Indian Institute of Technology Jodhpur
MAL1010, Dec'21-Mar'22
Assginment Sheet 9

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1. Let $F(x, y, z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of F at $(1, -1, 3)$ and the equations of the tangent plane and the normal line to the surface $F(x, y, z) = 7$ at $(1, -1, 3)$.
   Ans: Tangent Plane: $0.(x - 1) + 4(y + 1) + 6(z - 3) = 0$
   Normal line: $x = 1$; $3y - 2z + 9 = 0$.

2. Find $D_{\underline{u}} f(2, 2, 1)$, where $F(x, y, z) = 3x - 5y + 2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the surface $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$.
   Ans: $-2/3$

3. Show that the following functions have local minima at the indicated points.
   (i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7$, $(x_0, y_0) = (-1, 2)$.
   (ii) $f(x, y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3$, $(x_0, y_0) = (0, 0)$.

4. Analyze the following functions for local maxima, local minima and saddle points.
   (i) $f(x, y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$
   (ii) $f(x, y) = x^3 - 3xy^2$
   Ans: (i) $(0, 0)$ is a saddle point; $(\pm\sqrt{2}, 0)$ are local maxima; $(0, \pm\sqrt{2})$ are local minima.
   (ii) $(0, 0)$ is a saddle point.

5. Find the absolute maxima and absolute minimum of $f(x, y) = (x^2 - 4x)\cos y$ for $1 \le x \le 3$ and $-\pi/4 \le y \le \pi/4$.
   Ans: $f_{min} = -4$ at $(2, 0)$ and $f_{max} = -3/\sqrt{2}$ at $(3, \pm\pi/4)$.

6. Evaluate the minimum and maximum value of the function $f(x, y) = 2 - x^2 - 2y^2$ subject to the condition $g(x, y) = x^2 + y^2 - 1$.

7. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject the constraints $x + 2y + 3z = 6$ and $x + 3y + 4z = 9$.
   Ans: $f_{min} = 6$ at $(-1, 2, 1)$

8. Maximize the $f(x, y, z) = xyz$ subject to the constraints $x + y + z = 40$ and $x + y = z$.
   Ans: $f_{max} = 2000$ at $(10, 10, 20)$.

   Additional Practice Problems:

9. The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$. Ans: 4

10. Show that for the following function, partials exist at $(0, 0)$. But, $f(x, y)$ is discontinuous at $(0, 0)$.

$$f(x, y) := \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

11. Find second order partial derivatives, if $f(x,y) = x\cos y + ye^x$.

12. Show that each function satisfies Laplace equation. (i) $f(x,y,z) = x^2 + y^2 - 2z^2$
(ii) $f(x,y) = e^{-2y}\cos 2x$. (two-dimensional Laplace equation is: $(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0)$.

13. Find the derivative of $f(x,y) = x^2 \sin 2y$ at the point $(1, \pi/2)$ in the direction of $\bar{v} = 3\bar{i} - 4\bar{j}$.

14. Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P = (1,1,0)$ in the direction of $\bar{v} = 2\bar{i} - 3\bar{j} + 6\bar{k}$. In what directions does f change most rapidly at P, and what are the rates of change in these directions?

15. The surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ and $g(x,y,z) = x + z - 4 = 0$ meet in an ellipse E. Find the equation of the line tangent to E at the point $P_0(1,1,3)$.

16. Estimate how much the value of $f(x,y,z) = y\sin x + 2yz$ will change if the point $P(x,y,z)$ moves 0.1 from $P_0(0,1,0)$ stright toward $P_1(2,2,-2)$.