x-axis is the axis of the parabola.]

The equation of the parabola is of the form $y^2=-4ax$.

Since, the focus is (-2, 0), a = 2

∴ The equation of the parabola is $y^2 = -4 \times 2 \times x$,

$y^2 = -8x$

11. Vertex (0, 0) passing through (2, 3) and axis is along x-axis.

Solution:

We know that the vertex is (0, 0) and the axis of the parabola is the x-axis

The equation of the parabola is either of the form $y^2 = 4ax$ or $y^2 = -4ax$.

Given that the parabola passes through point (2, 3), which lies in the first quadrant.

So, the equation of the parabola is of the form $y^2 = 4ax$, while point (2, 3) must satisfy the equation $y^2 = 4ax$.

Then,

$3^2 = 4a(2)$

$3^2 = 8a$

$9 = 8a$

$a = 9/8$

Thus, the equation of the parabola is

$y^2 = 4\ (9/8)x$

$= 9x/2$

$2y^2 = 9x$

∴ The equation of the parabola is $2y^2 = 9x$

12. Vertex (0, 0), passing through (5, 2) and symmetric with respect to y-axis.

Solution:

We know that the vertex is (0, 0) and the parabola is symmetric about the y-axis.

The equation of the parabola is either of the form $x^2 = 4ay$ or $x^2 = -4ay$.

Given that the parabola passes through point (5, 2), which lies in the first quadrant.

So, the equation of the parabola is of the form $x^2 = 4ay$, while point (5, 2) must satisfy the equation $x^2 = 4ay$.

Then,

$5^2 = 4a(2)$

$25 = 8a$

$a = 25/8$

Thus, the equation of the parabola is

$x^2 = 4\ (25/8)y$

$x^2 = 25y/2$

$2x^2 = 25y$

∴ The equation of the parabola is $2x^2 = 25y$

EXERCISE 11.3

In each of the Exercises 1 to 9, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

1. $x^2/36 + y^2/16 = 1$

Solution:

Given:

The equation is $x^2/36 + y^2/16 = 1$

Here, the denominator of $x^2/36$ is greater than the denominator of $y^2/16$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

$a = 6$ and $b = 4$.

$c = \sqrt{(a^2 – b^2)}$

$= \sqrt{(36-16)}$

$= \sqrt{20}$

$= 2\sqrt{5}$

Then,

The coordinates of the foci are $(2\sqrt{5}, 0)$ and $(-2\sqrt{5}, 0)$.

The coordinates of the vertices are $(6, 0)$ and $(-6, 0)$

Length of major axis $= 2a = 2 (6) = 12$

Length of minor axis $= 2b = 2 (4) = 8$

Eccentricity, $e = c/a = 2\sqrt{5}/6 = \sqrt{5}/3$

Length of latus rectum $= 2b^2/a = (2×16)/6 = 16/3$

2. $x^2/4 + y^2/25 = 1$

Solution:

Given:

The equation is $x^2/4 + y^2/25 = 1$

Here, the denominator of $y^2/25$ is greater than the denominator of $x^2/4$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

$a = 5$ and $b = 2$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(25-4)}$

$= \sqrt{21}$

Then,

The coordinates of the foci are $(0, \sqrt{21})$ and $(0, -\sqrt{21})$.

The coordinates of the vertices are $(0, 5)$ and $(0, -5)$

Length of the major axis $= 2a = 2\ (5) = 10$

Length of the minor axis $= 2b = 2\ (2) = 4$

Eccentricity, $e = c/a = \sqrt{21}/5$

Length of latus rectum $= 2b^2/a = (2\times2^2)/5 = (2\times4)/5 = 8/5$

3. $x^2/16 + y^2/9 = 1$

Solution:

Given:

The equation is $x^2/16 + y^2/9 = 1$ or $x^2/4^2 + y^2/3^2 = 1$

Here, the denominator of $x^2/16$ is greater than the denominator of $y^2/9$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

$a = 4$ and $b = 3$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(16-9)}$

$= \sqrt{7}$

Then,

The coordinates of the foci are $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$.

The coordinates of the vertices are $(4, 0)$ and $(-4, 0)$

Length of the major axis $= 2a = 2\ (4) = 8$

Length of the minor axis $= 2b = 2\ (3) = 6$

Eccentricity, $e = c/a = \sqrt{7}/4$

Length of latus rectum $= 2b^2/a = (2\times3^2)/4 = (2\times9)/4 = 18/4 = 9/2$

4. $x^2/25 + y^2/100 = 1$

Solution:

Given:

The equation is $x^2/25 + y^2/100 = 1$

Here, the denominator of $y^2/100$ is greater than the denominator of $x^2/25$.

So, the major axis is along the y-axis, while the minor axis is along the x-axis.

On comparing the given equation with $x^2/b^2 + y^2/a^2 = 1$, we get

$b = 5$ and $a = 10$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(100-25)}$

$= \sqrt{75}$

$= 5\sqrt{3}$

Then,

The coordinates of the foci are $(0, 5\sqrt{3})$ and $(0, -5\sqrt{3})$.

The coordinates of the vertices are $(0, \sqrt{10})$ and $(0, -\sqrt{10})$

Length of the major axis = 2a = 2 (10) = 20

Length of the minor axis = 2b = 2 (5) = 10

Eccentricity, $e = c/a = 5\sqrt{3}/10 = \sqrt{3}/2$

Length of latus rectum $= 2b^2/a = (2\times5^2)/10 = (2\times25)/10 = 5$

5. $x^2/49 + y^2/36 = 1$

Solution:

Given:

The equation is $x^2/49 + y^2/36 = 1$

Here, the denominator of $x^2/49$ is greater than the denominator of $y^2/36$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

b = 6 and a =7

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(49-36)}$

$= \sqrt{13}$

Then,

The coordinates of the foci are $(\sqrt{13}, 0)$ and $(-\sqrt{3}, 0)$.

The coordinates of the vertices are (7, 0) and (-7, 0)

Length of the major axis = 2a = 2 (7) = 14

Length of the minor axis = 2b = 2 (6) = 12

Eccentricity, e = c/a = $\sqrt{13}/7$

Length of latus rectum = $2b^2/a = (2\times6^2)/7 = (2\times36)/7 = 72/7$

6. $x^2/100 + y^2/400 = 1$

Solution:

Given:

The equation is $x^2/100 + y^2/400 = 1$

Here, the denominator of $y^2/400$ is greater than the denominator of $x^2/100$.

So, the major axis is along the y-axis, while the minor axis is along the x-axis.

On comparing the given equation with $x^2/b^2 + y^2/a^2 = 1$, we get

b = 10 and a =20.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(400-100)}$

$= \sqrt{300}$

$= 10\sqrt{3}$

Then,

The coordinates of the foci are $(0, 10\sqrt{3})$ and $(0, -10\sqrt{3})$.

The coordinates of the vertices are (0, 20) and (0, -20)

Length of the major axis = 2a = 2 (20) = 40

Length of the minor axis = 2b = 2 (10) = 20

Eccentricity, e = c/a = $10\sqrt{3}/20 = \sqrt{3}/2$

Length of latus rectum = $2b^2/a = (2\times10^2)/20 = (2\times100)/20 = 10$

7. $36x^2 + 4y^2 = 144$

Solution:

Given:

The equation is $36x^2 + 4y^2 = 144$ or $x^2/4 + y^2/36 = 1$ or $x^2/2^2 + y^2/6^2 = 1$

Here, the denominator of $y^2/6^2$ is greater than the denominator of $x^2/2^2$.

So, the major axis is along the y-axis, while the minor axis is along the x-axis.

On comparing the given equation with $x^2/b^2 + y^2/a^2 = 1$, we get

$b = 2$ and $a = 6$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(36-4)}$

$= \sqrt{32}$

$= 4\sqrt{2}$

Then,

The coordinates of the foci are $(0, 4\sqrt{2})$ and $(0, -4\sqrt{2})$.

The coordinates of the vertices are $(0, 6)$ and $(0, -6)$

Length of the major axis $= 2a = 2\,(6) = 12$

Length of the minor axis $= 2b = 2\,(2) = 4$

Eccentricity, $e = c/a = 4\sqrt{2}/6 = 2\sqrt{2}/3$

Length of latus rectum $= 2b^2/a = (2\times 2^2)/6 = (2\times 4)/6 = 4/3$

8. $16x^2 + y^2 = 16$

Solution:

Given:

The equation is $16x^2 + y^2 = 16$ or $x^2/1 + y^2/16 = 1$ or $x^2/1^2 + y^2/4^2 = 1$

Here, the denominator of $y^2/4^2$ is greater than the denominator of $x^2/1^2$.

So, the major axis is along the y-axis, while the minor axis is along the x-axis.

On comparing the given equation with $x^2/b^2 + y^2/a^2 = 1$, we get

$b = 1$ and $a = 4$.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(16-1)}$

$= \sqrt{15}$

Then,

The coordinates of the foci are $(0, \sqrt{15})$ and $(0, -\sqrt{15})$.

The coordinates of the vertices are $(0, 4)$ and $(0, -4)$

Length of the major axis $= 2a = 2\ (4) = 8$

Length of the minor axis $= 2b = 2\ (1) = 2$

Eccentricity, $e = c/a = \sqrt{15}/4$

Length of latus rectum $= 2b^2/a = (2\times1^2)/4 = 2/4 = ½$

9. $4x^2 + 9y^2 = 36$

Solution:

Given:

The equation is $4x^2 + 9y^2 = 36$ or $x^2/9 + y^2/4 = 1$ or $x^2/3^2 + y^2/2^2 = 1$

Here, the denominator of $x^2/3^2$ is greater than the denominator of $y^2/2^2$.

So, the major axis is along the x-axis, while the minor axis is along the y-axis.

On comparing the given equation with $x^2/a^2 + y^2/b^2 = 1$, we get

a =3 and b =2.

$c = \sqrt{(a^2 - b^2)}$

$= \sqrt{(9-4)}$

$= \sqrt{5}$

Then,

The coordinates of the foci are $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$.

The coordinates of the vertices are (3, 0) and (-3, 0)

Length of the major axis = 2a = 2 (3) = 6

Length of the minor axis = 2b = 2 (2) = 4

Eccentricity, $e = c/a = \sqrt{5}/3$

Length of latus rectum $= 2b^2/a = (2\times2^2)/3 = (2\times4)/3 = 8/3$

In each of the following Exercises 10 to 20, find the equation for the ellipse that satisfies the given conditions:

10. Vertices (± 5, 0), foci (± 4, 0)

Solution:

Given:

Vertices (± 5, 0) and foci (± 4, 0)

Here, the vertices are on the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where ‘a’ is the semi-major axis.

Then, a = 5 and c = 4.

It is known that $a^2 = b^2 + c^2$.

So, $5^2 = b^2 + 4^2$

$25 = b^2 + 16$

$b^2 = 25 – 16$

$b = \sqrt{9}$

= 3

∴ The equation of the ellipse is $x^2/5^2 + y^2/3^2 = 1$ or $x^2/25 + y^2/9 = 1$

11. Vertices (0, ± 13), foci (0, ± 5)

Solution:

Given:

Vertices (0, ± 13) and foci (0, ± 5)

Here, the vertices are on the y-axis.

So, the equation of the ellipse will be of the form $x^2/b^2 + y^2/a^2 = 1$, where ‘a’ is the semi-major axis.

Then, $a = 13$ and $c = 5$.

It is known that $a^2 = b^2 + c^2$.

$13^2 = b^2+5^2$

$169 = b^2 + 15$

$b^2 = 169 – 125$

$b = \sqrt{144}$

$= 12$

$\therefore$ The equation of the ellipse is $x^2/12^2 + y^2/13^2 = 1$ or $x^2/144 + y^2/169 = 1$

12. Vertices (± 6, 0), foci (± 4, 0)

Solution:

Given:

Vertices (± 6, 0) and foci (± 4, 0)

Here, the vertices are on the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, $a = 6$ and $c = 4$.

It is known that $a^2 = b^2 + c^2$.

$6^2 = b^2+4^2$

$36 = b^2 + 16$

$b^2 = 36 – 16$

$b = \sqrt{20}$

$\therefore$ The equation of the ellipse is $x^2/6^2 + y^2/(\sqrt{20})^2 = 1$ or $x^2/36 + y^2/20 = 1$

13. Ends of major axis (± 3, 0), ends of minor axis (0, ±2)

Solution:

Given:

Ends of major axis (± 3, 0) and ends of minor axis (0, ±2)

Here, the major axis is along the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, a = 3 and b = 2.

∴ The equation for the ellipse $x^2/3^2 + y^2/2^2 = 1$ or $x^2/9 + y^2/4 = 1$

14. Ends of major axis (0, ±√5), ends of minor axis (±1, 0)

Solution:

Given:

Ends of major axis (0, ±√5) and ends of minor axis (±1, 0)

Here, the major axis is along the y-axis.

So, the equation of the ellipse will be of the form $x^2/b^2 + y^2/a^2 = 1$, where 'a' is the semi-major axis.

Then, a = √5 and b = 1.

∴ The equation for the ellipse $x^2/1^2 + y^2/(\sqrt{5})^2 = 1$ or $x^2/1 + y^2/5 = 1$

15. Length of major axis 26, foci (±5, 0)

Solution:

Given:

Length of major axis is 26 and foci (±5, 0)

Since the foci are on the x-axis, the major axis is along the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, $2a = 26$

$a = 13$ and $c = 5$.

It is known that $a^2 = b^2 + c^2$.

$13^2 = b^2+5^2$

$169 = b^2 + 25$

$b^2 = 169 – 25$

$b = \sqrt{144}$

$= 12$

∴ The equation of the ellipse is $x^2/13^2 + y^2/12^2 = 1$ or $x^2/169 + y^2/144 = 1$

16. Length of minor axis 16, foci (0, ±6).

Solution:

Given:

Length of minor axis is 16 and foci (0, ±6).

Since the foci are on the y-axis, the major axis is along the y-axis.

So, the equation of the ellipse will be of the form $x^2/b^2 + y^2/a^2 = 1$, where 'a' is the semi-major axis.

Then, $2b =16$

$b = 8$ and $c = 6$.

It is known that $a^2 = b^2 + c^2$.

$a^2 = 8^2 + 6^2$

$= 64 + 36$

$=100$

$a = \sqrt{100}$

$= 10$

∴ The equation of the ellipse is $x^2/8^2 + y^2/10^2 = 1$ or $x^2/64 + y^2/100 = 1$

17. Foci (±3, 0), a = 4

Solution:

Given:

Foci (±3, 0) and a = 4

Since the foci are on the x-axis, the major axis is along the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, c = 3 and a = 4.

It is known that $a^2 = b^2 + c^2$.

$a^2 = 8^2 + 6^2$

$= 64 + 36$

$= 100$

$16 = b^2 + 9$

$b^2 = 16 – 9$

$= 7$

∴ The equation of the ellipse is $x^2/16 + y^2/7 = 1$

18. b = 3, c = 4, centre at the origin; foci on the x axis.

Solution:

Given:

b = 3, c = 4, centre at the origin and foci on the x axis.

Since the foci are on the x-axis, the major axis is along the x-axis.

So, the equation of the ellipse will be of the form $x^2/a^2 + y^2/b^2 = 1$, where 'a' is the semi-major axis.

Then, $b = 3$ and $c = 4$.

It is known that $a^2 = b^2 + c^2$.

$a^2 = 3^2 + 4^2$

$= 9 + 16$

$=25$

$a = \sqrt{25}$

$= 5$

∴ The equation of the ellipse is $x^2/5^2 + y^2/3^2$ or $x^2/25 + y^2/9 = 1$

19. Centre at (0, 0), major axis on the y-axis and passes through the points (3, 2) and (1, 6).

Solution:

Given:

Centre at (0, 0), major axis on the y-axis and passes through the points (3, 2) and (1, 6).

Since the centre is at (0, 0) and the major axis is on the y- axis, the equation of the ellipse will be of the form $x^2/b^2 + y^2/a^2 = 1$, where 'a' is the semi-major axis.

The ellipse passes through points (3, 2) and (1, 6).

So, by putting the values $x = 3$ and $y = 2$, we get,

$3^2/b^2 + 2^2/a^2 = 1$

$9/b^2 + 4/a^2$…. (1)

And by putting the values $x = 1$ and $y = 6$, we get,

$1^1/b^2 + 6^2/a^2 = 1$

$1/b^2 + 36/a^2 = 1$ …. (2)

On solving equation (1) and (2), we get

$b^2 = 10$ and $a^2 = 40$.

∴ The equation of the ellipse is $x^2/10 + y^2/40 = 1$ or $4x^2 + y^2 = 40$

20. Major axis on the x-axis and passes through the points (4,3) and (6,2).

Solution:

Given:

Major axis on the x-axis and passes through the points (4, 3) and (6, 2).

Since the major axis is on the x-axis, the equation of the ellipse will be the form

$x^2/a^2 + y^2/b^2 = 1$…. (1) [Where 'a' is the semi-major axis.]

The ellipse passes through points (4, 3) and (6, 2).

So by putting the values x = 4 and y = 3 in equation (1), we get,

$16/a^2 + 9/b^2 = 1$ …. (2)

Putting, x = 6 and y = 2 in equation (1), we get,

$36/a^2 + 4/b^2 = 1$ …. (3)

From equation (2)

$16/a^2 = 1 – 9/b^2$

$1/a^2 = (1/16 (1 – 9/b^2))$ …. (4)

Substituting the value of $1/a^2$ in equation (3) we get,

$36/a^2 + 4/b^2 = 1$

$36(1/a^2) + 4/b^2 = 1$

$36[1/16 (1 – 9/b^2)] + 4/b^2 = 1$

$36/16 (1 – 9/b^2) + 4/b^2 = 1$

$9/4 (1 – 9/b^2) + 4/b^2 = 1$

$9/4 – 81/4b^2 + 4/b^2 = 1$

$-81/4b^2 + 4/b^2 = 1 – 9/4$

$(-81+16)/4b^2 = (4-9)/4$

$-65/4b^2 = -5/4$

$-5/4(13/b^2) = -5/4$

$13/b^2 = 1$

$1/b^2 = 1/13$

$b^2 = 13$

Now substituting the value of b^2 in equation (4) we get,

$1/a^2 = 1/16(1 – 9/b^2)$

$= 1/16(1 – 9/13)$

$= 1/16((13-9)/13)$

$= 1/16(4/13)$

$= 1/52$

$a^2 = 52$

Equation of ellipse is $x^2/a^2 + y^2/b^2 = 1$

By substituting the values of a^2 and b^2 in above equation we get,

$x^2/52 + y^2/13 = 1$

EXERCISE 11.4

In each of the Exercises 1 to 6, find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas.

1. $x^2/16 - y^2/9 = 1$

Solution:

Given:

The equation is $x^2/16 - y^2/9 = 1$ or $x^2/4^2 - y^2/3^2 = 1$

On comparing this equation with the standard equation of hyperbola $x^2/a^2 - y^2/b^2 = 1$,

We get a = 4 and b = 3,

It is known that $a^2 + b^2 = c^2$

So,

$c^2 = 4^2 + 3^2$

$= \sqrt{25}$

c = 5

Then,

The coordinates of the foci are (±5, 0).

The coordinates of the vertices are (±4, 0).

Eccentricity, e = c/a = 5/4

Length of latus rectum = $2b^2/a = (2 \times 3^2)/4 = (2\times9)/4 = 18/4 = 9/2$

2. $y^2/9 – x^2/27 = 1$

Solution:

Given:

The equation is $y^2/9 – x^2/27 = 1$ or $y^2/3^2 – x^2/27^2 = 1$

On comparing this equation with the standard equation of hyperbola $y^2/a^2 – x^2/b^2 = 1$,

We get a = 3 and b = √27,

It is known that $a^2 + b^2 = c^2$

So,

$c^2 = 3^2 + (\sqrt{27})^2$

= 9 + 27

$c^2 = 36$

$c = \sqrt{36}$

= 6

Then,

The coordinates of the foci are (0, 6) and (0, -6).

The coordinates of the vertices are (0, 3) and (0, – 3).

Eccentricity, e = c/a = 6/3 = 2

Length of latus rectum = $2b^2/a = (2 \times 27)/3 = (54)/3 = 18$

3. $9y^2 - 4x^2 = 36$

Solution:

Given:

The equation is $9y^2 - 4x^2 = 36$ or $y^2/4 - x^2/9 = 1$ or $y^2/2^2 - x^2/3^2 = 1$

On comparing this equation with the standard equation of hyperbola $y^2/a^2 - x^2/b^2 = 1$,

We get a = 2 and b = 3,

It is known that $a^2 + b^2 = c^2$

So,

$c^2 = 4 + 9$

$c^2 = 13$

$c = \sqrt{13}$

Then,

The coordinates of the foci are $(0, \sqrt{13})$ and $(0, -\sqrt{13})$.

The coordinates of the vertices are (0, 2) and (0, – 2).

Eccentricity, $e = c/a = \sqrt{13}/2$

Length of latus rectum = $2b^2/a = (2 \times 3^2)/2 = (2\times9)/2 = 18/2 = 9$

4. $16x^2 - 9y^2 = 576$

Solution:

Given:

The equation is $16x^2 - 9y^2 = 576$

Let us divide the whole equation by 576.We get

$16x^2/576 - 9y^2/576 = 576/576$

$x^2/36 - y^2/64 = 1$

On comparing this equation with the standard equation of hyperbola $x^2/a^2 - y^2/b^2 = 1$,

We get $a = 6$ and $b = 8$,

It is known that $a^2 + b^2 = c^2$

So,

$c^2 = 36 + 64$

$c^2 = \sqrt{100}$

$c = 10$

Then,

The coordinates of the foci are (10, 0) and (-10, 0).

The coordinates of the vertices are (6, 0) and (-6, 0).

Eccentricity, $e = c/a = 10/6 = 5/3$

Length of latus rectum $= 2b^2/a = (2 \times 8^2)/6 = (2\times64)/6 = 64/3$

5. $5y^2 - 9x^2 = 36$

Solution:

Given:

The equation is $5y^2 - 9x^2 = 36$

Let us divide the whole equation by 36. We get

$5y^2/36 - 9x^2/36 = 36/36$

$y^2/(36/5) - x^2/4 = 1$

On comparing this equation with the standard equation of hyperbola $y^2/a^2 - x^2/b^2 = 1$,

We get $a = 6/\sqrt{5}$ and $b = 2$,

It is known that $a^2 + b^2 = c^2$

So,

$c^2 = 36/5 + 4$

$c^2 = 56/5$

$c = \sqrt{(56/5)}$

$= 2\sqrt{14}/\sqrt{5}$

Then,

The coordinates of the foci are $(0, 2\sqrt{14}/\sqrt{5})$ and $(0, -2\sqrt{14}/\sqrt{5})$.

The coordinates of the vertices are $(0, 6/\sqrt{5})$ and $(0, -6/\sqrt{5})$.

Eccentricity, $e = c/a = (2\sqrt{14}/\sqrt{5}) / (6/\sqrt{5}) = \sqrt{14}/3$

Length of latus rectum $= 2b^2/a = (2 \times 2^2)/6/\sqrt{5} = (2\times4)/6/\sqrt{5} = 4\sqrt{5}/3$

6. $49y^2 - 16x^2 = 784$.

Solution:

Given:

The equation is $49y^2 - 16x^2 = 784$.

Let us divide the whole equation by 784, we get

$49y^2/784 - 16x^2/784 = 784/784$

$y^2/16 - x^2/49 = 1$

On comparing this equation with the standard equation of hyperbola $y^2/a^2 - x^2/b^2 = 1$,

We get $a = 4$ and $b = 7$,

It is known that $a^2 + b^2 = c^2$

So,

$c^2 = 16 + 49$

$c^2 = 65$

$c = \sqrt{65}$

Then,

The coordinates of the foci are $(0, \sqrt{65})$ and $(0, -\sqrt{65})$.

The coordinates of the vertices are (0, 4) and (0, -4).

Eccentricity, $e = c/a = \sqrt{65}/4$

Length of latus rectum $= 2b^2/a = (2 \times 7^2)/4 = (2\times49)/4 = 49/2$

In each Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions

7. Vertices (±2, 0), foci (±3, 0)

Solution:

Given:

Vertices (±2, 0) and foci (±3, 0)

Here, the vertices are on the x-axis.

So, the equation of the hyperbola is of the form $x^2/a^2 - y^2/b^2 = 1$

Since the vertices are (±2, 0), so, a = 2

Since the foci are (±3, 0), so, c = 3

It is known that, $a^2 + b^2 = c^2$

So, $2^2 + b^2 = 3^2$

$b^2 = 9 - 4 = 5$

∴ The equation of the hyperbola is $x^2/4 - y^2/5 = 1$

8. Vertices (0, ± 5), foci (0, ± 8)

Solution:

Given:

Vertices (0, ± 5) and foci (0, ± 8)

Here, the vertices are on the y-axis.

So, the equation of the hyperbola is of the form $y^2/a^2 - x^2/b^2 = 1$

Since the vertices are $(0, \pm5)$, so, $a = 5$

Since the foci are $(0, \pm8)$, so, $c = 8$

It is known that, $a^2 + b^2 = c^2$

So, $5^2 + b^2 = 8^2$

$b^2 = 64 - 25 = 39$

∴ The equation of the hyperbola is $y^2/25 - x^2/39 = 1$

9. Vertices (0, ± 3), foci (0, ± 5)

Solution:

Given:

Vertices $(0, \pm 3)$ and foci $(0, \pm 5)$

Here, the vertices are on the y-axis.

So, the equation of the hyperbola is of the form $y^2/a^2 - x^2/b^2 = 1$

Since the vertices are $(0, \pm3)$, so, $a = 3$

Since the foci are $(0, \pm5)$, so, $c = 5$

It is known that $a^2 + b^2 = c^2$

So, $3^2 + b^2 = 5^2$

$b^2 = 25 - 9 = 16$

∴ The equation of the hyperbola is $y^2/9 - x^2/16 = 1$

10. Foci (±5, 0), the transverse axis is of length 8.

Solution:

Given:

Foci (±5, 0) and the transverse axis is of length 8.

Here, the foci are on x-axis.

The equation of the hyperbola is of the form $x^2/a^2 – y^2/b^2 = 1$

Since the foci are (±5, 0), so, c = 5

Since the length of the transverse axis is 8,

2a = 8

a = 8/2

= 4

It is known that $a^2 + b^2 = c^2$

$4^2 + b^2 = 5^2$

$b^2 = 25 – 16$

= 9

∴ The equation of the hyperbola is $x^2/16 – y^2/9 = 1$

11. Foci (0, ±13), the conjugate axis is of length 24.

Solution:

Given:

Foci (0, ±13) and the conjugate axis is of length 24.

Here, the foci are on y-axis.

The equation of the hyperbola is of the form $y^2/a^2 – x^2/b^2 = 1$

Since the foci are (0, ±13), so, c = 13

Since the length of the conjugate axis is 24,

$2b = 24$

$b = 24/2$

$= 12$

It is known that $a^2 + b^2 = c^2$

$a^2 + 12^2 = 13^2$

$a^2 = 169 – 144$

$= 25$

∴ The equation of the hyperbola is $y^2/25 – x^2/144 = 1$

12. Foci (± 3√5, 0), the latus rectum is of length 8.

Solution:

Given:

Foci (± 3√5, 0) and the latus rectum is of length 8.

Here, the foci are on x-axis.

The equation of the hyperbola is of the form $x^2/a^2 – y^2/b^2 = 1$

Since the foci are (± 3√5, 0), so, $c = \pm 3\sqrt{5}$

Length of latus rectum is 8

$2b^2/a = 8$

$2b^2 = 8a$

$b^2 = 8a/2$

$= 4a$

It is known that $a^2 + b^2 = c^2$

$a^2 + 4a = 45$

$a^2 + 4a - 45 = 0$

$a^2 + 9a - 5a - 45 = 0$

$(a + 9) (a -5) = 0$

$a = -9$ or 5

Since a is non – negative, $a = 5$

So, $b^2 = 4a$

$= 4 \times 5$

$= 20$

∴ The equation of the hyperbola is $x^2/25 - y^2/20 = 1$

13. Foci (± 4, 0), the latus rectum is of length 12

Solution:

Given:

Foci (± 4, 0) and the latus rectum is of length 12

Here, the foci are on x-axis.

The equation of the hyperbola is of the form $x^2/a^2 - y^2/b^2 = 1$

Since the foci are (± 4, 0), so, $c = 4$

Length of latus rectum is 12

$2b^2/a = 12$

$2b^2 = 12a$

$b^2 = 12a/2$

$= 6a$

It is known that $a^2 + b^2 = c^2$

$a^2 + 6a = 16$

$a^2 + 6a - 16 = 0$

$a^2 + 8a - 2a - 16 = 0$

$(a + 8)(a - 2) = 0$

$a = -8$ or 2

Since a is non – negative, $a = 2$

So, $b^2 = 6a$

$= 6 \times 2$

$= 12$

∴ The equation of the hyperbola is $x^2/4 - y^2/12 = 1$

14. Vertices (±7, 0), e = 4/3

Solution:

Given:

Vertices $(\pm 7, 0)$ and $e = 4/3$

Here, the vertices are on the x- axis

The equation of the hyperbola is of the form $x^2/a^2 - y^2/b^2 = 1$

Since the vertices are $(\pm 7, 0)$, so, $a = 7$

It is given that $e = 4/3$

$c/a = 4/3$

$3c = 4a$

Substituting the value of a, we get

$3c = 4(7)$

$c = 28/3$

It is known that, $a^2 + b^2 = c^2$

$7^2 + b^2 = (28/3)^2$

$b^2 = 784/9 – 49$

$= (784 – 441)/9$

$= 343/9$

∴ The equation of the hyperbola is $x^2/49 – 9y^2/343 = 1$

15. Foci (0, ±√10), passing through (2, 3)

Solution:

Given:

Foci (0, ±√10) and passing through (2, 3)

Here, the foci are on y-axis.

The equation of the hyperbola is of the form $y^2/a^2 – x^2/b^2 = 1$

Since the foci are (±√10, 0), so, c = √10

It is known that $a^2 + b^2 = c^2$

$b^2 = 10 – a^2$ ………….. (1)

It is given that the hyperbola passes through point (2, 3)

So, $9/a^2 – 4/b^2 = 1$ … (2)

From equations (1) and (2), we get,

$9/a^2 – 4/(10-a^2) = 1$

$9(10 – a^2) – 4a^2 = a^2(10 –a^2)$

$90 – 9a^2 – 4a^2 = 10a^2 – a^4$

$a^4 – 23a^2 + 90 = 0$

$a^4 – 18a^2 – 5a^2 + 90 = 0$

$a^2(a^2 -18) -5(a^2 -18) = 0$

$(a^2 – 18) (a^2 -5) = 0$

$a^2 = 18$ or 5

In hyperbola, c > a i.e., $c^2 > a^2$

So, $a^2 = 5$

$b^2 = 10 – a^2$

= 10 – 5

= 5

∴ The equation of the hyperbola is $y^2/5 – x^2/5 = 1$

2Marks Questions & Answers

1. Show that the equation $x^2 + y^2 + 6x + 4y - 36 = 0$ represents a circle, also find its centre & radius?

Ans: It is of the form $x^2 + y^2 + 2gx + 2fy + 6 = 0$

Where 2g=−6, 2f=4 & $c = -36$

∴ g = −3, f = 2 & $c = -36$

Thus, center of the circle is (−g, −f) = (3, −2)

Radius of the circle is $\sqrt{g^2 + f^2 - c}$ = $\sqrt{9 + 4 + 36}$

=7 units

2. Find the equation of an ellipse whose foci are (±8, 0) & the eccentricity is 14?

Ans: Let the required equation of the ellipse be x2a2+y2b2=1, where $a^2>b^2$

Let the foci be (±c, 0), c=8

And e=$\frac{c}{a}$

⇒a= $\frac{c}{e}$

$\Rightarrow$a= $\frac{8}{\frac{1}{4}}$

$\Rightarrow$a=32

As c^2=$a^2 - b^2$

$\Rightarrow$$b^2$=$a^2 - c^2$

$\Rightarrow$$b^2$=1024−64

$\Rightarrow$$b^2$=960

$\therefore$$a^2$=1024

& b^2=960

Therefore the equation is $\frac{x^2}{1024} + \frac{y^2}{960} = 1$

3. Find the equation of an ellipse whose vertices are (0,±10)(0,±10)**&** e=$\frac{4}{5}$

Ans: Let equation be $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

Vertices are (0,±a), a=10

Let c^2=$a^2 - b^2$

So, e= $\frac{c}{a}$

$\Rightarrow$c=ae

$\Rightarrow$c=10×$\frac{4}{5}$

$\Rightarrow$c=8

Now, c^2=$a^2 - b^2$

$\Rightarrow$$b^2$=$a^2 - c^2$

$\Rightarrow$$b^2$=100−64

$\Rightarrow b^2$=36

So, a^2=$(10)^2$=100 & b^2=36

Therefore the equation is $\frac{x^2}{36}+\frac{y^2}{100}=1$

4. Find the equation of hyperbola whose length of latus rectum is 36 & foci are $(0, \pm 12)$

Ans: It is clear that c=12

Length of latus rectum =36=36

$\Rightarrow \frac{2b^2}{a}=36$

$\Rightarrow b^2$=18a

Now, $c^2=a^2-b^2$

$\Rightarrow a^2= c^2-b^2$

$\Rightarrow a^2$=144−18a

$\Rightarrow a^2$+18a−144=0

$\Rightarrow$ (a+24) (a−6)=0

$\Rightarrow$a=6 because a is non-negative.

Thus a^2=6^2=36 & b^2=108

Therefore, $\frac{x^2}{36}+\frac{y^2}{108}=1$

5. Find the equation of a circle drawn on the diagonal of the rectangle as its diameter, whose sides are x=6, x=−3, y=3& y=−1

Ans: Let ABCD be the given rectangle and AD=x=−3, BC=x=6, AB=y=−1 & CD=y=−3

Then A (−3,−1) and C(6,3).

The equation of the circle with AC as diameter is:

$(x+3)(x-6)+(y+1)(y-3)=0$

$\Rightarrow x^2+y^2-3x-2y-21=0$

6. Find the coordinates of the focus & vertex, the equations of the directrix & the axis & length of latus rectum of the parabola $x^2=-8y$
Ans: $x^2=-8y$ & $x^2=-4ay$
So, $4a=8$

$\Rightarrow a=2$

So it is downward parabola.

Foci is $F(0, -a)$ i.e. $F(0,-2)$.

Vertex is $O(0, 0)$

So, $y=a=2$

Its axis is $y-$ axis, whose equation is given by $x=0$
Length of latus rectum$=4a$ units.
$=4\times2$ units
$=8$ units

7. Show that the equation $6x^2+6y^2+24x-36y-18=0$ represents a circle. Also find its centre & radius.
Ans: $6x^2+6y^2+24x-36y-18=0$

So, $x^2+y^2+4x-6y+3=0$

Where, $2g=4$, $2f=-6$ & $c=3$

$\therefore g=2$, $f=-3$ & $c=3$

Thus, centre of circle is $(-g, -f)=(-2,3)$

Radius of circle $=\sqrt{4+9+9}=\sqrt{20}$

$=2\sqrt{5}$ units

8. Find the equation of the parabola with focus at $F(5,0)$ & directrix is $x=-5$

Ans: $F(5,0)$ lies on the right hand side of origin.

Thus, it is a right hand parabola.

Let the required equation be

y^2=4ax & a=5

Hence, y^2=20

9. Find the equation of the hyperbola with center at the origin, length of the transverse axis 66 & one focus at $(0, 4)$

Ans: Let its equation be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

It is clear that c=4.

Length of the transverse axis =6

⇒2a=6

⇒a=3

And, $c^2 = (a^2+b^2)$

$\Rightarrow b^2=c^2-a^2$

$\Rightarrow b^2=16-9$

$\Rightarrow b^2=7$

Thus, a^2=9 & b^2=7

Hence, equation is $\frac{x^2}{9} - \frac{x^2}{7} = 1.$

10. Find the equation of an ellipse whose vertices are $(0, \pm 13)$ & the foci are $(0,\pm 5)$

Ans: Let the equation be $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$

& a=13

Let foci be $(0,\pm c)$,

⇒c=5

$\therefore b^2 = a^2 - c^2$

$\Rightarrow b^2 = 169 - 25$

$\Rightarrow b^2 = 144$

$\Rightarrow a^2 = 169$

And $b^2 = 144$

Thus, equation is $\frac{x^2}{144} - \frac{y^2}{169} = 1$

Multiple Choice Questions

1) The length of the transverse axis is the distance between the ____.

a. Two vertices
b. Two Foci
c. Vertex and the origin
d. Focus and the vertex

Answer: (a) Two vertices

Explanation:

The length of the transverse axis is the distance between two vertices.

2) The parametric equation of the parabola $y^2 = 4ax$ is

a. $x = at$; $y = 2at$
b. $x = at^2$; $y = 2at$
c. $x = at^2$; $y^2 = at^3$
d. $x = at^2$; $y = 4at$

Answer: (b) $x = at^2$; $y = 2at$

Explanation:

The parametric equation of the parabola $y^2 = 4ax$ is $x = at^2$; $y = 2at$.

3) The centre of the circle $4x^2 + 4y^2 – 8x + 12y – 25 = 0$ is

a. (-2, 3)
b. (1, -3/2)
c. (-4, 6)
d. (4, -6)

Answer: (b) (1, -3/2)

Explanation:

Given circle equation: $4x^2 + 4y^2 – 8x + 12y – 25 = 0$

$\Rightarrow x^2 + y^2 – (8x/4) + (12y/4) – (25/4) = 0$

$\Rightarrow x^2 + y^2 - 2x + 3y - (25/4) = 0$ …(1)

As we know that the general equation of a circle is $x^2+y^2+2gx+2fy+c=0$, and the centre of the circle = (-g, -f)

Hence, by comparing equation (1) and the general equation,

2g = -2,

Thus, g = -1

2f = 3, thus, f = 3/2

Now, substitute the values in the centre of the circle (-g, -f), we get,

Centre = (1, -3/2).

Therefore, option (b) (1, -3/2) is the correct answer.

4) The equation of the directrix of the parabola $y^2+4y+4x+2=0$ is

a. x = 1
b. x = -1
c. x = 3/2
d. x = -3/2

Answer: (c) x = 3/2

Explanation:

Given equation: $y^2+4y+4x+2=0$

Rearranging the equation, we get

$(y+2)^2 = -4x+2$

$(y+2)^2 = -4(x - (½))$

Let $Y = y+2$ and $X = x-(½)$

So, $Y^2 = -4X$ …(1)

Hence, equation (1) is of the form $y^2 = -4ax$. …(2)

By comparing (1) and (2), we get a=1.

We know that equation of directrix is x= a

Now, substitute a = 1 and x = x-(½) in the directrix equation

$x - (½) = 1$

$x = 1+(½) = 3/2$.

Therefore, the equation of the directrix of the parabola $y^2+4y+4x+2=0$ is 3/2.

5) The number of tangents that can be drawn from (1, 2) to $x^2+y^2 = 5$ is

a. 0
b. 1
c. 2
d. More than 2

Answer: (b) 1

Explanation:

Given circle equation: $x^2+y^2= 5$

$x^2+y^2-5 =0$ …(1)

Now, substitute (1, 2) in equation (1), we get

Circle Equation: $(1)^2+(2)^2 -5 =0$

Equation of circle = 1+5-5 =0

This represents that the point lies on the circumference of a circle, and hence only one tangent can be drawn from (1, 2).

So, option (b) 1 is the correct answer.

6) The length of the latus rectum of $x^2 = -9y$ is equal to

a. 3 units
b. -3 units
c. 9/4 units
d. 9 units

Answer: (d) 9 units

Explanation:

Given parabola equation: $x^2 = -9y$ …(1)

Since the coefficient of y is negative, the parabola opens downwards.

The general equation of parabola is $x^2= -4ay$…(2)

Comparing (1) and (2), we get

$-4a = -9$

$a = 9/4$

We know that the length of latus rectum = 4a = 4(9/4) = 9.

Therefore, the length of the latus rectum of $x^2 = -9y$ is equal to 9 units.

7) For the ellipse $3x^2+4y^2 = 12$, the length of the latus rectum is:

a. 2/5
b. 3/5
c. 3
d. 4

Answer: (c) 3

Explanation:

Given ellipse equation: $3x^2+4y^2 = 12$

The given equation can be written as $(x^2/4) + (y^2/3) = 1$…(1)

Now, compare the given equation with the standard ellipse equation: $(x^2/a^2) + (y^2/b^2) = 1$, we get

$a = 2$ and $b = \sqrt{3}$

Therefore, $a > b$.

If $a>b$, then the length of latus rectum is $2b^2/a$

Substituting the values in the formula, we get

Length of latus rectum $= [2(\sqrt{3})^2] /2 = 3$

Therefore, the length of the latus rectum of the ellipse $3x^2+4y^2 = 12$ is 3.

8) The eccentricity of hyperbola is

a. $e =1$
b. $e > 1$
c. $e < 1$
d. $0 < e < 1$

Answer: (b) $e > 1$

Explanation:

The eccentricity of hyperbola is greater than 1. (i.e.) $e > 1$.

9) The focus of the parabola $y^2 = 8x$ is

a. (0, 2)
b. (2, 0)
c. (0, -2)
d. (-2, 0)

Answer: (b) (2, 0)

Explanation:

Given parabola equation $y^2 = 8x$ …(1)

Here, the coefficient of x is positive and the standard form of parabola is $y^2 = 4ax$ …(2)

Comparing (1) and (2), we get

$4a = 8$

$a = 8/4 = 2$

We know that the focus of parabolic equation $y^2 = 4ax$ is (a, 0).

Therefore, the focus of the parabola $y^2 = 8x$ is (2, 0).

Hence, option (b) (2, 0) is the correct answer.

10) In an ellipse, the distance between its foci is 6 and the minor axis is 8, then its eccentricity is

a. 1/2
b. 1/5
c. 3/5
d. 4/5

Answer: (c) 3/5

Explanation:

Given that the minor axis of ellipse is 8.(i.e) 2b = 8. So, b=4.

Also, the distance between its foci is 6. (i.e) 2ae = 6

Therefore, ae = 6/2 = 3

We know that $b^2 = a^2(1-e^2)$

$b^2 = a^2 – a^2e^2$

$b^2 = a^2 – (ae)^2$

Now, substitute the values to find the value of a.

$(4)^2 = a^2 -(3)^2$

$16 = a^2 – 9$

$a^2 = 16+9 = 25$.

So, $a = 5$.

The formula to calculate the eccentricity of ellipse is $e = \sqrt{[1-(b^2/a^2)]}$

$e = \sqrt{[1-(4^2/5^2)]}$

$e = \sqrt{[(25-16)/25]}$

$e = \sqrt{(9/25)} = 3/5$.

Hence, option (c) 3/5 is the correct answer.

Summary

In this Chapter the following concepts and generalizations are studied.

- A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.
- The equation of a circle with centre (h, k) and the radius r is $(x - h)^2 + (y - k)^2 = r^2$.
- A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point in the plane.
- The equation of the parabola with focus at (a, 0) $a > 0$ and directrix $x = -a$ is $y^2 = 4ax$.
- Latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola.
- Length of the latus rectum of the parabola $y^2 = 4ax$ is 4a.
- An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.
- The equation of an ellipse with foci on the x-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse.

- Length of the latus rectum of the ellipse :

$$\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}} + \frac{\boldsymbol{y^2}}{\boldsymbol{b^2}} = \mathbf{1} \text{ is } \frac{2b^2}{a}$$

- The eccentricity of an ellipse is the ratio between the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse.
- A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.
- The equation of a hyperbola with foci on the x-axis is :

$$\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}} - \frac{\boldsymbol{y^2}}{\boldsymbol{b^2}} = \mathbf{1}$$

- Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.
- Length of the latus rectum of the hyperbola :

$$\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}} - \frac{\boldsymbol{y^2}}{\boldsymbol{b^2}} = \mathbf{1} \text{ is } \frac{2b^2}{a}$$

- The eccentricity of a hyperbola is the ratio of the distances from the centre of the hyperbola to one of the foci and to one of the vertices of the hyperbola.