Divide-and-Conquer: Sorting Problem

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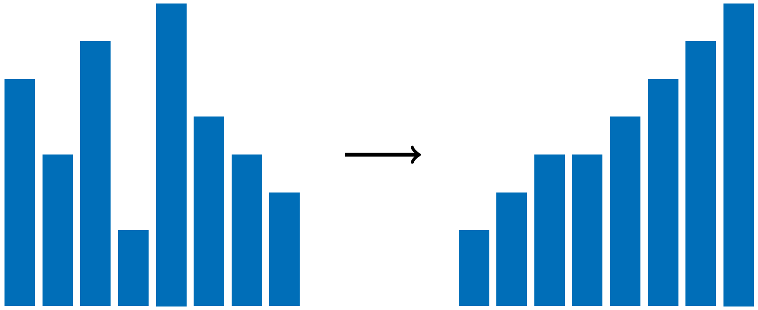
Algorithmic Design and Techniques
Algorithms and Data Structures

Outline

Sorting Problem



Sorting

Input: Sequence $A[1 \dots n]$.

Output: Permutation $A'[1 \dots n]$ of $A[1 \dots n]$ in non-decreasing order.

Why Sorting?

- Sorting data is an important step of many efficient algorithms.

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- Sorting data is an important step of many efficient algorithms.
- Sorted data allows for more efficient queries.

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Selection sort: example

8	4	2	5	2

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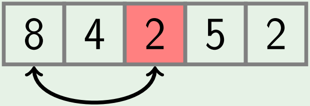
- Find a minimum by scanning the array

Selection sort: example



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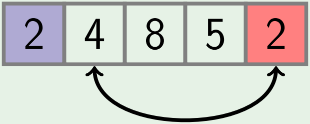

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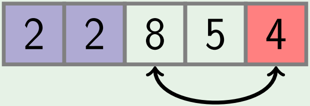

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$\texttt{SelectionSort}(A[1 \ldots n])$

$\texttt{for}\ i\ \texttt{from}\ 1\ \texttt{to}\ n\texttt{:}$
$\quad minIndex \leftarrow i$
$\quad \texttt{for}\ j\ \texttt{from}\ i+1\ \texttt{to}\ n\texttt{:}$
$\quad\quad \texttt{if}\ A[j] < A[minIndex]\texttt{:}$
$\quad\quad\quad minIndex \leftarrow j$
$\quad \{A[minIndex] = \min A[i \ldots n]\}$
$\quad \texttt{swap}(A[i], A[minIndex])$
$\quad \{A[1 \ldots i]\ \texttt{is in final position}\}$

$\texttt{SelectionSort}(A[1 \dots n])$

```
for i from 1 to n:
    minIndex ← i
    for j from i + 1 to n:
        if A[j] < A[minIndex]:
            minIndex ← j
    {A[minIndex] = min A[i ... n]}
    swap(A[i], A[minIndex])
    {A[1 ... i] is in final position}
```

Online visualization: selection sort

Lemma

The running time of `SelectionSort`$(A[1 \ldots n])$ is $O(n^2)$.

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Proof

n iterations of outer loop, at most n iterations of inner loop. $\square$

Too Pessimistic Estimate?

- As i grows, the number of iterations of the inner loop decreases: j iterates from $i + 1$ to n.

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- As i grows, the number of iterations of the inner loop decreases: j iterates from $i+1$ to n.
- A more accurate estimate for the total number of iterations of the inner loop is $(n-1)+(n-2)+\cdots+1$.
- We will show that this sum is $\Theta(n^2)$ implying that our initial estimate is actually tight.

Arithmetic Series

Lemma

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

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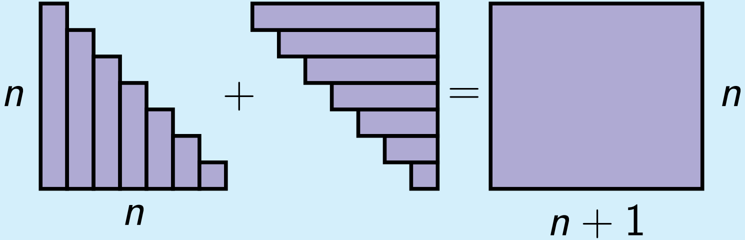
Proof

$$\begin{array}{cccc}
1 & 2 & \cdots & n \\
n & n-1 & \cdots & 1 \\
\hline
n+1 & n+1 & \cdots & n+1
\end{array} = n(n+1) \qquad \square$$

Alternative proof



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- Sorts in place: requires a constant amount of extra memory.
- There are many other quadratic time sorting algorithms: e.g., insertion sort, bubble sort.

Outline

Example: merge sort

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sort the halves recursively

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1	6	7	13

merge the sorted halves into one array

1	2	3	5	6	7	7	13

`MergeSort`$(A[1 \ldots n])$

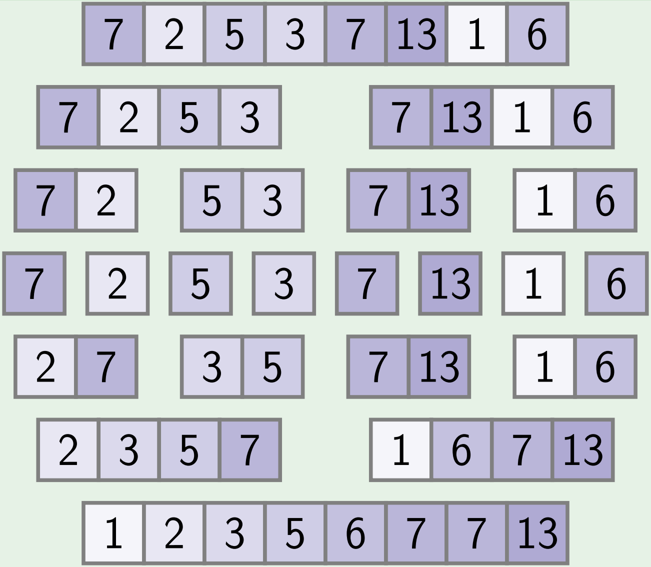
```
if n = 1:
    return A
m ← ⌊n/2⌋
B ← MergeSort(A[1 ... m])
C ← MergeSort(A[m + 1 ... n])
A′ ← Merge(B, C)
return A′
```

Merging Two Sorted Arrays

Merge($B[1 \ldots p], C[1 \ldots q]$)

```
{B and C are sorted}
D ← empty array of size p + q
while B and C are both non-empty:
  b ← the first element of B
  c ← the first element of C
  if b ≤ c:
    move b from B to the end of D
  else:
    move c from C to the end of D
move the rest of B and C to the end of D
return D
```

Merge sort: example



Lemma

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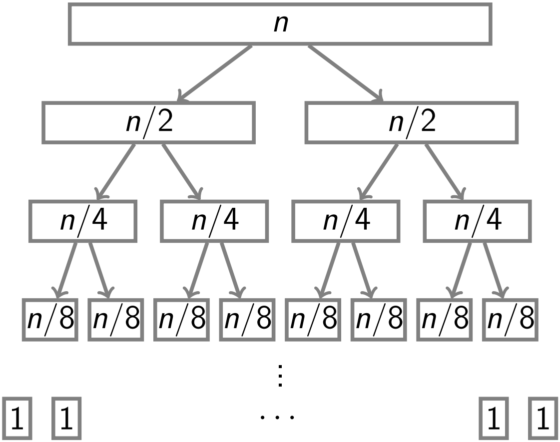
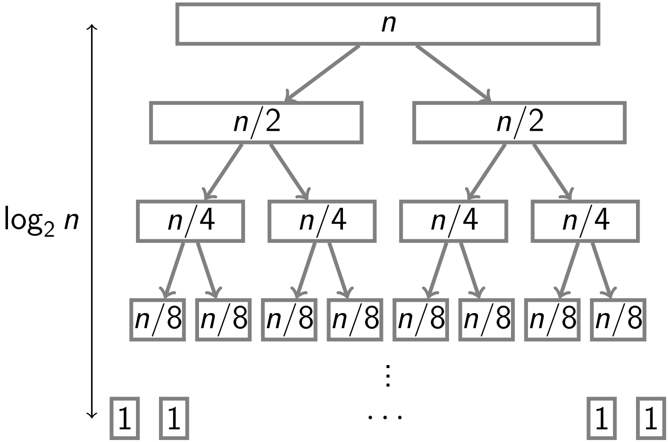
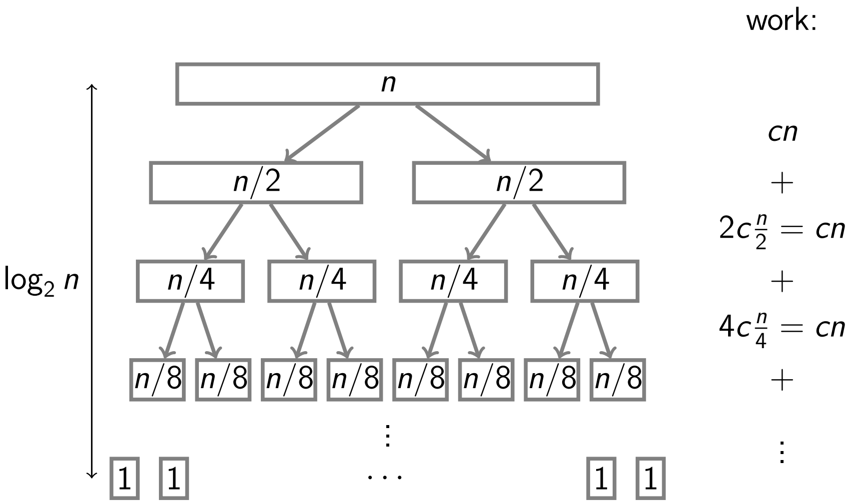
Proof

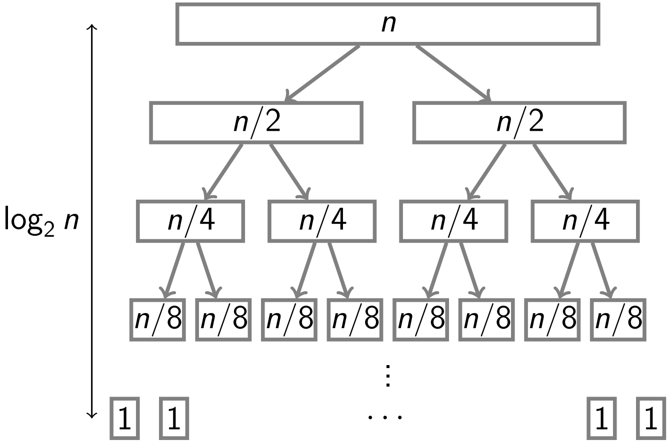
- The running time of merging B and C is $O(n)$.
- Hence the running time of `MergeSort`$(A[1 \dots n])$ satisfies a recurrence $T(n) \leq 2T(n/2) + O(n)$.


n
$n/2$ $n/2$
$n/4$ $n/4$ $n/4$ $n/4$
$n/8$ $n/8$ $n/8$ $n/8$ $n/8$ $n/8$ $n/8$ $n/8$
$\vdots$
1 1 $\cdots$ 1 1



n
$n/2$
$n/2$
$n/4$
$n/4$
$n/4$
$n/4$
$n/8$
$n/8$
$n/8$
$n/8$
$n/8$
$n/8$
$n/8$
$n/8$
$\log_2 n$
1
1
1
1


work:
n
$n/2$ $n/2$
$n/4$ $n/4$ $n/4$ $n/4$
$n/8$ $n/8$ $n/8$ $n/8$ $n/8$ $n/8$ $n/8$ $n/8$
$\vdots$
1 1 $\cdots$ 1 1
$\log_2 n$
cn
$+$
$2c\frac{n}{2} = cn$
$+$
$4c\frac{n}{4} = cn$
$+$
$\vdots$



work:

$$cn$$
$$+$$
$$2c\tfrac{n}{2} = cn$$
$$+$$
$$4c\tfrac{n}{4} = cn$$
$$+$$
$$\vdots$$

Total: $cn \log_2 n$

Outline

1. Problem Overview
2. Selection Sort
3. Merge Sort
4. Lower Bound for Comparison Based Sorting
5. Non-Comparison Based Sorting Algorithms

Definition

A comparison based sorting algorithm sorts objects by comparing pairs of them.

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Example

Selection sort and merge sort are comparison based.

Lemma

Any comparison based sorting algorithm performs $\Omega(n \log n)$ comparisons in the worst case to sort n objects.

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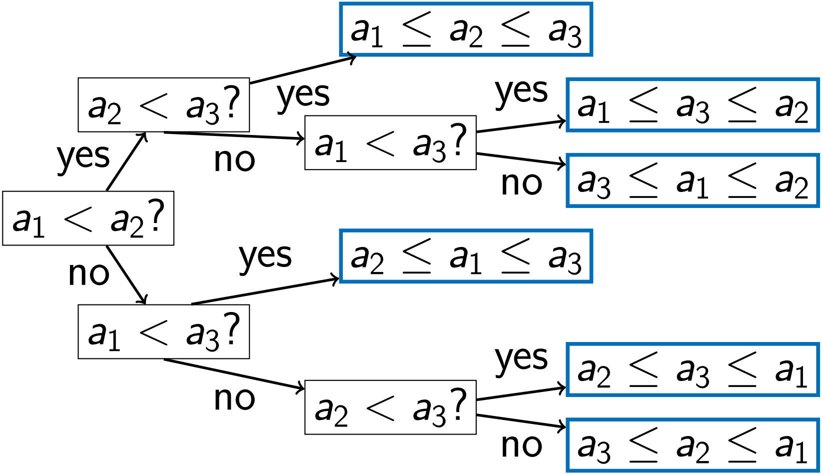
In other words

For any comparison based sorting algorithm, there exists an array $A[1 \dots n]$ such that the algorithm performs at least $\Omega(n \log n)$ comparisons to sort A.

Decision Tree



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Estimating Tree Depth

- the number of leaves ℓ in the tree must be at least $n!$ (the total number of permutations)
- the worst-case running time of the algorithm (the number of comparisons made) is at least the depth d
- $d \geq \log_2 \ell$ (or, equivalently, $2^d \geq \ell$)
- thus, the running time is at least

$$\log_2(n!) = \Omega(n \log n)$$

Lemma

$\log_2(n!) = \Omega(n \log n)$

Proof

$$
\begin{aligned}
\log_2(n!) &= \log_2(1 \cdot 2 \cdot \cdots \cdot n) \\
&= \log_2 1 + \log_2 2 + \cdots + \log_2 n \\
&\geq \log_2 \frac{n}{2} + \cdots + \log_2 n \\
&\geq \frac{n}{2} \log_2 \frac{n}{2} = \Omega(n \log n)
\end{aligned}
$$

$\square$

Outline

Example: sorting small integers

	1	2	3	4	5	6	7	8	9	10	11	12
A	2	3	2	1	3	2	2	3	2	2	2	1

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$\downarrow$

	1	2	3
$Count$	2	7	3

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A'	1	1	2	2	2	2	2	2	2	3	3	3

Example: sorting small integers

	1	2	3	4	5	6	7	8	9	10	11	12
A	2	3	2	1	3	2	2	3	2	2	2	1

we have sorted these numbers
without actually comparing them!

	1	2	3	4	5	6	7	8	9	10	11	12
A'	1	1	2	2	2	2	2	2	2	3	3	3

Counting Sort: Ideas

- Assume that all elements of $A[1 \dots n]$ are integers from 1 to M.

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Counting Sort: Ideas

- Assume that all elements of $A[1 \dots n]$ are integers from 1 to M.
- By a single scan of the array A, count the number of occurrences of each $1 \le k \le M$ in the array A and store it in $Count[k]$.
- Using this information, fill in the sorted array A'.

$\texttt{CountSort}(A[1 \ldots n])$

```
Count[1...M] ← [0,...,0]
for i from 1 to n:
   Count[A[i]] ← Count[A[i]] + 1
{k appears Count[k] times in A}
Pos[1...M] ← [0,...,0]
Pos[1] ← 1
for j from 2 to M:
   Pos[j] ← Pos[j − 1] + Count[j − 1]
{k will occupy range [Pos[k]...Pos[k + 1] − 1]}
for i from 1 to n:
   A′[Pos[A[i]]] ← A[i]
   Pos[A[i]] ← Pos[A[i]] + 1
```

Lemma

Provided that all elements of $A[1 \ldots n]$ are integers from 1 to M, $\texttt{CountSort}(A)$ sorts A in time $O(n + M)$.

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Remark

If $M = O(n)$, then the running time is $O(n)$.

Summary

- Merge sort uses the divide-and-conquer strategy to sort an n-element array in time $O(n \log n)$.
- No comparison based algorithm can do this (asymptotically) faster.
- One can do faster if something is known about the input array in advance (e.g., it contains small integers).