

Indian Institute of Technology Jodhpur
MAL1010, Dec'21
Assignment sheet 3

1. **Intermediate value property:** Let f be continuous on $[a, b]$, and let $f(a) < r < f(b)$. Then there exists x such that $a < x < b$ and $f(x) = r$.
 - (i) Show that the equation $(1 - x) \cos x = \sin x$ has at least one solution in $(0, 1)$.
 - (ii) Let $f : [0, 1] \rightarrow [0, 1]$ be continuous. Show that f has a fixed point in $[0, 1]$.
i.e., there exists $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.
 - (iii) Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Show that the range $\{f(x) : x \in [a, b]\}$ is a closed and bounded interval.
 - (iv) Show that a polynomial of odd degree has at least one real root.
2. (H.W.) Show that the converse of I.V.P. need not be true, by justifying with a counter example.
3. (H.W) If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function on a closed and bounded interval, then f is bounded and attains its bounds.
Prove that the converse of above result need not be true by justifying a counter example.
4. (H.W) Suppose f is a real valued function. Then give an example in each of the following cases.
 - (i) f is continuous and unbounded.
 - (ii) f is discontinuous and unbounded.
 - (iii) f is continuous and bounded. But it does not attain bounds.
5. Show that the function $f(x)$ is defined by

$$f(x) := \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable for all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not continuous at $x = 0$.

6. Let $f(0) = 0$ and $f'(0) = 1$. For a positive integer k , show that

$$\lim_{x \rightarrow 0} \frac{1}{x} \{f(x) + f(\frac{x}{2}) + \cdots + f(\frac{x}{k})\} = 1 + \frac{1}{2} + \cdots + \frac{1}{k}.$$

7. Show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one (real) root.
8. Let f and g be continuous functions on $[a, b]$ and differentiable (a, b) and let $f(a) = f(b) = 0$.
Prove that there is a point $c \in (a, b)$ such that $g'(c)f(c) + f'(c) = 0$.