Divide-and-Conquer: Searching in an Array

Neil Rhodes

Department of Computer Science and Engineering
University of California, San Diego

Algorithmic Design and Techniques
Algorithms and Data Structures

Outline

1. Main Idea of Divide-and-Conquer
2. Linear Search
3. Binary Search

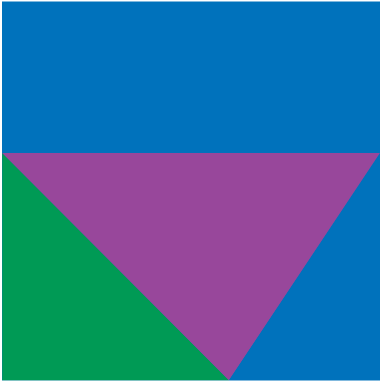
a problem to be solved

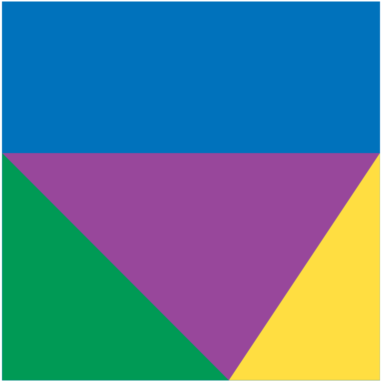
Divide: Break into non-overlapping subproblems of the same type

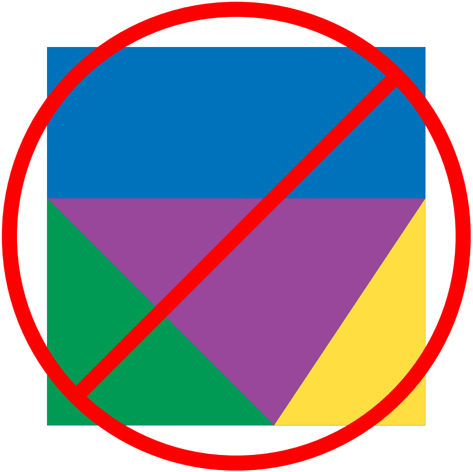
Divide: Break into non-overlapping subproblems of the same type

Divide: Break into non-overlapping subproblems of the same type

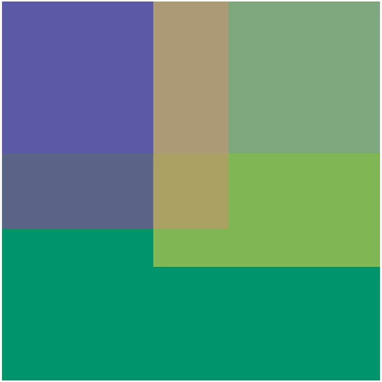
Divide: Break into non-overlapping subproblems of the same type

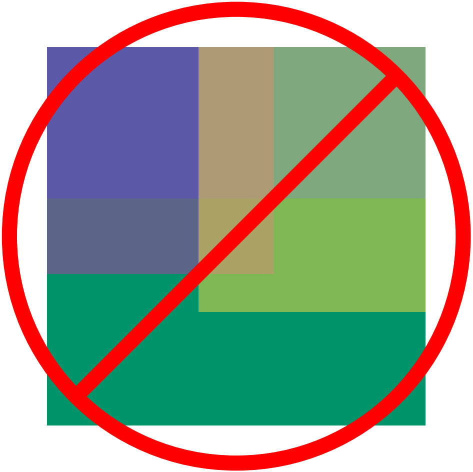






not the
same type





overlapping

Divide: break apart

Divide: break apart

Conquer: solve subproblems

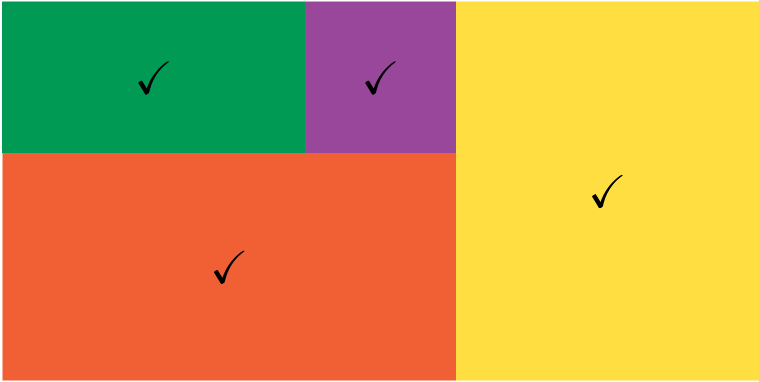
Conquer: solve subproblems

Conquer: solve subproblems

Conquer: solve subproblems

Conquer: solve subproblems

Conquer: combine



✓

1. Break into non-overlapping subproblems of the same type
2. Solve subproblems
3. Combine results

Outline

1. Main Idea of Divide-and-Conquer
2. Linear Search
3. Binary Search

Linear Search in Array

Ann	Pat	...	Joe	Bob

Linear Search in Array

Ann	Pat	...	Joe	Bob

Linear Search in Array

Ann	Pat	...	Joe	Bob

Linear Search in Array

Ann	Pat	...	Joe	Bob

Linear Search in Array

Ann	Pat	. . .	Joe	Bob

Linear Search in Array

Ann	Pat	...	Joe	Bob

Linear Search in Array

Ann	Pat	...	Joe	Bob

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
car	voiture	auto	Auto	auto
table	table	tavola	Tabelle	mesa

Searching in an array

Input: An array A with n elements.
A key k.

Output: An index, i, where $A[i] = k$.
If there is no such i, then
NOT_FOUND.

Recursive Solution

`LinearSearch`$(A, low, high, key)$

Recursive Solution

LinearSearch($A, low, high, key$)

```
if  high < low :
  return NOT_FOUND
if  A[low] = key :
  return low
```

Recursive Solution

LinearSearch$(A, low, high, key)$

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

Recursive Solution

LinearSearch($A, low, high, key$)

```
if high < low:
    return NOT_FOUND
if A[low] = key:
    return low
return LinearSearch(A, low + 1, high, key)
```

Definition

A recurrence relation is an equation recursively defining a sequence of values.

Definition

A recurrence relation is an equation recursively defining a sequence of values.

Fibonacci recurrence relation

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

Definition

A recurrence relation is an equation recursively defining a sequence of values.

Fibonacci recurrence relation

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

$\texttt{LinearSearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

```
if high < low:
  return NOT_FOUND
if A[low] = key:
  return low
return LinearSearch(A, low + 1, high, key)
```

$\texttt{LinearSearch}(A, low, high, key)$

```
if high < low:
    return NOT_FOUND
if A[low] = key:
    return low
return LinearSearch(A, low + 1, high, key)
```

Recurrence defining worst-case time:

$$T(n) = T(n-1) + c$$

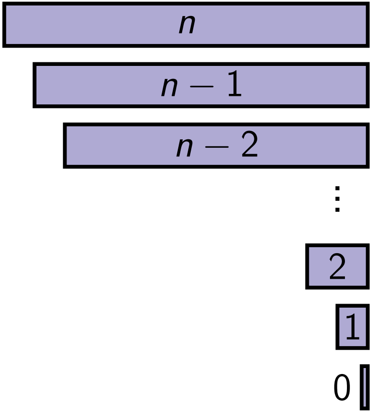
`LinearSearch`$(A, low, high, key)$

```
if high < low:
    return NOT_FOUND
if A[low] = key:
    return low
return LinearSearch(A, low + 1, high, key)
```

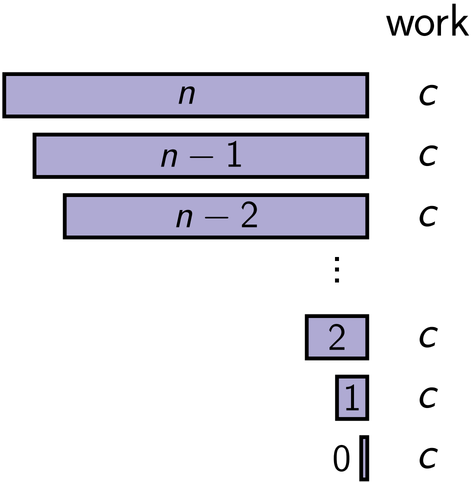
Recurrence defining worst-case time:

$$T(n) = T(n-1) + c$$
$$T(0) = c$$

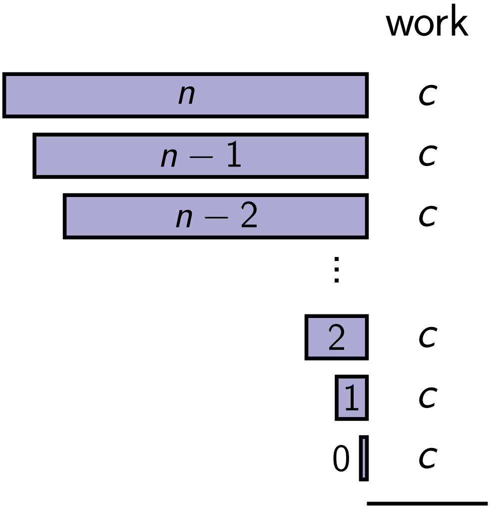
Runtime of Linear Search



Runtime of Linear Search



Runtime of Linear Search



Total: $\sum_{i=0}^{n} c = \Theta(n)$

Iterative Version

LinearSearchIt(A, low, $high$, key)

```
for i from low to high:
    if A[i] = key:
        return i
return NOT_FOUND
```

Summary

- Create a recursive solution

Summary

- Create a recursive solution
- Define a corresponding recurrence relation, T

Summary

- Create a recursive solution
- Define a corresponding recurrence relation, T
- Determine $T(n)$: worst-case runtime

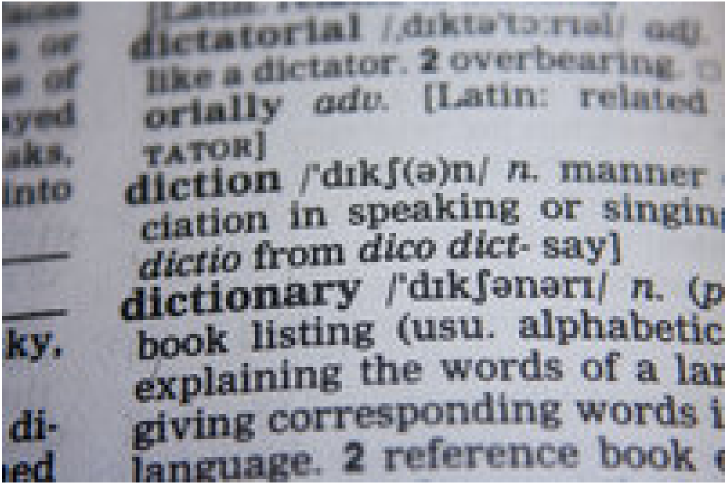
Summary

- Create a recursive solution
- Define a corresponding recurrence relation, T
- Determine $T(n)$: worst-case runtime
- Optionally, create iterative solution

Outline

1. Main Idea of Divide-and-Conquer
2. Linear Search
3. Binary Search

Searching Sorted Data



Searching in a sorted array

Input: A sorted array $A[low \dots high]$
($\forall low \leq i < high\colon A[i] \leq A[i+1]$).
A key k.

Output: An index, i, ($low \leq i \leq high$) where
$A[i] = k$.
Otherwise, the greatest index i,
where $A[i] < k$.
Otherwise ($k < A[low]$), the result is
$low - 1$.

Searching in a Sorted Array

Example

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$

$search(3) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$
$search(3) \to 1$
$search(4) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0$ $search(20) \to 4$

$search(3) \to 1$

$search(4) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \to 0 \quad search(20) \to 4$

$search(3) \to 1 \quad search(20) \to 5$

$search(4) \to 1$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$$\begin{array}{ll} search(2) \to 0 & search(20) \to 4 \\ search(3) \to 1 & search(20) \to 5 \\ search(4) \to 1 & search(60) \to 7 \end{array}$$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

Searching in a Sorted Array

Example

$search(2) \rightarrow 0$ $search(20) \rightarrow 4$

$search(3) \rightarrow 1$ $search(20) \rightarrow 5$

$search(4) \rightarrow 1$ $search(60) \rightarrow 7$

$search(70) \rightarrow 7$

3	5	8	20	20	50	60
1	2	3	4	5	6	7

BinarySearch(A, *low*, *high*, *key*)

`BinarySearch`$(A, \mathit{low}, \mathit{high}, \mathit{key})$

```
if high < low:
    return low − 1
```

`BinarySearch`$(A, low, high, key)$

```
if high < low:
    return low − 1
```

$mid \leftarrow \left\lfloor low + \frac{high - low}{2} \right\rfloor$

$\texttt{BinarySearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

```
if high < low:
    return low − 1
mid ← ⌊low + (high − low)/2⌋
if key = A[mid]:
    return mid
```

$$
\begin{array}{l}
\texttt{if } \mathit{high} < \mathit{low}\texttt{:} \\
\quad \texttt{return } \mathit{low} - 1 \\
\mathit{mid} \leftarrow \left\lfloor \mathit{low} + \frac{\mathit{high} - \mathit{low}}{2} \right\rfloor \\
\texttt{if } \mathit{key} = A[\mathit{mid}]\texttt{:} \\
\quad \texttt{return } \mathit{mid}
\end{array}
$$

BinarySearch$(A, low, high, key)$

```
if high < low:
  return low − 1
mid ← ⌊low + (high−low)/2⌋
if key = A[mid]:
  return mid
else if key < A[mid]:
  return BinarySearch(A, low, mid − 1, key)
```

$\texttt{BinarySearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

```
if high < low:
  return low − 1
mid ← ⌊low + (high−low)/2⌋
if key = A[mid]:
  return mid
else if key < A[mid]:
  return BinarySearch(A, low, mid − 1, key)
else:
  return BinarySearch(A, mid + 1, high, key)
```

Example: Searching for the key 50

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *high*

Example: Searching for the key 50

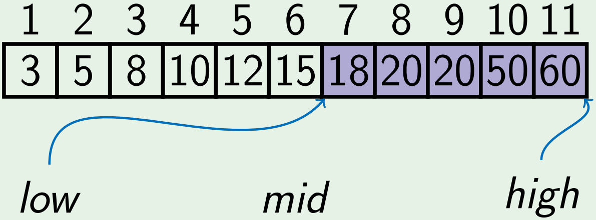
$\texttt{BinarySearch}(A, 1, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

BinarySearch$(A, 1, 11, 50)$
BinarySearch$(A, 7, 11, 50)$



Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$
$\texttt{BinarySearch}(A, 10, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$
$\texttt{BinarySearch}(A, 10, 11, 50)$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

low *mid* *high*

Example: Searching for the key 50

$\texttt{BinarySearch}(A, 1, 11, 50)$
$\texttt{BinarySearch}(A, 7, 11, 50)$
$\texttt{BinarySearch}(A, 10, 11, 50) \rightarrow 10$

1	2	3	4	5	6	7	8	9	10	11
3	5	8	10	12	15	18	20	20	50	60

Summary

Summary

- Break problem into non-overlapping subproblems of the same type.

Summary

- Break problem into non-overlapping subproblems of the same type.
- Recursively solve those subproblems.

Summary

- Break problem into non-overlapping subproblems of the same type.
- Recursively solve those subproblems.
- Combine results of subproblems.

$\texttt{BinarySearch}(A, \mathit{low}, \mathit{high}, \mathit{key})$

```
if high < low:
  return low - 1
mid ← ⌊low + (high - low)/2⌋
if key = A[mid]:
  return mid
else if key < A[mid]:
  return BinarySearch(A, low, mid - 1, key)
else:
  return BinarySearch(A, mid + 1, high, key)
```

Binary Search Recurrence Relation

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c$$

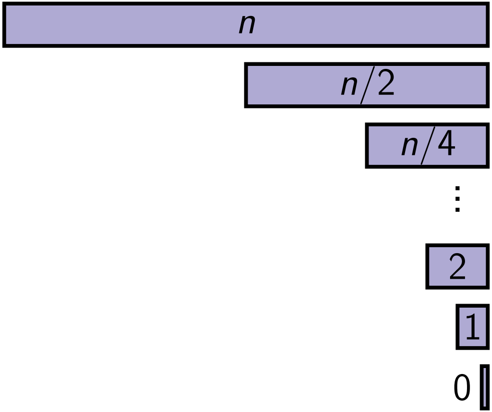
Binary Search Recurrence Relation

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c$$

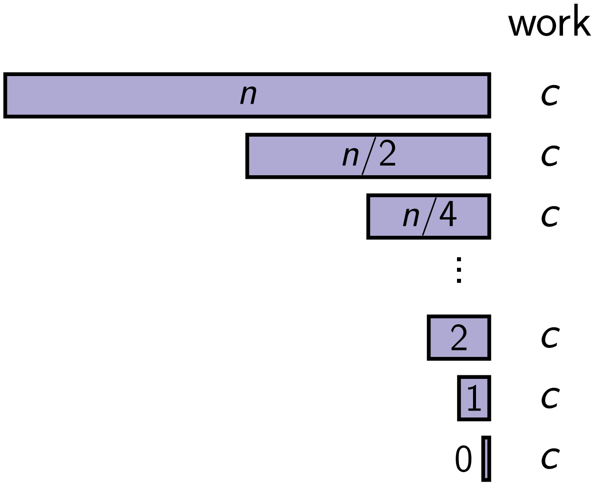
Binary Search Recurrence Relation

$$
\begin{aligned}
T(n) &= T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c \\
T(0) &= c
\end{aligned}
$$

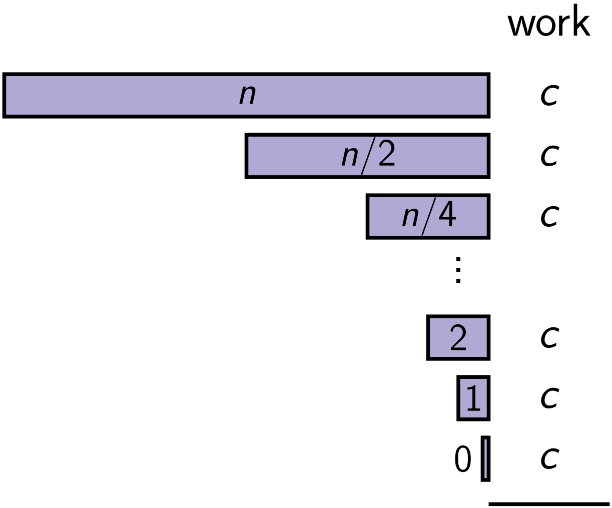
Runtime of Binary Search



Runtime of Binary Search



Runtime of Binary Search



Total: $\sum_{i=0}^{\log_2 n} c = \Theta(\log_2 n)$

Iterative Version

$\texttt{BinarySearchIt}(A, \mathit{low}, \mathit{high}, \mathit{key})$

$\texttt{while}\ \mathit{low} \leq \mathit{high}$:

$\quad \mathit{mid} \leftarrow \left\lfloor \mathit{low} + \frac{\mathit{high} - \mathit{low}}{2} \right\rfloor$

Iterative Version

$\texttt{BinarySearchIt}(A, \mathit{low}, \mathit{high}, \mathit{key})$

$\texttt{while}\ \mathit{low} \leq \mathit{high}$:

  $\mathit{mid} \leftarrow \left\lfloor \mathit{low} + \frac{\mathit{high} - \mathit{low}}{2} \right\rfloor$

  $\texttt{if}\ \mathit{key} = A[\mathit{mid}]$:

    $\texttt{return}\ \mathit{mid}$

Iterative Version

BinarySearchIt$(A, \mathit{low}, \mathit{high}, \mathit{key})$

while $\mathit{low} \leq \mathit{high}$:
  $\mathit{mid} \leftarrow \left\lfloor \mathit{low} + \frac{\mathit{high}-\mathit{low}}{2} \right\rfloor$
  if $\mathit{key} = A[\mathit{mid}]$:
    return mid
  else if $\mathit{key} < A[\mathit{mid}]$:
    $\mathit{high} = \mathit{mid} - 1$

Iterative Version

BinarySearchIt($A, low, high, key$)

```
while low ≤ high:
    mid ← ⌊low + (high − low)/2⌋
    if key = A[mid]:
        return mid
    else if key < A[mid]:
        high = mid − 1
    else:
        low = mid + 1
```

Iterative Version

$\texttt{BinarySearchIt}(A, \mathit{low}, \mathit{high}, \mathit{key})$

while $\mathit{low} \leq \mathit{high}$:
  $\mathit{mid} \leftarrow \left\lfloor \mathit{low} + \frac{\mathit{high} - \mathit{low}}{2} \right\rfloor$
  if $\mathit{key} = A[\mathit{mid}]$:
    return mid
  else if $\mathit{key} < A[\mathit{mid}]$:
    $\mathit{high} = \mathit{mid} - 1$
  else:
    $\mathit{low} = \mathit{mid} + 1$
return $\mathit{low} - 1$

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

Real-life Example

english (sorted)	**french** (sorted)	**italian** (sorted)	**german** (sorted)	**spanish** (sorted)
chair	chaise	casa	Haus	casa
house	bouton	foruncolo	Pickel	espenilla
pimple	maison	sedia	Sessel	silla

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted	**spanish** sorted
2	1
1	3
3	2

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted

2
1
3

spanish sorted

1
3
2

Real-life Example

english	french	italian	german	spanish
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Real-life Example

english	**french**	**italian**	**german**	**spanish**
house	maison	casa	Haus	casa
chair	chaise	sedia	Sessel	silla
pimple	bouton	foruncolo	Pickel	espenilla

english sorted
2
1
3

spanish sorted
1
3
2

Summary

Summary

The runtime of binary search is $\Theta(\log n)$.