Notes on Lexicographic Method for Hierarchical Multiobjective Programs

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1 Variable fixing over Constraint Addition Rule of Lexicographic Method for h-MOLP

Let us consider the following linear program

$$LPP := \min\{c^{'}x \ : \ Ax = b; \ l \leq x \leq u\}.$$

It is well studied that instead of explicitly adding $l \leq x \leq u$ as constraints in LPP one should prefer handling these constraints implicitly in a fashion similar to that used by the simplex method handling the non-negative constraints $x \geq 0$. This strategy avoids increasing of problem size and working with "compact basis" of size $m \times m$ only. [0]. In our case we are avoiding addition of constraints so that basis matrix size will always be $m \times m$.

Moreover, adding constraints make the problem ill-conditioned, leading to the problem with high condition number (in CPLEX it is referred as kappa value). Let us consider a toy example with two objective vectors in the fallowing hierarchical order: Objective vector $c^1 := (-0.333333, -0.666667)$ is of highest importance and objective vector $c^2 := (1,1)$ of least importance.

$$\begin{aligned} \mathbf{LP^1} := \ & \min\left(-0.333333\, x_1 - 0.666667\, x_2\right) \\ & \text{Subject to} \\ & \quad x_1 + 2x_2 = 3 \\ & \text{Bounds} \\ & \quad 0 \leq x_1 \leq 2 \\ & \quad 0 \leq x_2 \leq 2 \end{aligned}$$

and

$$\begin{aligned} \mathbf{LP^2} := \ & \min\left(x_1 + x_2\right) \\ & \text{Subject to} \\ & \quad x_1 + 2x_2 = 3 \\ & \quad -0.333333\, x_1 - 0.666667\, x_2 = -1.000000006 \\ & \text{Bounds} \\ & \quad 0 \leq x_1 \leq 2 \\ & \quad 0 \leq x_2 \leq 2 \end{aligned}$$

and

$$
\begin{aligned}
\textbf{varfix\_LP}^{\mathbf{2}} := \ & \min\,(x_1 + x_2) \\
& \text{Subject to} \\
& x_1 + 2x_2 = 3 \\
& \text{Bounds} \\
& 0 \le x_1 \le 0 \qquad (1) \\
& 0 \le x_2 \le 2 \qquad (2)
\end{aligned}
$$

In $\mathbf{LP^2}$ we have added the constraint 1 but in **varfix_LP**$^{\mathbf{2}}$ instead of adding the constraint 1, we fix the variable x_1 to zero. The optimal objective value of $\mathbf{LP^1}$, $\mathbf{LP^2}$ and **varfix_LP**$^{\mathbf{2}}$ are -1.000000006, 1.5 and 1.5.

In $\mathbf{LP^2}$, by changing rhs of 1 from -1.000000006 to -1.0 leads to change in optimal objective value from 1.5 to 2. The condition numbers of basis after solving $\mathbf{LP^2}$ and **varfix_LP**$^{\mathbf{2}}$ are 6.7e+8 and 1.0e+0 respectively. It says that a small change in the input can result in a big change in the computed solution of the model. High kappa value can cause various problems [0] in the quality of solution such as :

1. inconsistent result when presolving and input parameters are tuned
2. inaccuracy in the computed solution that contradicts the constraints in the model, etc.

2 Notations and Conventions

For a nonnegative integer t, we define $[t] := \{1, 2, \ldots, t\}$ if $t > 0$ and $[t] := \emptyset$ if $t = 0$. The dot product of two vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ is denoted by $u^{'}v$. For an $m \times n$ real matrix $M \in \mathbb{R}^{m \times n}$ we use $M^{'}$ to denote the transpose of M. We use M_i to denote the i^{th} column of M.

3 Linear Programming for Bounded Variables

Let us consider the following linear programming problem such that each decision variable is bounded below by a finite number:

$$LPP := \min_{x \in \mathbb{R}^n} \{c^{'}x \ : \ x \in S\}, \tag{3}$$

where

$$S := \{x \in \mathbb{R}^n \ : \ Ax = b, \ l \le x \le u\}. \tag{4}$$

Here, we assume that the real matrix $A \in \mathbb{R}^{m \times n}$ is of rank m. Moreover, $l_i < u_i$ for each $i \in [n]$.

Definition 3.1. *Let $S := \{x \in \mathbb{R}^n \ : \ Ax = b, \ l \le x \le u\}$ be a polyhedron as described above, and let $x^* \in \mathbb{R}^n$.*

(a) x^* *is a basic solution if:*

(i) *all equality constraints are active*

(ii) *out of the constraints that are active at* x^**, there are* n *of them that are linearly independent.*

(b) *if a basic solution satisfies all of the constraints, then it is called a basic feasible solution*

Notice that $x^* \in S$ is a basic feasible solution if and only if it is an extreme point of S.

Theorem 3.2. *Let* $S := \{x \in \mathbb{R}^n \ : \ Ax = b, \ l \leq x \leq u\}$ *be a polyhedron as described in Equation 4.* $x^* \in \mathbb{R}^n$ *is a basic solution if and only if* $Ax^* = b$*, and there exist an index set* $I_B \subseteq [n]$ *of cardinality* m *such that:*

(a) *The columns* $A_i, \ i \in I_B$ *are linearly independent.*

(b) *if* $i \notin I_B$*, then either* $x_i^* = l_i$ *or* $x_i^* = u_i$*.*

If x^* is a basic solution of S, the variables $x_i^*, \ i \in I_B$ are called the basic variables, and the remaining variables are called the nonbasic variables. The columns $A_i, \ i \in I_B$ are called the basic columns, and the remaining columns are called the nonbasic columns. The m basic columns written adjacent to each other, form a matrix, and is called the basis matrix B. Let $I_{N_1} := \{i \in [n] \ : \ x_i^* = l_i\}$ be the index set associated with the nonbasic variables at their lower bounds, and $I_{N_2} := \{i \in [n] \ : \ x_i^* = u_i\}$ be the index set associated with the nonbasic variables at their upper bounds. The columns $A_i, \ i \in I_{N_1}$ are the nonbasic columns associated with the index set I_{N_1}, and the columns $A_i, \ i \in I_{N_2}$ are the nonbasic columns associated with the index set I_{N_2}. The matrix associated with the nonbasic columns $A_i, \ i \in I_{N_1}$ is denoted by N_1, and the matrix associated with the nonbasic columns $A_i, \ i \in I_{N_2}$ is denoted by N_2.

Let x^* be a basic solution of S, and let B be an associated basis matrix. By representing the index set $[n]$ as $I_B \cup I_{N_1} \cup I_{N_2}$, one can partition the matrix A into $[B, N_1, N_2]$, the decision variable $x^{'}$ into $[x_B^{'}, \ x_{N_1}^{'}, \ x_{N_2}^{'}]$, the basic solution $x^{*^{'}}$ into $[x_B^{*^{'}}, \ x_{N_1}^{*^{'}}, \ x_{N_2}^{*^{'}}]$, and the cost vector $c^{'}$ into $[c_B^{'}, \ c_{N_1}^{'}, \ c_{N_2}^{'}]$.

Definition 3.3. *Let* x^* *be a basic solution of* S*. Let* B *be an associated basis matrix, and let* c_B *be the vector of costs associated with the basic variables. The reduced cost* $\bar{c}_i$ *for each* $i \in [n]$ *is defined as:*

$$\bar{c}_i := c_i - c_B^{'} B^{-1} A_i$$

Theorem 3.4. *Consider the linear programming problem as presented in Equation 3. Let* x^* *be a basic feasible solution of* S*. Let* B *be an associated basis matrix, and let* $\bar{c}$ *be the associated vector of reduced costs. Assume that* $\bar{c}_i \geq 0$ *for all* $i \in N_1$ *and* $\bar{c}_i \leq 0$ *for all* $i \in N_2$*. Then,* x^* *is an optimal solution.*

Proof. We will establish that $c^{'}x^* \le c^{'}y$ for all $y \in S$. Let $y \in S$, and let $d := y - x^*$. From $Ax^* = Ay = b$ we have $Ad = 0$. As $Ad = Bd_B + \sum_{i \in N_1 \cup N_2} A_i d_i$, we have

$$d_B = -\sum_{i \in N_1 \cup N_2} B^{-1} A_i d_i.$$

Now,

$$\begin{aligned} & c^{'}d \\ &= c^{'}_B d_B + \sum_{i \in N_1 \cup N_2} c_i d_i \\ &= c^{'}_B \Big(-\sum_{i \in N_1 \cup N_2} B^{-1} A_i d_i\Big) + \sum_{i \in N_1 \cup N_2} c_i d_i \\ &= \sum_{i \in N_1 \cup N_2} (c_i - c^{'}_B B^{-1} A_i) d_i \\ &= \sum_{i \in N_1} \bar{c}_i d_i + \sum_{i \in N_2} \bar{c}_i d_i \end{aligned}$$

For $i \in N_1$, we have $d_i = y_i - x^*_i = y_i - l_i \ge 0$. This implies that $\bar{c}_i d_i \ge 0$. For $i \in N_2$, we have $d_i = y_i - x^*_i = y_i - u_i \le 0$, which implies that $\bar{c}_i d_i \ge 0$. As a result, $c^{'}d \ge 0$, completing the proof. □

Definition 3.5. *A basis matrix is said to be optimal if $\bar{c}_i \ge 0$ for all $i \in N_1$ and $\bar{c}_i \le 0$ for all $i \in N_2$.*

4 Variable Fixing

Theorem 4.1. *Let $LP := \min_{x \in \mathbb{R}^n} \{c^{'}x \;:\; Ax = b,\ l \le x \le u\}$ be a linear programming problem, where $A \in \mathbb{R}^{m \times n}$ is a real matrix of rank m, and $l_i < u_i$ for each $i \in [n]$. We assume that the feasible set $S := \{x \in \mathbb{R}^n \;:\; Ax = b,\ l \le x \le u\}$ is non-empty. Let B be an optimal basis for problem LP. Let x^* and $\bar{c} := c^{'}_B B^{-1} A$ be the optimal solution and the reduced cost vector associated with B. Then the following are true.*

(i) $F := \{x \in S \;:\; c^{'}x = c^{'}x^*\}$, the set of optimal solutions of LP, is a face of S.

(ii) F can be represented as

$$S \cap \{x \in \mathbb{R}^n \;:\; x_i = l_i \ \ \forall i \in [n] \;:\; \bar{c}_i > 0\} \cap \{x \in \mathbb{R}^n \;:\; x_i = u_i \ \ \forall i \in [n] \;:\; \bar{c}_i < 0\}$$

(iii) In particular, $F = \{x \in \mathbb{R}^n \;:\; Ax = b,\ \tilde{l} \le x \le \tilde{u}\}$,

where $\tilde{l}_i,\ \tilde{u}_i$ for $i \in [n]$ is defined as:

$$\tilde{l}_i := \begin{cases} u_i, & \text{if } \bar{c}_i < 0 \\ l_i, & \text{otherwise,} \end{cases}$$

and

$$\tilde{u}_i := \begin{cases} l_i, & \text{if } \bar{c}_i > 0 \\ u_i, & \text{otherwise.} \end{cases}$$

Proof. (i) Note that x^* is an optimal solution of LP. This means that

$$F = \text{ arg min }_{x\in S} \; c^{'}x$$

In other words, F is the set of optimal solutions of LP. Moreover, it is straightforward to verify that F is a face of S.

(ii) As B is an optimal basis for problem LP and x^* is the optimal solution of LP associated with B, we have $x^*_{N_1} = l_{N_1},\; x^*_{N_2} = u_{N_2}$, and $x^*_B = B^{-1}b - B^{-1}N_1 l_{N_1} - B^{-1}N_2 u_{N_2}$. Now,

$$c^{'}x^* = c^{'}_B x^*_B + c^{'}_{N_1} x^*_{N_1} + c^{'}_{N_2} x^*_{N_2},$$

$$\Rightarrow c^{'}x^* = c^{'}_B\Big(B^{-1}b - B^{-1}N_1 l_{N_1} - B^{-1}N_2 u_{N_2}\Big) + c^{'}_{N_1} l_{N_1} + c^{'}_{N_2} u_{N_2}$$

$$\Rightarrow c^{'}x^* = c^{'}_B B^{-1}b + \bar{c}^{'}_{N_1} l_{N_1} + \bar{c}^{'}_{N_2} u_{N_2}$$

Let

$$R := \{x \in \mathbb{R}^n \;:\; l \le x \le u\}$$

$$E := \{x \in \mathbb{R}^n \;:\; Ax = b,\; c^{'}x = cx^*\}.$$

So,

$$F = E \cap R$$

By applying a suitable row operation on E we have

$$E = \{x \in \mathbb{R}^n \;:\; Ax = b,\; (c^{'} - c^{'}_B B^{-1}A)x = c^{'}x^* - c^{'}_B B^{-1}b\}.$$

Substituting the value of $c^{'}x^*$ we have

$$E = \{x \in \mathbb{R}^n \;:\; Ax = b,\; \bar{c}^{'}x = \bar{c}^{'}_{N_1} l_{N_1} + \bar{c}^{'}_{N_2} u_{N_2}\}.$$

Recall that $\bar{c}_B = 0_{m\times 1}$. Hence,

$$E = \{x \in \mathbb{R}^n \;:\; Ax = b,\; \bar{c}^{'}_{N_1} x_{N_1} + \bar{c}^{'}_{N_2} x_{N_2} = \bar{c}^{'}_{N_1} l_{N_1} + \bar{c}^{'}_{N_2} u_{N_2}\}.$$

As $F = E \cap R$, we have

$$F = \{x \in S \;:\; \bar{c}^{'}_{N_1} x_{N_1} + \bar{c}^{'}_{N_2} x_{N_2} = \bar{c}^{'}_{N_1} l_{N_1} + \bar{c}^{'}_{N_2} u_{N_2}\} \tag{5}$$

Now, we will complete Theorem 4.1(ii) by using six logical equivalent steps. Let $y \in F$. The first equivalence follows from Equation 5. Recall that $\bar{c}_i \ge 0$ for all $i \in I_{N_1}$ and $\bar{c}_i \le 0$ for all $i \in I_{N_2}$. This implies the second equivalence. As $\bar{c}_i = 0$ for $i \in I_B$, we have $\{i \in N_1 \;:\; \bar{c}_i > 0\} = \{i \in [n] \;:\; \bar{c}_i > 0\}$ and $\{i \in N_2 \;:\; \bar{c}_i < 0\} =$

$\{i \in [n] : \bar{c}_i < 0\}$. As a result, the third equivalence follows. Let $\alpha_i := y_i - l_i$ for all $i \in [n] : \bar{c}_i > 0$, and $\beta_i := y_i - u_i$ for all $i \in [n] : \bar{c}_i > 0$. The fourth equivalence follows from the definition of α_i and β_i. After simplification, we obtain the fifth equivalence. Recall that $l_i \leq y_i \leq u_i$ for all $i \in [n]$. This means that $\alpha_i \geq 0$ for all $i \in [n] : \bar{c}_i > 0$, and $\beta_i \leq 0$ for all $i \in [n] : \bar{c}_i < 0$.

$$y \in F$$

$$\Leftrightarrow y \in S \text{ and } \sum_{i \in I_{N_1}} \bar{c}_i y_i + \sum_{i \in I_{N_2}} \bar{c}_i y_i = \sum_{i \in I_{N_1}} \bar{c}_i l_i + \sum_{i \in I_{N_2}} \bar{c}_i u_i$$

$$\Leftrightarrow y \in S \text{ and } \sum_{i \in I_{N_1} : \bar{c}_i > 0} \bar{c}_i y_i + \sum_{i \in I_{N_2} : \bar{c}_i < 0} \bar{c}_i y_i = \sum_{i \in I_{N_1} : \bar{c}_i > 0} \bar{c}_i l_i + \sum_{i \in I_{N_2} : \bar{c}_i < 0} \bar{c}_i u_i$$

$$\Leftrightarrow y \in S \text{ and } \sum_{i \in [n] : \bar{c}_i > 0} \bar{c}_i y_i + \sum_{i \in [n] : \bar{c}_i < 0} \bar{c}_i y_i = \sum_{i \in [n] : \bar{c}_i > 0} \bar{c}_i l_i + \sum_{i \in [n] : \bar{c}_i < 0} \bar{c}_i u_i$$

$$\Leftrightarrow y \in S \text{ and } \sum_{i \in [n] : \bar{c}_i > 0} \bar{c}_i (l_i + \alpha_i) + \sum_{i \in [n] : \bar{c}_i < 0} \bar{c}_i (u_i + \beta_i) = \sum_{i \in [n] : \bar{c}_i > 0} \bar{c}_i l_i + \sum_{i \in [n] : \bar{c}_i < 0} \bar{c}_i u_i$$

$$\Leftrightarrow y \in S \text{ and } \sum_{i \in [n] : \bar{c}_i > 0} \bar{c}_i \alpha_i + \sum_{i \in [n] : \bar{c}_i < 0} \bar{c}_i \beta_i = 0$$

$$\Leftrightarrow y \in S \cap \{x \in \mathbb{R}^n : x_i = l_i \ \forall i \in [n] : \bar{c}_i > 0\} \cap \{x \in \mathbb{R}^n : x_i = u_i \ \forall i \in [n] : \bar{c}_i < 0\}$$

(iii) From the definition of $\tilde{l}$ and $\tilde{u}$ Theorem 4.1(iii) is true.

□

References

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