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HW-3

1). Using general polytrope relation,
a)

$$P = K\rho^{(1+\frac{1}{n})} = K\rho_c^{1+\frac{1}{n}}\theta^{1+n} \qquad [\rho = \rho_c \theta^n]$$

$$\frac{dP}{dr} = K\frac{4}{3}\rho^{1/3}\frac{d\rho}{dr}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

$n=3$: $\rho = \rho_c\theta^n \Rightarrow \rho_c\theta^3$, $r = a\xi$

$$\frac{d}{dr} = \frac{1}{a}\frac{d}{d\xi}$$

$$\frac{1}{a^2\xi^2}\frac{1}{a}\frac{d}{d\xi}\left(\frac{r^2}{\rho}\frac{1}{a}\frac{dP}{d\xi}\right) = -4\pi G\rho_c\theta^3$$

$$\frac{K}{\pi G\rho_c a^2}\rho_c^{1/3}\frac{1}{\xi^2}\frac{1}{d\xi}\left(\xi^2\frac{d\theta}{d\epsilon}\right) = -\theta^3$$

this will reduce to $n=3$ Lane-Emden Eq^n if extra term in LHS is 1

$$a = \sqrt{\frac{K}{\pi G\rho_c^{2/3}}}$$

b). For numerical evaluation, introduce ϕ as new intermediate variable

$$\frac{1}{\xi^2}\frac{1}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3 \qquad \phi = \frac{d\theta}{d\xi}$$

$$2\xi\phi + \xi^2 \frac{d\phi}{d\xi} = -\theta^3 \xi^2$$ ~~Lane-Emden~~

c). Gradient of θ vanishes smoothly,

$\Rightarrow \phi\big|_{\xi=0} = 0$ [Necessary to solve 1st ODE wrt ξ, θ]

$\theta\big|_{\xi=0} = 0$ [complete solution of θ wrt ξ]

d). $n=3$, $\xi = 6.90$

$$\alpha = \left[\frac{K \rho_c^{-2/3}}{\pi G}\right]^{1/2}$$

$$R = \alpha \xi_3 = 6.9 \times \frac{\sqrt{K}}{\sqrt{\pi G \rho_c^{2/3}}}$$

$$K = \left[\frac{3}{a}\left(\frac{k}{\mu m_H}\right)^4 \frac{1-\beta}{\beta^4}\right]^{1/3}$$

$$R = \frac{6.9}{\sqrt{\pi G \rho_c^{2/3}}} \underbrace{\left(\frac{3}{a}\left(\frac{k}{\mu m_H}\right)^4 \left(\frac{1-\beta}{\beta^4}\right)\right)^{1/6}}$$ $\left[\beta = \frac{P_{gas}}{P} = \text{constant}\right]$

$$= \frac{1}{\rho_c^{1/3} \mu^{1/3}} \left(\frac{1-\beta}{\beta^4}\right)^{1/6} \times 64605.388$$

$$\boxed{\Rightarrow R = 44.11 R_\odot / \mu^{2/3} \left(\frac{1-\beta}{\beta^4}\right)^{1/6} \frac{1}{\rho_c^{1/3}}}$$

e). $M = \int_0^R 4\pi r^2 dr \rho$

$r = \xi\alpha$, $dr = \alpha\, d\xi$ $\qquad \alpha \equiv$ scaling parameter

$$M = \int_0^R 4\pi \xi^2 \alpha^2 \alpha\, d\xi\, \rho = 4\pi\alpha^3 \int_0^{6.9\alpha} \xi^2 d\xi\, \rho_c \theta^3$$

$\rho = \rho_c \theta^3$ [$n=3$] $\qquad$ Polytrope

$$M = 4\pi\alpha^3 \rho_c \int_0^{\xi} \xi^2 \theta^3 d\xi$$

But, $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^3$.

$$\Rightarrow \xi^2 \frac{d\theta}{d\xi}\bigg|_0^R = 2.02$$

$$M = 4\pi(2.02)\left(\frac{K\rho_c^{-2/3}}{\pi G}\right)^{3/2}$$

$$25.371 \frac{\rho_c}{\rho_c} \frac{1}{(\pi G)^{3/2}} \sqrt{\frac{3}{a}\left(\frac{K}{\mu m_H}\right)^4 \left(\frac{1-\beta}{\beta^4}\right)}$$

$$25.371 \frac{1}{\rho_c (\pi G)^{3/2}} \frac{1}{\mu^2} \left(\frac{1-\beta}{\beta^4}\right)^{1/2} \sqrt{\frac{3}{a} \frac{K^4}{m_H^4}}$$

standard density of white dwarf: $\rho_c \sim 10^{10}\, kg\, m^{-3}$

$$a^{-1} = \frac{c}{4\sigma} \Rightarrow M = \frac{18.21}{\mu^2} M_\odot \left(\frac{1-\beta}{\beta^4}\right)^{1/2}$$

f) Solar composition: $X_H = 0.73$

Heavy element $= 0.02$

He, $Y = 0.25$

Mean particle density: $\frac{2\rho X}{m_H} + \frac{3\rho Y}{4 m_H} + \frac{\rho Z}{2 m_H} = \frac{\rho}{m_H \mu}$

$$n = \frac{\rho}{\mu m_H} \quad \mu = \left(2X + \frac{3Y}{4} + \frac{Z}{2}\right)^{-1} = 0.603$$

g) For sun, $\frac{\mu^2}{18.1} = \left(\frac{1-\beta}{\beta^4}\right)^{1/2}$

$$\frac{1-\beta}{\beta^4} = \left(\frac{\mu^2}{18.1}\right)^2 = 4.035 \times 10^{-4}$$

For numerical sol^n: $x = 0.9995$ and -13.8508

$x \neq -13.8503$ [Negative fraction not allowed]

$$\beta = \frac{P_{gas}}{P_{total}}$$

$\Rightarrow$ for sun, gas pressure is entirely from gas and no electron degeneracy takes place.

h) Gas pressure at centre of Sun: $P = \frac{k \rho T}{\mu m_H}$

$\mu = 0.603$ [assuming composition at centre to be same]

$$P_{gas} = \frac{1.38 \times 10^{-23} \times 1.6 \times 10^5 \times 1.6 \times 10^7}{0.603 \times 1.67 \times 10^{-27}}$$

$$= 3.51 \times 10^{16}\, N$$

Energy density at centre : $U = aT^4 = 4.95 \times 10^{13}$

$P_{rad} = \frac{U}{3} = 1.65 \times 10^{13} N$

other Pressures such as magnetic, degenerate (e^- or n^0) pressures have less contributions

$$\beta = \frac{P_{gas}}{P_{total}} = \frac{3.508 \times 10^{16}}{3.508 \times 10^{16} \times 1.6513 \times 10^{13}} = \frac{3.508}{3.50985} = 0.994529$$

i) ~~Both~~ Solutions of both (g) and (h) match with certain degree of magnitude. However, in both the approach to solve is different. In first, Lane-Emden eq^n is used which is valid for polytropes which is not correct for sun. In second, we assumed average molecular weight at the centre which is not true as centre has large amount of heavy elements which will increase value of μ.

2)

a) $$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\chi \rho}{T_c^3} \frac{L_r}{4\pi r^2}$$

For white dwarf : $M = 1 M_\odot$
$R = 0.01 R_\odot$

$$\frac{T - T_c}{R - R_c} = \frac{-3}{4ac} \frac{\chi \rho}{T_c^3} \frac{L_r}{4\pi r^2}$$

$$T - T_c = -\frac{3}{4ac} \frac{\chi \rho}{T_c^3} \frac{L}{4\pi R^2}$$

From opacity table,

surface temperature $T << T_c$

$\chi = 0.02\ m^2 kg^{-1}$

general mass luminosity relation : $L \propto M^{3.5}$

$L_W = 3.8 \times 10^{30} W$

Avg. central density $\sim \frac{M_\odot}{(0.01 R_\odot)^3} = 5.85 \times 10^9 kg\,m^{-3}$

$$T_c^4 = \frac{3}{4ac} \frac{\chi \rho}{4\pi R^2} L$$

$$T_c = \left(\frac{3c}{4ac} \times \frac{0.02 \rho L}{4\pi (0.01 R_\odot)^2} \right)^{1/4} = 5.61 \times 10^7 K$$

b). If we compare threshold temperatures of various elements:

~~For H~~

For H, $T_{Th} \sim 10^7$ (sun's core temperature)

For He, T_h should be $\sim 10^8$

For triple alpha process, rest reaction procedures (carbon, Oxygen), $T_{Th} \sim 10^9$

Since, these high temperatures are not observed in white dwarfs, we can conclude that it is mainly composed of Helium and small amount of Hydrogen (unignited).

c). For $T \to 0$,

Distribution function is discontinuous

$\Rightarrow$ Calculation will involved states $E < E_F$

In calculation we consider distribution function do not include density of states

$$n = \int f(p)\,dp\, 4\pi p^2 = \frac{2}{V}\int \frac{4\pi p^2}{h^3} V f(p)\,dp$$

$$g(p) = \frac{8\pi p^2 V}{h^3}, \quad \bar{f}(p) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

$$T \to 0 \quad \bar{f}(p) = \begin{cases} 1 & E < E_F \\ 0 & E > E_F \end{cases} \quad \text{with} \quad \lim_{T\to 0} \mu = E_F$$

$$n = \int_0^{p_F} \frac{8\pi p^2}{h^3}\,dp = \frac{8\pi}{h^3}\frac{p_F^3}{3}, \quad p_F\,(2mE_F)^{1/2}$$

In non-relativistic limit for simplicity,

$$n = \frac{8\pi}{h^3}(2mE_F)^{3/2} = 8\pi\left(\frac{2m}{h^2}\right)^{3/2} E_F^{3/2}$$

Consider WD is mainly helium,

mass density : $n \times 4m_H = \frac{3M}{4\pi R^3}$ or $n = \frac{3M}{16\pi R^3 m_H}$

For white dwarf, $M = M_\odot$, $R = 0.01 R_\odot$

$$n = \frac{1.397 \times 10^9}{4m_H} = 2.091 \times 10^{35}$$

$$2mE_F = \left(\frac{2.09 \times 10^{35}}{8\pi} h^3\right)^{2/3} = 1.803 \times 10^{-44}$$

$$E_F = 1.979 \times 10^{-11}\,J = 1.286 \times 10^8\,eV$$

~~For~~ $E_F = \frac{3}{2} K T_F$

$T_F = 9.560 \times 10^{11} K$

For temperature $T < T_F$, fermions can produce degenerate matter in WD.

$T_F = 9.56 \times 10^{11} K >> T$ of white dwarf, so we do not need lim $T \to 0$,

making distribution f^m $\breve{f}(\epsilon) \to 1$

sommerfeld expansion, $E = E_F \left(1 - \left(\frac{T}{T_F}\right)^2 \times F + \cdots \right)$

2nd term is small correction for $T << T_F$, so finite temp inside WD is not a problem.

d) ρ = mean density

μ = composition.

Gas pressure : $P_{gas} = \frac{\rho}{\mu m_n} KT$

Non-relativistic degenerate pressure : $P_{nr} = \frac{10^7}{\mu^{5/3}} \rho^{5/3}$

$$\frac{P_{gas}}{P_{nr}} = 10^{-7} \times \mu^{2/3} \frac{K}{\rho^{2/3}} T \times \frac{1}{m_n}$$

$\mu \simeq 2$, $\rho = \langle \rho \rangle = 1.397 \times 10^9$

$$\frac{P_{gas}}{P_{nr}} = 1.049 \times 10^{-9} T$$

$K \sim 10^{-23}$ in WD, with core temp $\sim 10^7$

P_{gas} is 10^{-2} order less than degenerate pressure.

$$P_r = \frac{1.24 \times 10^{10}}{\mu^{4/3}}$$

$$\frac{P_{gas}}{P_r} = (1.24 \times 10^{10})^{-1} \times \mu^{1/3} \frac{K}{\rho^{1/3}} \frac{T}{m_n} = 7.51 \times 10^{-16} T$$

If relativistic effect are considered (degenerate pressure)

For $T_c \sim 10^7$, P_{gas} is 10^{-3} order with degenerate pressure.

3). Planet distribution in the radius- period plane

→ Above equation is used to derive planet frequencies in the Baysian framework.

$$\langle p \rangle = \frac{\int_{R_{p,min}}^{R_{p,max}} \int_{P_{min}}^{P_{max}} (R_\odot/a)\, S(P, R_p)\, d\ln P\, d\ln R_p}{\int_{R_{p,min}}^{R_{p\,max}} \int_{P_{min}}^{P_{max}} d\ln P\, d\ln R_p}$$

→ Parameter space in the radius - period plane is divided into logarithmically equally spaced cells.

N_p = number of planet detection / cell

$\langle p \rangle$ = average detection efficiency.

$R_{p,min}$, $R_{p\,max}$, P_{min}, P_{max} ≡ Denote boundaries of the cell,

$R_\odot/a$ = transit geometric probability at semi-major axis a around sun-like host.

$S(P, R_p)$ ≡ sensitivity due to survey detection at given period, radius.

→ Planet radius ($R_\oplus$) is plotted against orbital period to derive results for planets in given survey.