Matrix Multiplication

$$\boxed{A_{r1\,c1} \times B_{r2\,c2} = Res_{r1 \times c2}}$$

$A_{3\times4}$ $B_{4\times2}$

$$A: \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 2 & -1 \\ 1 & 2 & -2 & 0 \end{bmatrix} \times B: \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \boxed{2} & 3 \\ -3 & 0 \\ 3 & 0 \end{bmatrix}_{3\times2} \end{array}$$

$\underline{\underline{C}}$

C[0, 0] = Multiply 0th row of A with 0th col in B.

$$\begin{bmatrix} 1 & 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

C[0, 1] = Multiply 0th row of A with 1st col in B

$$A_{r1c1} * B_{r2c2} \Rightarrow C_{r1c2}$$

$$Res[i][j] = (\underbrace{i^{th} \text{ row in } A}_{c1}) * (\underbrace{j^{th} \text{ col in } B}_{r2})$$

$c1 \longleftrightarrow r2$

$$\underline{\underline{c_1 = r_2}}$$

# of iterations to find Res[i, j] = $\underline{\underline{c_1}}$ (OR r2)

Total # of iterations to multiply A & B =

$(r_1 \times c_2) \times c_1$ OR $(r_1 \times c_2) \times r_2$

$\Rightarrow A_{r_1 c_1} \times B_{r_2 c_2} \Rightarrow \text{Cost} = r_1 \times c_1 \times c_2$

$A_{l \times m} * B_{m \times n} \Rightarrow C_{l \times n}$

there are l * n values and for each value m iterations are there so total is l * n * m

$\text{Cost} = l \times m \times n$

Ex $\underset{3\times5}{M_1} \quad \underset{5\times7}{M_2} \quad \underset{7\times4}{M_3} \Rightarrow M_{3\times4}$

I $\underbrace{[\underset{3\times5}{M_1} \; \underset{5\times7}{M_2}]}_{3\times7} \times \underset{7\times4}{M_3}$

$\text{Cost} = 105 + 84 = 189$

II $\underset{3\times5}{M_1} \times \underbrace{[\underset{5\times7}{M_2} \times \underset{7\times4}{M_3}]}_{5\times4}$

$\text{Cost} = 140 + 60 = 200$

Q: Given N matrices, find the min cost to multiply these N matrices.

N=4

M_1 M_2 M_3 M_4

$a_1 b_1$ $a_2 b_2$ $a_3 b_3$ $a_4 b_4$

a_1 b_1 b_2 b_3 b_4 $\Rightarrow$

N matrices $\Rightarrow$ dimensions [N+1]

N=5 $\Rightarrow$ arr[6]

	0	1	2	3	4	5
Arr:	2	3	4	3	4	5

M_1 : A[0] x A[1]

M_2 : A[1] x A[2]

M_3 : A[2] x A[3]

M_4 : A[3] x A[4]

M_5 : A[4] x A[5]

Obs 1:

$$\boxed{M_i \Rightarrow A[i-1] \times A[i]}$$

$$\boxed{M_x \Rightarrow V[x-1] \times V[x]}$$

$$\underline{\underline{V}} : \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$$

$$V : [\ 2 \quad 3 \quad 4 \quad 3 \quad 4 \quad 5\ ]$$

	Res. dimension
Multiply matrices from 1 to 3	$V[0] \times V[3]$
M_1: $V[0] \times V[1]$, M_2: $V[1] \times V[2]$, M_3: $V[2] \times V[3]$	
Multiply matrices from 2 to 4	$V[1] \times V[4]$
M_2: $V[1] \times V[2]$, M_3: $V[2] \times V[3]$, M_4: $V[3] \times V[4]$	

Obs 2:

$$\boxed{\text{Multiply matrices from } i \text{ to } j \Rightarrow V[i-1] \times V[j]}$$

Dimensions of the resultant matrix if we multiply matrices from $\underline{\underline{i \text{ to } j}}$.

$\underline{\underline{N}} \Rightarrow M_1 \quad M_2 \quad M_3 \quad M_4 \quad \ldots \quad \underline{\underline{M_N}}$

$\Rightarrow$ Matrix Chain Multiplication ($\underline{\underline{MCM}}$)

$\underline{\underline{N=5}}$

$M_1 \quad M_2 \quad M_3 \quad M_4 \quad M_5$

mcm(1-5) : Min cost to multiply matrices 1 to 5.

$\text{mcm}(\overset{i}{1} - \overset{j}{5})$

$\boxed{\text{mcm}(1-1)}$ + mcm(2-5)
=0

mcm(1-3) + mcm(4-5)

$\text{mcm}(\overset{i}{1} - \overset{i+1}{2}) + \boxed{\text{mcm}(\overset{i+2}{3} - \overset{j}{5})}$

mcm(1-4) + m(5-5)

mcm(3-3) + mcm(4-5)

mcm(3-4) + mcm(5-5)

⇒ Optimal substructure
⇒ Overlapping subproblems } ✓

* dp[i, j] ⇒ Min cost to multiply matrices from $\underline{\underline{i \text{ to } j}}$.

$M_i \quad M_{i+1} \quad M_{i+2} - - - - - M_{j-1} \quad M_j$

$$dp[i,j] = \underline{\underline{\text{Min}}} \begin{cases} dp[i, \overset{(k)}{i}] + dp[\overset{k+1}{i+1}, j] + c_1 \rightarrow V[i-1] \times V[k] \times V[j] \\ dp[i, i+1] + dp[i+2, j] + c_2 \\ dp[i, i+2] + dp[i+3, j] + c_3 \\ \vdots \\ dp[i, j-1] + dp[j, j] \end{cases}$$

```
for( k = i ; k<j ; k++ ) {
    ans = min(ans, dp[i,k] + dp[k+1, j] + V[i-1] × V[k] × V[j])
}
```

dimension of res

$M_i \cdots M_k \Rightarrow V[i-1] \times V[k]$

$M_{k+1} \cdots M_j \Rightarrow V[k] \times V[j]$

$$\underbrace{[M_i \cdots M_k]}_{A} \; \underbrace{[M_{k+1} \cdots M_j]}_{B}$$

A: $V[i-1] \times V[k]$ B: $V[k] \times V[j]$

Cost $= V[i-1] \times V[k] \times V[j]$

DP Expression

$$dp[i,j] = \min \left[\forall_{k=i}^{k<j} \; dp(i,k) + dp(k+1,j) + v[i-1] \times v[k] \times v[j] \right]$$

$i == j \Rightarrow$ for loop won't even execute once.

$\Downarrow$

Base Case

$$dp(i,i) = 0$$

dp table

final ans : $dp[1][N]$

$\Rightarrow$ int $dp[N+1][N+1]$

$N=4 \Rightarrow M_1, M_2, M_3, M_4$

	0	1	2	3	4
0	0	0	0	0	0
1	0	0			0
2	0	0	0		
3	0	0	0	0	
4	0	0	0	0	0

$i > j$

```
int minCost ( int N, int v[N+1] ){
    int dp[N+1][N+1] = {-1};
    return mcm ( 1, N, dp, v );
}
```

↳ min Cost to multiply matrices from 1 to N ($i = 1$, $j = N$)

```
int mcm(int i, int j, int dp[][], int v[]) {
    if(i==j) return 0;

    if (dp[i,j] == -1) {
        cost = INT_MAX;
        for( k=i; k<j ; k++){
            a = mcm(i, k, dp, v);
            b = mcm( k+1, j, dp , v);
            c = v[i-1] x v[k] x v[j]
            cost = min (cost, a+b+c);
        }
        dp[i][j] = cost;
    }
    return dp[i][j];
}
```

⇒ Recursion + Memoization.

TC : # of dp states * TC of each dp state.

$\underbrace{\text{\# of dp states}}_{N^2} * \underbrace{\text{TC of each dp state}}_{N}$

TC : $\underline{\underline{O(N^3)}}$

SC : $\underline{\underline{O(N^2)}}$

$M_1 \; M_2 \; M_3 \; M_4 \; M_5 \Rightarrow (M_1)(M_2 M_3 M_4)(M_5)$

mcm(1-1) + mcm(2-4) + mm(5-5) +

1-5

(1-1), 2-5

1-2, 3-5

1-3, 4-5

1-4, 5-5

2-2, 3-5

2-3, 4-5

2-4, 5-5