Indian Institute of Technology Jodhpur
MAL1010, Dec'21-Mar'22
Assignment Sheet 4

Q.1. Use Mean value theorem to prove that

$$|\sin x - \sin y| \leq |x - y| \quad \forall \ x, y \in \mathbb{R}.$$

Q.2. Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be continuous. Suppose $f^{'}(x)$ exists and $f^{'}(x) < 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, then show that $f(x) = x$ for every $x \in (-a, a)$.

Q.3. Cauchy Mean Value Theorem: Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $g^{'}(x) \neq 0$ for all $x \in (a, b)$. Then prove that there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - f(a)} = \frac{f^{'}(c)}{g^{'}(c)}.$$

Q.4. Using Cauchy mean value theorem, show that $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$.

Q.5. Let f be continuous on $[a, b], a > 0$ and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{bf(a) - af(b)}{b - a} = f(c) - cf^{'}(c).$$

Q.6. Use Taylor's Theorem with $n = 2$ to approximate $(1 + x)^{1/3}, \ \ x > -1$.

Q.7. Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that

$$x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k}x^{2k} \ < \ \log(1 + x) \ < \ x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$

Q.8. Find the extreme values by first derivative test, for a given $f(x) = \frac{1}{x^4 - 2x^2 + 7}, \ \ x \in \mathbb{R}$.

Q.9. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of maxima/minima, points of inflection and asymptotes.
(i) $f = x(6 - 2x)^2$ (ii) $f(x) = x^5 - 5x^4$ (iii) $f = x^{1/5}$
(iv) $f = x^{2/5}$ (v) $f = x^4$ (vi) $f = x^{1/3}$
(vii) $f(x) = \frac{x^2 - 3}{x - 2}$ (viii) $f(x) = \frac{x^3}{3x^2 + 1}$

Q.10. Give an example of $f : [0, 1] \to \mathbb{R}$ such that f is

(i) strictly increasing and convex.
(ii) strictly increasing and concave.
(iii) strictly decreasing and convex.
(iv) strictly decreasing and concave.