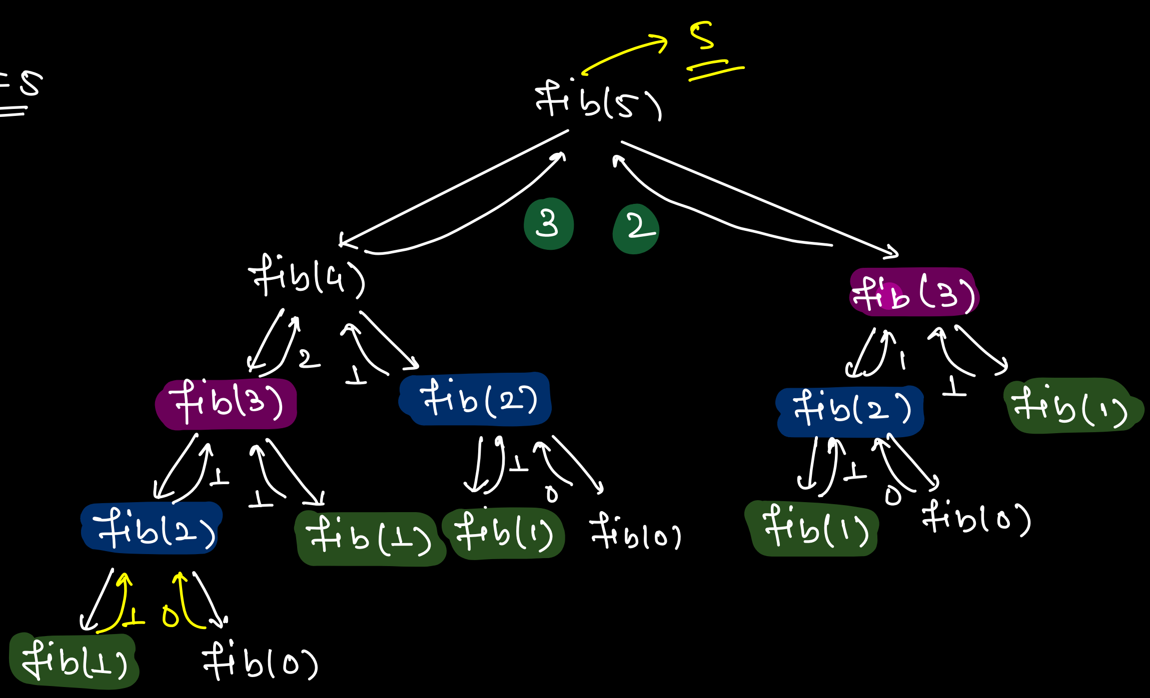
Dynamic Programming

fibonacci Series

N →	0	1	2	3	4	5	6	7	...
	0	1	1	2	3	5	8	13	...

```
int fib(int N) {
    if(N<=1) return N;
    return fib(N-1) + fib(N-2);
}
```

N=5



TC : $O(2^N)$

$$T(N) = T(N-1) + \underbrace{T(N-2)}_{\simeq T(N-1)} + 1$$

$$= 2T(N-1) + 1 \Rightarrow 2^N$$

TC of recursive funⁿ =

$\underbrace{\text{No. of fun}^n\text{ calls}}_{\approx 2^N} * \underbrace{\text{TC of each fun}^n\text{ call}}_{1}$.

$\approx 2^N * 1$

$\Rightarrow \underline{\underline{O(2^N)}}$

⇒

1) Solve a problem using subproblems. ⇒ Optimal Substructure
2) Solving subproblems more than once.
   ↳ Overlapping Subproblems.

D.P

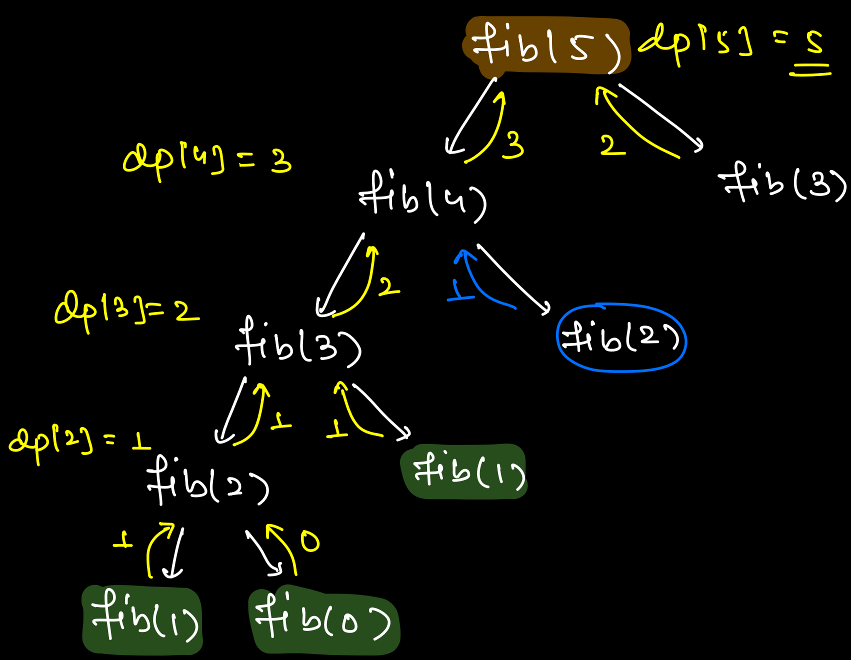
↳ Solving a subproblem exactly once.

```
int dp[N+1] = {-1};
int fib(int N) {
    if (N<=1) dp[N]=N, return N;
    if( dp[N] == -1) { // Not calculated yet
        dp[N] = fib(N-1) + fib(N-2);
    }
    return dp[N];
}
```

TC: $N \times 1 \Rightarrow O(N)$

SC: $O(\underset{\text{dp[]}}{N} + \underset{\text{Stack}}{N})$

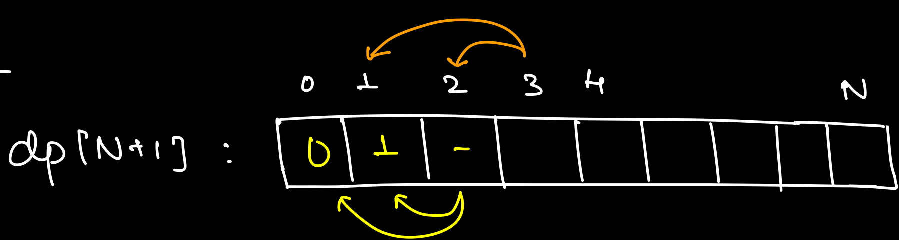
0	1	2	3	4	5
~~-1~~	~~-1~~	~~-1~~	~~-1~~	~~-1~~	~~-1~~
0	1	1	2	3	5



⇒ Top Down DP. fib(5) → fib(4) → fib(3) → - - -

⇒ Recursion + Memory : Memoization DP

#



dp[N] = dp[N-1] + dp[N-2] ⇒ Dp Expression.

```
dp[0] = 0, dp[1] = 1;
for( i = 2 ; i <= N ; i++ ) {
        dp[i] = dp[i-1] + dp[i-2]
}
return dp[N];
```

TC : O(N)

SC : O(N)

dp[]

⇒ Bottom Up DP

⇒ Iterative + Table {dp[]/Memory} ⇒ Tabulation DP

Steps.

1. Optimal Substructure.
2. Overlapping subproblems.
   → Same subproblem is repeating lot of times.
3. dp State : What dp table should contain.
4. dp Expression : How to calculate dp state using smaller subproblems.
5. Base Cases : Values for which dp expression won't work.
6. Code.
7. TC & SC analysis.
8. Optimization → TC, SC

```
int fib(int N) {
    int a = 0, b = 1;
    int c;
    fib( i=2; i<=N; i++) {
        c = a+b;
        a = b;
        b = c;
    }
    return c;
}
```

a	b	c
0	1	1
1	1	2
1	2	3
2	3	5
⋮	⋮	

$$\boxed{\begin{array}{l} TC : O(N) \\ SC : O(1) \end{array}}$$

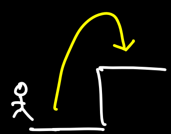
Q.* N Stairs.

Amazon / SAP

Given N stairs, in how many ways we can go from $0^{th} \longrightarrow N^{th}$

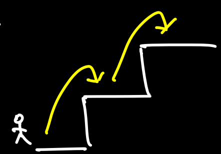
Note: from i^{th} floor $\longrightarrow$ $i+1$ OR $i+2$ floors.

$N=1$

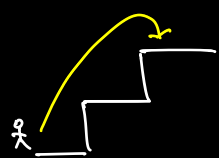


$\Rightarrow 1$

$N=2$

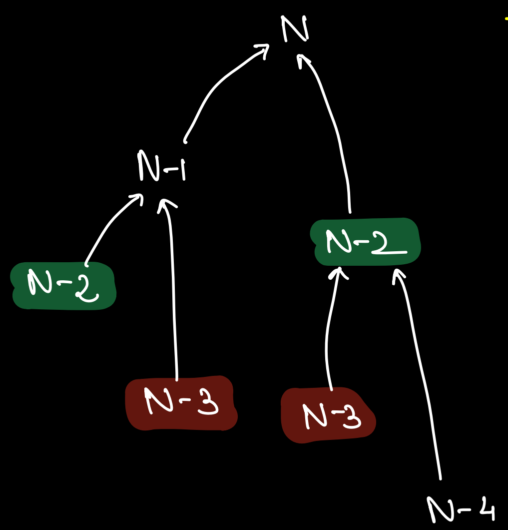


$\Rightarrow$ 2 ways.



$N=3 \Rightarrow \{1\ 1\ 1\}$
$\{1\ 2\}$
$\{2, 1\}$
$\Rightarrow$ 3 ways.

$N=4 \Rightarrow \{1\ 1\ 1\ 1\}$
$\{1\ 1\ 2\}$
$\{1\ 2\ 1\}$
$\{2\ 1\ 1\}$
$\{2\ 2\}$
$\Rightarrow$ 5 ways.

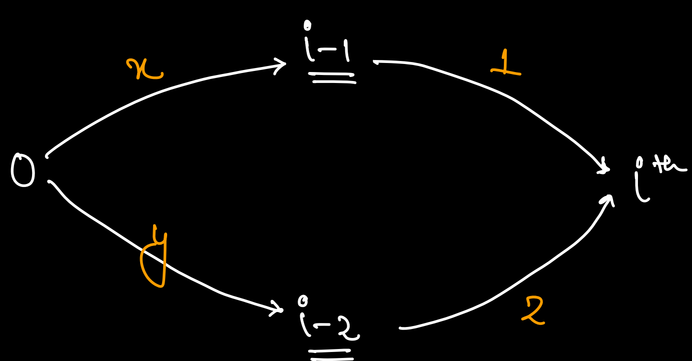


→ Optimal substructure
→ Overlapping subproblems. } DP

⇒ Dp State

dp[i]: # of ways to reach i^{th} floor.

⇒ Dp Expression



$$\boxed{dp[i] = x + 2y}$$

$dp[i] = dp[i-1] + 2\, dp[i-2]$

$dp[1] = 1, \; dp[2] = 2$

$dp[3] = 2 + 2 \times 1 = 4$

X

$\underline{\underline{N=4}}$

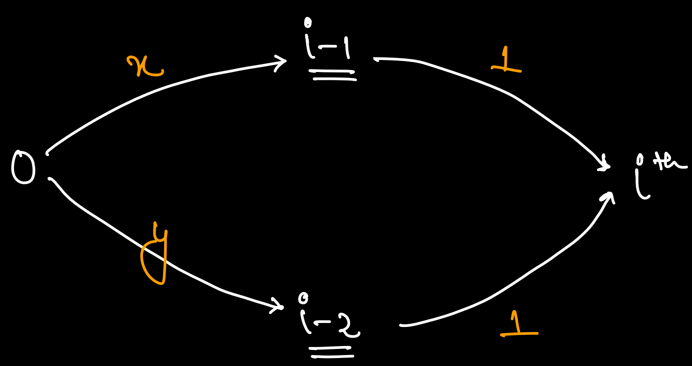
$\{1\ 1\ 1\ 1\} \Rightarrow 0 \to 1 \to 2 \to 3 \to 4$

$\{1\ 1\ 2\} \Rightarrow 0 \to 1 \to 2 \longrightarrow 4$

$\{1\ 2\ 1\} \Rightarrow 0 \to 1 \longrightarrow 3 \longrightarrow 4$

$\{2\ 1\ 1\} \Rightarrow 0 \longrightarrow 2 \to 3 \to 4$

$\{2\ 2\} \Rightarrow 0 \longrightarrow 2 \longrightarrow 4$



$$dp[i] = x * 1 + y * 1$$

$$\boxed{dp[i] = dp[i-1] + dp[i-2]}$$

$dp[1] = 1,\ dp[2] = 2$

$dp[3] = dp[1] + dp[2]$

$= \underline{\underline{3}}$

Base Case : $i=1$, $i=0$

$dp[2] = dp[1] + \underline{dp[0]}$

$2 = 1 + dp[0]$

$\underbrace{dp[0]}_{\downarrow} = 1$ ✓

# of ways to reach ground floor.

TC: O(N)

SC: O(N) ⟶ O(1)

Q. 2 faced dice $\begin{cases} \to 1 \\ \to 2 \end{cases}$

We can roll this dice as many times as we want.

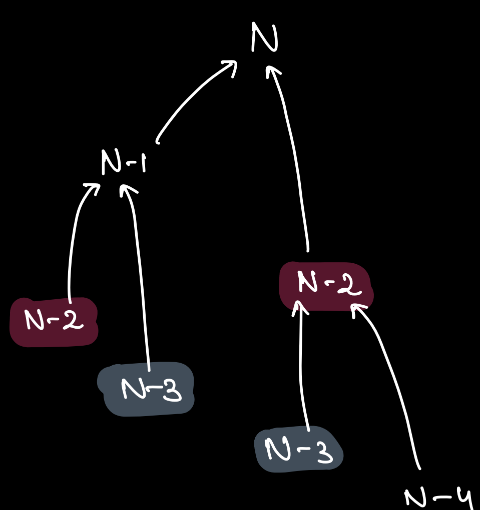
# of ways to make sum = N.

$N = 1 \Rightarrow 1$ way

$N = 2 \Rightarrow \{1\ 1\} \Rightarrow 2$ ways.
$\{2\}$

$N = 3 \Rightarrow \{1\ 1\ 1\}$
$\{1\ 2\}$ $\Rightarrow 3$ ways.
$\{2\ 1\}$

$N = 4 \Rightarrow \left.\begin{matrix} \{1\ 1\ 1\ 1\} \\ \{1\ 1\ 2\} \\ \{1\ 2\ 1\} \\ \{2\ 1\ 1\} \\ \{2\ 2\} \end{matrix}\right\}$ 5 ways.



$\begin{cases} \to \text{Optimal substructure} \\ \to \text{Overlapping subproblems.} \end{cases}$

$\Rightarrow$ DP

dp[i] : # of ways to make sum = i.

dp[i] = dp[i-1] + dp[i-2].

Q. 6 faced dice → 1, 2, 3, 4, 5, 6

We can roll this dice as many times as we want

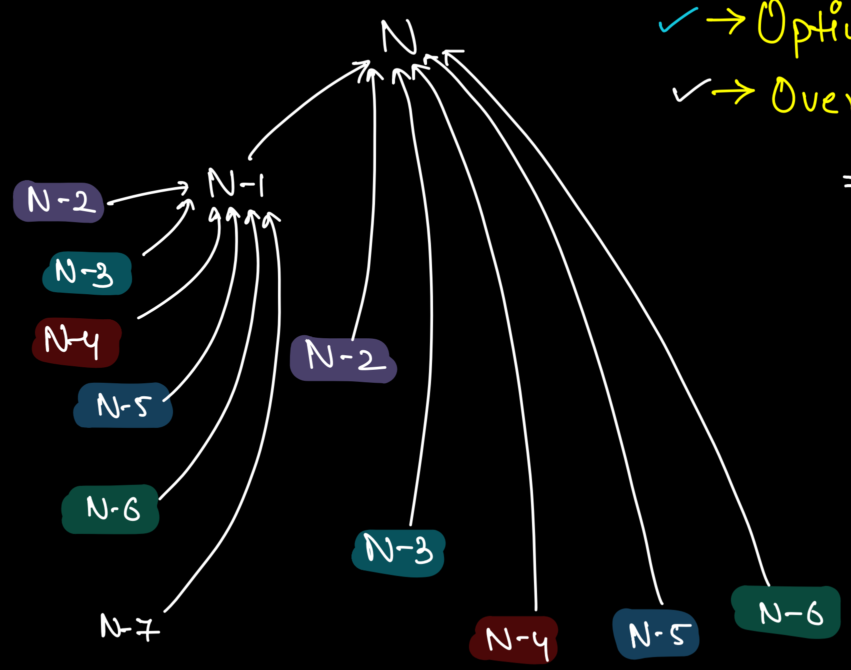
# of ways to make sum = N.

N = 1 ⇒ 1

N = 2 ⇒ {1 1} ⇒ 2 ways.
{2}

N = 3 ⇒ {1 1 1}
{1 2} ⇒ 4 ways.
{2 1}
{3}

N = 4 ⇒ {1 1 1 1}
{1 1 2}
{1 2 1} ⇒ 8 ways.
{2 1 1}
{2 2}
{1 3}
{3 1}
{4}



✓ → Optimal substructure

✓ → Overlapping subproblems.

⇒ DP

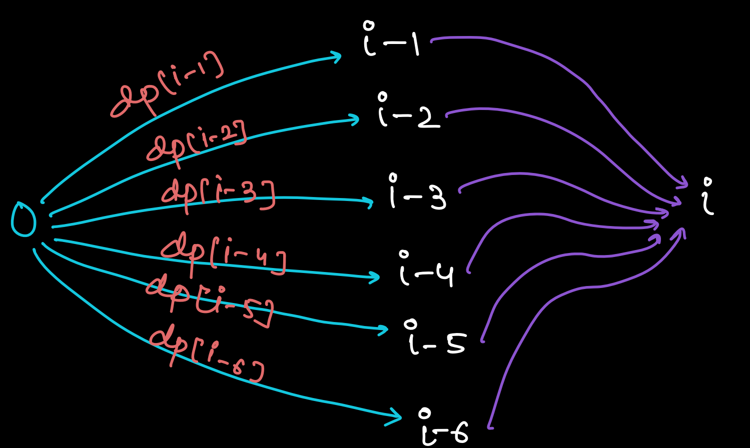
dp State.

dp[i] : # of ways to make sum = i.

dp table.

dp[N+1]

dp Expression.



$$dp[i] = dp[i-1] + dp[i-2] + dp[i-3] + dp[i-4] + dp[i-5] + dp[i-6]$$

Base Case

$i = 0, 1, 2, 3, 4, 5$

$dp[0] = 1$ $\qquad dp[3] = 4$

$dp[1] = 1$ $\qquad dp[4] = 8$

$dp[2] = 2$ $\qquad dp[5] = 16$

$\Rightarrow$

$$dp[i] = \sum_{\substack{j=1 \\ i>=j}} dp[i-j]$$

$$dp[1] = dp[1-1]$$

$$\boxed{1 = dp[0]}$$

$$dp[2] = dp[2-1] + dp[2-2]$$

$$= dp[1] + dp[0]$$

$$= 1 + 1 = 2$$

$$dp[3] = dp[3-1] + dp[3-2] + dp[3-3]$$

$$= 2 + 1 + 1$$

$$= 4$$

Code :

```
int dp[N+1];
dp[0] = 1;
for ( i = 1 ; i <= N; i++) {
    sum = 0
    for ( j = 1; j <= i && j <= 6; j++) {
        sum += dp[i-j];
    }
    dp[i] = sum;
}
return dp[N];
```

$i-j >= 0$

6

TC: # of dp states * No. of Iterations of 1 dp state

: N * 6

: O(N)

SC: O(N)

Can we optimise SC?

Keep only 7 Variables] TODO

→ Bottom Up DP

$$ways(N) = ways(N-1) + ways(N-2) + ways(N-3) + ways(N-4) + ways(N-5) + ways(N-6)$$

{ Smaller Subproblems. } Optimal Substructure. ✓

TODO Solve this using Memoization.

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