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Assignment - 1

1) a) Net force on m,

$$\frac{GmM}{r^2} = \frac{mv^2}{r}$$

$$\Rightarrow M = \frac{v^2 r}{G}$$

G = gravitational constant
M = mass of milky way
r = radius of sun's orbit
v = orbital speed of sun.

$$M = \frac{v^2 r}{G} = \frac{8 \times 10^{3} \times 3 \times 10^{16} \times (220 \times 10^{3})^3}{6.6 \times 10^{-11}} \, kg$$

$$= 1.78 \times 10^{41} \, kg$$

$$\approx 10^{11} M_\odot$$

Actual mass of milky way $= 10^{12} M_\odot$

b) Reason of discrepancy: Assumption that whole mass of milky way is inside the sun's orbit, however discrepancy is not major.

2) a) $\frac{l_2}{l_1} = (100)^{1/5 (m_2 - m_1)}$

$$\Rightarrow m_1 - m_2 = 2.5 \log (l_2/l_1)$$

Using definition of absolute magnitude,

$$m_1 - m_2 = 2.5 \log \left(\frac{d^2}{(10pc)^2} \right)$$

$m_2 = M$ (absolute magnitude)

$$\Rightarrow m - M = 5 \log_{10} d/10$$

$$2 - M = 5 \log 4/10$$

$$M = 2 - 5 \log 0.4$$

$$= 2 + 1.99 = 3.99$$

$$\frac{l_2}{3.9 \times 10^{26}} = 100^{1/5 (5 - 3.99)}$$

$$= 100^{\frac{1}{5}(1.01)}$$

$$\Rightarrow l_2 = 3.9 \times 10^{26} \times 2.535 = 9.89 \times 10^{26} \, W$$

3).
a). $\theta = 1.22\frac{\lambda}{D} = 1.22 \times \frac{1}{10\times10^3}$ radian

$= \frac{1.22}{10^4} \times \frac{18\times36\times10^4}{\pi} = 25.477''$

In optical band, ~~next~~ we see it in yellow region ~~5~~ 550nm

$\frac{1.22}{10^4} = 1.22 \times \frac{550\times10^{-2}}{D} \Rightarrow D = 5.5\,mm$

~~optic~~ GMRT should have atleast 5.5mm diameter for visible range.

c). $D = 5.5\,mm$ is a very small aperture and will limit the flux entering, to overcome this we have to increase the exposure time to ~~~~ collect more flux to get brighter image with good resolution.

4). a) Plank's law : frequency $= \nu$
Temperature $= T$

$$U_\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/k_BT}-1}$$

$$U(T) = \int_0^\infty \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/k_BT}-1}\,d\nu$$ [For all frequency at temperature T]

$$x = \frac{h\nu}{k_BT}$$

$$U(T) = \frac{8\pi k_B^4}{h^3c^3}T^4\int_0^\infty \frac{x^3}{e^x-1}\,dx$$

$$U(T) = a_BT^4 \quad : \quad a_B = \frac{8\pi k_B^4}{c^3h^3}\int_0^\infty \frac{x^3\,dx}{e^x-1}$$

b) From the surface of arbitary object, total emitted flux :

$$F = \int_\nu\int I_\nu \cos\theta\, d\Omega\, d\nu \qquad ①$$

Total energy passing through surface of area dA, at angle θ

$$dE_\nu d\nu = U_\nu d\nu dV = U_\nu \cos\theta \, c \, dt \, dA \, d\nu$$
$$= I_\nu \cos\theta \, dt \, dA \, d\Omega \, d\nu$$

$[dV = c \, dt \cos\theta \, d\Omega]$

I_ν = specific intensity

U_ν = specific energy density

$I_\nu = \frac{c}{4\pi} U_\nu$ (Isotropic)

$$\Rightarrow F = \int_\nu \int \frac{c}{4\pi} U_\nu \cos\theta \, d\Omega \, d\nu$$

Using $U(T) = a_B T^4$

$$\Rightarrow F = \frac{c a_B}{4\pi} T^4 \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$\Rightarrow F = \frac{c a_B}{4} T^4$$

$$\Rightarrow \boxed{F = \sigma T^4} \qquad \sigma = \frac{c a_B}{4}$$

c). $U_\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{\frac{h\nu}{k_B T}} - 1}$

$$\frac{d(U_\nu)}{d\nu} = \frac{8\pi h}{c^3} \left[\frac{3\nu^2}{e^{\frac{h\nu}{k_B T}} - 1} - \nu^3 \frac{h}{k_B T} e^{h\nu/k_B T} \right]$$

$\frac{d(U_\nu)}{d\nu} = 0$ [U_ν is max^m]

$$\Rightarrow \frac{3\nu^2}{e^{h\nu/k_B T} - 1} = \frac{\nu^3 \frac{h}{k_B T} e^{h\nu/k_B T}}{(e^{h\nu - k_B} - 1)^2}$$

$$\Rightarrow 3(e^{\frac{h\nu}{k_B T}} - 1) = \frac{h\nu}{k_B T} e^{h\nu/k_B T}$$

$$\Rightarrow 3e^{h\nu/k_B T} - 3 - \frac{h\nu}{k_B T} e^{h\nu/k_B T} = 0$$

$$x = \frac{\nu}{T} = \frac{\nu_{max}}{T}$$

$\Rightarrow$ $3e^{hx/k_B} - 3 - \frac{hx}{k_B} e^{hx/k_B} = 0$

$x = \frac{\nu_{max}}{T} \approx 5.88 \times 10^{10}\ Hz\ K^{-1}$

d) $\lambda = \frac{c}{\nu}$, $U_\lambda = \frac{8\pi h}{\lambda^3} \frac{1}{e^{hc/kT\lambda} - 1}$

$\frac{dU_\lambda}{d\lambda} = 0$, we get $3 = \frac{8\pi h^2 c}{\lambda k T} \frac{e^{hc/\lambda kT}}{(e^{hc/\lambda kT} - 1)}$

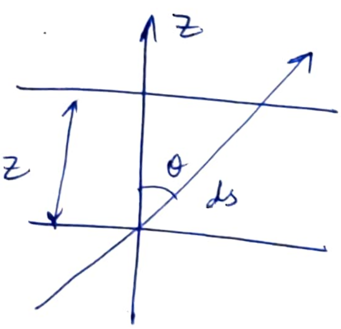
cont. Take $\frac{hc}{\lambda kT} = t$

$3(e^t - 1) = 8\pi h e^t$

$\frac{\nu_{max}}{T} = \frac{hc}{\lambda_{min} T} = 5.06 \times 10^{10}$

$\lambda_{min} T = \frac{6.62 \times 10^{-34} \times 3 \times 10^{16} \times 10^{10}}{1.80} = 3.37 \times 10^{-16}\ mK$

6)



$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu}$ — ①

$\theta = \cos^{-1}\mu$

In this case, I_ν is $f(z, \mu)$

In presence in matter,

$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$ — ② [J_ν = emission coefficient]

$\mu \frac{dI_\nu(z,\mu)}{dz} = j_\nu - \alpha_\nu I_\nu$

$\Rightarrow \frac{\mu\, dI_\nu}{\alpha_\nu\, dz} = j_\nu - I_\nu$

optical depth $= d\tau_\nu = -\alpha_\nu dz$

& $S_\nu = \frac{-j_\nu}{\alpha_\nu}$

$$\mu \frac{\partial I_\nu}{\partial \tau_\nu} = I_\nu - S_\nu$$

$$\mu \frac{\partial}{\partial \tau}\left(I_\nu e^{-\frac{\tau_\nu}{\mu}}\right) = -S e^{-\frac{\tau_\nu}{\mu}}$$

$$I_\nu e^{-t_\nu/\mu}\Big|_{\tau_{\nu,0}}^{\tau_\nu} = -\int_{\tau_{\nu_0}}^{\tau_\nu} \frac{S_\nu}{\mu} e^{-\frac{t_\nu}{\mu}} dt_\nu$$

$$0 \le \mu \le 1 : I_\nu = \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_0-\tau_0)/\mu} \frac{dt_\nu}{\mu}.$$

$$-1 \le \mu \le 0 : I_\nu = \int_0^{\tau_\nu} S_\nu e^{-(\tau_\nu - t_\nu)/(1-\mu)} \frac{dt_\nu}{-\mu}$$

Some func^n: $B_\nu(T(\tau_\nu)) \equiv B_\nu(\tau_\nu)$

$$S_\nu(t_\nu) = B_\nu(t_\nu) + (t_\nu - \tau_\nu)\frac{dB_\nu}{d\tau_\nu} - \ldots$$

Considering 1st 2 terms.

$$\boxed{I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu}}$$

7) $S = \frac{3F}{4\pi}(\tau + q)$

$$I_\nu(\tau, \mu) = \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - \tau_\nu)/\mu} \frac{dt_\nu}{\mu}$$

since $\frac{dF}{d\tau} = 0 \Rightarrow J = S$

$$I(\tau, \mu \ge 0) = \int_\tau^\infty S e^{-(t-\tau)/\mu} \frac{dt}{\mu} \quad -②$$

$$I(\tau, \mu) = \frac{3F}{4\pi} \int_\tau^\infty e^{-(t-\tau)/\mu} \left(\frac{dt}{\mu}\right)(t+q)$$

$$= \frac{3F}{4\pi} \int_\tau^\infty t e^{-(t-\tau)/\mu} \frac{dt}{\mu} + q \int_\tau^\infty e^{-(t-\tau)/\mu} \frac{dt}{\mu}$$

$$I = \frac{3F}{4\pi} \int_0^\infty (\mu x + \tau) e^{-x} dx + q \int_0^\infty e^{-x} dx$$

$$I(\tau, \mu) = \frac{3F}{4\pi}[\mu + \tau + q]$$

b) $\tau = 0 \Rightarrow I(0,\mu) = \frac{3F}{4\pi}\left[\mu + 9\right]$

c). $T^4 = \frac{3}{4} T_{eff}^4 (\tau + 4/3)$

For blackbody, $T = T_{eff}$

$T_{eff}^4 = 3/4\, T_{eff}^4 (\tau + 2/3)$

$\rightarrow \tau = 2/3$

d). $\tau = 2/3$

Probability of escaping a photon $= e^{-\tau} = e^{-2/3} = 0.513$

8). a) $\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + S_\nu$

$-\frac{dI_\nu}{ds} = j_\nu$

$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j(s')\,ds'$

$I_\nu(s) = \int_{s_0}^{s} S_\nu(s')\,ds'$

$s = \tan\theta$

$ds = b\, d\tan\theta$

$I_\nu(s) = 2 j_\nu b \int_0^{\cos^{-1} b/R} d\tan\theta$

$= j_\nu b \tan\theta \Big|_0^{\cos^{-1} b/R}$

$I_\nu(s) = 2 j_\nu b \,|\tan\theta|, \cos^{-1}(b/R)$

$= 2 j_\nu b \left(\frac{\sqrt{1 - b^2/R^2}}{b/R}\right) = 2 j_\nu \sqrt{R^2 - b^2}$

Intensity $= \sqrt{R^2 - b^2}$

b). Emission coefficient :

$j_\nu = \frac{P_\nu}{4\pi}$

Total power emitted $= 4/3\, \pi r^3 \rho$ ①

flux = σT_{eff}^4

Total power = $4\pi R^2 \times \sigma \times T_{eff}^4$

$\Rightarrow T_{eff}^4 = \frac{R f}{3\sigma}$

$\Rightarrow T_{eff} = \left(\frac{R f}{3\sigma}\right)^{1/4}$

c) $\tau_\nu >> 1$, $\frac{dI_\nu}{ds} = -I_\nu + S_\nu$

Kirchhoff's law : $S_\nu = B_\nu(T) = I_\nu(T)$

At a distance B from centre, intensity doesn't depend on ray path.

$I_\nu = B_\nu$

object is blackbody & $T = T_{eff}$

5)

a) $U_\nu = \frac{I_\nu 4\pi}{c}$

$B_\nu = \frac{2h\nu^3}{c^2} \frac{d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}$ for $h\nu << 1$

$= \frac{2h\nu^3}{c^2} \frac{d\nu}{\left(1 + \frac{h\nu}{kT} - 1\right)}$

$\Rightarrow B_\nu = \frac{2\nu^2 d\nu kT}{c^2} = \frac{2kT\nu^2 d\nu}{c^2}$

b) For $h\nu >> T$

$B_\nu = \frac{2h\nu^3}{c^2} e^{-h\nu/kT} d\nu$

$= \frac{2h\nu^3}{c^2} \frac{d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}$

$\frac{dB_\nu}{dT} = \frac{2h\nu^3 d\nu}{c^2} \frac{\frac{h\nu}{k} e^{h\nu/kT}}{\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right)^2}$

$\frac{dB_\nu}{dT}$ always > 0

as $T \to 0$, $B_\nu \to 0$

$T \to \infty$, $B_\nu \to \infty$