Part2:

1. **Covariance and Correlation matrices of U:**

Part II.

1. Find the correlation and covariance matrices of U.

Given: $f_{XY}(x,y) = \begin{cases} x + \frac{3}{2}y^2 & , 0 < x, y \leq 1 \\ 0 & , \text{otherwise} \end{cases}$

Random vector, $U = \begin{pmatrix} X \\ Y \end{pmatrix}$

Solution:

Marginal Pdfs of X and Y

$$\Rightarrow f_X(x) = \int_0^1 f_{XY}(x,y)\, dy$$

$$= \int_0^1 x + \frac{3}{2}y^2\, dy$$

$$= x + \frac{3}{2}\left[\frac{y^3}{3}\right]_0^1 \text{, for } 0 < x < 1$$

$$\boxed{f_X(x) = x + \frac{1}{2} \text{ , for } 0 < x < 1}$$

$$\Rightarrow f_Y(y) = \int_0^1 f_{XY}(x,y)\, dx$$

$$= \int_0^1 x + \frac{3}{2}y^2\, \cancel{dy}\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \frac{3}{2}y^2$$

$$\boxed{f_Y(y) = \frac{1}{2} + \frac{3}{2}y^2 \text{ , for } 0 < y < 1}$$

$$E[X] = \int_0^1 x\, f_X(x)\, dx$$

$$= \int_0^1 x\left[x + \frac{1}{2}\right] dx$$

$$= \int_0^1 x^2 + \frac{x}{2}\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$\boxed{E[X] = 7/12}$$

$$E[X^2] = \int_0^1 x^2 f_X(x)\, dx$$

$$= \int_0^1 x^3 + \frac{x^2}{2}\, dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{6}$$

$$\boxed{E[X^2] = \frac{5}{12}}$$

$$Var(X) = E[X^2] - [E[X]]^2$$

$$= \frac{5}{12} - \left[\frac{7}{12}\right]^2$$

$$= \frac{5}{12} - \frac{49}{144}$$

$$= \frac{5\times 12 - 49}{144}$$

$$= \frac{11}{144}$$

$$E[Y] = \int_0^1 y\, f_Y(y)\, dy$$

$$= \int_0^1 y\left[\frac{1}{2} + \frac{3}{2}y^2\right] dy$$

$$= \int_0^1 \frac{y}{2} + \frac{3y^3}{2}\, dy$$

$$= \left[\frac{y^2}{4} + \frac{3y^4}{8}\right]_0^1$$

$$= \frac{1}{4} + \frac{3}{8} = \frac{5}{8}$$

$$\boxed{E[Y] = 5/8}$$

$$E[Y^2] = \int_0^1 y^2 f_Y(y)\, dy$$

$$= \int_0^1 \frac{y^2}{2} + \frac{3}{2}y^4\, dy$$

$$= \left[\frac{y^3}{6} + \frac{3y^5}{10}\right]_0^1$$

$$= \frac{1}{6} + \frac{3}{10}$$

$$= \frac{10+18}{60} = \frac{28}{60}$$

$$= \frac{14}{30}$$

$$\boxed{E[Y^2] = 7/15}$$

$$Var(Y) = E[Y^2] - [E[Y]]^2$$

$$= \frac{7}{15} - \left(\frac{5}{8}\right)^2$$

$$= \frac{7}{15} - \frac{25}{64}$$

$$= \frac{448 - 375}{960}$$

$$= \frac{73}{960}$$

$$E[XY] = \int_0^1 \int_0^1 xy \; f_{XY}(x,y) \, dx \, dy$$

$$= \int_0^1 \int_0^1 xy \left[x + \frac{3}{2} y^2\right] dx \, dy$$

$$= \int_0^1 \int_0^1 x^2 y + \frac{3}{2} x y^3 \, dx \, dy$$

$$= \int_0^1 \left[\frac{x^3 y}{3} + \frac{3x^2 y^3}{4}\right]_0^1 dy$$

$$= \int_0^1 \frac{y}{3} + \frac{3y^3}{4} \, dy$$

$$= \left[\frac{y^2}{6} + \frac{3y^4}{16}\right]_0^1$$

$$= \frac{1}{6} + \frac{3}{16} = \frac{8+9}{48}$$

$$\boxed{E[XY] = \frac{17}{48}}$$

We have, $\text{cov}(X, Y) = E[XY] - E[X]\,E[Y]$

$$= \frac{17}{48} - \left[\left(\frac{7}{12}\right)\left(\frac{5}{8}\right)\right]$$

$$= -\frac{1}{96}$$

The correlation matrix is given as,

$$R_U = E[UU^T] = \begin{bmatrix} E[X^2] & E[XY] \\ E[YX] & E[Y^2] \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{12} & 17/48 \\ 17/48 & 7/15 \end{bmatrix}$$

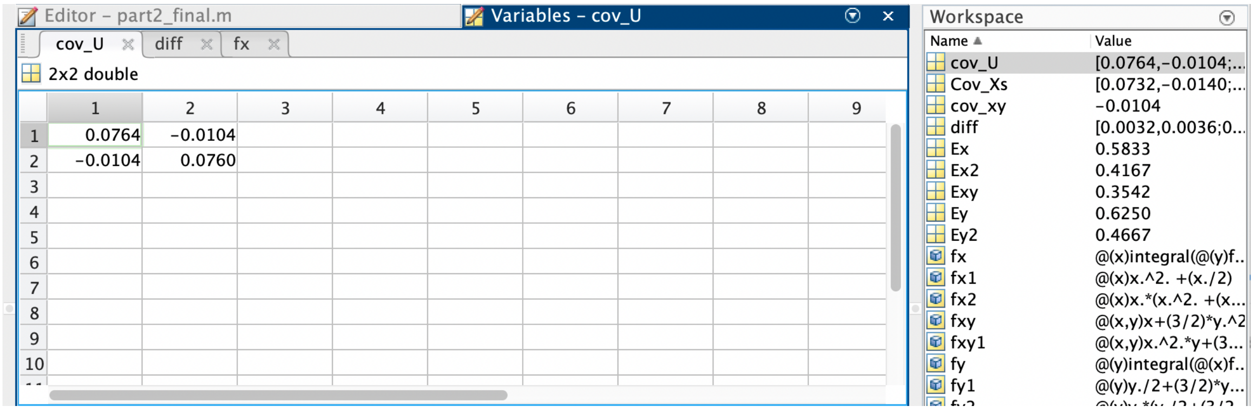
The covariance matrix is given as,

$$C_U = \begin{bmatrix} \text{var}(X) & \text{cov}(X,Y) \\ \text{cov}(Y,X) & \text{var}(Y) \end{bmatrix} = \begin{bmatrix} 11/144 & -1/96 \\ -1/96 & 73/960 \end{bmatrix}$$

2and 3) Generation of 1000-sample vector series Xs with same covariance matrix as U:

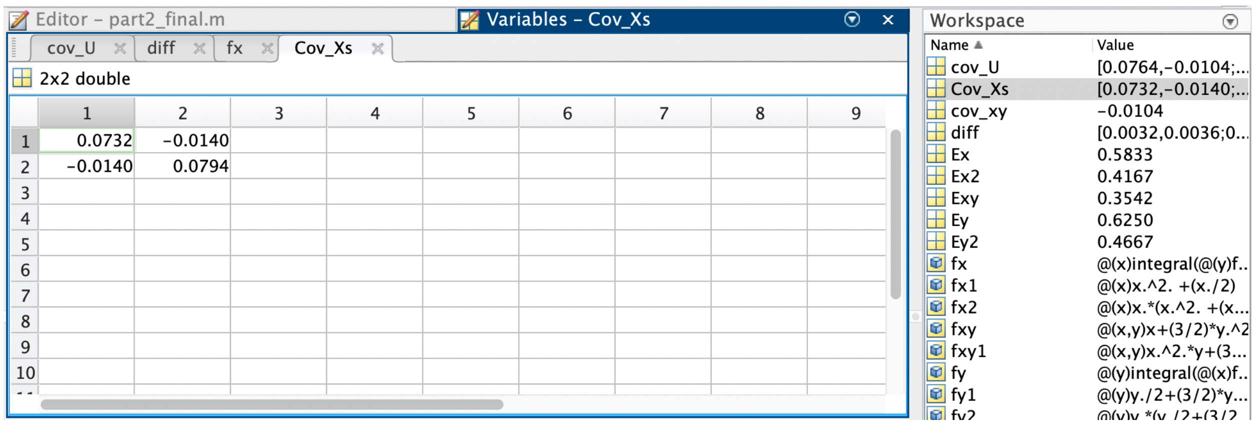
```
clear all;

fxy = @(x,y) x+(3/2)*y.^2;
% marginal pdf
fx = @(x)integral(@(y)fxy,0,1);
fy = @(y)integral(@(x)fxy,0,1);

%calculate E[X]
fx1= @(x) x.^2.+(x./2);
Ex= integral(fx1,0,1);

%calculate E[X^2]
fx2= @(x) x.*(x.^2.+(x./2));
Ex2= integral(fx2,0,1);

%calculate var[X]
Var_X=Ex2-(Ex^2);

%calculate E[Y]
fy1= @(y) y./2 +(3/2)*y.^3;
Ey= integral(fy1,0,1);

%calculate E[Y^2]
fy2= @(y) y.*(y./2 +(3/2)*y.^3);
Ey2= integral(fy2,0,1);

%calculate var[Y]
Var_y=Ey2-(Ey^2);

%calculate E[XY]
fxy1=@(x,y) x.^2.*y+(3/2)*x.*y.^3;
Exy= integral2(fxy1,0,1,0,1);
cov_xy= Exy-(Ex*Ey);

%correlation matrix of U
R_U=[Ex2 Exy; Exy Ey2];
%covariance matrix of U
cov_U=[Var_X cov_xy; cov_xy Var_y];
disp(cov_U);
%cholesky factorization and Xs generation
Upper_tri=chol(cov_U,"upper");
Lower_tri=chol(cov_U,"lower");
Xs=Upper_tri'*randn(2,500);
Cov_Xs=(cov(Xs'));
disp(Xs);

diff=cov_U-Cov_Xs;
```

Covariance of U:



	1	2
1	0.0764	-0.0104
2	-0.0104	0.0760

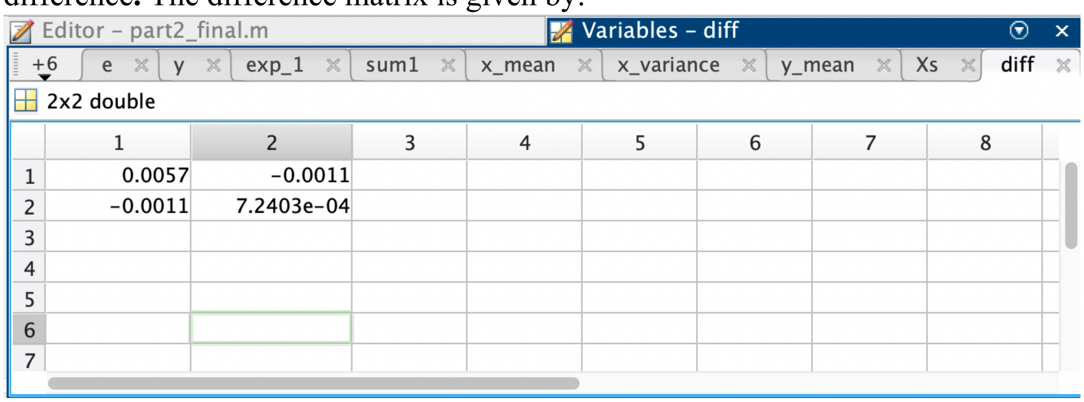
Covariance of Xs:



	1	2
1	0.0732	-0.0140
2	-0.0140	0.0794

4)

By comparing the covariance matrix of U and Xs, we can see that both are almost similar with negligible variations. When we calculate the difference matrix for these two, we get very minute difference**.** The difference matrix is given by:



	1	2
1	0.0057	-0.0011
2	-0.0011	7.2403e-04

Now taking the sample size as 2000: We have:

Cov_Xs =

0.0752	-0.0079
-0.0079	0.0789

```
>> cov_U

cov_U =

    0.0764   -0.0104
   -0.0104    0.0760
```

We can clearly see that as the sample size is increased, the covariance matrix of U and Xs are even more similar with very minute difference. Thus, by increasing the sample number, we can improve the covariance matrices based on requirements.

<u>Program files for Part2:</u>
Part2_final.m