Indian Institute of Technology Jodhpur
MAL1010, Dec'21
Assignment Sheet-2

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1. Use the comparision test to determine the convergence or divergence of the following series.

   (a) $\sum_{n=1}^{\infty} \frac{n}{n^2-\sin^2 n}$ (b) $\sum_{n=1}^{\infty} \frac{e^{-n}}{n+\cos^2 n}$ (c) $\sum_{n=1}^{\infty} \frac{n^{2.5}-2}{n^4+6}$

   (d) $\sum_{n=1}^{\infty} \frac{3^{(-1/n)}}{n^3}$ (e) $\sum_{n=1}^{\infty} \frac{2^{(1/n)}}{5n}$

2. Use the D'Alembert's ratio test to determine the convergence or divergence of the following series.

   (a) $\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}$ (b) $\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2\, 3^{(1+n)}}$ (c) $\sum_{n=1}^{\infty} \frac{4^{(1-2n)}}{n^2+1}$

   (d) $\sum_{n=0}^{\infty} \frac{(2n-1)!}{3n!}$ (e) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 2^n$

3. Use the Cauchy's n^{th} root test to determie the convergence or divergence of the following series.

   (a) $\sum_{n=1}^{\infty} \left(\frac{3n+1}{4-\sqrt{3n}}\right)^{2n}$ (b) $\sum_{n=0}^{\infty} \frac{n^{(1-4n)}}{6^{2n}}$ (c) $\sum_{n=4}^{\infty} \frac{(-5)^{1+3n}}{(2)^{7n-2}}$

   (d) $\sum_{n=2}^{\infty} \left(\frac{5n^2-2n+1}{3n^2+n-3}\right)^{-n}$ (e) $\sum_{n=1}^{\infty} \left(\frac{5-2n}{7+3n}\right)^{n/3}$

4. Discuss the convergence of the following alternative series.

   (a) $\sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{n^3+4n+1}$ (b) $\sum_{n=1}^{\infty} \frac{1}{(-2)^n\,(6n+1)}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^{n-2}}{3^n+3n}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n}{2n-1}$
   (e) $\frac{1}{1^2} - \frac{1}{1} + \frac{1}{2^2} - \frac{1}{2} + \frac{1}{3^2} - \frac{1}{3} + \cdots$

   $\sum_{n=0}^{\infty} a_n$, where
   $$a_n := \begin{cases} \frac{-1}{n}, & n \;\; is \;\; odd \\ 2^{-n}, & n \;\; is \;\; even \end{cases}$$

5. Define a function $f(x)$ such that
   $$f(x) := \begin{cases} 2x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

   Show that f is continuous at 0 using the $\epsilon - \delta$ definition and sequential characterization.

6. Show that the following function is not continuous at $x = 0$, using sequential characterization.

$$f(x) := \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

7. Show that the following function is continuous only at $x = 0$.

$$f(x) := \begin{cases} 0, & x \in \mathbb{Q} \\ x, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every $x, y \in \mathbb{R}$, we have $|f(x) - f(y)| \leq |x - y|$. Show that f is continuous.

9. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every point $c \in \mathbb{R}$.