Indian Institute of Technology Jodhpur
MAL1010, Dec'21
Tutorial Sheet 1

Real Numbers and Sequences

1. Mark the following statements as True/False and justify your answers.
   (a) The set $\{2/m : m \in \mathbb{N}\}$ is bounded above.
   (b) Union of intervals is also an interval.
   (c) Nonempty intersection of intervals is also an interval.
   (d) Nonempty intersection of open intervals is also an open interval.
   (e) Nonempty intersection of closed intervals is also a closed interval.
   (f) sup of S, where $S \subseteq \mathbb{R}$ is an interior point of S.
   (g) Limit point of a set S is an interior point of S.

2. Consider $\mathcal{D} \subseteq \mathcal{R}$ and $f : \mathcal{D} \to \mathcal{R}$ defined by the following. Determine whether f is bounded above on $\mathcal{D}$. If yes, find an upper bound for f on $\mathcal{D}$. Also, determine whether f is bounded below on $\mathcal{D}$. If yes, find a lower bound for f on D. Also, determine whether f attains its upper bound or lower bound.
   (i) $\mathcal{D}$ =$(-1, 1)$ and $f(x) = x^2 - 1$, (ii) $\mathcal{D}$ =$(-1, 1)$ and $f(x) = x^3 - 1$, (iii) $\mathcal{D}$ =$(-1, 1]$ and $f(x) = x^2 - 2x - 3$, (iv) $\mathcal{D} = \mathcal{R}$ and $f(x) = \frac{1}{1+x^2}$.

3. Prove that the absolute value function, $f : \mathcal{R} \to \mathcal{R}$ defined by $f(x) = |x|$, is not a rational function.

4. Prove that the following numbers are irrational:
   $(i)\sqrt{3}$, $(ii)\sqrt[3]{2}$ $(iii)\sqrt{2} + \sqrt{3}$.

5. Prove that the function $f : [0, \infty) \to \mathcal{R}$ defined by the following is an algebraic function: $(i)f(x) = \sqrt{x}$, $(ii)f(x) = \sqrt{x} + \sqrt{2x}$, $(iii)f(x) = \sqrt{x} + \sqrt[3]{x}$.

6. Using $\epsilon - n_0$ definition prove the following.
   (i) $\lim\limits_{n\to\infty} \frac{10}{n} = 0$
   (ii) $\lim\limits_{n\to\infty} \frac{5}{3n+1} = 0$
   (iii) $\lim\limits_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0$

7. Show that the following limits exist and find them.
   (i) $\lim\limits_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$
   (ii) $\lim\limits_{n\to\infty} \frac{n!}{n^n}$
   (iii) $\lim\limits_{n\to\infty} \left(\frac{n^3+3n^2+1}{n^4+8n^2+2}\right)$

(iv) $\lim_{n\to\infty} n^{1/n}$

(v) $\lim_{n\to\infty} \left(\frac{\cos \pi\sqrt{n}}{n^2}\right)$

8. Use Sandwich theorem to find the limit, for the following sequences.

   (i) $\{x_n\} = \frac{\sin n}{n}$

   (ii) $\{x_n\} = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$

   (iii) $\{x_n\} = (a^n + b^n)^{1/n}$ where $0 < a < b$.

9. Determine whether the sequences are increasing or decreasing:

   (i) $\{\frac{n}{n^2+1}\}_{n\geq 1}$

   (ii) $\{\frac{1-n}{n^2}\}_{n\geq 2}$

10. Let $\{x_n\}$ be a sequence, such that $x_n > 0$ for all n and $\lim_{n\to\infty} \frac{x_n+1}{x_n} = \lambda$. Then prove that

    (i) if $\lambda < 1$ then $\lim_{n\to\infty} x_n = 0$

    (ii) if $\lambda > 1$ then $\lim_{n\to\infty} x_n = \infty$

11. Every convergent sequence is bounded. Is converse of the statement true? Justify your answer.

12. Suppose $\{x_n\}$ is a bounded and increasing sequence. Then prove that the sup of the set $\{x_n : n \in \mathbb{N}\}$ is the limit of $\{x_n\}$.

13. **Cauchy sequence:** A sequence $\{x_n\}_{n\geq 1}$ is said to be cauchy if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon \;\; \forall\, m, n \geq n_0$.

    Prove that every convergent sequence is Cauchy. Conversely, every Cauchy sequence in $\mathbb{R}$ is also convergent. (This is an equivalent way of stating the **Completeness property** of real numbers).

14. **Bolzano-Weierstrass theorem:** Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.