Binomial theorem:

$x+y$ is called as binomial

$x+y+z$ is called as trinomial

$x+y+z+\dots$ is called as multinomial,

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2+2xy+y^2$

$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

$(x+y)^4 = \dots$

$(x+y)^5 = \dots$

$n=0$ 1 $\longrightarrow$ 1

$n=1$ 1   1 $\longrightarrow (1+1) = 2 = 2^1$

$n=2$ 1   2   1 $= (1+2+1) = 4 = 2^2$

$n=3$ 1   3   3   1   Sum $= 8 = 2^3$

$n=4$ 1   4   6   4   1   Sum $= 16 = 2^4$

$n=5$ 1   5   10   10   5   1   Sum $= 32$

This method is called as pascal's triangle.

$$(x+y)^4 = 1\cdot x^4 + 4x^3\cdot y + 6x^2y^2 + 4xy^3 + 1\cdot y^4.$$

$$(x+y)^5 = 1\cdot x^5 + 5\cdot x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$\boxed{nC_r = \frac{n!}{r!\cdot(n-r)!} = nC_{n-r}}$$

where $n!$ = product of first n natural no.s

and $0! = 1$

Properties of nC_r:

(i) $nC_r = nC_{n-r}$

(ii) $nC_0 = nC_n = 1$

(iii) $nC_1 = nC_{n-1} = n$

(iv) $nC_2 = \frac{n!}{(n-2)!\cdot 2!} = \frac{n\cdot(n-1)}{1\cdot 2} = nC_{n-2}$

$$nC_3 = \frac{n\cdot(n-1)(n-2)}{1\cdot 2\cdot 3} = nC_{n-3}$$

(v) $${}^{n}C_r = \frac{n}{r}\cdot {}^{(n-1)}C_{r-1}$$

Pf: $${}^{n}C_r = \frac{n!}{r!\cdot(n-r)!} = \frac{n\cdot(n-1)!}{r\cdot(r-1)!\cdot(n-r)!}$$

$$= \frac{n}{r}\cdot\frac{(n-1)!}{(r-1)!\,((n-1)-(r-1))!}$$

$$= \frac{n}{r}\cdot {}^{(n-1)}C_{r-1}.$$

(vi) $$\boxed{{}^{n}C_r + {}^{n}C_{r+1} = {}^{(n+1)}C_{r+1}}$$

$${}^{n}C_r + {}^{n}C_{r+1} = \frac{n!}{(n-r)!\cdot r!} + \frac{n!}{(n-r-1)!\,(r+1)!}$$

$$= \frac{n!}{r!\cdot(n-r-1)!}\left[\frac{1}{(n-r)\cdot 1} + \frac{1}{1\cdot(r+1)}\right]$$

$$= \frac{n!}{r!\cdot(n-r-1)!}\cdot\frac{(n-\cancel{r}+\cancel{r}+1)}{(n-r)\cdot(r+1)}$$

$$= \frac{(n+1)!}{(r+1)!\,(n-r)} = {}^{(n+1)}C_{r+1}$$

14 cricket probables $\rightarrow$ 11 players team.

$^{14}C_{11}$ ways we can choose 11 players from 14 members

with kohli

From 13 players remaining 10 we can choose in $^{13}C_{10}$ ways

without kohli

from 13 players we can choose 11 players in $^{13}C_{11}$ ways.

$$^{13}C_{10} + {^{13}C_{11}} = {^{14}C_{11}}$$

$$^{n}C_{r} + {^{n}C_{r+1}} = {^{(n+1)}C_{r+1}}$$

From (n+1) objects (r+1) objects we can select in $^{(n+1)}C_{r+1}$ ways.

If a particular object always included then it can be done in $^{n}C_{r}$ ways.

& If same particular object always not included in $^{n}C_{r+1}$ ways.

$$\Rightarrow \quad {^{n}C_{r}} + {^{n}C_{r+1}} = {^{(n+1)}C_{r+1}}$$

$$^{14}C_{11} = \{K, R, ..\} + \{K, R^c ..\} + \{R, K^c, ..\} + \{K^c, R^c, ..\}$$

$$= \underbrace{^{12}C_{9} + {^{12}C_{10}}}_{^{13}C_{10}} + \underbrace{^{12}C_{10} + {^{12}C_{11}}}_{^{13}C_{11}} = {^{14}C_{11}}$$

$$^{(n+2)}C_{r+2} = {}^nC_r + {}^nC_{r+1} + {}^nC_{r+1} + {}^nC_{r+2}$$

(vii) $$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}$$

(viii) $$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{\cancel{n!}}{(n-r)!\cdot r!} \cdot \frac{(r+1)!\cdot(n-r-1)!}{\cancel{n!}}$$

$$= \frac{(r+1)\cdot \cancel{r!}\cdot \cancel{(n-r-1)!}}{\cancel{r!}\cdot(n-r)\cdot \cancel{(n-r-1)!}}$$

$$= \frac{r+1}{n-r}.$$

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}.$$ → max subscript

difference of other combination

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-(r-1)}{r} = \frac{n-r+1}{r}$$

$1 \quad 1 \rightarrow {}^1C_0 \quad {}^1C_1$

$1 \quad 2 \quad 1 \rightarrow {}^2C_0 \quad {}^2C_1 \quad {}^2C_2$

$1 \quad 3 \quad 3 \quad 1 \quad {}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3$

Coefficients of $(x+y)^4 = {}^4C_0, {}^4C_1, {}^4C_2, {}^4C_3, {}^4C_4.$

$(x+y)^4 = {}^4C_0 \cdot x^4 + {}^4C_1 \cdot x^3 \cdot y^1 + {}^4C_2 \cdot x^2 \cdot y^2 + {}^4C_3 \cdot x^1 \cdot y^3 + {}^4C_4 \cdot y^4.$

$||^y \quad (x+y)^n = {}^nC_0 \cdot x^n + {}^nC_1 \cdot x^{n-1} \cdot y + {}^nC_2 \, x^{n-2} \cdot y^2 + {}^nC_3 \cdot x^{n-3} \cdot y^3$
$+ \cdots + {}^nC_r \cdot x^{n-r} \cdot y^r + \cdots + {}^nC_{n-1} \, x \, y^{n-1} + {}^nC_n \cdot y^n$

$$\boxed{(x+y)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^{n-r} \cdot y^r}$$

This Proof we can do by mathematical induction.

Properties:

① $nC_r \cdot x^{n-r} \cdot y^r$ is called as general term of binomial expansion. And it is $(r+1)^{th}$ term of expansion. and it is denoted by T_{r+1}.

$$T_{r+1} = nC_r \cdot x^{n-r} \cdot y^r$$

② no. of terms in $(x+y)^n$ expansion $= n+1$

③ In $(x+y)^n$ expansion exponents of x are decreasing from n to 0 and exponents of y are increasing from 0 to n and sum of exponents of x, y in each term $= n$.

④ $nC_0, nC_1, nC_2 \ldots nC_n$ are called as binomial coefficients of expansion &

$$nC_0 + nC_1 + nC_2 + \ldots + nC_n = 2^n$$

$$(x+y)^n = \sum_{r=0}^{n} nC_r \cdot x^{n-r} \cdot y^r$$

Sub $x=1, y=1$ in expansion then

$$\sum_{r=0}^{n} nC_r \cdot (1)^{n-r} \cdot (1)^r = (1+1)^n$$

$$\Rightarrow \boxed{\sum_{r=0}^{n} nC_r = 2^n.}$$

(5) $(x-y)^n$ expansion sub $y=-y$ in for $(x+y)^n$ expansion

$$\Rightarrow (x-y)^n = nC_0 \cdot x^n - nC_1 \cdot x^{n-1} \cdot y + nC_2 \cdot x^{n-2} \cdot y^2 - nC_3 \cdot x^{n-3} \cdot y^3 + \cdots + nC_r \cdot x^{n-r} \cdot (-y)^r + \cdots + nC_n (-y)^n.$$

Sub $x=y=1$

$$(1-1)^n = nC_0 - nC_1 + nC_2 - nC_3 \cdots$$

$$\Rightarrow nC_0 - nC_1 + nC_2 - nC_3 + \cdots = 0$$

$$\Rightarrow nC_0 + nC_2 + nC_4 + \cdots = nC_1 + nC_3 + nC_5 + \cdots$$

$$\Rightarrow \boxed{nC_0 + nC_2 + nC_4 + \cdots = nC_1 + nC_3 + nC_5 + \cdots = 2^{n-1}}$$

(6) 1st from ending = $(n+1)^{th}$ term from starting
2nd " " = n^{th} " " starting

r^{th} from ending of expansion

$= (n+2-r)^{th}$ term from starting.

$$\boxed{T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot y^r}$$

⑦ middleterm:

$(x+y)^{10}$ middle term is 6^{th} term

& $10C_5 \cdot x^5 \, y^5$

If n is even then $\left(\frac{n}{2}\right)+1^{th}$ term

is middle term of $(x+y)^n$.

And middle term $= {}^nC_{\frac{n}{2}} \cdot x^{\frac{n}{2}} \cdot y^{\frac{n}{2}}$.

$(a+y)^3$ middle terms are 2, 3 terms

$(x+y)^5$ middle terms are 3, 4 term.

If n is odd then $(x+y)^n$ middle terms are.

$\frac{n+1}{2}$, $\frac{n+3}{2}$ terms

$$T_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}} \cdot x^{\frac{n+1}{2}} \cdot y^{\frac{n-1}{2}}$$

$$T_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}} \cdot x^{\frac{n-1}{2}} \cdot y^{\frac{n+1}{2}}.$$

(8) greatest binomial coefficient:

$^5C_0, \; ^5C_1, \; ^5C_2, \; ^5C_3, \; ^5C_4, \; ^5C_5$

$\downarrow \; 1, \quad \downarrow \; 5, \quad \downarrow \; 10, \quad \downarrow \; 10, \quad \downarrow \; 5, \quad \downarrow \; 1$

$^6C_0 \quad ^6C_1 \quad ^6C_2 \quad ^6C_3 \quad ^6C_4 \quad ^6C_5 \quad ^6C_6$

$\downarrow \; 1 \quad \downarrow \; 6 \quad \downarrow \; 15 \quad \downarrow \; 20 \quad \downarrow \; 15 \quad \downarrow \; 6 \quad \downarrow \; 1$

If n is even then only one greatest binomial coefficient is possible and it is $^nC_{\frac{n}{2}}$.

If n is odd then 2 greatest binomial coefficients are possible and they are

$^nC_{\frac{n-1}{2}}, \quad ^nC_{\frac{n+1}{2}}$.

Expand $(2x-3y)^6$

$$(2x-3y)^6 = {}^6C_0 \cdot (2x)^6 - {}^6C_1 \cdot (2x)^5 \cdot (3y) + {}^6C_2 \cdot (2x)^4 \cdot (3y)^2$$
$$- {}^6C_3 \cdot (2x)^3 \cdot (3y)^3 + {}^6C_4 \cdot (2x)^2 (3y)^4 - {}^6C_5 \cdot 2x \cdot (3y)^5$$
$$+ {}^6C_6 \cdot (3y)^6$$

Find the Coefficient of x^5 in the expansion of $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$

General term

$$T_{r+1} = {}^nC_r \cdot (x)^{n-r} \cdot (y)^r$$

$$= {}^{10}C_r \cdot \left(\frac{x}{2}\right)^{10-r} \cdot \left(-\frac{3}{x^2}\right)^r$$

$$= {}^{10}C_r \cdot x^{10-r} \cdot \frac{1}{x^{2r}} \cdot \frac{(-3)^r}{2^{10-r}}$$

$$= {}^{10}C_r \cdot x^{10-3r} \cdot \frac{(-3)^r}{2^{10-r}}$$

As x^5 coefficient is required

$$10-3r = 5$$
$$\Rightarrow 3r = 5 \Rightarrow r = \frac{5}{3}.$$

As $r \in W$ $r = \frac{5}{3}$ is not possible

$\Rightarrow$ No x^5 term in the expansion.

$\Rightarrow$ x^5 Coefficient $= 0$.

Find x^{-5} Coefficient in above expansion

$$10 - 3r = -5 \Rightarrow r = 5$$

$\Rightarrow$ $T_{5+1} = 6^{th}$ term has x^{-5} term.

$$\Rightarrow T_{5+1} = {}^{10}C_5 \cdot x^{-5} \cdot \frac{(-3)^5}{2^5} =$$

$$\text{Coefficient of } x^{-5} = -\left(\frac{3}{2}\right)^5 \cdot {}^{10}C_5$$

Find the Constant term (Independent from x)

in the expansion of $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{2022}$

$$T_{r+1} = {}^{2022}C_r \cdot \left(\frac{x}{2}\right)^{2022-r} \cdot \left(-\frac{3}{x^2}\right)^r$$

$$= {}^{2022}C_r \cdot \frac{(-3)^r}{2^{2022-r}} \cdot x^{2022-3r}$$

As constant term is required, exponent of $x = 0$

$\Rightarrow \quad 2022 - 3r = 0$

$\Rightarrow \quad r = \frac{2022}{3} = 674$

$\Rightarrow$ 675^{th} term is constant in expansion

$\Rightarrow \quad T_{675} = {}^{2022}C_{674} \cdot \frac{(-3)^{674}}{2^{2022-674}}$

$= {}^{2022}C_{674} \cdot \frac{3^{674}}{2^{1348}}$

Find the term which is independent of x in the expansion of $(1+x+2x^3)\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$

$(1+x+2x^3)\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$

$= (1+x+2x^3) \sum_{r=0}^{9} {}^9C_r \cdot \left(\frac{3x^2}{2}\right)^{9-r} \cdot \left(\frac{-1}{3x}\right)^r$

$= (1+x+2x^3)\left(\sum_{r=0}^{9} {}^9C_r \cdot \left(\frac{3}{2}\right)^{9-r} \cdot \left(-\frac{1}{3}\right)^r \cdot x^{18-3r}\right)$

$= (1+x+2x^3)\left(\text{constant} + \frac{1}{x}\text{ term} + \frac{1}{x^3}\text{ term} + \ldots\right)$

For constant term in $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$ expansion

$18 - 3r = 0 \Rightarrow r = 6$

$\Rightarrow$ constant term in $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$ is $9C_6 \cdot \left(\frac{3}{2}\right)^{9-6} \cdot \left(-\frac{1}{3}\right)^6$

$\Rightarrow \quad 9C_6 \cdot \frac{1}{2^3} \cdot \frac{1}{3^3} = \frac{\cancel{9} \times \cancel{8} \times 7}{1 \times 2 \times 3} \times \frac{1}{\cancel{8}} \times \frac{1}{\cancel{27}\,3}$

$= \frac{7}{18}$

For $\frac{1}{x}$ term in expansion $18 - 3r = -1$

$\Rightarrow 3r = 19$

$\Rightarrow r = \frac{19}{3}$

As $r = \frac{19}{3}$ is not possible $\frac{1}{x}$ coefficient $= 0$

For $\frac{1}{x^3}$ term in expansion $18 - 3r = -3$

$\Rightarrow 3r = 21$

$\Rightarrow r = 7$

$\Rightarrow \frac{1}{x^3}$ coefficient is

$9C_7 \cdot \left(\frac{3}{2}\right)^2 \cdot \left(-\frac{1}{3}\right)^7$

$= \frac{9 \times \cancel{8}}{\cancel{2}} \times \frac{3^2}{\cancel{2^2}} \times \frac{-1}{3^7}$

$$= -\frac{1}{3^3} = -\frac{1}{27}$$

$$(1+x+2x^3)\left(\frac{7}{18} + 0\cdot\frac{1}{x} - \frac{1}{27}\cdot\frac{1}{x^3} + \cdots\right)$$

$$= \frac{7}{18} - \frac{2}{27} = \frac{21-4}{54} = \frac{17}{54}$$

Find the number of terms in the expansion of $(x+y)^n + (x-y)^n$, $(x+y)^n - (x-y)^n$ when n is even, odd.

$$(x+y)^n + (x-y)^n = 2\left({}^nC_0 \cdot x^n + {}^nC_2 \cdot x^{n-2} \cdot y^2 + \cdots + y^n\right)$$

Total terms when n is even $= \frac{n}{2} + 1$

If n is odd $= 2\left({}^nC_0 \cdot x^n + {}^nC_2 \cdot x^{n-2} \cdot y^2 + \cdots + {}^nC_{n-1} \cdot x y^{n-1}\right)$

$= \frac{n+1}{2}$ terms

$$(x+y)^n - (x-y)^n = 2\left({}^nC_1 \cdot x^{n-1} y + {}^nC_3\, x^{n-3} \cdot y^3 + \cdots\right)$$

2nd, 4th, 6th terms, , , ,

If n is even then total terms $= \frac{n}{2}$.

If n is odd " " $= \frac{n+1}{2}$.

$(x+y)^n + (x-y)^n$ — n is even: $\frac{n}{2}+1$; n is odd: $\frac{n+1}{2}$

$(x+y)^n - (x-y)^n$ — n is even: $\frac{n}{2}$; n is odd: $\frac{n+1}{2}$.

Find the number of rational terms in the expansion of $\left(2^{1/5} + 3^{1/10}\right)^{55}$.

$$\left(2^{\frac{1}{5}} + 3^{\frac{1}{10}}\right)^{55} = \sum_{r=0}^{55} {}^{55}C_r \cdot \left(2^{\frac{1}{5}}\right)^{55-r} \cdot \left(3^{\frac{1}{10}}\right)^{r}$$

$$= \sum_{r=0}^{55} {}^{55}C_r \cdot 2^{11-\left(\frac{r}{5}\right)} \cdot (3)^{\frac{r}{10}}.$$

To get rational terms $\frac{r}{5}, \frac{r}{10}$ must be integers. $\Rightarrow r = 0, 10, 20, 30, 40, 50.$

$\Rightarrow$ number of rational terms $= 6$

number of irrational terms $= 50$

Find the number of rational & irrational terms in the expansion of $\left(3^{1/6} + 2^{1/8}\right)^{100}$

$$\left(3^{\frac{1}{6}} + 2^{\frac{1}{8}}\right)^{100} = \sum 100C_r \cdot \left(3^{\frac{1}{6}}\right)^{100-r} \cdot \left(2^{\frac{1}{8}}\right)^{r}$$

$\frac{100-r}{6}$, $\frac{r}{8}$ both are integers,

If $r = 0, 8$ $\frac{100-r}{6}$ is not integer.

$r = 16$ $\frac{100-16}{6} = \frac{84}{6} = 14$ is integer

$r = 16+24$ $\frac{100-(16+24)}{6} = \frac{100-16}{6} + \frac{24}{6}$ integer

$r = 16,\ 16+24,\ 16+48,\ 16+72$ are giving integers

$\Rightarrow$ If $r = 16, 40, 64, 88$ then terms are rational

$\Rightarrow$ 4 terms are rational and 97 terms are irrational.

Find the coefficient of x^{48} in $(1+x)^{101}(1-x+x^2)^{100}$

$$(1+x)\left((1+x)(1-x+x^2)\right)^{100} = (1+x)\left(1+x^3\right)^{100}$$

$$= (1+x)\left(1 + 100C_1 \cdot x^3 + 100C_2 \cdot (x^3)^2 + \cdots + 100C_r \cdot (x^3)^r + \cdots\right)$$

$3r = 48 \Rightarrow r = 16$

$$\Rightarrow \underbrace{(1+x)\left(1 + {}^{100}C_1 x^3 \cdots + {}^{100}C_{15} \cdot x^{45} + {}^{100}C_{16} \cdot x^{48} + {}^{100}C_{17} x^{51} \cdots\right)}$$

x^{48} Coefficient $= 1 \cdot {}^{100}C_{16} = {}^{100}C_{16}$.

Find the remainder when 5^{99} is divided by 26.

$$5^{99} = (5^2)^{49} \cdot 5$$

$$= (26-1)^{49} \times 5$$

$$= \left((26)^{49} - {}^{49}C_1 \cdot (26)^{48} + \cdots + 49 \cdot 26 + (-1)^{49}\right) \cdot 5$$

$$= (26k - 1) \cdot 5$$

$$= 26(5k) - 5.$$

Remainder is $-5 = (-26) + \underline{\underline{21}}$

21 is the remainder.

Find last two digits of 27^{256}

$$27^{256} = (3^3)^{256} = 3^{3 \times 256} = (3^4)^{3 \times 64}$$

$$= (81)^{192} = (80+1)^{192} = (1+80)^{192}$$

$$= 1 + 192 \cdot 80 + {}^{192}C_2 \cdot (80)^2 + \cdots$$

$$= 1 + 15360 + 100K$$

$$= 100k + 15361$$

$$= 100t + 61$$

$\Rightarrow$ last two digits are 61.

Find the value of $\left\{ \frac{3^{2021}}{28} \right\}$

$$3^{2021} = 9 \times 3^{2019} = 9 \times (3^3)^{673} = 9 \times (27)^{673}$$

$$3^{2021} = 9 \times (28-1)^{673} = 9(28t - 1)$$

$$= 28 \times 9t - 9$$

$$= 28 \times 9t - 28 + 19$$

$$\Rightarrow 3^{2021} = 28K + 19 \quad K \in \mathbb{N}$$

$$\Rightarrow \frac{3^{2021}}{28} = K + \frac{19}{28}$$

$$\Rightarrow \left\{ \frac{3^{2021}}{28} \right\} = \frac{19}{28}.$$

$x^4 - x^3 + 2x^2 + 3x + 1$

$$\frac{\Sigma\alpha^3 - \Sigma(\alpha\beta\gamma)^3}{\Sigma(\alpha\beta)^3 - 30(\alpha\beta\gamma\delta)^3}.$$

Find eq where roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ in give polynomial

Replace x with $x^{1/3}$

$$x^{4/3} - x + 2x^{2/3} + 3x^{1/3} + 1 = 0$$

$$x - 1 = x^{4/3} + 2x^{2/3} + 3x^{1/3}$$

$$(x-1) = x^{1/3}(x + 2 \cdot x^{1/3} + 3)$$

$$(x-1)^3 = x \cdot ((x+3) + 2 \cdot x^{1/3})^3$$

$$(x-1)^3 = x \cdot \left[(x+3)^3 + 8 \cdot x + 3 \cdot (x+3) 2x^{1/3} (x+3+2x^{1/3})\right]$$

$$(x-1)^3 = x\left[(x+3)^3 + 8x + 3(x+3) \cdot 2\cancel{x^{1/3}} \cdot \frac{x-1}{\cancel{x^{1/3}}}\right]$$

$(x-1)^3 = x\left((x+3)^2 + 8x + 3(x+3)(x-1)\right)$

eq roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$.

Find the coefficient of x^{20} in the expansion of $\sum_{r=0}^{100} 100C_r (2x-3)^r \cdot (2-x)^{100-r}$

$= (2x-3 + 2-x)^{100}$

$= (x-1)^{100} = (1-x)^{100}$

x^{20} coefficient is $100C_{20} \cdot (-1)^{20} = 100C_{20}$

$= 100C_{80}$

$\boxed{x^r \text{ Coefficient in } (1+x)^n = nC_r}$

$\boxed{x^r \text{ Coefficient in } (1-x)^n = nC_r \cdot (-1)^r}$

Find the Coefficient of x^{22} in the expansion of

$1 + (1+x) + (1+x)^2 + (1+x)^3 + \dots + (1+x)^{2021}$

Method 1:

$a = 1, \quad r = 1+x, \quad n = 2022$

$$1 + (1+x) + \cdots + (1+x)^{2021} = \frac{1 \cdot \left((1+x)^{2022} - 1\right)}{(1+x) - 1}$$

$$= \frac{(1+x)^{2022} - 1}{x}$$

x^{22} coefficient in $\dfrac{(1+x)^{2022} - 1}{x}$

$= x^{23}$ coefficient in $(1+x)^{2022} - 1$

$= x^{23}$ " " $(1+x)^{2022}$

$= {}^{2022}C_{23}$

x^k coefficient in $1 + (1+x) + (1+x)^2 + \cdots + (1+x)^n$

$= {}^{(n+1)}C_{k+1}$

Method 2:

x^{22} coefficient in $1 + (1+x) + \cdots + (1+x)^{2021}$

$= x^{22}$ in $(1+x)^{22} + (1+x)^{23} + \cdots + (1+x)^{2021}$

$= x^{22}$ in $(1+x)^{22} + x^{22}$ in $(1+x)^{23} + \cdots + x^{22}$ in $(1+x)^{2021}$

$= {}^{22}C_{22} + {}^{23}C_{22} + {}^{24}C_{22} + \cdots + {}^{2021}C_{22}$

$\boxed{{}^{n}C_{r} + {}^{n}C_{r+1}} = {}^{(n+1)}C_{r+1}$

$= {}^{23}C_{23} + {}^{23}C_{22} + {}^{24}C_{22} + \cdots + {}^{2021}C_{22}$

$= {}^{24}C_{23} + {}^{24}C_{22} + {}^{25}C_{22} + \cdots + {}^{2021}C_{22}$

$= {}^{25}C_{23} + {}^{25}C_{22} + \cdots + {}^{2021}C_{22}$

$= {}^{2021}C_{23} + {}^{2021}C_{22}$

$= {}^{2022}C_{23}$

If 2^{nd}, 3^{rd}, 4^{th} terms of $(x+a)^n$ are 240, 720, 1080 then find x, a, n.

${}^{n}C_{1} \cdot x^{n-1} \cdot a = 240$ — (i)

$\frac{n(n-1)}{2} \cdot x^{n-2} \cdot a^2 = 720 \Rightarrow n(n-1)\, x^{n-2} \cdot a^2 = 720 \times 2$ — (ii)

$\frac{n(n-1)(n-2)}{6} \cdot x^{n-3} \cdot a^3 = 1080 \Rightarrow n(n-1)(n-2)\, x^{n-3} \cdot a^3 = 1080 \times 6$ — (iii)

$\frac{(ii)}{(i)}$: $\frac{n(n-1) \cdot x^{n-2} \cdot a^2}{n \cdot x^{n-1} \cdot a} = \frac{720 \times 2}{240}$

$\Rightarrow (n-1) \cdot \frac{a}{x} = 6$

$\frac{(iii)}{(ii)}$. $\quad (n-2) \cdot \frac{a}{x} = \frac{\cancel{1080}^{9} \times \cancel{6}^{3}}{\cancel{720} \times \cancel{4}_{2}}$

$$(n-2) \cdot \frac{a}{x} = \frac{9}{2}$$

$$(n-2) \cdot \frac{\cancel{6}^{2}}{n-1} = \frac{\cancel{9}^{3}}{2}$$

$$4n - 8 = 3n - 3$$

$$\Rightarrow n = 5$$

$$4 \cdot \frac{a}{x} = 6 \Rightarrow x = \frac{2a}{3}.$$

$$5 \cdot \left(\frac{2a}{3}\right)^4 \cdot a = 240 \Rightarrow 16 \cdot \frac{a^4 \cdot a}{81} = 48$$

$$\Rightarrow a^5 = 243$$

$$\Rightarrow a = 3$$

$$\Rightarrow x = 2.$$

*** Let n be any positive integer such that $(9+4\sqrt{5})^n = I + f$ where I is integer & $0 < f < 1$ then show that I is odd & $(I+f)(1-f) = 1$

$$I + f = (9+4\sqrt{5})^n \quad \text{— (i)}$$

let $G = (9-4\sqrt{5})^n$ — (ii)

$$9-4\sqrt{5} = \sqrt{81}-\sqrt{80} < 1$$

$$0 < 9-4\sqrt{5} < 1$$

$$\Rightarrow \quad 0 < (9-4\sqrt{5})^n < 1$$

$$\Rightarrow \quad 0 < G < 1$$

By adding (i), (ii)

$$I+f+G = (9+4\sqrt{5})^n + (9-4\sqrt{5})^n$$

$$= \left({}^nC_0 \cdot 9^n + {}^nC_1 \cdot \cancel{9^{n-1} \cdot 4\sqrt{5}} + {}^nC_2 \cdot 9^{n-2} \cdot (4\sqrt{5})^2 + {}^nC_3 \cdot 9^{n-3} \cdot \cancel{(4\sqrt{5})^3} + \cdots\right)$$

$$+ \left({}^nC_0 \cdot 9^n - {}^nC_1 \cdot \cancel{9^{n-1} \cdot 4\sqrt{5}} + {}^nC_2 \cdot 9^{n-2} (4\sqrt{5})^2 - {}^nC_3 \cdot 9^{n-3} \cdot \cancel{(4\sqrt{5})^3} + \cdots\right)$$

$$\Rightarrow I+f+G = 2\left({}^nC_0 \cdot 9^n + {}^nC_2 \cdot 9^{n-2} \cdot (4\sqrt{5})^2 + \cdots\right)$$

$I + f + G$ is even integer
↓
integer

$\Rightarrow f + G$ is integer.

$0 < f < 1$ & $0 < G < 1$

$\Rightarrow 0 < f + G < 2$

$\Rightarrow \underline{f + G = 1}$

$\Rightarrow I + 1$ is even integer

$\Rightarrow I$ is odd $\Rightarrow \underbrace{\left[(9 + 4\sqrt{3})^n\right]}_{\text{g.i.f}}$ is odd

$$(I+f)(1-f) = (I+f) \cdot G$$
$$= (9+4\sqrt{3})^n (9-4\sqrt{3})^n$$
$$= (81-80)^n = 1$$

let $(5\sqrt{5}+11)^n = I + f$ & n is odd

then prove $(I+f) \cdot f$ is even.

I is integer and $0 < f < 1$.

$I + f = (5\sqrt{5}+11)^n$

let $G = (5\sqrt{5}-11)^n$

$5\sqrt{5}-11$

$= \sqrt{125} - \sqrt{121} < 1$

$\Rightarrow 0 < (5\sqrt{5}-11)^n < 1$

$\Rightarrow 0 < G < 1$

$I + f + G = 2\left[(5\sqrt{5})^n + \ldots\right]$

$I + f + G$ is not a integer.

$I + f - G = (5\sqrt{5}+11)^n - (5\sqrt{5}-11)^n$

$= 2\left({}^nC_1 \cdot (5\sqrt{5})^{n-1} \cdot 11 + {}^nC_3 \cdot (5\sqrt{5})^{n-3} \cdot 11^3 + \ldots \right)$

$= 2 \times \text{Integer} \quad \left[\because (5\sqrt{5})^{\text{even}} = \text{integer}\right]$

$\Rightarrow I + f - G$ is even integer. & I is integer

$\Rightarrow f - G$ is integer.

$0 < f < 1$ & $0 < G < 1$

$\Rightarrow -1 < f - G < 1$

$\Rightarrow f - G = 0$

$\Rightarrow \underline{f = G}$

$\Rightarrow I$ is even integer.

$(I+f) \cdot f = (I+f) \cdot G$

$$= (5\sqrt{5}+11)^n (5\sqrt{5}-11)^n$$

$$= (125-121)^n$$

$$= 4^n$$

$\Rightarrow (I+f)f$ is even integer.

Find the integral part of $(\sqrt{2}+1)^6$

$I+f = (\sqrt{2}+1)^6, \quad G = (\sqrt{2}-1)^6$

$0 < f < 1, \quad 0 < G < 1$

$0 < f+G < 2$

$$I+f+G = (\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$$

$$\Rightarrow I+f+G = 2\left(6C_0 \cdot (\sqrt{2})^6 + 6C_2 \cdot (\sqrt{2})^4 + 6C_4 \cdot (\sqrt{2})^2 + 6C_6 \cdot 1\right)$$

$\Rightarrow I+f+G$ is even no $\Rightarrow f+G$ is integer

$\Rightarrow f+G = 1$

$$\Rightarrow I+1 = 2(8 + 15\cdot 4 + 15\cdot 2 + 1)$$

$$\Rightarrow I+1 = 2\times 99 = 198$$

$$\Rightarrow I = 197 \quad \Rightarrow \left[(\sqrt{2}+1)^6\right] = 197.$$

$(\sqrt{3}+1)^{20} = I + f$ then show that

2^{10} divides $I+1$

$0 < f < 1, \quad 0 < G < 1$

let $G = (\sqrt{3}-1)^{20}$

$f + G = 1$

as $f+G$ is integer

$\Rightarrow I + f + G = (\sqrt{3}+1)^{20} + (\sqrt{3}-1)^{20}$

$\Rightarrow I + 1 = \left((\sqrt{3}+1)^2\right)^{10} + \left((\sqrt{3}-1)^2\right)^{10}$

$(\sqrt{3}+1)^2$

$= 3+1+2\sqrt{3}$

$= 4+2\sqrt{3}$

$(\sqrt{3}-1)^2 = 4-2\sqrt{3}$

$= (4+2\sqrt{3})^{10} + (4-2\sqrt{3})^{10}$

$= 2^{10}(2+\sqrt{3})^{10} + (2-\sqrt{3})^{10} \cdot 2^{10}$

$= 2^{10}\left[(2+\sqrt{3})^{10} + (2-\sqrt{3})^{10}\right]$

$= 2^{10}\left[2 \cdot \left({}^{10}C_0 \cdot 2^{10} + {}^{10}C_2 \cdot 2^8 \cdot 3 + \cdots\right)\right]$

$= 2^{11} \times \text{Integer}$

$\Rightarrow 2^{11}$ divides $I+1$

Numerically greatest term

$$(1+2)^6 = 1 + 6C_1 \cdot 2 + 6C_2 \cdot 2^2 + 6C_3 \cdot 2^3 + 6C_4 \cdot 2^4 + 6C_5 \cdot 2^5 + 2^6$$

$$= 1 + 12 + 60 + 160 + \underline{240} + 192 + 64$$

240 is called as numerically greatest term in $(1+2)^6$ expansion.

In $(1-2)^6$ expansion also 240 is numerically greatest term

In $(1+x)^n$ expansion numerically greatest coefficient is $nC_{\frac{n}{2}}$ if n is even

$nC_{\frac{n-1}{2}}, nC_{\frac{n+1}{2}}$ if n is odd

Def: In the binomial expansion of $(1+x)^n$ the r^{th} term t_r is called as numerically greatest term if

$$|T_k| \leq |T_r| \quad \text{for all } k = 1, 2, 3 \ldots n+1$$

Theorem: let $n \in \mathbb{N}$ and $x \in \mathbb{R} - \{0\}$

and $m = \left[\frac{(n+1)|x|}{|x|+1}\right] \rightarrow$ g.i.f.

* If $\frac{(n+1)|x|}{|x|+1}$ is not a integer then T_{m+1} is numerically greatest term, in $(1+x)^n$

* If $\frac{(n+1)|x|}{|x|+1}$ is an integer then T_m, T_{m+1} are numerically greatest terms. in $(1+x)^n$

Pf: let T_r is numerically greatest term. then $|T_{r-1}| \leq |T_r| \geq |T_{r+1}|$

$$|T_{r-1}| \leq |T_r|$$

$$|{}^nC_{r-2} \cdot x^{r-2}| \leq |{}^nC_{r-1} \cdot x^{r-1}|$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} \leq \frac{|x|^{r-1}}{|x|^{r-2}}$$

$$\Rightarrow \quad \frac{r-1}{n-r+2} \le |x|$$

$$\Rightarrow \quad (r-1) \le |x|\left((n+1)-(r-1)\right)$$

$$\Rightarrow \quad (r-1) \le (n+1)|x| - (r-1)|x|$$

$$\Rightarrow \quad (r-1)(1+|x|) \le (n+1)|x|$$

$$\Rightarrow \quad r-1 \le \frac{(n+1)|x|}{1+|x|}$$

$$\Rightarrow \quad r \le \frac{(n+1)|x|}{|x|+1} + 1 \quad \text{——} \; (i)$$

$$|T_r| \ge |T_{r+1}|$$

$$\Rightarrow \quad |{}^nC_{r-1} \cdot x^{r-1}| \ge |{}^nC_r \cdot x^r|$$

$$\Rightarrow \quad \frac{{}^nC_{r-1}}{{}^nC_r} \ge \frac{|x|^r}{|x|^{r-1}}$$

$$\Rightarrow \quad \frac{r}{n-r+1} \ge |x|$$

$$\Rightarrow \quad r \ge (n+1)|x| - r|x|$$

$$\Rightarrow \quad r(1+|x|) \ge (n+1)|x|$$

$\Rightarrow \quad r \geq \dfrac{(n+1)|x|}{|x|+1} \quad - (ii)$

From (i), (ii)

$$r \in \left[\frac{(n+1)|x|}{|x|+1}, \frac{(n+1)|x|}{|x|+1} + 1 \right]$$

If $\dfrac{(n+1)|x|}{|x|+1}$ is an integer

Then $\left[\dfrac{(n+1)|x|}{|x|+1} \right] = m$

then $r \in [m, m+1] \Rightarrow T_m, T_{m+1}$ are numerically greatest terms.

If $\dfrac{(n+1)|x|}{|x|+1}$ is not an integer.

Then $\left[\dfrac{(n+1)|x|}{|x|+1}, \dfrac{(n+1)|x|}{|x|+1} + 1 \right]$ interval

has only one integer & that integer

is $\left[\dfrac{(n+1)|x|}{|x|+1} \right] + 1$

let $\left[\frac{(n+1)|x|}{|x|+1}\right] = m$ Then $r = m+1$

then T_{m+1} is numerically greatest term.

Find the numerically greatest term in the expansion of $(3-5x)^{15}$ when $x = \frac{1}{5}$.

$$(3-5x)^{15} = 3^{15}\left(1-\frac{5x}{3}\right)^{15}$$

Here $n = 15$, $x = -\frac{5x}{3}$

$$|x| = \left|\frac{-5\cdot\frac{1}{5}}{3}\right| = \frac{1}{3}$$

$$\frac{(n+1)|x|}{|x|+1} = \frac{(15+1)\frac{1}{3}}{\frac{1}{3}+1} = 4.$$

$\frac{(n+1)|x|}{|x|+1} = 4$ is an integer

$\Rightarrow$ T_4, T_5 both are numerically greatest terms.

$$T_4 = {}^{15}C_3 \cdot \left(-\frac{5x}{3}\right)^3 \cdot 3^{15}$$

$$= {}^{15}C_3 \cdot \left(-5\cdot\frac{1}{5}\right)^3 \cdot \frac{1}{3^3} \cdot 3^{15}$$

$$= -\,{}^{15}C_3 \cdot 3^{12}.$$

$$T_5 = {}^{15}C_4 \cdot \left(\frac{5x}{3}\right)^4 \cdot 3^{15}$$

$$= {}^{15}C_4 \cdot 3^{11}$$

$${}^{15}C_3 \cdot 3^{12} = \frac{15 \times 14 \times 13}{1 \times 2 \times 3} \times 3^{12}$$

$$= 15 \times 7 \times 13 \times 3^{11}$$

$${}^{15}C_4 \cdot 3^{11} = \frac{15 \times \cancel{14}^{7} \times 13 \times \cancel{12}}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \times 3^{11}$$

$$= 15 \times 7 \times 13 \times 3^{11}$$

Find the numerically greatest term in $(3+2x)^{50}$ when $x = \frac{1}{5}$.

$$3^{50}\left(1 + \frac{2x}{3}\right)^{50}$$

$$\frac{(n+1)\left|\frac{2x}{3}\right|}{\left|\frac{2x}{3}\right| + 1} = \frac{51 \cdot \frac{2}{15}}{\frac{2}{15} + 1}$$

$$= \frac{51 \cdot 2}{17} = 6$$

$\Rightarrow$ T_6, T_7 are numerically greatest terms

$(3+2x)^{51}$ when $x = \frac{1}{5}$.

$3^{51}\left(1+\frac{2x}{3}\right)^{51}$

$$\frac{(n+1)\left|\frac{2x}{3}\right|}{\left|\frac{2x}{3}\right|+1} = \frac{52 \cdot \frac{2}{15}}{\frac{2}{15}+1} = \frac{104}{17}$$

$$= 6.1$$

$$\Rightarrow \left[\frac{(n+1)\left(\frac{2x}{3}\right)}{\left(\frac{2x}{3}\right)+1}\right] = 6$$

$\Rightarrow$ T_7 is numerically greatest.

In $(1+x)^{20}$ expansion 6^{th} term is numerically greatest then find possible values of x.

$$\frac{(n+1)|x|}{|x|+1} \in [5, 6]$$

$\rightarrow T_5, T_6$ are greatest

$\rightarrow T_6, T_7$ are greatest?

Ex. $5.2 \rightarrow T_6$ is greatest

$$\frac{21 \cdot |x|}{|x|+1} \in [5, 6]$$

$$\frac{|x|}{|x|+1} \in \left[\frac{5}{21}, \frac{6}{21}\right]$$

$$\frac{|x|}{|x|+1} \geq \frac{5}{21} \Rightarrow 21|x| \geq 5|x|+5$$

$$\Rightarrow 16|x| \geq 5$$

$$\Rightarrow |x| \geq \frac{5}{16}$$

$$\frac{|x|}{|x|+1} \leq \frac{6}{21} \Rightarrow 21|x| \leq 6|x|+6$$

$$\Rightarrow 15|x| \leq 6$$

$$\Rightarrow |x| \leq \frac{6}{15}$$

$$\Rightarrow |x| \in \left[\frac{5}{16}, \frac{6}{15}\right]$$

In $(1+x)^{2n}$ expansion if exactly middle term is numerically greatest then find the possible values of x.

$n+1$ th term is numerically greatest

$$\Rightarrow \frac{(2n+1)|x|}{|x|+1} \in [n, n+1]$$

$$\frac{(2n+1)|x|}{|x|+1} \geq n$$

$$\Rightarrow 2n|x|+|x| \geq n|x|+n$$

$$\Rightarrow (n+1)|x| \geq n \Rightarrow |x| \geq \frac{n}{n+1}$$

$$\frac{(2n+1)|x|}{|x|+1} \leq n+1$$

$$\Rightarrow \quad (2n+1)|x| \leq (n+1)|x| + (n+1)$$

$$\Rightarrow \quad n|x| \leq n+1$$

$$\Rightarrow \quad |x| \leq \frac{n+1}{n}$$

$$\Rightarrow \quad |x| \in \left[\frac{n}{n+1} \quad \frac{n+1}{n}\right]$$

$$\Rightarrow \quad x \in \left[-\left(\frac{n+1}{n}\right), -\frac{n}{n+1}\right] \cup \left[\frac{n}{n+1}, \frac{n+1}{n}\right]$$

Trinomial expansion.

$$(x+\overline{y+z})^n = {}^nC_0 \cdot x^n + {}^nC_1 \cdot x^{n-1}(y+z) + {}^nC_2 \cdot x^{n-2}(y+z)^2 + {}^nC_3 \cdot x^{n-3}(y+z)^3 + \cdots + {}^nC_r \cdot x^{n-r} \cdot (y+z)^r + \cdots + (y+z)^n$$

No. of terms in this expansion

$$= 1 + 2 + 3 + 4 + \cdots + (n+1)$$

$$= \frac{(n+1)(n+2)}{2} = {}^{(n+2)}C_2$$

$$(x+y+z)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^{n-r} \cdot (y+z)^r$$

$$= \sum_{r=0}^{n} {}^nC_r \cdot x^{n-r} \cdot \sum_{K=0}^{r} {}^rC_K \cdot y^{r-K} \cdot z^K$$

$$= \sum_{r=0}^{n} \sum_{K=0}^{r} {}^nC_r \cdot x^{n-r} \cdot {}^rC_K \cdot y^{r-K} \cdot z^K$$

$$= \sum_{r=0}^{n} \sum_{K=0}^{r} \frac{n!}{(n-r)!\,\cancel{r!}} \cdot \frac{\cancel{r!}}{(r-K)!\,K!} \cdot x^{n-r} \cdot y^{r-K} \cdot z^K$$

$$= \sum_{r=0}^{n} \sum_{k=0}^{r} \frac{n!}{(n-r)!\,(r-k)!\,k!} x^{n-r} \cdot y^{r-k} \cdot z^{k}$$

let $n-r = p$, $r-k = q$ $k = s$

then $$= \sum_{\substack{0 \le p,q,s \le n \\ p+q+s=n}} \frac{n!}{p!\,q!\,s!} x^{p} y^{q} z^{s}.$$

$$\boxed{(x+y+z)^{n} = \sum_{\substack{0 \le p,q,s \le n \\ p+q+s=n}} \frac{n!}{p!\,q!\,s!} x^{p} \cdot y^{q}.}$$

$$(x+y)^{n} = \sum_{r=0}^{n} {}^{n}C_{r} \cdot x^{n-r} \cdot y^{r}$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!} x^{n-r} \cdot y^{r}$$

$$= \sum_{\substack{0 \le p,q \le n \\ p+q=n}} \frac{n!}{p!\,q!} x^{p} \cdot y^{q},$$

$$(x+y+z+w)^n = \sum_{\substack{0 \le p,q,r,s \le n \\ p+q+r+s=n}} \frac{n!}{p!q!r!s!} x^p y^q z^r w^s$$

And no. of terms in this expansion $= {}^{(n+3)}C_3$.

$n = 6$

$6+0+0+0$	$\rightarrow$	4 terms
$5+1+0+0$	$\rightarrow$	12 terms.
$4+2+0+0$	$\rightarrow$	12 terms
$3+3+0+0$	$\rightarrow$	6 terms
$3+2+1+0$	$\rightarrow$	24 terms
$2+2+2+0$	$\rightarrow$	4 terms
$4+1+1+0$	$\rightarrow$	12 terms
$2+2+1+1$	$\rightarrow$	6 terms
$3+1+1+1$	$\rightarrow$	4 terms.
		84

$${}^9C_3 = \frac{9\times8\times7}{6}$$
$$= 12\times7$$
$$= 84.$$

$$(x+\overline{y+z+w})^n = {}^nC_0 \cdot x^n + {}^nC_1 \cdot x^{n-1}(y+z+w)$$
$$+ {}^nC_2 \cdot x^{n-2}(y+z+w)^2 + {}^nC_3 \cdot x^{n-3} \cdot (y+z+w)^3 + \cdots$$
$$+ (y+z+w)^n$$

$$= 1 + 3 + 6 + {}^5C_2 + {}^6C_2 + \cdots + {}^{(n+2)}C_2$$

$$= {}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2 + \cdots + {}^{(n+2)}C_2$$

$$= {}^3C_3 + {}^3C_2 + {}^4C_2 + {}^5C_2 + \cdots + {}^{(n+2)}C_2$$

$$= {}^4C_3 + {}^4C_2 + {}^5C_2 \cdots + {}^{(n+2)}C_2$$

$$= {}^5C_3 + {}^5C_2 + \cdots + {}^{(n+2)}C_2$$

$$= {}^{(n+2)}C_3 + {}^{(n+2)}C_2$$

$$= {}^{(n+3)}C_3 .$$

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{0 \le a_1, a_2 \ldots a_r \le n \\ a_1 + a_2 + \cdots + a_r = n}} \frac{n!}{a_1! \, a_2! \ldots a_r!} \, x_1^{a_1} \cdot x_2^{a_2} \cdots x_r^{a_r}$$

And no. of terms in this expansion

$$= {}^{n+r-1}C_{r-1}$$

$$\frac{n!}{p!\,q!\,s!} = \frac{(p+q+s)!}{p!\,q!\,s!}$$

$p+q+s=n$

$$= \frac{(p+q+s)!}{(p+q)!\,.\,s!} \cdot \frac{(p+q)!}{p!\,.\,q!}$$

$$= \underline{(p+q+s)\,C_s} \cdot \underline{(p+q)C_p} \text{ is}$$

always integer.

① Find the value of $\displaystyle\sum_{\substack{\alpha+\beta+\gamma=11 \\ 0\le\alpha,\beta,\gamma\le 11}} \frac{11!}{\alpha!\,\beta!\,\gamma!} = ?$

$$\sum_{\substack{\alpha+\beta+\gamma=11 \\ 0\le\alpha,\beta,\gamma\le 11}} \frac{11!}{\alpha!\,\beta!\,\gamma!} \cdot 1^{\alpha} \cdot 1^{\beta} \cdot 1^{\gamma} = (1+1+1)^{11} = 3^{11}$$

② Find no. of terms in the expansion

of $(x+2y-3z)^{12}$

Ans: $(n+2)C_2 = {}^{14}C_2 = \frac{14\times 13}{2} = 91$

③ Find no. of terms in $(x^2+2x+3)^{12}$

Ans: 25 terms,

$$(x^2+2x+3)^{12} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{24} \cdot x^{24}$$

$$(x^2+2x+3)^{12} = \sum_{\substack{p+q+r=12 \\ 0\le p,q,r\le 12}} (x^2)^p \cdot (2x)^q \cdot (3)^r \cdot \frac{12!}{p!\,q!\,r!}$$

$$\sum_{\substack{0\le p,q,r\le 12 \\ p+q+r=12}} (x^2)^p (2x)^q \cdot (1)^r \frac{12!}{p!\,q!\,r!} = (x^2+2x+1)^{12}$$

$$= (x+1)^{24}$$

$$= 25 \text{ terms}$$

$$\sum_{\substack{p+q+r=12 \\ 0\le p,q,r\le 12}} \cdot x^{2p+q} \cdot 2^q \cdot 3^r \cdot \frac{12!}{p!\,q!\,r!}$$

If $p=0$, $q = 0,1,2,3,\ldots 12$. if $r = 12-q$.

Then it covers $x^0, x^1, x^2, \ldots x^{12}$ all terms.

$q=0$, $p = 7, 8, 9, 10, 11, 12$ gives $x^{14}, x^{16}, x^{18}, \ldots x^{24}$.

$q=1$, $p=6$, $r=5$ covers x^{13}

$q=1, \; p=7, \; r=4 \quad x^{15}$

$q=1, \; p=8, \; r=3 \quad x^{17}.$

$q=1, \; p=11, \; r=0 \quad x^{23}$ term

$\Rightarrow \; x^0, x^1, \ldots \; x^{24}$ all terms are present in $(x^2+2x+3)^{12}$ expansion.

Find x^5 Coefficient in $(x^2+2x+3)^{12}$ expansion.

$$(x^2+2x+3)^{12} = \sum_{\substack{0\le \alpha,\beta,\gamma \le 12 \\ \alpha+\beta+\gamma=12}} \frac{12!}{\alpha!\,\beta!\,\gamma!} (x^2)^{\alpha} \cdot (2x)^{\beta} \cdot (3)^{\gamma}$$

$$= \sum_{\substack{\alpha+\beta+\gamma=12 \\ 0\le \alpha,\beta,\gamma\le 12}} \frac{12!}{\alpha!\,\beta!\,\gamma!} \cdot 2^{\beta} \cdot 3^{\gamma} \cdot \; x^{2\alpha+\beta}$$

Here $2\alpha+\beta = 5$

$\Rightarrow$ (i) $\alpha=0, \; \beta=5, \; \gamma=7 \rightarrow \frac{12!}{0!\,5!\,7!} \cdot 2^5 \cdot 3^7$

(ii) $\alpha=1, \; \beta=3, \; \gamma=8 \rightarrow \frac{12!}{1!\,3!\,8!} \; 2^3 \cdot 3^8$

(iii) $\alpha=2, \; \beta=1, \; \gamma=9 \rightarrow \frac{12!}{2!\,1!\,9!} \cdot 2^1 \cdot 3^9$

$\Rightarrow$ x^5 coefficient in $(x^2+2x+3)^{12}$ is

$$\frac{12!}{5!\,7!}\,2^5\cdot 3^7 + \frac{12!}{3!\,8!}\,2^3\cdot 3^8 + \frac{12!}{2!\,9!}\cdot 2\cdot 3^9$$

Find the coefficient of x^3 in the expansion of $\left(x^2+2x-\frac{1}{x}\right)^{10}$

$$\sum_{\substack{0\le p,q,r\le 10\\ p+q+r=10}} \frac{10!}{p!\,q!\,r!}\cdot (x^2)^p\cdot (2x)^q\cdot \left(-\frac{1}{x}\right)^r$$

$$= \sum_{p+q+r=10} \frac{10!}{p!\,q!\,r!}\cdot 2^q\cdot(-1)^r \quad x^{2p+q-r}$$

$2p+q-r = 3, \quad p+q+r=10$

If $p=0$ $\quad q-r=3, \quad q+r=10 \Rightarrow 2q=13$ is not possible

If $p=1$ $\quad q-r=1, \quad q+r=9 \Rightarrow q=5, r=4$

If $p=2$ $\quad q-r=-1, \quad q+r=8 \Rightarrow 2q=7$ is not possible

If $p=3$ $\quad q-r=-3, \quad q+r=7 \Rightarrow 2q=4 \Rightarrow q=2$ $\quad r=5$

If $p=5$ $\quad q-r=-7, \quad q+r=5 \Rightarrow q$ is negative

$(p, q, r) = (1, 5, 4), (3, 2, 5)$ are possibilities

$$\frac{10!}{5!\,4!} \cdot 2^5 \cdot (-1)^4 + \frac{10!}{3!\,2!\,5!} \cdot (2^2) \cdot (-1)^5$$

$$= \underline{\underline{30240}}.$$

Find the coefficient of $a^8 b^7 c^5$ in the expansion of $(ab + bc + ca)^{10}$

$$(ab + bc + ca)^{10} = (abc)^{10}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{10}$$

$a^8 b^7 c^5$ coefficient in $(abc)^{10}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{10}$

$= a^{-2} b^{-3} \cdot c^{-5}$ term coefficient in $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{10}$

$$= \frac{10!}{2!\,3!\,5!} \cdot \left(\frac{1}{a}\right)^2 \cdot \left(\frac{1}{b}\right)^3 \cdot \left(\frac{1}{c}\right)^5.$$

$$(ab + bc + ca)^{10} = \sum_{p+q+r=10} \frac{10!}{p!\,q!\,r!} (ab)^p \cdot (bc)^q \cdot (ca)^r$$

$$= \sum_{p+q+r=10} \frac{10!}{p!\,q!\,r!}\, a^{p+r} \cdot b^{p+q} \cdot c^{q+r}$$

$$= \sum_{p+q+r=10} \frac{10!}{p!\,q!\,r!} \cdot a^{10-q} \cdot b^{10-r} \cdot c^{10-p}$$

$10-q = 8 \Rightarrow q = 2$

$10-r = 7 \Rightarrow r = 3$

$10-p = 5 \Rightarrow p = 5$

$\Rightarrow$ required term coefficient $= \dfrac{10!}{5!\,3!\,2!}$

let $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r \cdot x^r = a_0 + a_1 x + a_2 x^2 + \dots + a_{2n} x^{2n}$

then (i) $a_0 + a_1 + a_2 + \dots + a_{2n} = ?$

(ii) $a_0 - a_1 + a_2 - \dots + a_{2n} = ?$

(iii) $a_0 + a_2 + a_4 + \dots + a_{2n} = ?$

(iv) $a_1 + a_3 + a_5 + \dots = ?$

(v) show that $a_r = a_{2n-r}$

(vi) show that $a_0^2 - a_1^2 + a_2^2 - a_3^2 \dots + a_{2n}^2 = a_n$.

Replace x with 1 then

$$a_0 + a_1 + a_2 + \dots + a_{2n} = (1+1+1)^n = 3^n$$

Replace x with (-1) then

$$a_0 - a_1 + a_2 - a_3 + \dots + a_{2n} = (1-1+1)^n = 1$$

By adding above 2 results.

$$2(a_0 + a_2 + \dots + a_{2n}) = 3^n + 1$$

$$\Rightarrow \quad a_0 + a_2 + a_4 + \dots + a_{2n} = \frac{3^n + 1}{2}$$

By subtracting First 2 results

$$2(a_1 + a_3 + a_5 + \dots) = 3^n - 1$$

$$\Rightarrow \quad a_1 + a_3 + a_5 + \dots = \frac{3^n - 1}{2}.$$

$$(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_{2n} \cdot x^{2n} \text{ --- (i)}$$

Replace x with $\frac{1}{x}$,

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + a_{2n} \cdot \frac{1}{x^{2n}}$$

$$\frac{(x^2+x+1)^n}{\cancel{x^{2n}}} = \frac{a_0 \cdot x^{2n} + a_1 \cdot x^{2n-1} + a_2 \cdot x^{2n-2} + \cdots + a_{2n}}{\cancel{x^{2n}}}$$

$$\Rightarrow (x^2+x+1)^n = a_0 \cdot x^{2n} + a_1 x^{2n-1} + \cdots + a_{2n} \quad \text{— (ii)}$$

From (i), (ii)

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n} = a_0 x^{2n} + a_1 x^{2n-1} + a_2 x^{2n-2} + \cdots + a_{2n}$$

As these two polynomials are equal

corresponding coefficients are equal

$$\Rightarrow a_0 = a_{2n}, \quad a_1 = a_{2n-1}, \quad a_2 = a_{2n-2} \ldots a_r = a_{2n-r}$$

$$a_r = a_{2n-r} \quad \text{for } r = 0, 1, 2, \ldots 2n.$$

Replace x with $-\frac{1}{x}$ in given expansion

$$(1+x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}$$

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + \cdots + a_{2n} \cdot \frac{1}{x^{2n}}.$$

$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots + a_{2n}^2$ is constant term in

$$\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}\right)\left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + \cdots + \frac{a_{2n}}{x^{2n}}\right)$$

$\Rightarrow a_0^2 - a_1^2 + a_2^2 - a_3^2 \dots + a_{2n}^2$ is constant term in

$$(1+x+x^2)^n \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n$$

$$= \text{constant in } \frac{(1+x+x^2)^n (x^2-x+1)^n}{x^{2n}}$$

$$= x^{2n} \text{ in } (1+x+x^2)^n (1-x+x^2)^n$$

$$= x^{2n} \text{ in } \left((1+x+x^2)(1-x+x^2)\right)^n$$

$$= x^{2n} \text{ in } \left((1+x^2)^2 - (x)^2\right)^n$$

$$= x^{2n} \text{ in } (1+x^2+x^4)^n = a_n$$

$\left[\because (1+x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots + a_{2n} x^{2n}\right]$

Replace x with x^2

$$(1+x^2+x^4)^n = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_n x^{2n} + \dots + a_{2n} x^{4n}$$

$\Rightarrow x^{2n}$ coefficient in $(1+x^2+x^4)^n = a_n$.

In above problem find the value of

$$1 \cdot a_0 + 3 \cdot a_1 + 5 \cdot a_2 + 7 \cdot a_3 + \dots + (4n+1) \cdot a_{2n}.$$

let $S = 1 \cdot a_0 + 3 \cdot a_1 + 5 \cdot a_2 + \dots + (4n-1)a_{2n-1} + (4n+1)a_{2n}$ — (i)

$S = (4n+1) \cdot a_{2n} + (4n-1) \cdot a_{2n-1} + \dots + 3 \cdot a_1 + 1 \cdot a_0$

Here $a_r = a_{2n-r}$ for $r = 0, 1, 2, \dots 2n$

$\Rightarrow \quad S = (4n+1) \cdot a_0 + (4n-1)a_1 + \dots + 3 \cdot a_{2n-1} + 1 \cdot a_{2n}$ — (ii)

add (i), (ii)

$$2S = (4n+2)(a_0 + a_1 + a_2 + \dots + a_{2n})$$

$$\Rightarrow \quad S = (2n+1)(a_0 + a_1 + a_2 + \dots + a_{2n})$$

$$= (2n+1) \cdot 3^n$$

other method

$$x(1+x^2+x^4)^n = a_0 x + a_1 x^3 + a_2 x^5 + \dots + a_{2n} x^{4n+1}$$

Differentiate both sides w.r.t x

$$(1+x^2+x^4)^n + x \cdot n(1+x^2+x^4)^{n-1} \cdot (2x+4x^3)$$

$$= a_0 + 3a_1 x^2 + 5a_2 x^3 + \dots + (4n+1)a_{2n} x^{4n}$$

Sub $x = 1$,

$$a_0 + 3a_1 + 5a_2 + \dots + (4n+1)a_{2n} = 3^n + 3^{n-1} \cdot 6n$$

$$= 3^n + 3^n \cdot 2n$$

$$= 3^n (2n+1)$$

Problems on Binomial Coefficients:

$$(1+x)^n = nC_0 + nC_1 \cdot x + nC_2 \cdot x^2 + nC_3 \cdot x^3 + \dots + nC_n \cdot x^n$$

Here $nC_0, nC_1, nC_2, \dots nC_n$ are called as binomial coefficients and these are denoted by $C_0, C_1, C_2, C_3, \dots C_n$,

$$C_r = nC_r = \frac{n!}{(n-r)!\, r!}$$

Standard Results:

$$C_0 + C_1 + C_2 + C_3 + \dots + C_n = \sum_{r=0}^{n} C_r = 2^n$$

$$C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n \cdot C_n = \sum_{r=0}^{n} (-1)^r C_r = 0$$

$$C_0 + C_2 + C_4 + C_6 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

Find the value of $C_0 + 2 \cdot C_1 + 2^2 \cdot C_2 + 2^3 \cdot C_3 + \dots + 2^n \cdot C_n = ?$

$$(1+x)^n = C_0 + C_1 \cdot x + C_2 \cdot x^2 + \dots + C_n \cdot x^n$$

Sub $x = 2$

$$(1+2)^n = 3^n = C_0 + C_1 \cdot 2 + C_2 \cdot 2^2 + \dots + C_n \cdot 2^n$$

$$C_0 + 3 \cdot C_1 + 3^2 \cdot C_2 + \dots + 3^n \cdot C_n = (1+3)^n = 4^n$$

Find the value of $\sum_{r=0}^{n} r.C_r$

let

$S = 0.C_0 + 1.C_1 + 2.C_2 + 3.C_3 + \cdots + n.C_n = \sum_{r=0}^{n} r.C_r$

$S = n.C_n + (n-1)C_{n-1} + \cdots + 0.C_0$

$C_0 = C_n, \quad C_1 = C_{n-1} \quad \cdots \quad C_n = C_0 \quad C_r = C_{n-r}$

By adding above 2 results

$2S = n.C_0 + n.C_1 + n.C_2 + \cdots + n.C_n.$

$2S = n(C_0 + C_1 + \cdots + C_n)$

$2S = n.2^n$

$\Rightarrow \quad S = n.2^{n-1}$

Method 2:

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n . x^n$

differentiate both sides w.r.t x

$n(1+x)^{n-1} = C_1 + 2.C_2.x + 3.C_3.x^2 + \cdots + n.C_n.x^{n-1}$

sub $x = 1$

$\Rightarrow \quad C_1 + 2.C_2 + 3.C_3 + \cdots + n.C_n = n.2^{n-1}$

Method 3:

$$\sum_{r=0}^{n} r \cdot C_r = \sum_{r=0}^{n} r \cdot {}^nC_r$$

$$= \sum_{r=1}^{n} r \cdot {}^nC_r \quad [\because \text{ at } r=0 \text{ term is zero}]$$

$$= \sum_{r=1}^{n} \cancel{r} \cdot \frac{n}{\cancel{r}} \cdot {}^{(n-1)}C_{r-1}$$

$$\boxed{{}^nC_r = \frac{n}{r} \cdot {}^{(n-1)}C_{r-1}} = n \cdot \sum_{r=1}^{n} {}^{(n-1)}C_{r-1}$$

$$= n\left({}^{(n-1)}C_0 + {}^{(n-1)}C_2 + {}^{(n-1)}C_3 + \cdots + {}^{(n-1)}C_{n-1}\right)$$

$$= n \cdot 2^{n-1} \quad \left[\because (1+1)^{n-1} = {}^{(n-1)}C_0 + {}^{(n-1)}C_1 + \cdots + {}^{(n-1)}C_{n-1}\right]$$

$$\therefore \boxed{\sum_{r=0}^{n} r \cdot C_r = n \cdot 2^{n-1}}$$

Find the value of $\sum_{r=0}^{n} r \cdot (r-1) \cdot C_r$

$$\sum_{r=0}^{n} r(r-1) \cdot C_r = \sum_{r=2}^{n} r \cdot (r-1) \cdot {}^nC_r \cdot \quad [\because \text{ at } r=0,1 \text{ terms are zero}]$$

$$= \sum_{r=2}^{n} \cancel{r}\cdot(r-1)\ \frac{n}{\cancel{r}}\cdot {}^{(n-1)}C_{r-1}$$

$$= \sum_{r=2}^{n} \cancel{(r-1)}\cdot n \cdot \frac{n-1}{\cancel{r-1}}\cdot {}^{(n-2)}C_{r-2}$$

$$= n(n-1)\sum_{r=2}^{n} {}^{(n-2)}C_{r-2}$$

$$= n(n-1)\left[{}^{(n-2)}C_0 + {}^{(n-2)}C_1 + {}^{(n-2)}C_2 + \dots + {}^{(n-2)}C_{n-2}\right]$$

$$= n(n-1)\cdot 2^{n-2}.$$

<u>method2:</u>

$$(1+x)^n = C_0 + C_1\cdot x + C_2\cdot x^2 + \dots + C_n\cdot x^n$$

$$n(1+x)^{n-1} = C_1 + 2\cdot C_2\cdot x + 3\cdot C_3\cdot x^2 + \dots + n\cdot C_n\cdot x^{n-1}$$

$$n(n-1)(1+x)^{n-2} = 2\cdot 1\cdot C_2 + 3\cdot 2\cdot C_3\cdot x + 4\cdot 3\cdot C_4\cdot x^2 + \dots + n\cdot(n-1)\cdot C_n\cdot x^{n-2}$$

Sub $x = 1$

$$n(n-1)\,2^{n-2} = 1\cdot 2\cdot C_2 + 2\cdot 3\cdot C_3 + \dots + (n-1)\cdot n\cdot C_n$$

$$= \sum_{r=2}^{n} r\cdot(r-1)\cdot C_r$$

$$= \sum_{r=0}^{n} r\cdot(r-1)\cdot C_r$$

11) $\sum_{r=0}^{n} r \cdot (r-1) \cdot (r-2) \cdot C_r = n(n-1)(n-2) \cdot 2^{n-3}$

$$\sum_{r=0}^{n} r \cdot (r-1)(r-2) \ldots (r-k) \cdot C_r = n(n-1)(n-2) \ldots (n-k) \cdot 2^{n-(k+1)}$$

$(k < n)$

$$\boxed{\begin{aligned} &\sum_{r=0}^{n} (-1)^r \cdot r \cdot C_r = 0 \\ &\sum_{r=0}^{n} (-1)^r \cdot r \cdot (r-1) \cdot C_r = 0 \\ &\sum_{r=0}^{n} (-1)^r \cdot r \cdot (r-1) \cdot (r-2) \cdot C_r = 0 \end{aligned}}$$

Pf: $(1+x)^n = C_0 + C_1 \cdot x + C_2 \cdot x^2 + \ldots + C_n \cdot x^n$

$n(1+x)^{n-1} = C_1 + 2 \cdot C_2 \cdot x + 3 \cdot C_3 \cdot x^2 + \ldots + n \cdot C_n \cdot x^{n-1}$

Sub $x = -1$

$0 = C_1 - 2 \cdot C_2 + 3 \cdot C_3 - 4 \cdot C_4 + \ldots + (-1)^{n-1} \cdot n \cdot C_n$

$-C_1 + 2 \cdot C_2 - 3 \cdot C_3 + \ldots + (-1)^r \cdot r \cdot C_r + \ldots + (-1)^n \cdot n \, C_n = 0$

$$\sum_{r=0}^{n} (-1)^r \cdot r \cdot C_r = 0$$

(ii) sub $x=-1$ in

$$n(n-1)(1+x)^{n-2} = 2.C_2 + 3.2.C_3 x + 4.3.C_4.x^2 + \ldots + n.(n-1)C_n.x^{n-2}$$

Find the value of

$$a.C_0 + (a+d).C_1 + (a+2d).C_2 + \ldots + (a+nd).C_n$$

$$= a(C_0 + C_1 + C_2 + \ldots + C_n) + d(1.C_1 + 2.C_2 + \ldots + n.C_n)$$

$$= a.2^n + d.n.2^{n-1}$$

Method 2:

$$S = a.C_0 + (a+d).C_1 + \ldots + (a+nd)C_n$$

$$S = (a+nd).C_n + (a+(n-1)d).C_{n-1} + \ldots + a.C_0$$

$$2S = (2a+nd)(C_0 + C_1 + C_2 + \ldots + C_n)$$

$$2S = (2a+nd).2^n \Rightarrow S = (2a+nd).2^{n-1}$$

Find the value of $\sum_{r=0}^{n} r^2 \cdot C_r$

Method 1:

$$(1+x)^n = C_0 + C_1 \cdot x + C_2 x^2 + \cdots + C_r \cdot x^r + \cdots + C_n \cdot x^n$$

$$n(1+x)^{n-1} = C_1 + 2 \cdot C_2 \cdot x + 3 \cdot C_3 \cdot x^2 + \cdots + r \cdot C_r \cdot x^{r-1} + \cdots + n \cdot C_n \cdot x^{n-1}$$

multiply both sides with x

$$n \cdot x(1+x)^{n-1} = C_1 \cdot x + 2 \cdot C_2 \cdot x^2 + 3 \cdot C_3 \cdot x^3 + \cdots + r \cdot C_r \cdot x^r + \cdots + n \cdot C_n \cdot x^n$$

differentiate both sides w.r.t x

$$n \cdot (n-1)(1+x)^{n-2} \cdot x + n(1+x)^{n-1} = 1^2 \cdot C_1 + 2^2 \cdot C_2 \cdot x + 3^2 \cdot C_3 \cdot x^2 + \cdots + r^2 \cdot C_r \cdot x^{r-1} + \cdots + n^2 \cdot C_n \cdot x^{n-1}$$

sub $x=1$ both sides.

$$n(n-1)2^{n-2} + n \cdot 2^{n-1} = 1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \cdots + n^2 \cdot C_n$$

$$2^{n-2}\left[n^2 - n + 2n\right] = \sum r^2 \cdot C_r$$

$$\Rightarrow \sum r^2 \cdot C_r = n(n+1)2^{n-2}$$

Method 2:

$$\sum_{r=0}^{n} r^2 \cdot {}^nC_r = \sum_{r=0}^{n} (r^2 - r + r) \cdot {}^nC_r$$

$$= \sum_{r=0}^{n} (r^2 - r) \cdot {}^nC_r + r \cdot {}^nC_r$$

$$= \sum_{r=0}^{n} r(r-1) \cdot {}^nC_r + \sum_{r=0}^{n} r \cdot {}^nC_r$$

$$= n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1}$$

$$= n(n+1)\, 2^{n-2}$$

method 3:

$$\sum_{r=0}^{n} r^2 \cdot {}^nC_r = \sum_{r=1}^{n} r^{\cancel{2}} \cdot \frac{n}{\cancel{r}} \cdot {}^{(n-1)}C_{r-1}$$

$$= n \cdot \sum_{r=1}^{n} r \cdot {}^{(n-1)}C_{r-1}$$

$$= n \sum_{r=1}^{n} \left((r-1) + 1\right) {}^{(n-1)}C_{r-1}$$

$$= n \cdot \left[\sum_{r=1}^{n} (r-1) \cdot {}^{(n-1)}C_{r-1} + \sum_{r=1}^{n} {}^{(n-1)}C_{r-1}\right]$$

$$= n\cdot\left(\sum_{r=2}^{n} \cancel{(r-1)}\cdot\frac{n-1}{\cancel{r-1}}\,{}^{(n-2)}C_{r-2}\right) + n\cdot 2^{n-1}$$

$$= n(n-1)\cdot 2^{n-2} + n\cdot 2^{n-1}$$

$$= n(n+1)\,2^{n-2}$$

$$\sum_{r=0}^{n} (-1)^r\cdot r^2\cdot C_r = 0$$

Pf:

$$\sum_{r=0}^{n} (-1)^r\cdot r^2\cdot C_r = \sum_{r=0}^{n} (-1)^r\cdot\left(r(r-1)+r\right)\cdot C_r$$

$$= \sum_{r=0}^{n} (-1)^r\cdot r(r-1)\,C_r + \sum_{r=0}^{n} (-1)^r\cdot r\cdot C_r$$

$$= 0 + 0$$

$$= 0.$$

Find the value of $\sum_{r=0}^{n} r^3\cdot C_r$.

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n\cdot x^n$$

$$n(1+x)^{n-1} = C_1 + 2\cdot C_2\cdot x + 3\cdot C_3 x^2 + \cdots + n\cdot C_n\cdot x^{n-1}$$

$$nx(1+x)^{n-1} = C_1\cdot x + 2\cdot C_2\cdot x^2 + 3\cdot C_3\cdot x^3 + \cdots + n\cdot C_n\cdot x^n$$

$$n(1+x)^{n-1} + n(n-1)\,x(1+x)^{n-2} = C_1 + 2^2\cdot C_2\cdot x + 3^2\cdot C_3\cdot x^2 + \cdots + n^2\cdot C_n\cdot x^{n-1}$$

$$nx(1+x)^{n-1} + n(n-1)\,x^2(1+x)^{n-2} = C_1\cdot x + 2^2\cdot C_2\cdot x^2 + 3^2\cdot C_3\cdot x^3 + \cdots + n^2\cdot C_n\cdot x^n$$

differentiate both sides w.r.t x

$$n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2} + n(n-1)2x(1+x)^{n-2} + n(n-1)(n-2)x^2(1+x)^{n-3}$$
$$= 1^3 C_1 + 2^3 \cdot C_2 \cdot x + 3^3 \cdot C_3 \cdot x^2 + \cdots + n^3 \cdot C_n \cdot x^{n-1}$$

sub $x=1$ in this expansion then

$$\sum_{r=0}^{n} r^3 \cdot C_r = n \cdot 2^{n-1} + n(n-1) \cdot 2^{n-2} + 2n(n-1)2^{n-2}$$
$$+ n(n-1)(n-2)2^{n-3}$$
$$= n \cdot 2^{n-1} + 3n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3}.$$
$$= 2^{n-3}\left[4n + 6n(n-1) + n(n-1)(n-2)\right]$$
$$= 2^{n-3} \cdot n\left[4 + 6n - 6 + n^2 - 3n + 2\right]$$
$$= 2^{n-3} \cdot n(n^2 + 3n)$$
$$= n^2(n+3)2^{n-3}.$$

sub $x=-1$ in above expansion

$$\sum_{r=0}^{n} (-1)^r \cdot r^3 \cdot C_r = 0 \quad (\text{here } n>3.)$$

method 2:

$$r(r-1)(r-2) = r(r^2-3r+2)$$
$$= r^3 - 3r^2 + 2r$$

$$\Rightarrow r^3 - 3r^2 + 2r = r(r-1)(r-2)$$

$$\Rightarrow r^3 = r(r-1)(r-2) + 3r^2 - 2r$$

$$\Rightarrow r^3 = r(r-1)(r-2) + 3(r^2 - r + r) - 2r$$

$$\Rightarrow r^3 = r(r-1)(r-2) + 3r(r-1) + r$$

$$\Rightarrow \sum_{r=0}^{n} r^3 \cdot {}^nC_r = \sum_{r=0}^{n} \Big(r(r-1)(r-2) + 3r(r-1) + r \Big) \cdot {}^nC_r$$

$$= \sum_{r=0}^{n} r(r-1)(r-2) \cdot {}^nC_r + 3\sum_{r=0}^{n} r(r-1)\,{}^nC_r + \sum_{r=0}^{n} r\,{}^nC_r$$

$$= n(n-1)(n-2) \cdot 2^{n-3} + 3n(n-1)\,2^{n-2} + n \cdot 2^{n-1}$$

$$= n^2(n+3)\,2^{n-3}$$

For $\sum_{r=0}^{n} r^4\, {}^nC_r$

$$r(r-1)(r-2)(r-3) = r(r^3 - 6r^2 + 11r - 6)$$

$$= r^4 - 6r^3 + 11r^2 - 6r$$

$$\Rightarrow r^4 = r(r-1)(r-2)(r-3) + 6r^3 - 11r^2 + 6r$$

$$= r(r-1)(r-2)(r-3) + 6\Big(r(r-1)(r-2) + 3r(r-1) + r\Big) - 11\big(r(r-1) + r\big) + 6r$$

$$\Rightarrow r^4 = r(r-1)(r-2)(r-3) + 6r(r-1)(r-2) + 7r(r-1) + r$$

$$\Rightarrow \sum_{r=0}^{n} r^4 \cdot {}^nC_r = n(n-1)(n-2)(n-3)\,2^{n-4} + 6\cdot n(n-1)(n-2)\,2^{n-3} + 7n(n-1)\,2^{n-2} + n\cdot 2^{n-1}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \cdots$$

$$\Rightarrow \cos\theta + i\sin\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= e^{i\theta}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta}$$

$$= \cos n\theta + i\sin n\theta$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

This is called as demoivers theorem.

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$$

$$\left[\cos\theta - i\sin\theta = e^{-i\theta}\right]$$

$$(\cos\theta + i\sin\theta)^n = {}^nC_0 \cdot \cos^n\theta + {}^nC_1 \cdot \cos^{n-1}\theta\, i\sin\theta + {}^nC_2 \cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3 \cdot \cos^{n-3}\theta\,(i\sin\theta)^3 + \dots + {}^nC_n (i\sin\theta)^n$$

$$\cos n\theta + i\sin n\theta = {}^nC_0 \cdot \cos^n\theta + i\, {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta - i\, {}^nC_3 \cos^{n-3}\theta \sin^3\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta \dots$$

$$= \left({}^nC_0 \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta \dots\right)$$
$$+ i\left({}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + {}^nC_5 \cos^{n-5}\theta \sin^5\theta \dots +\right.$$

$$x + iy = a + ib \Rightarrow x = a,\ y = b$$

By comparing real and imaginary parts.

$$\cos n\theta = {}^nC_0 \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta + \dots$$

$$\Rightarrow \cos n\theta = \cos^n\theta\left[{}^nC_0 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta \dots\right]$$
$$= \cos^n\theta\left[1 - S_2 + S_4 - S_6 + \dots\right]$$

$$\sin n\theta = {}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \cdots -$$

$$\sin n\theta = \cos^n\theta \left[{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + {}^nC_5 \tan^5\theta \cdots \right]$$

$$= \cos^n\theta \left[S_1 - S_3 + S_5 - \cdots \right]$$

$$\therefore \tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{S_1 - S_3 + S_5 - S_7 + \cdots}{1 - S_2 + S_4 - S_6 + \cdots}$$

let $x = \cos\theta + i\sin\theta$

then $\frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$$= \frac{\cos\theta - i\sin\theta}{(\cos\theta)^2 - (i\sin\theta)^2}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta$$

$x = \cos\theta + i\sin\theta$, $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\boxed{x + \frac{1}{x} = 2\cos\theta} \qquad \boxed{x - \frac{1}{x} = 2i\sin\theta}$$

$$x^n + \frac{1}{x^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$\boxed{x^n + \frac{1}{x^n} = 2\cos n\theta}$$

$$x^n - \frac{1}{x^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n$$

$$\boxed{x^n - \frac{1}{x^n} = 2i\sin n\theta}$$

$$\cos^3\theta = \frac{(2\cos\theta)^3}{8} = \frac{\left(x + \frac{1}{x}\right)^3}{8}$$

$$= \frac{1}{8}\cdot\left[\left(x^3 + \frac{1}{x^3}\right) + 3\cancel{x}\cdot\frac{1}{\cancel{x}}\left(x + \frac{1}{x}\right)\right]$$

$$= \frac{1}{8}\cdot\left[2\cos 3\theta + 3\cdot 2\cos\theta\right]$$

$$= \frac{\cos 3\theta + 3\cos\theta}{4}$$

$$\sin^5\theta = (\sin^2\theta)^2\cdot\sin\theta$$

$$= \left(\frac{1 - \cos 2\theta}{2}\right)^2\cdot\sin\theta$$

$$= \left(\frac{1 - 2\cos 2\theta + \cos^2 2\theta}{4}\right)\cdot\sin\theta$$

$$= \left(\frac{1 - 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}}{4}\right)\cdot\sin\theta$$

$$= \left(\frac{3 - 4\cos 2\theta + \cos 4\theta}{8}\right) \cdot \sin\theta$$

$$= \frac{3\sin\theta - 4\sin\theta\cos 2\theta + \cos 4\theta \sin\theta}{8}$$

$$= \frac{3\sin\theta - 2\cdot(\sin 3\theta - \sin\theta) + \frac{1}{2}\cdot(\sin 5\theta - \sin 3\theta)}{8}$$

$$= \frac{6\sin\theta - 4\sin 3\theta + 4\sin\theta + \sin 5\theta - \sin 3\theta}{16}$$

$$= \frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{16}$$

$$\sin^5\theta = \frac{(2i\sin\theta)^5}{32 \cdot i^5} = \frac{\left(x - \frac{1}{x}\right)^5}{32i}$$

$$= \frac{x^5 - 5C_1 \cdot x^4 \cdot \frac{1}{x} + 5C_2\, x^3 \cdot \frac{1}{x^2} - 5C_3 \cdot x^2 \cdot \frac{1}{x^3} + 5C_4\, x \cdot \frac{1}{x^4} - 5C_5 \cdot \frac{1}{x^5}}{32i}$$

$$= \frac{\left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)}{32i}$$

$$= \frac{2i \sin 5\theta - 5 \cdot 2i \sin 3\theta + 10 \cdot 2i \sin\theta}{32i}$$

$$= \frac{\sin 5\theta - 5 \sin 3\theta + 10 \sin\theta}{16}$$

$$\cos^8\theta = \frac{(2\cos\theta)^8}{2^8}$$

$$= \frac{1}{2^8} \cdot \left(x + \frac{1}{x}\right)^8$$

$$= \frac{1}{2^8} \cdot \Big(\left(x^8 + \frac{1}{x^8}\right) + 8C_1\left(x^6 + \frac{1}{x^6}\right) + 8C_2\left(x^4 + \frac{1}{x^4}\right)$$
$$+ 8C_3\left(x^2 + \frac{1}{x^2}\right) + 8C_4 \Big)$$

$$\cos^8\theta = \frac{2\cos 8\theta + 8(2\cos 6\theta) + 28 \cdot 2\cos 4\theta + 56 \cdot (2\cos 2\theta) + 70}{2^8}$$

$$\cos^{2n}\theta = \frac{2\cdot\cos 2n\theta + 2nC_1\, 2\cos(2n-2)\theta + 2nC_2\cdot 2\cos(2n-4)\theta + \dots + 2nC_{n-1}2\cos 2\theta + 2nC_n}{2^{2n}}$$

Find

$C_0 + C_3 + C_6 + C_9 + \dots$

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \dots$$

Replace x with $1, \omega, \omega^2$

$$(1+1)^n = C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + \dots$$

$$(1+\omega)^n = C_0 + C_1\omega + C_2\omega^2 + C_3\omega^3 + C_4\omega^4 + C_5\omega^5 + C_6\omega^6 + \dots$$

$$(1+\omega^2)^n = C_0 + C_1\omega^2 + C_2\omega^4 + C_3\omega^6 + C_4\omega^8 + C_5\omega^{10} + C_6\omega^{12} + \dots$$

$$2^n + (1+\omega)^n + (1+\omega^2)^n = 3\cdot C_0 + C_1(1+\omega+\omega^2) + C_2(1+\omega^2+\omega^4) + C_3(1+\omega^3+\omega^6) + C_4(1+\omega^4+\omega^8) + C_5(1+\omega^5+\omega^{10}) + C_6(1+\omega^6+\omega^{12}) + \dots$$

$$2^n + (1+\omega)^n + (1+\omega^2)^n = 3(C_0 + C_3 + C_6 + \dots)$$

$$\Rightarrow C_0 + C_3 + C_6 + C_9 + \dots = \frac{2^n + (1+\omega)^n + (1+\omega^2)^n}{3}$$

$$\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \qquad \omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$\Rightarrow 1+\omega = \frac{1}{2} + \frac{i\sqrt{3}}{2} \qquad 1+\omega^2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$

$\Rightarrow 1+\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, \quad 1+\omega^2 = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

$\Rightarrow (1+\omega)^n = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n = \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$

$(1+\omega^2)^n = \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^n = \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}$

$\Rightarrow (1+\omega)^n + (1+\omega^2)^n = 2\cos\frac{n\pi}{3}.$

$\Rightarrow C_0 + C_3 + C_6 + C_9 + \cdots = \dfrac{2^n + 2\cos\frac{n\pi}{3}}{3}$

Find the value of $C_1 + C_4 + C_7 + C_{10} + \cdots$

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots$

$x^2(1+x)^n = C_0 x^2 + C_1 x^3 + C_2 x^4 + C_3 x^5 + C_4 x^6 + \cdots$

Here replace x with $1, \omega, \omega^2$ and add three results.

$2^n + \omega^2(1+\omega)^n + \omega^4(1+\omega^2)^n = 3(C_1 + C_4 + C_7 + \cdots)$

$3(C_1 + C_4 + C_7 + \cdots) = 2^n + \omega^2 \cdot (-\omega^2)^n + \omega(-\omega)^n$

$= 2^n + (-1)^n\left[\omega^{n+1} + (\omega^2)^{n+1}\right]$

$$= 2^n - \left[(-\omega)^{n+1} + (-\omega^2)^{n+1}\right]$$

$$= 2^n - \left[\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^{n+1} + \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{n+1}\right]$$

$$= 2^n - 2\cos(n+1)\frac{\pi}{3}$$

$$\Rightarrow C_1 + C_4 + C_7 + \cdots = \frac{2^n - 2\cos(n+1)\frac{\pi}{3}}{3}$$

$C_2 + C_5 + C_8 + - -$

In $x(1+x)^n$ expansion sub $x = 1, \omega, \omega^2$

and add.

Fourth roots of unity are $x^4 = 1$ roots

$$x^4 - 1 = (x^2-1)(x^2+1)$$

$$= (x-1)(x+1)(x-i)(x+i)$$

$\pm 1, \pm i$ are called as 4^{th} roots of unity.

Find $C_0 + C_4 + C_8 + C_{12} + \cdots$

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots$$

sub $x = i$

$$(1+i)^n = C_0 + C_1 i + C_2 i^2 + C_3 i^3 + C_4 i^4 + \cdots$$

$$(\sqrt{2})^n \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^n = C_0 + C_1 i - C_2 - C_3 i + C_4 + C_5 i + \cdots$$

$$(\sqrt{2})^n \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n = \left(C_0 - C_2 + C_4 - C_6 + \cdots\right) + i\left(C_1 - C_3 + C_5 - C_7 + \cdots\right)$$

$$(\sqrt{2})^n \left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = \left(C_0 - C_2 + C_4 - C_6 + \cdots\right) + i\left(C_1 - C_3 + C_5 - C_7 + \cdots\right)$$

Compare real & imaginary terms

$$\boxed{\begin{aligned} C_0 - C_2 + C_4 - C_6 + \cdots &= (\sqrt{2})^n \cos\frac{n\pi}{4} \\ C_1 - C_3 + C_5 - C_7 + \cdots &= (\sqrt{2})^n \sin\frac{n\pi}{4}. \end{aligned}}$$

$$C_0 + C_2 + C_4 + C_6 + C_8 + \cdots = 2^{n-1}$$

$$C_0 - C_2 + C_4 - C_6 + C_8 + \cdots = 2^{n/2} \cos\frac{n\pi}{4}.$$

$$2(C_0 + C_4 + C_8 + \cdots) = 2^{n-1} + 2^{\frac{n}{2}} \cos\frac{n\pi}{4}$$

$$\Rightarrow \quad C_0 + C_4 + C_8 + \cdots = 2^{n-2} + 2^{\frac{n-2}{2}} \cdot \cos\frac{n\pi}{4}.$$

$$\text{||ly} \quad C_2 + C_6 + C_{10} + \cdots = 2^{n-2} - 2^{\frac{n-2}{2}} \cos\frac{n\pi}{4}.$$

$$C_1 + C_5 + C_9 + \cdots = 2^{n-2} + 2^{\frac{n-2}{2}} \sin\frac{n\pi}{4}$$

$$C_3 + C_7 + C_{11} + \cdots = 2^{n-2} - 2^{\frac{n-2}{2}} \sin\frac{n\pi}{4},$$

other method:

For $C_0 + C_4 + C_8 + \cdots$ substitute $x = 1, -1, i, -i$

In $(1+x)^n$ expansion

and add 4 eqs

For $C_1 + C_5 + C_9 + \cdots$ substitue $x = 1, -1, i, -i$

In $x^3(1+x)^n$ expansion

and add 4 results.

Find x^{50} in $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998}$

$+ \cdots + 1001 \cdot x^{1000}$

$$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 50 \cdot x^{49} \cdot (1+x)^{951}$$
$$+ 51 \cdot x^{50} (1+x)^{950} + \cdots$$

$$= {}^{1000}C_{50} + 2 \cdot {}^{999}C_{49} + 3 \cdot {}^{998}C_{48} + \cdots + 50 \cdot {}^{951}C_1$$
$$+ 51 \cdot {}^{950}C_0 \quad {}^{951}C_0$$

$$= \left({}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + \cdots + {}^{952}C_2 + {}^{951}C_1 + {}^{950}C_0\right)$$
$$+ \left({}^{999}C_{49} + {}^{998}C_{48} \quad + {}^{952}C_2 + {}^{951}C_1 + {}^{950}C_0\right)$$
$$+ \left({}^{998}C_{48} + \cdots \quad + {}^{951}C_1 + {}^{950}C_0\right)$$
$$+ \quad \cdot$$
$$+ \left({}^{951}C_0 + {}^{950}C_0\right)$$
$$+ {}^{950}C_0$$

$${}^{1001}C_{50} + {}^{1000}C_{49} + {}^{999}C_{48} + \cdots + {}^{952}C_1 + {}^{952}C_0$$

$$= {}^{1002}C_{50}.$$

$\sum r^4 \, C_r = ?$

$$r^4 = r(r-1)(r-2)(r-3) + A\,r(r-1)(r-2) + B\,r(r-1) + C\,r + D$$

Sub $r=0$ in above eq, $0 = D$

Sub $r=1$ in above eq, $1 = C + D \Rightarrow C = 1$

Sub $r=2$ $\quad 16 = 2B + 2C + D \Rightarrow 2B = 14$

$\Rightarrow B = 7$

Sub $r=3$ $\quad 81 = 0 + 6A + 6B + 3C + D$

$\Rightarrow 6A + 42 + 3 = 81$

$\Rightarrow 6A = 36$

$\Rightarrow A = 6$

$$\Rightarrow r^4 = r(r-1)(r-2)(r-3) + 6r(r-1)(r-2) + 7r(r-1) + r$$

$$\downarrow$$

$$\sum_{r=0}^{n} r^4 \, C_r = n(n-1)(n-2)(n-3) \cdot 2^{n-4} + 6n(n-1)(n-2)\,2^{n-3} + 7n(n-1)\,2^{n-2} + 2^{n-1}$$

Integration Problems:

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_r x^r + \cdots + C_n x^n$$

integrate both sides w.r.t x

$$\int_0^x (1+x)^n \cdot dx = \int_0^x (C_0 + C_1 x + \cdots + C_n x^n) \cdot dx$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right|_0^x = \left.\left(C_0 x + C_1 \frac{x^2}{2} + C_2 \cdot \frac{x^3}{3} + \cdots + C_n \cdot \frac{x^{n+1}}{n+1}\right)\right|_0^x$$

$$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0 x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \cdots + C_n \cdot \frac{x^{n+1}}{n+1} - 0$$

$$\Rightarrow C_0 \cdot x + C_1 \cdot \frac{x^2}{2} + C_2 \cdot \frac{x^3}{3} + \cdots + C_n \cdot \frac{x^{n+1}}{n+1} = \frac{(1+x)^{n+1} - 1}{n+1}$$

$$\boxed{\sum_{r=0}^{n} \frac{x^{r+1}}{r+1} \cdot C_r = \frac{(1+x)^{n+1} - 1}{n+1}} \quad \text{— (i)}$$

Find $C_0 \cdot 2 + C_1 \cdot \frac{2^2}{2} + C_2 \cdot \frac{2^3}{3} + \cdots + C_n \cdot \frac{2^{n+1}}{n+1}$,

Sub $x = 2$ in above result then

$$\sum_{r=0}^{n} C_r \cdot \frac{2^{r+1}}{r+1} = \frac{(1+2)^{n+1} - 1}{n+1} = \frac{3^{n+1} - 1}{n+1}$$

Find the value of $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}$.

$$\sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{2^{n+1} - 1}{n+1}$$

Pf: Sub $x = 1$ in above result (i).

Other method:

$$\sum_{r=0}^{n} \frac{C_r}{r+1} = \sum_{r=0}^{n} \frac{1}{r+1} \cdot {}^nC_r$$

$$= \sum_{r=0}^{n} \frac{1}{n+1} \cdot \left(\frac{n+1}{r+1} \cdot {}^nC_r \right)$$

$$= \frac{1}{n+1} \sum_{r=0}^{n} {}^{(n+1)}C_{r+1}$$

$$= \frac{1}{n+1} \left[{}^{(n+1)}C_1 + {}^{(n+1)}C_2 + {}^{(n+1)}C_3 + \cdots + {}^{(n+1)}C_{n+1} \right]$$

$$= \frac{1}{n+1} \cdot \left[2^{n+1} - {}^{(n+1)}C_0 \right] = \frac{2^{n+1} - 1}{n+1}$$

Find the value of $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \cdots + (-1)^n \cdot \frac{C_n}{n+1}$

Sub $x = -1$ in eq (i)

$$\sum_{r=0}^{n} \frac{(x)^{r+1}}{r+1} \cdot C_r = \frac{(1+x)^{n+1} - 1}{n+1}$$

$$\sum_{r=0}^{n} \frac{(-1)^{r+1}}{r+1} \cdot C_r = \frac{0-1}{n+1}$$

$$(-1) \sum_{r=0}^{n} (-1)^r \cdot \frac{C_r}{r+1} = -\frac{1}{n+1}$$

$$\sum_{r=0}^{n} (-1)^r \cdot \frac{C_r}{r+1} = \frac{1}{n+1}$$

Other method:

$$\sum_{r=0}^{n} (-1)^r \cdot \frac{C_r}{r+1} = \frac{1}{n+1} \sum_{r=0}^{n} (-1)^r \cdot \frac{n+1}{r+1} \cdot {}^nC_r$$

$$= \frac{1}{n+1} \cdot \sum_{r=0}^{n} (-1)^r \cdot (n+1)C_{r+1}$$

$$= \frac{1}{n+1} \left[(n+1)C_1 - (n+1)C_2 + (n+1)C_3 - (n+1)C_4 \cdots \right]$$

$$(n+1)C_0 - (n+1)C_1 + (n+1)C_2 \cdots = 0$$

$$\Rightarrow (n+1)C_0 = 1 = (n+1)C_1 - (n+1)C_2 + (n+1)C_3 \cdots$$

$$\Rightarrow \sum_{r=0}^{n} (-1)^r \cdot \frac{C_r}{r+1} = \frac{1}{n+1}$$

Find (i) $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \frac{C_6}{7} + \cdots$

(ii) $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots$

let (i) = A , (ii) = B

$$A + B = \frac{2^{n+1} - 1}{n+1} , \quad A - B = \frac{1}{n+1}$$

$$\Rightarrow 2A = \frac{2^{n+1}}{n+1} \Rightarrow A = \frac{2^n}{n+1}$$

$$2B = \frac{2^{n+1} - 2}{n+1} \Rightarrow B = \frac{2^n - 1}{n+1}$$

Find $\sum_{r=0}^{n} \frac{C_r}{(r+1)(r+2)}$

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$$

After integrating from 0 to x

$$\frac{(1+x)^{n+1} - 1}{n+1} = C_0 \cdot x + C_1 \cdot \frac{x^2}{2} + C_2 \cdot \frac{x^3}{3} + \dots + C_n \cdot \frac{x^{n+1}}{n+1}$$

once again integrate this result from 0 to 1

$$\int_0^1 \frac{(1+x)^{n+1} - 1}{n+1} \cdot dx = \int_0^1 C_0 \cdot x + C_1 \cdot \frac{x^2}{2} + \dots + C_n \cdot \frac{x^{n+1}}{n+1}$$

$$\left. \frac{\frac{(1+x)^{n+2}}{n+2} - x}{n+1} \right|_0^1 = \left. C_0 \cdot \frac{x^2}{2} + C_2 \cdot \frac{x^3}{2 \cdot 3} + \dots + C_n \cdot \frac{x^{n+2}}{(n+2)(n+1)} \right|_0^1$$

$$\frac{\frac{2^{n+2}}{n+2} - 1}{n+1} - \frac{\frac{1}{n+2}}{n+1} = \frac{C_0}{1 \cdot 2} + \frac{C_2}{2 \cdot 3} + \dots + \frac{C_n}{(n+1)(n+2)}$$

$$\Rightarrow \sum_{r=0}^{n} \frac{C_r}{(r+1)(r+2)} = \frac{2^{n+2} - (n+2) - 1}{(n+1)(n+2)}$$

other method:

$$\sum_{r=0}^{n} \frac{C_r}{(r+1)(r+2)} = \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n} \frac{n+1}{r+1} \cdot \frac{n+2}{r+2} \cdot {}^nC_r$$

$$= \frac{1}{(n+1)(n+2)} \cdot \sum_{r=0}^{n} {}^{(n+2)}C_{r+2}$$

$$= \frac{1}{(n+1)(n+2)} \left[{}^{(n+2)}C_2 + {}^{(n+2)}C_3 + \dots + {}^{(n+2)}C_{n+2} \right]$$

$$= \frac{1}{(n+1)(n+2)} \left[{}^{(n+2)}C_0 + {}^{(n+2)}C_1 + {}^{(n+2)}C_2 + {}^{(n+2)}C_3 + \dots + {}^{(n+2)}C_{n+2} - {}^{(n+2)}C_0 - {}^{(n+2)}C_1 \right]$$

$$= \frac{1}{(n+1)(n+2)} \cdot \left[2^{n+2} - {}^{(n+2)}C_0 - {}^{(n+2)}C_1 \right]$$

$$\sum_{r=0}^{n} \frac{C_r}{(r+1)(r+2)(r+3)} = \frac{1}{(n+1)(n+2)(n+3)} \sum_{r=0}^{n} {}^{(n+3)}C_{r+3}$$

$$= \frac{2^{n+3} - {}^{(n+3)}C_0 - {}^{(n+3)}C_1 - {}^{(n+3)}C_2}{(n+1)(n+2)(n+3)}$$

$$\sum_{r=0}^{n} (-1)^r \cdot \frac{C_r}{(r+1)(r+2)}$$

$$= \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n} (-1)^r \cdot {}^{(n+2)}C_{r+2}$$

$$= \frac{1}{(n+1)(n+2)} \left({}^{(n+2)}C_2 - {}^{(n+2)}C_3 + {}^{(n+2)}C_4 + \cdots + (-1)^n \, {}^{(n+2)}C_{n+2} \right)$$

$$= \frac{1}{(n+1)(n+2)} (n+1) = \frac{1}{n+2}.$$

$$\Big[\because {}^{(n+2)}C_0 - {}^{(n+2)}C_1 + {}^{(n+2)}C_2 - {}^{(n+2)}C_3 + \cdots = 0$$

$$\Rightarrow {}^{(n+2)}C_2 - {}^{(n+2)}C_3 + {}^{(n+2)}C_4 + \cdots = {}^{(n+2)}C_1 - {}^{(n+2)}C_0$$

$$= n+1 \Big]$$

$$\sum_{r=0}^{n} (-1)^r \frac{C_r}{(r+1)(r+2)(r+3)}$$

Ans:

$$\frac{{}^{(n+3)}C_0 - {}^{(n+3)}C_1 + {}^{(n+3)}C_2}{(n+1)(n+2)(n+3)}$$

$$\sum_{r=0}^{n} (-1)^{n} \cdot \frac{{}^{(n+3)}C_{r+3}}{(n+1)(n+2)(n+3)}$$

$$\frac{{}^{(n+3)}C_3 - {}^{(n+3)}C_4 + {}^{(n+3)}C_5 \cdots + (-1)^{n}\,{}^{(n+3)}C_{n+3}}{(n+1)(n+2)(n+3)}$$

$${}^{(n+3)}C_0 - {}^{(n+3)}C_1 + {}^{(n+3)}C_2 - {}^{(n+3)}C_3 + {}^{(n+3)}C_4 - \cdots = 0$$

$$\Rightarrow {}^{(n+3)}C_0 - {}^{(n+3)}C_1 + {}^{(n+3)}C_2 = {}^{(n+3)}C_3 - {}^{(n+3)}C_4 + {}^{(n+3)}C_5 + \cdots$$

$$\boxed{\int f(x)\cdot g(x)\,dx = f(x)\int g(x)\,dx - \int f'(x)\cdot\left(\int g(x)\,dx\right)dx}$$

Ex: $\int x \cdot \sin x \, dx = x\int \sin x \, dx - \int \frac{d}{dx}(x)\cdot\left(\int \sin x \, dx\right)dx$

$$= x(-\cos x) - \int 1\cdot(-\cos x)\,dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + K$$

Find $\sum_{r=0}^{n} \frac{C_r}{r+2}$.

i.e. Find $\frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \cdots + \frac{C_n}{n+2}$.

Ans: $\frac{2^{n+1}-1}{n+1} - \frac{2^{n+2}-(n+3)}{(n+1)(n+2)}$

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n \cdot x^n$

multiply both sides with x

$x(1+x)^n = C_0 x + C_1 x^2 + C_2 \cdot x^3 + \cdots + C_n \cdot x^{n+1}$

$\int_0^1 \left(C_0 x + C_1 x^2 + \cdots + C_n \cdot x^{n+1}\right) = \int_0^1 x(1+x)^n \cdot dx$

$\left(C_0 \cdot \frac{x^2}{2} + C_1 \cdot \frac{x^3}{3} + \cdots + C_n \cdot \frac{x^{n+2}}{n+2}\right)_0^1 = \int_0^1 (1+x-1)(1+x)^n dx$

$\frac{C_0}{2} + \frac{C_1}{3} + \cdots + \frac{C_n}{n+2} = \int_0^1 (1+x)^{n+1} - (1+x)^n \, dx.$

$= \left. \frac{(1+x)^{n+2}}{n+2} - \frac{(1+x)^{n+1}}{n+1} \right|_0^1$

$$\sum_{r=0}^{n} \frac{C_r}{r+2} = \left(\frac{2^{n+2}}{n+2} - \frac{2^{n+1}}{n+1}\right) - \left(\frac{1}{n+2} - \frac{1}{n+1}\right)$$

$$\int x(1+x)^n\,dx = x\int (1+x)^n\,dx - \int \frac{d}{dx}(x)\cdot\left(\int (1+x)^n\,dx\right)dx$$

$$= \frac{x\cdot(1+x)^{n+1}}{n+1} - \int 1\cdot\frac{(1+x)^{n+1}}{n+1}\,dx$$

$$= \frac{x(1+x)^{n+1}}{n+1} - \frac{(1+x)^{n+2}}{(n+2)(n+1)}$$

$$\Rightarrow \int_0^1 x(1+x)^n\,dx = \left[x\frac{(1+x)^{n+1}}{n+1} - \frac{(1+x)^{n+2}}{(n+2)(n+1)}\right]_0^1$$

$$= \left(\frac{2^{n+1}}{n+1} - \frac{2^{n+2}}{(n+1)(n+2)}\right) - \left(0 - \frac{1}{(n+1)(n+2)}\right)$$

$$= \frac{2^{n+1}}{n+1} - \frac{2^{n+2}}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$$

Other Method:

$$\sum_{r=0}^{n} \frac{C_r}{r+2} = \sum_{r=0}^{n} \frac{(r+1)}{(r+1)(r+2)} \cdot C_r$$

$$= \sum_{r=0}^{n} \frac{(r+2)-1}{(r+1)(r+2)} \cdot C_r$$

$$= \sum_{r=0}^{n} \left(\frac{1}{r+1} - \frac{1}{(r+1)(r+2)} \right) C_r$$

$$= \sum_{r=0}^{n} \frac{C_r}{r+1} - \sum_{r=0}^{n} \frac{C_r}{(r+1)(r+2)}$$

$$= \left(\frac{2^{n+1} - (n+1)C_0}{n+1} \right) - \left(\frac{2^{n+2} - (n+2)C_0 - (n+2)C_1}{(n+1)(n+2)} \right)$$

$$= \left(\frac{2^{n+1} - 1}{n+1} \right) - \left(\frac{2^{n+2} - (n+3)}{(n+1)(n+2)} \right)$$

Day 6 Adv

3. $(a\alpha^2 x^2 + 2b\alpha x + c)^n$ $\quad ac > b^2$

Sum of coefficients $= (a\alpha^2 + 2b\alpha + c)^n$

$D = 4b^2 - 4ac < 0$

$\Rightarrow \quad a\alpha^2 + 2b\alpha + c > 0$ if $a > 0$

< 0 if $a < 0 \quad \forall x$

Ans: ABCD

$x^5 + 10x^2 + x + 5 = 0$ has one root is α

Then

(i) $[\alpha] = -3$, ✓

(ii) number of real roots $= 3$

(iii) number of roots between $(-2, -1)$ is 3 ✗

(iv) equation has atleast one positive root ✗

$f(x) = x^5 + 10x^2 + x + 5$

$\Rightarrow f'(x) = 5x^4 + 20x + 1$

$f'(x) > 0 \quad \underline{\forall x > 0} \Rightarrow f(x)$ is increasing in $(0, \infty)$.

$f(0) = 5$.

$\Rightarrow$ no positive root for the eq

$f''(x) = 20x^3 + 20$

$\Rightarrow$ $f''(x) = 0$ have only one real root.

$\Rightarrow$ $f(x) = 0$ have maximum three real root

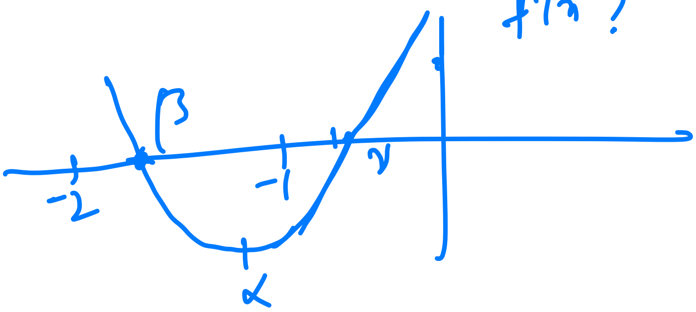
$f(-3) = -243 + 90 - 3 + 5 < 0$

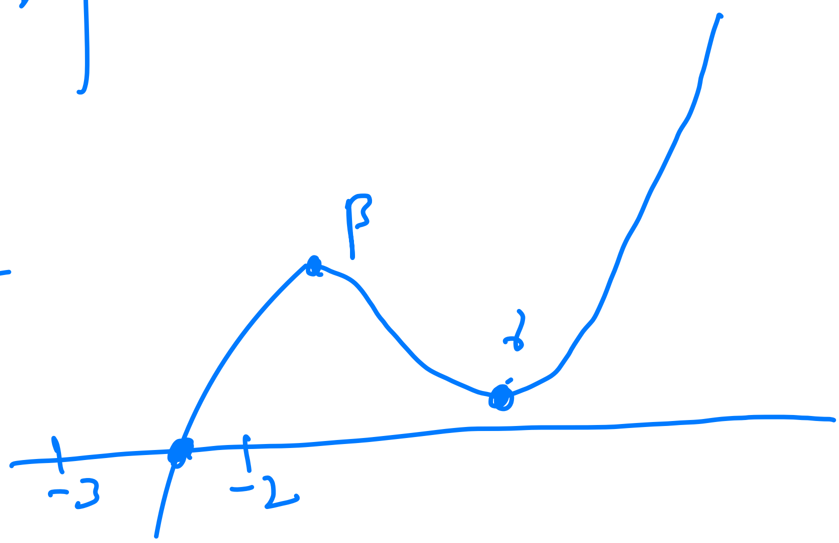
$f(-2) = -32 + 40 - 2 + 5 > 0$

$\Rightarrow$ There is a root in $(-3, -2)$.

$5x^4 + 20x + 1 \Rightarrow$ has 2 negative roots

$f'(x)$?





$f(x) = x^5 + 10x^2 + x + 5$

$\gamma \in (-1, 0)$

$f(\gamma) = \gamma^5 + 10\gamma^2 + \gamma + 5 > 0$

If the equation $x^2 - 3px + 2q = 0$, $x^2 - 3ax + 2b = 0$

have a common root. α, β $\alpha, \frac{1}{\beta}$.

Then $(2q - 2b)^2 = ?$

ⓐ $36pa \cdot (q-b)^2$ ⓑ $18pa\,(q-b)^2$ ⓒ $36bq\,(p-a)^2$ ⓓ $18bq\,(p-a)^2$

$x^2 - 3px + 2q = 0$

$x^2 - 3ax + 2b = 0$

$$\alpha = \frac{2q - 2b}{-3a + 3p} \Rightarrow \alpha = \frac{2}{3}\frac{q-b}{(p-a)}$$

$$\alpha \cdot \beta = 2q$$

$$\frac{\cancel{2}}{3} \cdot \left(\frac{q-b}{p-a}\right) \cdot \beta = pq$$

$$\Rightarrow \beta = \frac{3q \cdot (p-a)}{q-b}.$$

$$\frac{\alpha}{\beta} = 2b$$

$$\frac{9}{3}\cdot\frac{(q-b)}{(p-a)}\cdot\frac{(q-b)}{3q(p-a)} = 2b$$

$$\Rightarrow \quad 2(q-b)^2 = 18bq(p-a)^2$$

$$\Rightarrow \quad (2q-2b)^2 = 36bq(p-a)^2$$

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots + (-1)^r\cdot{}^nC_r = 28 \quad \text{Then } n = ?$$

$${}^{(n+1)}C_0 - {}^nC_1 \qquad\qquad {}^nC_r + {}^nC_{r+1} = {}^{(n+1)}C_{r+1}$$

$$\cancel{{}^{(n-1)}C_0} - \left(\cancel{{}^{(n-1)}C_0} + \cancel{{}^{(n-1)}C_1}\right) + \left(\cancel{{}^{(n-1)}C_1} + \cancel{{}^{(n-1)}C_2}\right) - \left(\cancel{{}^{(n-1)}C_2} + {}^{(n-1)}C_3\right)$$

$$+ (-1)^r\left({}^{(n-1)}C_{r-1} + {}^{(n-1)}C_r\right)$$

$$\Rightarrow \quad (-1)^r\cdot{}^{(n-1)}C_r = 28 = {}^8C_2$$

$$n = 9, \quad r = 2$$

$$t_{100} = \sum_{r=0}^{100}\frac{1}{({}^{100}C_r)^5} \qquad S_{100} = \sum_{r=0}^{100}\frac{r}{({}^{100}C_r)^5} \quad \text{Then}$$

$$\frac{100\, t_{100}}{S_{100}} = ?$$

$$S_{100} = \sum_{r=0}^{100} \frac{100-r}{\left({}^{100}C_{100-r}\right)^5}$$

$$= \sum_{r=0}^{100} \frac{100-r}{\left({}^{100}C_r\right)^5}$$

$$= 100 \sum_{r=0}^{100} \frac{1}{\left({}^{100}C_r\right)^5} - \sum_{r=0}^{100} \frac{r}{\left({}^{100}C_r\right)^5}$$

$$S_{100} = 100 \cdot t_{100} - S_{100}$$

$$\Rightarrow \quad 2S_{100} = 100 \cdot t_{100}$$

$$\Rightarrow \quad \frac{100 \cdot t_{100}}{S_{100}} = 2$$

Show that

$${}^nC_0 \cdot {}^{2n}C_n - {}^nC_1 \cdot {}^{(2n-1)}C_n + {}^nC_2 \cdot {}^{(2n-2)}C_n + \cdots + (-1)^n \cdot {}^nC_n \, {}^nC_n = 1$$

L.H.S $= {}^nC_0 \cdot x^n$ Coefficient in $(1+x)^{2n} - {}^nC_1 \cdot x^n$ in $(1+x)^{2n-1}$ $+ {}^nC_2 \cdot x^n$ in $(1+x)^{2n-2} + \cdots + (-1)^n \cdot {}^nC_n \cdot x^n$ in $(1+x)^n$

$= x^n$ Coefficient in $\left({}^nC_0 \cdot (1+x)^{2n} - {}^nC_1 \cdot (1+x)^{2n-1} + {}^nC_2 (1+x)^{2n-2} + \cdots + (-1)^n \cdot {}^nC_n \cdot (1+x)^n \right]$

$= x^n$ Coefficient in $(1+x)^n \left({}^nC_0 \cdot (1+x)^n - {}^nC_1 \cdot (1+x)^{n-1} + {}^nC_2 \cdot (1+x)^{n-2} + \cdots + (-1)^n \cdot {}^nC_n \cdot (1+x)^{n-n} \right)$

$= x^n$ Coefficient in $(1+x)^n \left((1+x) - 1 \right)^n$

$= x^n$ Coefficient in $x^n (1+x)^n$

$= 1$

${}^nC_0 \cdot {}^{2n}C_n - {}^nC_1 \cdot {}^{(2n-2)}C_n + {}^nC_2 \cdot {}^{(2n-4)}C_n + \cdots$

${}^nC_0 \cdot x^n$ in $(1+x)^{2n} - {}^nC_1 \cdot x^n$ in $(1+x)^{2n-2} + {}^nC_2 \cdot x^n$ in $(1+x)^{2n-4} + \cdots$

$= x^n$ in $\left({}^nC_0 \cdot (1+x)^{2n} - {}^nC_1 \cdot (1+x)^{2n-2} + {}^nC_2 \cdot (1+x)^{2n-4} + \cdots \right)$

$= x^n$ in $\left({}^nC_0 \cdot ((1+x)^2)^n - {}^nC_1 \cdot ((1+x)^2)^{n-1} + {}^nC_2 \cdot ((1+x)^2)^{n-2} + \cdots \right)$

$= x^n$ in $\left((1+x)^2 - 1 \right)^n$

$= x^n$ in $\left(x^2 + 2x \right)^n$

$= x^n$ in $x^n (x+2)^n = \underline{\underline{2^n}}$

$n = 2^3 \times 3^2$

$\{2^0, 2^1, 2^2, 2^3\}$ $\{3^0, 3^1, 3^2\}$

$4 \times 3 = \underline{\underline{12}}$

total factors

$2^0 \cdot 3^0, \quad 2^0 \cdot 3^1, \quad 2^0 \cdot 3^2, \quad 2^1 \cdot 3^0, \quad \ldots \quad 2^3 \cdot 3^2$

$\underline{\underline{(3+1) \times (2+1)}}$

$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ $\quad p_i$ are prime

no. of factors $\quad (\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \ldots (\alpha_k+1)$

$n = (21)^{52} = 3^{52} \times 7^{52}$

number of factors $= (52+1)(52+1) = 53^2$

$= 2809$

$21^{52} = (20+1)^{52} = (1+20)^{52}$

$= 1 + 52 \cdot 20 + 100k$

$= \underline{\underline{100k + 41}}$

$\boxed{2^5 \times 3^8 \times 5^7}$

Total factors $\cdot \quad (5+1) \times (8+1) \times (7+1)$

$\rightarrow$ even factors $= \quad 5 \times (8+1) \times (7+1)$

$\{2^1, 2^2, 2^3, 2^4, 2^5\}$

$\rightarrow$ multiples of 10 -

$\{2^1, 2^2, 2^3, 2^4, 2^5\} \times \{3^0, 3^1, \ldots 3^8\} \times \{5^1, 5^2, 5^3 \ldots 5^7\}$

$5 \times 9 \times 7$

${}^{74}C_{37} - 2$ is divisible by.

${}^{2n}C_n - 2$ when $n = 37$

$\cancel{({}^{37}C_0^2 + {}^{37}C_1^2 + {}^{37}C_2^2 \ldots + {}^{37}C_{36}^2 + {}^{37}C_{37}^2) - 2}$

$\Rightarrow {}^{74}C_{37} - 2 = {}^{37}C_1^2 + {}^{37}C_2^2 + {}^{37}C_3^2 + \ldots + {}^{37}C_{36}^2$

$r \neq 0, 37$ ${}^{37}C_r$ is always integer and divisible by 37.

${}^{74}C_{37} - 2$ is divisible by 37^2.

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n$

$\frac{f(x)}{1-x} = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n$ If $a_0 = 1$, $a_0, a_1, a_2 \ldots$ G.P.

$b_1 = 3$ then $b_{10} = ?$

$|x| < 1$

$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$

$$\frac{f(x)}{1-x} = (a_0 + a_1 x + a_2 x^2 + \dots)(1 + x + x^2 + \dots)$$

$$b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots + b_{10} x^{10} + \dots$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots + (a_0 + a_1 + \dots + a_{10})x^{10}$$

$a_0 = 1$ $\quad a_0 + a_1 = b_1 = 3$

$\Rightarrow a_1 = 2 \Rightarrow r = 2$

$$b_{10} = 1 + 2 + 2^2 + \dots + 2^9 = 2^{10} - 1$$

Find the remainder when $1^5 + 2^5 + 3^5 + \dots + 100^5$ is divided by 4.

$x^n + a^n$ is always divisible by $x + a$ when n is odd

$1^5 + 99^5$ is always divisible by $1 + 99 = 100$

$2^5 + 98^5$ " " 100

$3^5 + 97^5$ " " 100

$\vdots$

$49^5 + 51^5$ " " 100

50^5 " " 100

$\Rightarrow 1^5 + 2^5 + \dots + 99^5 + 100^5$ is divisible by 100

$$\sum_{k=1}^{25} (4k)^5 + (4k-1)^5 \; (4k-2)^5 + (4k-3)^5$$

$$\sum_{k=1}^{5} \underline{4t} + \underline{4t_1} - 1 + \underline{4t_2} - \underline{32} + \underline{4t_3} - 243$$

$$= \sum_{k=1}^{5} 4x - \underline{\underline{244}}$$ is always multiple of 4

$$(1+x+x^2+x^3+x^4)^{496} = a_0 + a_1 x + \cdots + a_{1984} x^{1984}$$

(i) Find G.C.D of the coefficients

$(a_3, a_8, a_{13}, \ldots \; a_{1983})$

(ii) show that $10^{347} > a_{992} > 10^{340}$

$$a_3 = \frac{496!}{x_1! \, x_2! \, x_3! \, x_4! \, x_5!} 1^{x_1} x^{x_2} (x^2)^{x_3} (x^3)^{x_4} (x^4)^{x_5}$$

$$\sum x_i = 496$$

$$x_2 + 2x_3 + 3x_4 + 4x_5 = 3$$

$x_2 = 3, \; x_3 = x_4 = x_5 = 0$

$x_2 = 1, \; x_3 = 1, \; x_4 = x_5 = 0$

$x_4 = 1, \; x_2 = x_3 = x_5 = 0$

Binomial theorem for negative index & rational index:

$$(1+x)^n = nC_0 + nC_1 x + nC_2 x^2 + nC_3 x^3 + \cdots$$

$$\Rightarrow (1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

$$+ \cdots + \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!} x^r + \cdots \quad (1)$$

This expansion is valid for negative and rational index and it is valid only for $|x| < 1$

And number of terms in this expansion = infinite.

Negative index:

Sub $n = -1$ in above expansion

$$(1+x)^{-1} = 1 - x + \frac{(-1)(-2)}{2} \cdot x^2 + \frac{(-1)(-2)(-3)}{3!} \cdot x^3$$

$$+ \cdots \frac{(-1)(-2)(-3)\cdots(-r)}{r!} \cdot x^r + \cdots$$

$$\Rightarrow (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^r x^r + \cdots$$

Other method:

$1-x+x^2-x^3+\dots$ terms are in G.P with

$a=1,\quad r=-x$

Sum of infinite terms $= \dfrac{1}{1-(-x)} = \dfrac{1}{1+x}$

$= (1+x)^{-1}$

$$\Rightarrow \boxed{\begin{aligned} 1-x+x^2-x^3+\dots &= (1+x)^{-1} \\ 1+x+x^2+x^3+\dots &= (1-x)^{-1} \end{aligned}}$$

$(1+x)^{-2}$:

Sub $n=-2$ in the equation (i)

then

$$(1+x)^{-2} = 1+(-2)x+\frac{(-2)(-2-1)}{2!}.x^2+\frac{(-2)(-2-1)(-2-2)}{3!}x^3$$

$$+\dots+\frac{(-2)(-2-1)(-2-2)\dots(-2-(r-1))}{r!}.x^r+\dots$$

$$\Rightarrow (1+x)^{-2} = 1-2x+\frac{2.3}{2!}.x^2+\frac{2.3.4}{3!}.(-1).x^3+\dots$$

$$+\frac{2.3.4.5\dots(r+1)}{r!}.(-1)^r.x^r+\dots$$

$$\Rightarrow \boxed{\begin{aligned}(1+x)^{-2} &= 1-2x+3x^2-4x^3+5x^4 \cdots \\ (1-x)^{-2} &= 1+2x+3x^2+4x^3+5x^4+\cdots\end{aligned}}$$

Other method :

$$(1+x)^{-1} = 1-x+x^2-x^3+\cdots$$

differentiate both sides w.r.t x

$$-1(1+x)^{-2}\cdot 1 = -1+2x-3x^2+4x^3-5x^4+\cdots$$

$$\Rightarrow (1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4+\cdots$$

$$(1+x)^{-3} = \underset{{}^{2}C_2}{1} - \underset{{}^{3}C_2}{3x} + \underset{{}^{4}C_2}{6x^2} - \underset{{}^{5}C_2}{10x^3} + \cdots + \underset{{}^{(r+2)}C_2}{\frac{(r+1)(r+2)}{2}\cdot(-x)^r} + \cdots$$

$$(1-x)^{-3} = 1+3x+6x^2+10x^3+\cdots + \frac{(r+1)(r+2)}{2}x^r+\cdots$$

$$(1+x)^{-n} = 1 - nx + \frac{(-n)(-n-1)}{2!}\cdot x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}\cdot x^3$$

$$+ \dots + \frac{(-n)(-n-1)(-n-2)\dots(-n-(r-1))}{r!}\cdot x^r + \dots$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}\cdot x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots$$

$$+ (-1)^r \cdot \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}\cdot x^r + \dots$$

$$(1+x)^{-n} = 1 - {}^{n}C_1 x + {}^{(n+1)}C_2 x^2 - {}^{(n+2)}C_3 x^3 + {}^{(n+3)}C_4 x^4 + \dots$$

$$+ (-1)^r \cdot {}^{(n+r-1)}C_r x^r + \dots$$

$***$ $$\boxed{(1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r \cdot {}^{(n+r-1)}C_r \cdot x^r}$$

$***$ $$\boxed{(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{(n+r-1)}C_r\, x^r}$$

$$(1-x)^{-12} = 1 + {}^{12}C_1 \cdot x + {}^{13}C_2 x^2 + {}^{14}C_3 x^3 + {}^{15}C_4 x^4 + \dots$$

$$+ {}^{111}C_{100}\, x^{100} + \dots$$

$(1+x)^n \rightarrow$ All terms are positive $(+ \cdot + = +)$

$(1-x)^n \rightarrow$ Alternatively $+,-$ terms will come $(-,+ = -)$

$(1+x)^{-n} \rightarrow$ Alternatively $+,-$ terms will come $(+ \cdot - = -)$

$(1-x)^{-n} \rightarrow$ All terms are positive $((-1)\cdot(-1) = +)$

Rational Index:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$+ \cdot + \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!}x^r + \cdots$$

Sub $n = \frac{p}{q}$.

$$(1+x)^{p/q} = 1 + \frac{p}{q}\cdot x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{2!}\cdot x^2 + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{3!}\cdot x^3 + \cdots$$

$$+ \cdot + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)\cdots\left(\frac{p}{q}-(r-1)\right)}{r!}\cdot x^r + \cdots$$

$$\Rightarrow (1+x)^{p/q} = 1 + p\cdot\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}\cdot\left(\frac{x}{q}\right)^2 + \frac{p(p-q)(p-2q)}{3!}\cdot\left(\frac{x}{q}\right)^3 + \cdots$$

$$+ \cdots + \frac{p(p-q)(p-2q)\cdots(p-(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \cdots$$

$$\Rightarrow (1+x)^{p/q} = 1 + \sum_{r=1}^{\infty} \frac{p(p-q)(p-2q)\ldots(p-(r-1)q)}{r!} \cdot \left(\frac{x}{q}\right)^r$$

$$(1-x)^{p/q} = 1 - p\cdot\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}\cdot\left(\frac{x}{q}\right)^2 - \frac{p(p-q)(p-2q)}{3!}\cdot\left(\frac{x}{q}\right)^3 + \ldots$$

$$(1-x)^{-p/q} = 1 + p\cdot\frac{x}{q} + \frac{p(p+q)}{2!}\cdot\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

$$+ \frac{p(p+q)(p+2q)\ldots(p+(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \ldots$$

Find the value of $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots\infty$

$$\frac{1.3}{4.8} = \frac{1.(1+2)}{2!}\cdot\left(\frac{1}{4}\right)^2$$

$$\frac{1.3.5}{4.8.12} = \frac{1.3.5}{4.1 \cdot 4.2 \cdot 4.3} = \frac{1(1+2)(1+2.2)}{3!}\cdot\left(\frac{1}{4}\right)^3$$

$$1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots$$

$$= 1 + 1.\left(\frac{1}{4}\right) + \frac{1.(1+2)}{2!}\left(\frac{1}{4}\right)^2 + \frac{1.(1+2)(1+2.2)}{3!}\cdot\left(\frac{1}{4}\right)^3 + \ldots$$

By comparing with above last result

$$p = 1,\; q = 2,\quad \frac{x}{q} = \frac{1}{4}$$

$$\Rightarrow x = \frac{1}{2}$$

$$\Rightarrow \left(1-\frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2} = 2^{1/2} = \sqrt{2}$$

Find $\frac{3\cdot 5}{5\cdot 10} + \frac{3\cdot 5\cdot 7}{5\cdot 10\cdot 15} + \frac{3\cdot 5\cdot 7\cdot 9}{5\cdot 10\cdot 15\cdot 20} + \cdots$

$$\frac{3(3+2)}{2!}\left(\frac{1}{5}\right)^2 + \frac{3(3+2)(3+2\cdot 2)}{3!}\left(\frac{1}{5}\right)^3 + \cdots$$

$$p = 3 \quad q = 2 \qquad \frac{x}{q} = \frac{1}{5} \Rightarrow x = \frac{2}{5}$$

$$1 + 3\left(\frac{1}{5}\right) + \frac{3(3+2)}{2!}\left(\frac{1}{5}\right)^2 + \cdots = \left(1-\frac{2}{5}\right)^{-3/2}$$

$$\Rightarrow \frac{3\cdot 5}{5\cdot 10} + \frac{3\cdot 5\cdot 7}{5\cdot 10\cdot 15} + \cdots = \left(\frac{3}{5}\right)^{-3/2} - \frac{8}{5}$$

$$= \left(\frac{5}{3}\right)^{3/2} - \frac{8}{5}.$$

Prove That $2^K \cdot {}^nC_0 \cdot {}^nC_K - 2^{K-1}\, {}^nC_1 \cdot {}^{(n-1)}C_{K-1} + 2^{K-2} \cdot {}^nC_2 \cdot {}^{(n-2)}C_{K-2} + \cdots + (-1)^K \cdot {}^nC_K \cdot {}^{(n-K)}C_0 = {}^nC_K.$

$${}^nC_0 \cdot x^K \text{ in } (1+2x)^n - {}^nC_1 \cdot x^{K-1} \text{ in } (1+2x)^{n-1} + {}^nC_2 \cdot x^{K-2} \text{ in } (1+2x)^{n-2} + \cdots + (-1)^K \cdot {}^nC_K \cdot x^{K-K} \text{ in } (1+2x)^{n-K}$$

$$\Rightarrow x^K \text{ term in } \left({}^nC_0 \cdot (1+2x)^n - {}^nC_1 \cdot x \cdot (1+2x)^{n-1} + {}^nC_2 \cdot (1+2x)^{n-2} \cdot x^2 + \cdots + (-1)^K \cdot {}^nC_K \cdot x^K \cdot (1+2x)^{n-K}\right)$$

$$= x^K \text{ term in } \left({}^nC_0 \cdot (1+2x)^n - {}^nC_1 \cdot x\, (1+2x)^{n-1} + \cdots + (-1)^K\, {}^nC_K \cdot x^K (1+2x)^{n-K} + (-1)^K\, {}^nC_{K+1} \cdot x^{K+1} (1+2x)^{n-(K+1)} + {}^nC_{K+2} \cdot x^{K+2} \cdot (1+2x)^{n-(K+2)} + \cdots + {}^nC_n x^n\right)$$

$$= x^K \text{ term } \left((1+2x) - x\right)^n$$

$$= x^K \text{ in } (1+x)^n$$

$$= {}^nC_K.$$

$(1+x+x^2)^{10} = a_0 + a_1 x + a_2 x^2 + \dots + a_{20} x^{20}$

then find the value of

$^{10}C_0 \cdot a_9 - {}^{10}C_1 \cdot a_8 + {}^{10}C_2 \cdot a_7 + \dots - {}^{10}C_9 \cdot a_0$

$\left(a_0 + a_1 x + a_2 x^2 + \dots + a_7 x^7 + a_8 x^8 + a_9 x^9 + \dots\right)\left({}^{10}C_0 \cdot x^0 - {}^{10}C_1 \cdot x^1 + {}^{10}C_2 x^2 - {}^{10}C_3 \cdot x^3 + \dots - {}^{10}C_9 \cdot x^9 + {}^{10}C_{10} \cdot x^{10}\right)$

$\Rightarrow {}^{10}C_0 \cdot a_9 - {}^{10}C_1 \cdot a_8 + {}^{10}C_2 \cdot a_7 \dots - {}^{10}C_9\, a_0$ is

x^9 coefficient in $(1+x+x^2)^{10}(1-x)^{10}$

$= (1-x^3)^{10}$

$= {}^{10}C_0 - {}^{10}C_1 \cdot x^3 + {}^{10}C_2 \cdot x^6 - {}^{10}C_3 \cdot x^9 + \dots$

Given result $= -{}^{10}C_3$.

$(100)^{101}$, $(101)^{100}$ which is greater in these 2 numbers

$(101)^{100} = (100+1)^{100}$

$$= {}^{100}C_0 \cdot (100)^{100} + {}^{100}C_1 \cdot (100)^{99} + {}^{100}C_2 (100)^{98} + {}^{100}C_3 (100)^{97}$$

$$+ \cdot {}^{100}C_{99} \, 100 + {}^{100}C_{100} \cdot 1$$

$$= (100)^{100} + (100)^{100} + \frac{100 \times 99}{2} \times (100)^{98} + \frac{100 \times 99 \times 98}{6} \times (100)^{97}$$

$$+ \cdot \cdot + ((100)^2 + 1)$$

$$< (100)^{100} + (100)^{100} + (100)^2 (100)^{98} + (100)^3 (100)^{97} + \cdot \cdot + (100)^{100}$$

$$< (100)^{100} \cdot 100$$

$$< (100)^{101}$$

$$\Rightarrow \underline{(101)^{100} < (100)^{101}}.$$

Show that $\displaystyle\sum_{r=0}^{n} (-1)^r \cdot \frac{{}^nC_r}{{}^{(r+3)}C_r} = \frac{3}{n+3}$

$${}^{(r+3)}C_r = {}^{(r+3)}C_3 = \frac{(r+3)(r+2)(r+1)}{6}$$

$$\frac{{}^nC_r}{{}^{(r+3)}C_r} = \frac{6 \cdot {}^nC_r}{(r+1)(r+2)(r+3)}$$

$$= \frac{6}{(n+1)(n+2)(n+3)} \cdot \frac{n+1}{r+1} \cdot \frac{n+2}{r+2} \cdot \frac{n+3}{r+3} \cdot {}^nC_r$$

$$= \frac{6}{(n+1)(n+2)(n+3)} \cdot {}^{(n+3)}C_{r+3}.$$

$$\Rightarrow \sum_{r=0}^{n} (-1)^r \cdot \frac{{}^nC_r}{{}^{(r+3)}C_r} = \sum_{r=0}^{n} (-1)^r \cdot \frac{6}{(n+1)(n+2)(n+3)} \cdot {}^{(n+3)}C_{r+3}$$

$$= \frac{6}{(n+1)(n+2)(n+3)} \cdot \left[{}^{(n+3)}C_3 - {}^{(n+3)}C_4 + {}^{(n+3)}C_5 \cdots + (-1)^n \cdot {}^{(n+3)}C_{n+3} \right]$$

$${}^{(n+3)}C_0 - {}^{(n+3)}C_1 + {}^{(n+3)}C_2 - {}^{(n+3)}C_3 + {}^{(n+3)}C_4 + \cdots = 0$$

$$\Rightarrow {}^{(n+3)}C_3 - {}^{(n+3)}C_4 + {}^{(n+3)}C_5 \cdots = 1 - (n+3) + \frac{(n+3)(n+2)}{2}$$

$$= \frac{2 - 2n - \cancel{6} + n^2 + 5n + \cancel{6}}{2}$$

$$\frac{n^2 + 3n + 2}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$= \frac{6}{(n+1)(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{2} = \frac{3}{n+3}$$

$$\sum_{i=0}^{n-1} \sum_{j=1+i}^{n+1} {}^nC_i \cdot {}^{(n+1)}C_j$$

Sub $j = k+1$

then $k = i, 1, 2, \ldots n$

$$= \sum_{i=0}^{n-1} \sum_{k=i}^{n} {}^nC_i \cdot {}^{(n+1)}C_{k+1}.$$

$$= \sum_{i=0}^{n-1} \sum_{k=i}^{n} {}^nC_i \left({}^nC_k + {}^nC_{k+1} \right)$$

$$= \sum_{i=0}^{n-1} \sum_{k=i}^{n} {}^nC_i \cdot {}^nC_k + \sum_{i=0}^{n-1} \sum_{k=i}^{n} {}^nC_i \cdot {}^nC_{k+1}$$

$$= \frac{2^{2n} + {}^{2n}C_n}{2} - 1 + {}^nC_0 \cdot \left({}^nC_1 + {}^nC_2 \cdots + {}^nC_n \right)$$

$$+ {}^nC_1 \cdot \left({}^nC_2 + {}^nC_3 \cdots + {}^nC_n \right)$$

$$+ {}^nC_2 \left({}^nC_3 + {}^nC_4 \cdots + {}^nC_n \right)$$

$$+ \cdots + {}^nC_{n-1} \cdot {}^nC_n$$

$$= \frac{2^{2n} + {}^{2n}C_n}{2} - 1 + \frac{2^{2n} - {}^{2n}C_n}{2}$$

$$= 2^{2n} - 1 \quad = \underline{\underline{4^n - 1}}$$

Sub $n = 3$

$$\sum_{i=0}^{2} \sum_{j=1+i}^{4} {}^3C_i \cdot {}^4C_j$$

$$= {}^3C_0 \cdot ({}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) + {}^3C_1 ({}^4C_2 + {}^4C_3 + {}^4C_4) + {}^3C_2 ({}^4C_3 + {}^4C_4)$$

$$= 1 \cdot (15) + 3 \cdot (6 + 4 + 1) + 3(1 + 4)$$

$$= 15 + 33 + 15 = \underline{\underline{63}}$$

$$2^{2 \cdot 3} - 1 = 2^6 - 1 = \underline{\underline{63}}$$

$$\left[\begin{array}{l} 4^n - 2^n + n \\ 4^3 - 2^3 + 3 = 64 - 8 + 3 \\ \qquad = \underline{\underline{59}} \end{array} \right]$$

$x + y + z + 5t = 15$ no. of positive integral solutions.

$$\underset{1,2,3,\ldots 15}{x} + y + z + \underbrace{5t}_{5,10,15,20,\ldots} = 15$$

$3 + 4 + 3 + 5$

x^{15} Coefficient in $(x+x^2+x^3+\dots)(x^1+x^2+x^3+x^4\dots)(x+x^2+x^3\dots)$

$(x^5+x^{10}+x^{15}+x^{20}+\dots)$

x^{15} coefficient in $(x+x^2+x^3+\dots)^3(x^5+x^{10})$

x^{15} coeff in $x^3 \cdot x^5 \cdot (1+x+x^2+\dots)^3(1+x^5)$

$= x^7$ in $(1-x)^{-3} \cdot (1+x^5)$

$= x^7$ in $(1+3C_1 \cdot x+4C_2x^2+5C_3x^3+6C_4x^4+\dots+9C_7x^7+\dots)$
$(1+x^5)$

$= 4C_2 + 9C_7$

$= 6+36 = \underline{\underline{42}}$

$\left(x^2+1+\frac{1}{x^2}\right)^n$

replace $x^2 = t$

$\left(t+1+\frac{1}{t}\right)^n$

$t^0, t^1, t^2 \dots t^n, \frac{1}{t}, \frac{1}{t^2} \dots \frac{1}{t^n}$

no. of terms $= 2n+1$.

Constant term in $\left(t+1+\frac{1}{t}\right)^n$

$$\left(1+\left(t+\frac{1}{t}\right)\right)^n$$

$$= {}^nC_0 + {}^nC_1 \cdot \left(t+\frac{1}{t}\right) + {}^nC_2 \cdot \left(t+\frac{1}{t}\right)^2 + {}^nC_3 \cdot \left(t+\frac{1}{t}\right)^3 + \cdots$$

$$+ {}^nC_n \cdot \left(t+\frac{1}{t}\right)^n$$

$${}^nC_0 + {}^nC_2 \cdot 2 + {}^nC_4 \cdot {}^4C_2 + {}^nC_6 \cdot {}^6C_3 + \cdots$$

$$> \left({}^nC_0 + {}^nC_2 + {}^nC_4 + {}^nC_6 + \cdots\right) \quad > 2^{n-1}$$

Sub $x^2 = t$, $x^{2n-2} = t^{n-1}$

$$\left(t+1+\frac{1}{t}\right)\left(t+1+\frac{1}{t}\right)\left(t+1+\frac{1}{t}\right)\cdots\left(t+1+\frac{1}{t}\right)$$

n terms.

$t + t + \cdots + t \qquad nt$

$$P = \sum_{r=1}^{50} \frac{{}^{(50+r)}C_r \cdot (2r-1)}{{}^{50}C_r \cdot (50+r)}$$

$$S = \sum_{r=1}^{50} \frac{{}^{(50+r)}C_r}{{}^{50}C_r} \cdot \left(\frac{2r-1}{50+r} \right)$$

$$\frac{2r-1}{50+r} = 1 - \frac{(51-r)}{50+r}$$

$$= \sum_{r=1}^{50} \frac{{}^{(50+r)}C_r}{{}^{50}C_r} - \frac{{}^{(50+r)}C_r}{{}^{50}C_r} \cdot \frac{51-r}{50+r}$$

$$\frac{{}^{(50+r)}C_r}{{}^{50}C_r} \cdot \frac{51-r}{50+r}$$

$$= \frac{50+\cancel{r}}{r} \cdot \frac{{}^{(49+r)}C_{r-1}}{50+\cancel{r}} \cdot \frac{51-r}{{}^{50}C_r}$$

$$= \frac{51-r}{r} \cdot \frac{{}^{(49+r)}C_{r-1}}{{}^{50}C_r}$$

$$\left(\frac{50-r+1}{r} \right) \cdot \frac{{}^{(49+r)}C_{r-1}}{{}^{50}C_r}$$

$$= \frac{\cancel{{}^{50}C_r}}{{}^{50}C_{r-1}} \cdot \frac{{}^{49+r}C_{r-1}}{\cancel{{}^{50}C_r}}$$

$$= \frac{(49+r)C_{r-1}}{50C_{r-1}}$$

$$\sum_{r=1}^{50} \frac{(50+r)C_r}{50C_r} - \frac{(49+r)C_{r-1}}{50C_{r-1}}$$

$$= 100C_{50} - 1$$

show that

$$\sum_{k=0}^{n} \frac{(n+k)C_k}{2^k} = 2^n$$

$$nC_0 \cdot \frac{1}{2^0} + (n+1)C_1 \cdot \frac{1}{2^1} + (n+2)C_2 \cdot \frac{1}{2^2} + \dots + (n+n)C_n \cdot \frac{1}{2^n}$$

$= x^0$ coefficient in $\left(1+\frac{x}{2}\right)^n + x^1$ coefficient in $\left(1+\frac{x}{2}\right)^{n+1}$
$+ x^2$ coefficient in $\left(1+\frac{x}{2}\right)^{n+2} + \dots + x^n$ coeffi $\left(1+\frac{x}{2}\right)^{2n}$

$= x^n$ coefficient in $\left(x^n \cdot \left(1+\frac{x}{2}\right)^n + x^{n-1} \cdot \left(1+\frac{x}{2}\right)^{n+1} + x^{n-2} \cdot \left(1+\frac{x}{2}\right)^{n+2} + \dots + x^0 \cdot \left(1+\frac{x}{2}\right)^{2n}\right)$

$= x^n$ coefficient in $\displaystyle x^n\left(1+\frac{x}{2}\right)^n \cdot \frac{\left(\left(\frac{1+\frac{x}{2}}{x}\right)^{n+1} - 1\right)}{\left(\frac{1+\frac{x}{2}}{x}\right) - 1}$

$$= x^n \text{ Coeff. } \frac{\cancel{x^n}\left(1+\frac{x}{2}\right)^n\left(\left(1+\frac{x}{2}\right)^{n+1} - x^{n+1}\right)}{\cancel{x^{n+1}}\,\frac{\left(1-\frac{x}{2}\right)}{\cancel{x}}}$$

$$= x^n \text{ Coeff. } \frac{\left(1+\frac{x}{2}\right)^{2n+1} - x^{n+1}\left(1+\frac{x}{2}\right)^n}{\left(1-\frac{x}{2}\right)}$$

$$= x^n \text{ Coeff. } \left(1+\frac{x}{2}\right)^{2n+1} \cdot \left(1-\frac{x}{2}\right)^{-1}$$

$$= x^n \text{ Coeff } \left(1+\frac{x}{2}\right)^{2n+1} \cdot \left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\cdots\right)$$

$$= \left({}^{(2n+1)}C_0 + {}^{(2n+1)}C_1 \cdot \frac{x}{2} + {}^{(2n+1)}C_2 \left(\frac{x}{2}\right)^2 + \cdots + {}^{(2n+1)}C_n \cdot \left(\frac{x}{2}\right)^n + \cdots\right)$$
$$\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\cdots+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^{n-1}+\left(\frac{x}{2}\right)^n+\cdots\right)$$

$$= {}^{(2n+1)}C_0 \cdot \frac{1}{2^n} + {}^{(2n+1)}C_1 \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} + {}^{(2n+1)}C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{n-2} + \cdots + {}^{(2n+1)}C_n \cdot \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{2^n}\left[{}^{(2n+1)}C_0 + {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2 + \cdots + {}^{(2n+1)}C_n\right]$$

$$= \frac{2^{2n}}{2^n} = 2^n.$$

$S = (2n+1)C_0 + (2n+1)C_1 + \dots + (2n+1)C_n$ $\qquad {}^nC_r = {}^nC_{n-r}$

$S = (2n+1)C_{2n+1} + (2n+1)C_{2n} + (2n+1)C_{n+1}$

$2S = (2n+1)C_0 + \dots + (2n+1)C_{2n+1}$

$\Rightarrow 2S = 2^{2n+1} \Rightarrow \underline{\underline{S = 2^{2n}}}$

Find the value of $\sum_{r=0}^{100} {}^{100}C_r (\sin rx)$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow \sin\theta =$ Imaginary part of $e^{i\theta}$

$= I.p \text{ of } (e^{i\theta})$

$\sin rx = I.p \text{ of } e^{irx}$

$\sum_{r=0}^{100} {}^{100}C_r \cdot \sin rx = \sum_{r=0}^{100} {}^{100}C_r \cdot I.p \text{ of } e^{irx}$

$= I.p \text{ of } \sum_{r=0}^{100} {}^{100}C_r \cdot e^{irx}$

$= I.p \text{ of } \sum_{r=0}^{100} {}^{100}C_r \cdot (e^{ix})^r$

$= I.p \text{ of } (1 + e^{ix})^{100}$

$$= I.P \text{ of } (1 + \cos x + i \sin x)^{100}$$

$$= I.P \text{ of } \left(2\cos^2\frac{x}{2} + i \cdot 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)^{100}$$

$$= I.P \text{ of } 2^{100}\cdot\left(\cos\frac{x}{2}\right)^{100}\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)^{100}$$

$$= I.P \text{ of } 2^{100}\cdot\left(\cos\frac{x}{2}\right)^{100}\left(\cos\left(\frac{x}{2}\cdot 100\right) + i\sin\left(\frac{x}{2}\cdot 100\right)\right)$$

$$= I.P \text{ of } 2^{100}\left(\cos\frac{x}{2}\right)^{100}\left(\cos 50x + i\sin 50x\right)$$

$$\sum_{r=0}^{100} {}^{100}C_r \cdot \sin rx = 2^{100}\cdot\left(\cos\frac{x}{2}\right)^{100}\cdot\sin(50x).$$

$$\sum_{r=0}^{100} {}^{100}C_r \cos rx = ?$$

$$e^{irx} = \cos rx + i \sin rx$$

$$\Rightarrow \quad \cos rx = R.P \text{ of } e^{irx}$$

$$\sum_{r=0}^{100} {}^{100}C_r \cdot \cos rx = R.P \text{ of } \sum_{r=0}^{100} {}^{100}C_r \cdot \left(e^{ix}\right)^r$$

$$= 2^{100}\cdot\left(\cos\frac{x}{2}\right)^{100}\cdot\left(\cos 50x\right)$$

Prove that

$$\frac{{}^nC_0}{x} - \frac{{}^nC_1}{x+1} + \frac{{}^nC_2}{x+2} - \cdots + (-1)^n \frac{{}^nC_n}{x+n}$$

$$= \frac{n!}{x(x+1)(x+2)\ldots(x+n)}$$

where $n \in \mathbb{N}$ and x is not a negative integer

$$\sum_{r=0}^{n} (-1)^r \cdot \frac{{}^nC_r}{x+r}$$

$$\left({}^nC_0 - {}^nC_1 \cdot y + {}^nC_2 \cdot y^2 - \cdots\right) y^{x-1} = (1-y)^n$$

$$\int {}^nC_0 \cdot y^{x-1} - {}^nC_1 \cdot y^x + {}^nC_2 \cdot y^{x+1} \cdots + {}^nC_n \cdot y^{n+x-1} = \int (1-y)^n \cdot y^{x-1}$$

$$\left.\frac{{}^nC_0 \cdot y^x}{x} - \frac{{}^nC_1 \cdot y^{x+1}}{x+1} + \cdots + \frac{{}^nC_n \cdot y^{n+x}}{n+x} \cdot (-1)^n\right|_0^1 = \int_0^1 (1-y)^n \cdot y^{x-1} \cdot dy$$

$$\frac{{}^nC_0}{x} - \frac{{}^nC_1}{x+1} + \cdots + \frac{{}^nC_n}{x+n} \cdot (-1)^n = \int_0^1 (1-y)^n \cdot y^{x-1} \cdot dy$$

$$\int f \cdot g \, dx = f \int g(x)\, dx - \int f'(x) \cdot \left(\int g(x)\, dx\right) dx$$

$$\int_0^1 (1-y)^n \cdot y^{x-1} \, dy = (1-y)^n \cdot \int y^{x-1} \cdot dy - \int \left(n(1-y)^{n-1} \cdot \int y^{x-1} dy\right) dy$$

$$= (1-y)^n \cdot \frac{y^x}{x} \Big|_0^1 - \int_0^1 n(1-y)^{n-1} \cdot \frac{y^x}{x} \cdot dy$$

$$= 0 - \frac{n}{x} \cdot \int_0^1 (1-y)^{n-1} \cdot y^x \, dy$$

$$= -\frac{n}{x} \cdot -\frac{n-1}{x+1} \cdot \int_0^1 (1-y)^{n-2} \cdot y^{x+1} \cdot dy$$

$$= \frac{n(n-1)(n-2)(n-3)\cdots}{x(x+1)(x+2)\cdots} \cdot \int_0^1 y^{x+n-1} \cdot dy$$

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x+1)$$

Sub $x = -1$ both sides $\quad 1 = A$

Sub $x = -2$ both sides $\quad 1 = -B$

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\frac{1}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$1 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Sub $x=-1$ $\quad (2-1)(3-1).\ A = 1$

$$2A = 1 \quad \Rightarrow A = \frac{1}{2}$$

Sub $x=2$ $\quad B(2-1)(2-3) = 1 \quad \Rightarrow -B = 1$

$$B = -1$$

$$S_n = nC_1 - \left(1+\frac{1}{2}\right)nC_2 + \left(1+\frac{1}{2}+\frac{1}{3}\right)nC_3 \cdots + (-1)^{n-1}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).nC_n$$

$$S_n = \sum_{r=1}^{n}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{r}\right).\ nC_r\cdot(-1)^{r-1}$$

$$= \sum_{r=1}^{n} nC_r\cdot(-1)^{r-1}.\ \int_0^1 (1+x+x^2+x^3+\cdots+x^{r-1})\,dx$$

$$\left[\because \int_0^1 x^{r-1} = \frac{x^r}{r}\Big|_0^1 = \frac{1}{r}\right]$$

$$= \sum_{r=1}^{n} nC_r\cdot(-1)^{r-1}.\ \int_0^1\left(\frac{1-x^r}{1-x}\right).\,dx$$

$$= \sum_{r=1}^{n} nC_r\cdot(-1)^{r-1}\left[\int_0^1 \frac{1}{1-x}.\,dx - \int_0^1 \frac{x^r}{1-x}\,dx\right]$$

$$\left[\because \int k\cdot f(x)\,dx = k\int f(x)\,dx\right]$$

$$\left(\int_0^1 \frac{1}{1-x}\,dx\right)\cdot \sum_{r=1}^{n} {}^nC_r\cdot(-1)^{r-1} - \sum_{r=1}^{n} {}^nC_r\cdot(-1)^{r-1}\int_0^1 \frac{x^r}{1-x}\,dx$$

$$\left(\int_0^1 \frac{1}{1-x}\,dx\right)\left({}^nC_1 - {}^nC_2 + {}^nC_3 \cdots + (-1)^{n-1}\cdot {}^nC_n\right) + \int_0^1 \frac{\sum_{r=1}^{n} {}^nC_r\cdot(-1)^r\cdot x^r}{1-x}\,dx$$

$${}^nC_0 - {}^nC_1 + {}^nC_2 \cdots = 0$$

$$\Rightarrow {}^nC_1 - {}^nC_2 + {}^nC_3 \cdots = -1$$

$$= 1\cdot\int_0^1 \left(\frac{1}{1-x}\right)dx + \int_0^1 \frac{(1-x)^n - 1}{1-x}\,dx$$

$$= \cancel{\int_0^1 \frac{1}{1-x}\,dx} + \int_0^1 \frac{(1-x)^n}{1-x}\,dx - \cancel{\int_0^1 \frac{1}{1-x}\,dx}$$

$$\therefore \int_0^1 \frac{(1-x)^n}{1-x}\,dx = \int_0^1 (1-x)^{n-1}\,dx$$

$$= -\left.\frac{(1-x)^n}{n}\right|_0^1$$

$$= -\left(0 - \frac{1}{n}\right)$$

$$= 1/n.$$

$$\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2}{\sqrt{2x^2+1}+\sqrt{2x^2-1}}\right)^6$$

degree = ?

$$\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\sqrt{2x^2+1}-\sqrt{2x^2-1}\right)^6$$

$$= 6C_0\cdot\left(\sqrt{2x^2+1}\right)^6+6C_2\cdot\left(\sqrt{2x^2+1}\right)^4\left(\sqrt{2x^2-1}\right)^2$$
$$+6C_4\cdot\left(\sqrt{2x^2+1}\right)^2\left(\sqrt{2x^2-1}\right)^4+6C_6\cdot\left(\sqrt{2x^2-1}\right)^6$$

$$6C_0\cdot(2x^2+1)^3+6C_2\cdot(2x^2+1)^2(2x^2-1)$$

$$\downarrow$$

$$\underline{\underline{6}}$$

$$\left(3^{-x/4}+3^{\frac{5x}{4}}\right)^n$$

$$\Sigma nC_r = 2^n = 64$$

$$\underline{\underline{n=6}}$$

$$\left(3^{-x/4}+3^{\frac{5x}{4}}\right)^6$$

$$\left(3^{-\frac{x}{4}}\right)^3\cdot\left(3^{\frac{5x}{4}}\right)^3 \quad 6C_3 = 6C_2\cdot\left(3^{-x/4}\right)^4\cdot\left(3^{\frac{5x}{4}}\right)^2+5$$

$$\Rightarrow \underline{\underline{x=0}}$$

The no. of positive integral solutions of $x^4 - y^4 = 3789108$

$$(x-y)(x+y)(x^2+y^2) = 3789108$$

$$= 4 \times 947277$$

$$= 4 \times 9 \times 105253$$

As L.H.S is divisible by 8
R.H.S is divisible by 4. No solutions for this

$$(1+px+x^2)^n = 1 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}$$

$$(np+pr)\, a_r = (r+1)\, a_{r+1} + (r-1)\, a_{r-1}$$

$$n(1+px+x^2)^{n-1} \cdot (p+2x) = a_1 + 2a_2 x + 3a_3 x^2 \cdots + (r+1)\, a_{r+1} \cdot x^r \cdots$$

Compare x^r coefficients both sides.

$$n(1+px+x^2)^n(p+2x) = (a_1+2a_2x+3a_3x^2 \dots)(1+px+x^2)$$

$$n(p+2x)(1+a_1x+a_2x^2+\dots) = (a_1+2a_2x+3a_3x^2\dots)(1+px+x^2)$$

$$ {}^{s}C_r = \begin{cases} \dfrac{s!}{r!\,(s-r)!} & r \le s \\ 0 & r > s \end{cases}$$

$$g(m,n) = \sum_{p=0}^{m+n} \frac{f(m,n,p)}{{}^{(n+p)}C_p}$$

$$f(m,n,p) = \sum_{i=0}^{p} ({}^{m}C_i)\,({}^{n+i}C_p)\,({}^{p+n}C_{p-i})$$

$$f(m,n,p) = \sum_{i=0}^{p} \frac{m!}{(m-i)!\,i!} \cdot \frac{\cancel{(n+i)}!}{p!\,(n+i-p)!} \cdot \frac{(p+n)!}{(p-i)!\,\cancel{(n+i)}!}$$

$$= \sum_{i=0}^{p} \frac{m!}{(m-i)!\,i!} \cdot \frac{(p+n)!}{p!\,(n+i-p)!\,(p-i)!}$$

$$= \frac{(p+n)!}{p!} \cdot \sum_{i=0}^{p} \frac{1}{(n+i-p)!\,(p-i)!} \cdot \frac{m!}{(m-i)!\,i!}$$

$$= \frac{(p+n)!}{p!} \cdot \sum_{i=0}^{p} \frac{{}^{n}C_{(p-i)}}{n!} \cdot {}^{m}C_{i}$$

$$= \frac{(p+n)!}{p!\,n!} \cdot \sum_{i=0}^{p} {}^{m}C_{i} \cdot {}^{n}C_{p-i}$$

$${}^{m}C_{0} \cdot {}^{n}C_{p} + {}^{m}C_{1} \cdot {}^{n}C_{p-1} + {}^{m}C_{2} \cdot {}^{n}C_{p-2} + \dots + {}^{m}C_{p} \cdot {}^{n}C_{0} \dots$$

$$(1+x)^{m}(1+x)^{n} = \left({}^{m}C_{0} + {}^{m}C_{1}x + {}^{m}C_{2}x^{2} + \dots + {}^{m}C_{p}x^{p} + \dots\right)$$
$$\left(1 + {}^{n}C_{1}x + {}^{n}C_{2}x^{2} + \dots + {}^{n}C_{p}x^{p} \dots\right)$$

$$\Rightarrow {}^{m}C_{0} \cdot {}^{n}C_{p} + {}^{m}C_{1}\,{}^{n}C_{p-1} + \dots = x^{p} \text{ coeff(s) in } (1+x)^{m+n}$$
$$= {}^{(m+n)}C_{p}.$$

$$\Rightarrow f(m,n,p) = \frac{(p+n)!}{p!\,n!} \cdot {}^{(m+n)}C_{p}$$
$$= {}^{(p+n)}C_{p} \cdot {}^{(m+n)}C_{p}.$$

$$g(m,n) = \sum_{p=0}^{m+n} \frac{f(m,n,p)}{{}^{(p+n)}C_{p}}$$

$$= \sum_{p=0}^{m+n} \frac{\cancel{{}^{(p+n)}C_{p}} \cdot {}^{(m+n)}C_{p}}{\cancel{{}^{(p+n)}C_{p}}}$$

$= 2^{m+n}$

p is prime number $\boxed{n < p < 2n}$

$N = {}^{2n}C_n$.

(a) p divides N (b) p^2 divides N

(c) p cannot divide N (d) None

$${}^{2n}C_n = \frac{2n!}{n! \cdot n!} \quad \text{is a integer.}$$

$$= \frac{2n \times (2n-1) \times (2n-2) \times \cdots \times (n+1)}{1 \times 2 \times 3 \times \cdots \times n} \quad \text{is a integer}$$

as $p \in (n, 2n)$ p is one of the term in numerator.

$$= \frac{2n \times (2n-1) \times (2n-2) \times \cdots p \times \cdots \times (n+1)}{n!}$$

is a integer.

$= p \times$ Integer $\Rightarrow {}^{2n}C_n$ is divisible by p.

Prove that

$$\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$$

and show $[\sqrt{n}+\sqrt{n+1}] = [\sqrt{4n+1}]$

$$(\sqrt{n}+\sqrt{n+1})^2 - (\sqrt{4n+1})^2$$

$$= n + (n+1) + 2\sqrt{n}\sqrt{n+1} - 4n - 1$$

$$= -2n + 2\sqrt{n^2+n}$$

$$= 2\left(\sqrt{n^2+n} - n\right)$$

$$= \frac{2(\cancel{n^2}+n-\cancel{n^2})}{\sqrt{n^2+n}+n} = \frac{2n}{\sqrt{n^2+n}+n} > 0.$$

$$\Rightarrow (\sqrt{n}+\sqrt{n+1})^2 > (\sqrt{4n+1})^2$$

$$\Rightarrow \sqrt{n}+\sqrt{n+1} > \sqrt{4n+1}$$

$$x = 4n+1 \Rightarrow x+1 = 4n+2$$

$$= 2(2n+1)$$

$$\Rightarrow \lim_{n\to\infty} (\sqrt{n}+\sqrt{n+1})^2 - (\sqrt{4n+1})^2 = \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+n}+n} = 1$$

$\Rightarrow$ let $\sqrt{n}+\sqrt{n+1} = \sqrt{x+1}$

then $\sqrt{4n+1} = \sqrt{x}$

as $x+1$ can not be perfect square

$$[\sqrt{x}] = [\sqrt{x+1}]$$

$$\Rightarrow \left[\sqrt{4n+1}\right] = \left[\sqrt{n}+\sqrt{n+1}\right]$$

$$\underbrace{\sqrt{4n+1}} < \sqrt{n}+\sqrt{n+1} < \underbrace{\sqrt{4n+2}}$$

$$4n+1 < 2n+1+2\sqrt{n^2+n} < 4n+2$$

$$1 < 1-2n+2\sqrt{n^2+n} < 2$$

$$0 < 2\sqrt{n^2+n}-2n < 1$$

$$0 < 2\frac{n}{\sqrt{n^2+n}+n} < 1$$

$$\sqrt{n} < \sqrt{\frac{4n+1}{2}} < \sqrt{n+1}$$

$$2\sqrt{n} < \sqrt{n+\sqrt{n+1}} < 2\sqrt{n+1}$$

$$2\sqrt{n} < \sqrt{4n+1} < 2\sqrt{n+1}$$

$$2\sqrt{n} < \sqrt{n}+\sqrt{n+1} < 2\sqrt{n+1}$$

$$\Rightarrow \left[\sqrt{4n+1}\right] = \left[\sqrt{n+\sqrt{n+1}}\right]$$

$$\sum_{0\le i<j\le n}\sum j\cdot {}^nC_i$$

$$\sum_{i=0}^{n-1}\sum_{j=i+1}^{n} j\cdot {}^nC_i$$

$$= \sum_{i=0}^{n-1} {}^nC_i\left((i+1)+(i+2)+(i+3)+\dots+n\right)$$

$$= \sum_{i=0}^{n-1} {}^nC_i\left((1+2+3+\dots+n)-(1+2+\dots+i)\right)$$

$$= \sum_{i=0}^{n-1} {}^nC_i\left(\frac{n(n+1)}{2}-\frac{i(i+1)}{2}\right)$$

$$= \frac{n(n+1)}{2} \cdot \sum_{i=0}^{n-1} {}^nC_i - \frac{1}{2} \sum_{i=0}^{n-1} (i^2+i) \cdot {}^nC_i$$

$$= \frac{n(n+1)}{2} \cdot (2^n - 1) - \frac{1}{2} \cdot \left(\sum_{i=0}^{n-1} (i^2 - i) \cdot {}^nC_i + 2i\,{}^nC_i \right)$$

$$= \frac{n(n+1)}{2} \cdot (2^n - 1) - \frac{1}{2} \cdot \left[n(n-1) 2^{n-2} - (n^2 - n) + 2 \cdot (n \cdot 2^{n-1} - n) \right]$$

$$(2x^2+3x+4)^{10} = \sum_{r=0}^{20} a_r x^r \quad \text{then find } \frac{a_7}{a_{13}}.$$

$$(2x^2+3x+4)^{10} = a_0 + a_1 x + a_2 x^2 + \dots + a_7 x^7 + \dots + a_{13} x^{13} + \dots + a_{20} x^{20}$$

Replace x with $\frac{2}{x}$ both sides.

$$\left(2 \cdot \left(\frac{2}{x}\right)^2 + 3 \cdot \frac{2}{x} + 4\right)^{10} = a_0 + a_1 \cdot \frac{2}{x} + \dots + a_7 \cdot \left(\frac{2}{x}\right)^7 + \dots + a_{13} \left(\frac{2}{x}\right)^{13} + \dots$$

$$\frac{2^{10} (4 + 3x + 2x^2)^{10}}{x^{20}} = a_0 + a_1 \cdot \frac{2}{x} + \dots + a_7 \cdot \left(\frac{2}{x}\right)^7 + \dots + a_{13} \cdot \left(\frac{2}{x}\right)^{13} + \dots$$

$$(2x^2+3x+4)^{10} = \frac{x^{20}}{2^{10}} \cdot \left(a_0 + \frac{2a_1}{x} + \dots + \frac{2^7 \cdot a_7}{x^7} + \frac{2^{13} \cdot a_{13}}{x^{13}} \right)$$

$$(a_0 + a_1 x + a_7 x^7 + \dots + a_{13} x^{13} + \dots) = \frac{a_0 \cdot x^{20}}{2^{10}} + \dots + \frac{a_7 \cdot x^{13}}{2^3} + \dots 2^3 \cdot a_{13} \cdot x^7$$

$$a_7 = 2^3 \cdot a_{13}$$

$$\Rightarrow \frac{a_7}{a_{13}} = 8.$$

n is an even positive integer $K = \frac{3n}{2}$

$$\sum_{r=1}^{K} (-3)^{r-1} \cdot {}^{3n}C_{2r-1}$$

(a) 0 (b) 1 (c) 2^n (d) 2^{3n}

let $n = 2t$.

$$\sum_{r=1}^{3t} (-3)^{r-1} \cdot {}^{6t}C_{2r-1}$$

$$^{6t}C_1 \cdot 1 + {}^{6t}C_3 \cdot (-3) + {}^{6t}C_5 \cdot (3)^2 + {}^{6t}C_7 \cdot (-3)^3 + \ldots$$

$$= {}^{6t}C_1 \cdot 1 - {}^{6t}C_3 \cdot (\sqrt{3})^2 + {}^{6t}C_5 \cdot (\sqrt{3})^4 - {}^{6t}C_7 \cdot (\sqrt{3})^6 + \ldots$$

$$= \frac{1}{\sqrt{3}} \cdot \left[{}^{6t}C_1 \cdot (\sqrt{3}) - {}^{6t}C_3 \cdot (\sqrt{3})^3 + {}^{6t}C_5 \cdot (\sqrt{3})^5 \ldots \right]$$

$(1+\sqrt{3}i)^{6t}$ expansion

$$= 1 + {}^{6t}C_1 \cdot \sqrt{3}i + {}^{6t}C_2 \cdot (\sqrt{3}i)^2 + {}^{6t}C_3 \cdot (\sqrt{3}i)^3 + \cdots$$

$$= \left(1 - {}^{6t}C_2 (\sqrt{3})^2 + \cdots\right) + i\left({}^{6t}C_1 \cdot \sqrt{3} - {}^{6t}C_3 (\sqrt{3})^3 + \cdots\right)$$

$$= \frac{1}{\sqrt{3}} \cdot \text{Imaginary part of } (1+\sqrt{3}i)^{6t}$$

$$= \frac{1}{\sqrt{3}} \text{ Imaginary part of } 2^{6t} \cdot \left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{6t}$$

$$= \frac{1}{\sqrt{3}} \cdot 2^{6t} \cdot \text{I.p of } \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{6t}$$

$$= \frac{2^{6t}}{\sqrt{3}} \cdot \text{I.p of } \left(\cos 6 \cdot \frac{\pi}{3} t + i\sin\frac{\pi}{3} \cdot 6t\right)$$

$$= \frac{2^{6t}}{\sqrt{3}} \cdot \text{I.p of } \left(\cos 2\pi t + i \sin 2\pi t\right)$$

$$= \frac{2^{6t}}{\sqrt{3}} \cdot \sin(2\pi t) = 0.$$

$${}^nC_0 + {}^nC_1 - {}^nC_2 - {}^nC_3 + {}^nC_4 + {}^nC_5 - {}^nC_6 - {}^nC_7 + \cdots = 0$$

iff $n =$

(a) $4k$
(b) $4k+1$
(c) $4k+2$
(d) $4k+3$

$${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \cdots = 2^{n/2}\cos\frac{n\pi}{4}$$

$${}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 \cdots = 2^{n/2}\sin\frac{n\pi}{4}.$$

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots$$

Sub $x = i$

$$(1+i)^n = C_0 + C_1 i + C_2 i^2 + C_3 i^3 + C_4 i^4 + \cdots$$

$$= \left(C_0 - C_2 + C_4 - C_6 + \cdots\right) + i\left(C_1 - C_3 + C_5 - C_7 \cdots\right)$$

$$C_0 - C_2 + C_4 - C_6 + \cdots = \text{R.P of } (1+i)^n$$

$$= \text{R.P of } (\sqrt{2})^n\left(\frac{1+i}{\sqrt{2}}\right)^n$$

$$= \text{R.P of } (\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$

$$= \text{R.P of } (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$$\Rightarrow 2^{n/2}\left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right] = 0 \checkmark$$

$$\Rightarrow \tan\frac{n\pi}{4} = -1$$

$$\frac{n\pi}{4} = k\pi - \frac{\pi}{4}$$

$$\Rightarrow n = 4k-1 \,/\, 4k+3.$$