The Shear Coefficient in Timoshenko's Beam Theory

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The equations of Timoshenko's beam theory are derived by integration of the equations of three-dimensional elasticity theory. A new formula for the shear coefficient comes out of the derivation. Numerical values of the shear coefficient are presented and compared with values obtained by other writers.

The range of applicability of the one-dimensional theory of beams can be extended by taking account of transverse shear deformations and, in the case of vibrating beams, rotary inertia. The equations which include these effects are generally referred to as Timoshenko's beam equations [1, 2][1] and they have received considerable attention in the literature. In these equations the effective transverse shear strain is taken as equal to the average shear stress on a cross section divided by the product of the shear modulus and the shear coefficient K. The coefficient K is a dimensionless quantity, dependent on the shape of the cross section, which is introduced to account for the fact that the shear stress and shear strain are not uniformly distributed over the cross section.

According to the commonly accepted definition, K is the ratio of the average shear strain on a section to the shear strain at the centroid. The analysis which leads to this definition is available in reference [3], as well as numerical values of K based upon it. Recently, however, Leibowitz and Kennard [4] have criticized the accepted definition, pointing out an unwarranted assumption in the underlying analysis. The author agrees with the criticism of Leibowitz and Kennard and the present paper is a response to their suggestion that further study of the shear coefficient is needed.

There is other evidence that the commonly accepted definition of K is not entirely satisfactory. Several writers [5, 6, 7] have pointed out that the customary values of K lead to unsatisfactory results when Timoshenko's beam equations are used to calculate the high-frequency spectrum of vibrating beams, and have advocated that K should be adjusted arbitrarily so that better results are obtained. It must be noted, however, that these writers do not dispute the basic definition of K. Instead they point out that the distribution of shear strain over a cross section depends on the mode of vibration of the beam and therefore varies with frequency. The unsatisfactory results, they claim, arise from using static strain distributions as a basis for calculating K instead of the strain distributions which occur in high-frequency motion.

In this paper a fresh derivation of Timoshenko's beam equations is presented. Our approach is to derive the beam equations by integration of the equations of three-dimensional elasticity theory. While this technique is familiar in plate and shell theory, it apparently has not been used previously in connection with beams. A new formula for the shear coefficient comes out of the derivation, and numerical values of K are calculated for a number of cross sections. Finally, the new values of K are compared with those obtained by other authors.

[1] Numbers in brackets designate References at end of paper.

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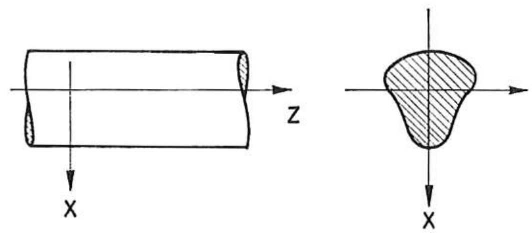

Fig. 1 General uniform beam

Derivation of Timoshenko's Beam Theory

We consider a uniform beam as illustrated in Fig. 1. The z-axis is taken to coincide with the line of centroids of the cross sections. To exclude complications arising from coupling with torsional deflections it will be assumed that the cross section and applied loads are symmetric about the x-z plane which is therefore the plane of deflection of the beam. The beam is acted upon by body forces F_x, F_y, and by surface tractions T_x, T_y, applied to the lateral faces. For simplicity it will be assumed that the z-components of body force and surface traction are zero.

A basic quantity in the beam equations is the transverse deflection W. This quantity requires precise definition since the cross section of the beam inevitably distorts to a small extent and all points of the cross section do not undergo the same displacement. Often W is taken as the displacement of the centroid. Here we follow a different course and define W as the mean deflection of the cross section. Thus

$$W = (1/A)\iint u_x dxdy \tag{1}$$

where A is the area of the cross section, u_x is the x-component of displacement of a point of the beam, and the integration extends over the cross section. The advantage of this definition becomes apparent in the first equilibrium equation for the beam, which we now derive.

The equation of equilibrium of an element of the beam with regard to forces in the x-direction is

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + F_x = \rho \frac{\partial^2 u_x}{\partial t^2} \tag{2}$$

where σ_{xx}, and so on, are the components of stress and ρ the mass density of the beam. Integration of (2) over the cross section yields

$$\iint \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x\right) dxdy + \frac{\partial}{\partial z} \iint \sigma_{xz} dxdy = \rho \frac{\partial^2}{\partial t^2} \iint u_x dxdy \tag{3}$$

Let

$$Q = \iint \sigma_{xz} dxdy \tag{4}$$

and we recognize that Q is the total transverse shear force acting on a cross section. Also let

$$p = \iint \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x \right) dxdy \tag{5}$$

Transformation by the divergence theorem gives

$$\begin{aligned} p &= \oint (n_x \sigma_{xx} + n_y \sigma_{xy}) ds + \iint F_x dxdy \\ &= \oint T_x ds + \iint F_x dxdy \end{aligned} \tag{6}$$

where n_x, n_y, are the components of the unit normal to the boundary of the cross section and ds is an element of arc of the boundary. From (6) we recognize that p is the total transverse load per unit length applied to the beam. Therefore (3) may be written

$$\frac{\partial Q}{\partial z} + p = \rho A \frac{\partial^2 W}{\partial t^2} \tag{7}$$

which is the first equation of Timoshenko's beam theory.

Timoshenko's equations also contain a quantity Φ which has been given various definitions and interpretations. Here we define Φ by the relation

$$\Phi = (1/I) \iint x u_z dxdy \tag{8}$$

where I is the moment of inertia of the cross section about the y-axis and u_z is the z-component of displacement of an element of the beam. We may interpret Φ as the mean angle of rotation of a cross section about the neutral axis. If cross sections remained plane as the beam bends, than u_z would be proportional to x, and Φ would be exactly equal to the angle of rotation of each cross section. In reality each cross section warps to some extent in addition to rotating. However, Φ may be regarded as the mean angle of rotation of the cross section in the sense that Φ is the angle of inclination of the plane which most nearly coincides with the position of the warped cross section. The foregoing definition of Φ arises naturally in the moment-equilibrium equation which we now consider.

To derive the moment-equilibrium equation of the beam we begin with the equation of equilibrium of an element of the beam with regard to forces in the z-direction,

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2} \tag{9}$$

Multiplication of (9) by x and integration over the cross section yields

$$\iint x \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} \right) dxdy + \frac{\partial}{\partial z} \iint x \sigma_{zz} dxdy = \rho \frac{\partial^2}{\partial t^2} \iint x u_z dxdy \tag{10}$$

Let

$$M = \iint x \sigma_{zz} dxdy \tag{11}$$

so that M is the bending moment acting at any section of the beam. Making use of integration by parts and the divergence theorem, we find

$$\begin{aligned} \iint x \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} \right) dxdy &= \iint \left(\frac{\partial (x\sigma_{xz})}{\partial x} + \frac{\partial (x\sigma_{yz})}{\partial y} - \sigma_{xz} \right) dxdy \\ &= \oint x (n_x \sigma_{xz} + n_y \sigma_{yz}) ds - \iint \sigma_{xz} dxdy \\ &= -Q \end{aligned} \tag{12}$$

since the line integral vanishes as a result of the assumption that the surface tractions have no z-component. Hence (10) becomes

$$\frac{\partial M}{\partial z} - Q = \rho I \frac{\partial^2 \Phi}{\partial t^2} \tag{13}$$

which is the second equation of Timoshenko's beam theory.

In order to derive the relation between W, Φ, and the shear distortion of the beam we define residual displacements v_x, v_z, by

$$u_x = W + v_x, \qquad u_z = U + x\Phi + v_z \tag{14}$$

where

$$U = (1/A) \iint u_z dxdy \tag{15}$$

is the mean displacement of the cross section in the z-direction. It follows from (14) that

$$\iint v_x dxdy = \iint v_z dxdy = \iint x v_z dxdy = 0 \tag{16}$$

The residual displacement v_z essentially represents the warping of the cross section. The stress-strain relation

$$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \frac{\sigma_{xz}}{G} \tag{17}$$

may now be written

$$\frac{\partial W}{\partial z} + \Phi = \frac{\sigma_{xz}}{G} - \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \tag{18}$$

Integration of (18) over the cross section then yields, in view of (16),

$$\frac{\partial W}{\partial z} + \Phi = \frac{1}{AG} \iint \left(\sigma_{xz} - G \frac{\partial v_z}{\partial x} \right) dxdy \tag{19}$$

We pass over, for the moment, the task of evaluating the integral in (19) and turn instead to the relation between the bending moment and curvature for the beam.

To obtain the moment-curvature relation we begin with the stress-strain relation

$$E \frac{\partial u_z}{\partial z} = \sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}) \tag{20}$$

where E is Young's modulus and ν is Poisson's ratio. Multiplication of (20) by x and integration over the cross section yields

$$E \frac{\partial}{\partial z} \iint x u_z dxdy = \iint x \sigma_{zz} dxdy - \nu \iint x(\sigma_{xx} + \sigma_{yy}) dxdy$$

or

$$EI \frac{\partial \Phi}{\partial z} = M - \nu \iint x(\sigma_{xx} + \sigma_{yy}) dxdy \tag{21}$$

To sum up the results obtained so far, it has been found that the motion of the beam satisfies the following four equations;

$$\frac{\partial Q}{\partial z} + p = \rho A \frac{\partial^2 W}{\partial t^2} \tag{22a}$$

$$\frac{\partial M}{\partial z} - Q = \rho I \frac{\partial^2 \Phi}{\partial t^2} \tag{22b}$$

$$EI \frac{\partial \Phi}{\partial z} = M - \nu \iint x(\sigma_{xx} + \sigma_{yy}) dxdy \tag{22c}$$

$$\frac{\partial W}{\partial z} + \Phi = \frac{1}{AG} \iint \left(\sigma_{xz} - G \frac{\partial v_z}{\partial x} \right) dxdy \tag{22d}$$

So far, no approximation beyond the customary assumptions of the linear theory of elasticity has been introduced. However, equations (22) are not yet in a viable form and to make further progress it is necessary to introduce two assumptions about the distribution of stress and strain within the beam.

The first assumption concerns the integral in (22c). In order to obtain the moment-curvature relation in its customary form we must assume that the stresses σ_{xx}, σ_{yy} are so small in comparison with σ_{zz} that the integral in (22c) may be neglected. This leaves

$$EI\,\frac{\partial \Phi}{\partial z} = M \tag{23}$$

The second assumption concerns the distribution of shear stress within the beam and is needed in order to evaluate the integral in (22d). Now exact solutions, exclusive of end effects, for the shear stresses in a uniform beam are known in two particular cases of loading—a cantilever beam under a single transverse load at the tip, and a uniformly loaded beam. The analysis of a tip-loaded cantilever constitutes the classical flexure problem and is a standard chapter in textbooks on the theory of elasticity [8, 10, 11]. It is also known that the distribution of transverse shear stress in a uniformly loaded beam is exactly the same as in a tip-loaded cantilever [9]. In both cases the shear stresses σ_{xz}, σ_{yz} and the displacement u_z are given by [8]

$$\begin{aligned}
\sigma_{xz} &= -\frac{Q}{2(1+\nu)I}\left(\frac{\partial \chi}{\partial x} + \frac{\nu x^2}{2} + \frac{(2-\nu)y^2}{2}\right)\\
\sigma_{yz} &= -\frac{Q}{2(1+\nu)I}\left(\frac{\partial \chi}{\partial y} + (2+\nu)xy\right)\\
u_z &= xf(z) - \frac{Q}{EI}(\chi + xy^2)
\end{aligned}\tag{24}$$

where $\chi(x, y)$ is a harmonic function which satisfies the boundary condition

$$\frac{\partial \chi}{\partial n} = -n_x\left(\frac{\nu x^2}{2} + \frac{(2-\nu)y^2}{2}\right) - n_y(2+\nu)xy \tag{25}$$

on the boundary of the cross section. Solutions for χ for a variety of cross sections are available in standard textbooks and in the literature. The function $f(z)$ in (24) is a polynomial whose exact form depends on the end conditions of the beam and which, for our purposes, need not be known.

We note that in a tip-loaded cantilever the shear force Q is constant, while in a uniformly loaded beam Q varies linearly along the beam. Since relations (24) are exact in the cases of constant and linearly varying Q it is natural to adopt them as approximations to the shear stresses and displacements for general loadings, including dynamic loadings. It seems reasonable to assume that this approximation is valid provided Q does not vary too rapidly along the length of the beam. A more thorough justification of this assumption for static loadings can be found in [15].

Assuming then, that the shear stresses and warping displacement are given by (24), we find

$$\begin{aligned}
v_z = \frac{Q}{EI}\bigg(-\chi - xy^2 &+ \frac{1}{A}\iint(\chi + xy^2)dxdy\\
&+ \frac{x}{I}\iint x(\chi + xy^2)dxdy\bigg)
\end{aligned}\tag{26}$$

We may now evaluate the integral in (22d) obtaining

$$\begin{aligned}
\iint\left(\sigma_{xz} - G\frac{\partial y_z}{\partial x}\right)dxdy = \frac{Q}{2(1+\nu)I}\bigg(&\frac{\nu(I_1 - I)}{2}\\
&- \frac{A}{I}\iint x(\chi + xy^2)dxdy\bigg)
\end{aligned}\tag{27}$$

where

$$I_1 = \iint y^2 dxdy \tag{28}$$

is the moment of inertia of the cross section about the x-axis. Substitution of (27) into (22d) then yields

$$\frac{\partial W}{\partial z} + \Phi = \frac{Q}{KAG} \tag{29}$$

where K is the quantity given by

$$K = \frac{2(1+\nu)I}{\dfrac{\nu(I_1 - I)}{2} - \dfrac{A}{I}\displaystyle\iint x(\chi + xy^2)dxdy} \tag{30}$$

This, then, is our formula for the shear coefficient K. Numerical values of K for certain cross sections are calculated in the next section. The final form of the equations of Timoshenko's beam theory are

$$\begin{aligned}
\frac{\partial Q}{\partial z} + p &= \rho A\,\frac{\partial^2 W}{\partial t^2}\\
\frac{\partial M}{\partial z} - Q &= \rho I\,\frac{\partial^2 \Phi}{\partial t^2}\\
EI\,\frac{\partial \Phi}{\partial z} &= M\\
\frac{\partial W}{\partial z} + \Phi &= \frac{Q}{KAG}
\end{aligned}\tag{31}$$

Calculation of Shear Coefficients for Various Cross Sections

Circle

The function χ for a circle of radius a is given by Love [8] as

$$\chi = -\left(\frac{3}{4} + \frac{\nu}{2}\right)a^2x + \frac{1}{4}(x^3 - 3xy^2)$$

The value of K then follows from (30) as

$$K = \frac{6(1+\nu)}{7+6\nu} \tag{32}$$

Hollow Circle

For a circular tube of inner radius b and outer radius a the function χ is given by Love [8] as

$$\chi = -\left(\frac{3}{4} + \frac{\nu}{2}\right)\left((a^2 + b^2)r + \frac{a^2b^2}{r}\right)\cos\theta + \frac{r^3}{4}\cos 3\theta$$

where r, θ are polar coordinates. The value of K is then

$$K = \frac{6(1+\nu)(1+m^2)^2}{(7+6\nu)(1+m^2)^2 + (20+12\nu)m^2} \tag{33}$$

where $m = b/a$ is the ratio of inner to outer radius. For a thin-walled tube (33) reduces to

$$K = \frac{2(1+\nu)}{4+3\nu}$$

Rectangle

Referring once again to Love, we find that the function χ for a rectangular cross section is

$$\begin{aligned}
\chi = (-(1+\nu)a^2 + \nu b^2/3)x &+ (2+\nu)(x^3 - 3xy^2)/6\\
&+ \frac{4\nu b^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-)^n\sinh(n\pi x/b)}{n^3\cosh(n\pi a/b)}\cos(n\pi y/b)
\end{aligned}$$

where the length of the side parallel to the x-axis is $2a$ and the length of the side parallel to the y-axis is $2b$. The value of K then follows as

$$K = \frac{10(1+\nu)}{12+11\nu} \tag{34}$$

It is remarkable that K is independent of the aspect ratio of the rectangle.

Ellipse

The function χ for an ellipse, again from Love, is

$$\chi = -\frac{a^2(2(1+\nu)a^2+b^2)}{3a^2+b^2}\,x + \frac{2a^2+b^2+\nu(a^2-b^2)/2}{9a^2+3b^2}\,(x^3-3xy^2)$$

where a is the semiaxis in the x-direction and b is the semiaxis in the y-direction. Here a may be greater than, less than, or equal to, b. The value of K then is

$$K = \frac{12(1+\nu)(3a^4+a^2b^2)}{(40+37\nu)a^4+(16+10\nu)a^2b^2+\nu b^4} \tag{35}$$

Semicircle

For a semicircle of radius a the flexure function has been calculated by the author as

$$\chi = \frac{(2-\nu)}{6}(x^3-3xy^2) - \left(\frac{4a}{3\pi}\right)^2 x + \frac{2(1-\nu)}{3\pi}\,r^2\cos 2\theta + \frac{(3+2\nu)a^3}{2\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n(4n^2+1)}\left(\frac{r}{a}\right)^{2n}\cos 2n\theta - \frac{3(1-2\nu)a^3}{2\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n(4n^2-9)}\left(\frac{r}{a}\right)^{2n}\cos 2n\theta$$

where x, y are rectangular coordinates with origin at the centroid and r, θ are polar coordinates with origin at the center of the semicircle. When K is calculated from (30) and numerical values are inserted in the infinite series which occur, the final result obtained is

$$K = \frac{1+\nu}{1.305+1.273\nu} \tag{36}$$

Thin-Walled Sections

The shear stress and warping in thin-walled sections can be calculated if some simplifying assumptions, based on the thinness of the wall, are introduced. Let us consider a section such as shown in Fig. 2. The arc length along the section is denoted by s, the slope angle of the tangent to the section is θ, and t is the variable thickness of the section. In thin-walled sections the shear stress τ, Fig. 2, follows the contour and may be assumed to be uniform across the thickness. Moreover, τ can be calculated directly from requirements of equilibrium using well-known methods [12]. To obtain the warping displacement it is advantageous to put

$$\psi = -\chi - xy^2 \tag{37}$$

so that the third equation of (24) becomes

$$u_z = xf(z) + (Q/EI)\psi \tag{38}$$

For thin-walled sections ψ will be a function of s only. Now

$$\tau = \sigma_{xz}\cos\theta + \sigma_{yz}\sin\theta \tag{39}$$

and substitution of (37) into (24) and the result into (39) yields

$$\frac{d\psi}{ds} = \frac{2(1+\nu)I}{Q}\,\tau + \frac{\nu}{2}\left((x^2-y^2)\cos\theta + 2xy\sin\theta\right) \tag{40}$$

Integration of (40) then gives ψ. In turn, the shear coefficient is obtained from (30) which, in view of (37), may be written

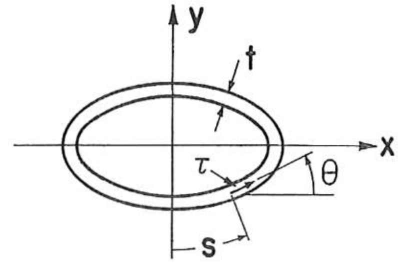

Fig. 2 Thin-walled cross section

$$K = \frac{2(1+\nu)I}{\dfrac{\nu(I_1-I)}{2} + \dfrac{A}{I}\displaystyle\oint x\psi t\,ds} \tag{41}$$

Calculations such as outlined in the foregoing have been carried out for some common thin-walled sections. Although straightforward, the calculations are tedious and will not be detailed here. The final results are presented in Fig. 3, which summarizes all our calculations of K.

Comparison With Other Evaluations of K

Table 1 compares our values of K with those obtained by other authors.

Table 1

Source	K for rectangle	K for circle
Timoshenko [3]	0.667	0.750
Mindlin [5]	0.822	0.847
Goodman [7], ($\nu = 1/3$)	0.870	...
Roark [13]	0.833	0.900
Formula (30), $\nu = 0$	0.833	0.857
Formula (30), $\nu = 0.3$	0.850	0.886
Formula (30), $\nu = 1/2$	0.870	0.900

With the exception of Timoshenko's values, the foregoing results all agree fairly well with each other. The agreement is particularly interesting in view of the vastly different methods used by the various authors. The values of Mindlin and of Goodman were based on high-frequency vibration modes while those of Roark were derived with only static deflections in mind. Mindlin chose K so that the frequency of the first thickness-shear mode, as calculated from the Timoshenko equations, agrees with the frequency given by the three-dimensional equations for small elastic vibrations. Goodman chose K so that the frequency equation obtained from the Timoshenko equations is correct in the limit of zero wavelength. Roark's values are based on the work of Newlin and Trayer [14] who calculate a K for static deflection of a simple beam by means of elementary strain-energy methods. All these calculations are indirect in the sense that they make no use of the definition of K as the ratio of the average shear strain of the shear strain at the centroid. The only values of K based directly on the definition are Timoshenko's and, as Table 1 indicates, these values are exceptional. This is in harmony with Leibowitz and Kennard's [4] criticism of the customary definition of K.

The nature of our approximation regarding the shear stresses suggests that our values of K are most satisfactory for static and long-wavelength, low-frequency deformations of beams. Comparison of our values with those of Mindlin and Goodman then gives an indication of the variation of the effective value of K with frequency.

References

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2 S. P. Timoshenko, "On the Transverse Vibrations of Bars of Uniform Cross-Section," *Philosophical Magazine*, vol. 43, 1922, pp. 125–131.

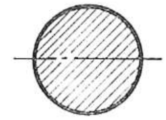
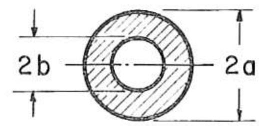
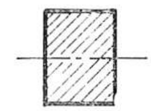
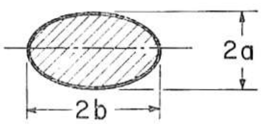
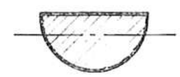
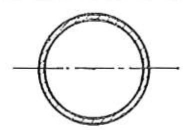
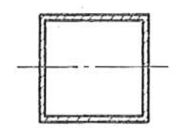
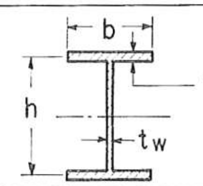
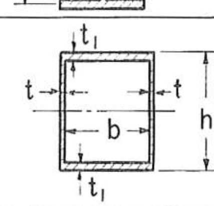
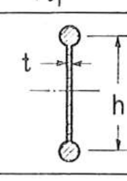
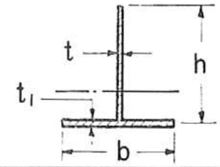
CIRCLE $K = \frac{6(1+\nu)}{7+6\nu}$	
HOLLOW CIRCLE $K = \frac{6(1+\nu)(1+m^2)^2}{(7+6\nu)(1+m^2)^2 + (20+12\nu)m^2}$ WHERE $m = b/a$	
RECTANGLE $K = \frac{10(1+\nu)}{12+11\nu}$	
ELLIPSE $K = \frac{12(1+\nu)a^2(3a^2+b^2)}{(40+37\nu)a^4 + (16+10\nu)a^2b^2 + \nu b^4}$ a MAY BE EITHER > OR < b	
SEMICIRCLE $K = \frac{1+\nu}{1.305+1.273\nu}$	
THIN-WALLED ROUND TUBE $K = \frac{2(1+\nu)}{4+3\nu}$	
THIN-WALLED SQUARE TUBE $K = \frac{20(1+\nu)}{48+39\nu}$	
THIN-WALLED I-SECTION $K = \frac{10(1+\nu)(1+3m)^2}{(12+72m+150m^2+90m^3) + \nu(11+66m+135m^2+90m^3) + 30n^2(m+m^2) + 5\nu n^2(8m+9m^2)}$ WHERE $m = 2bt_F/ht_W$, $n = b/h$	
THIN-WALLED BOX SECTION $K = \frac{10(1+\nu)(1+3m)^2}{(12+72m+150m^2+90m^3) + \nu(11+66m+135m^2+90m^3) + 10n^2((3+\nu)m+3m^2)}$ WHERE $m = bt_1/ht$, $n = b/h$	
SPAR-AND-WEB SECTION $K = \frac{10(1+\nu)(1+3m)^2}{(12+72m+150m^2+90m^3) + \nu(11+66m+135m^2+90m^3)}$ WHERE $m = 2A_S/ht$, A_S = AREA OF ONE SPAR	
THIN-WALLED T-SECTION $K = \frac{10(1+\nu)(1+4m)^2}{(12+96m+276m^2+192m^3) + \nu(11+88m+248m^2+216m^3) + 30n^2(m+m^2) + 10\nu n^2(4m+5m^2+m^3)}$ WHERE $m = bt_1/ht$, $n = b/h$	

Fig. 3 Formulas for shear coefficient; ν = Poisson's ratio; neutral axis is shown as a chain-dotted line

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