Mathematical Programming 28 (1984) 218–225
North-Holland

ANALYSIS OF A LINEARIZATION HEURISTIC FOR SINGLE-MACHINE SCHEDULING TO MAXIMIZE PROFIT

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Received 21 September 1981
Revised manuscript received 27 May 1983

We consider the problem of scheduling n jobs without preemption on a single machine to maximize total profit, where profit is given by a nonincreasing, concave separable function of job starting times. A heuristic is given in which jobs are sequenced optimally relative to a specific linear approximation of the profit function. This heuristic always obtains at least $\frac{2}{3}$ of the optimal profit, and examples exist where the heuristic obtains only $\frac{2}{3}$ of the optimal profit. A large class of alternative linearizations is considered and shown to give arbitrarily bad results.

Key words: Production/Scheduling-One Machine, Deterministic.

Introduction

This paper is a worst-case study of a class of heuristics for scheduling n jobs on a single machine to maximize a nonlinear profit function. Define

$$t_j = \text{process time of job } j \text{ (assumed integral)},$$

$$x_j = \text{starting time of job } j,$$

$$p_j(x_j) = \text{profit received if job } j \text{ is started at time } x_j,$$

$$T = \sum_{j=1}^{n} t_j,$$

$$X = \{x \mid x_{j_1} = 0 \text{ and } x_{j_{k+1}} = x_{j_k} + t_{j_k},\ k = 1, \ldots, n-1 \text{ for some permutation } j_1, \ldots, j_n \text{ of } 1, \ldots, n\}.$$

The problem we consider is

$$Z = \max_{x \in X} \sum_{j=1}^{n} p_j(x_j) \tag{P}$$

where we assume that $p_j(\cdot)$ is nonincreasing-concave with $p_j(T) \geqslant 0$. This problem

* Work supported in part by NSF Grant ECS 82-05438 to the University of Pennsylvania and ONR Contract N00014-81-C-0302.

is NP-hard (Lenstra et al. [10]) and hence the study of heuristics is of interest. Smith [13] has shown that if all $p_j(\cdot)$ are linear, an optimal solution is obtained by sequencing jobs in order of decreasing $(p_j(0)-p_j(T))/t_j$. We use this result to obtain a heuristic for (P) by sequencing jobs optimally relative to linear approximations of the $p_j(\cdot)$. The linear approximation we use is dynamic in the sense that the approximation for job j changes as other jobs are selected by the heuristic for processing before job j.

In Section 1 we give a precise description of our linearization rule and the associated heuristic. We exhibit a class of examples for which the profit achieved by this heuristic approaches $\frac{2}{3}Z$ as n approaches infinity. Let Z_H denote the heuristic profit. Section 2 is concerned with establishing a lower bound on Z_H/Z. We first show in Section 2.1 that Z_H/Z is minimized by examples in which all $p_j(\cdot)$ are piecewise linear with at most three pieces. In Section 2.2 we show that $Z_H \geqslant \frac{1}{2}Z$ using a proof based on Lagrangian relaxation. Finally, in Section 2.3 we state that $Z_H \geqslant \frac{2}{3}Z$. The proof of this result is contained in Fisher and Krieger [7] but is omitted here because it is long and complicated. The referees of this paper have also reviewed reference [7].

In Section 3 we consider a class of plausible alternative linearizations and show that they are arbitrarily bad in the sense that there is a set of examples over which Z_H/Z approaches 0.

Problem (P) is equivalent to

$$\min_{x\in X} \sum_{j=1}^{n} c_j(x_j) \qquad (\mathrm{P_{min}})$$

where $c_j(x_j)$ is a convex, nondecreasing delay cost and we assume that a revenue $r_j \geqslant c_j(T)$ is received for performing job j. Simply set $p_j(x_j)=r_j-c_j(x_j)$ to obtain the equivalence. Several papers have been concerned with optimization methods for a slight generalization of $(\mathrm{P_{min}})$ in which nonconvex $c_j(\cdot)$ are allowed. Successful computational experience has been obtained with dynamic programming (Baker and Schrage [1]), Lagrangian relation (Fisher [4]), and branch and bound based on an assignment relaxation (Rinnooy Kan et al. [12]). Lawler [9] has given a pseudopolynomial algorithm (in the sense that running time is $O(n^4T)$) for the case where we are given a due date d_j for job j and $c_j(x_j)=\max(0, x_j+p_j-d_j)$.

Although (P) and $(\mathrm{P_{min}})$ are equivalent from the perspective of optimization, they are not equivalent in terms of heuristic performance. Although the linearization heuristic presented here could be applied to $(\mathrm{P_{min}})$, the lower bound we establish on Z_H/Z for the maximization case does not imply an upper bound on Z_H/Z in the minimization case. Indeed, for $(\mathrm{P_{min}})$ it is possible that $Z=0$, so the ratio Z_H/Z is not useful for worst-case analysis. From this point of view, the maximization model is very appropriate for the analysis conducted here. Fisher [5] describes other examples where heuristics have different worst-case performance on minimization and maximization versions of problems.

Carroll [2] has developed a heuristic for minimizing tardiness in a general job shop with multiple machines. Carroll's heuristic also applies to the single-machine tardiness problem, and with appropriate setting of a parameter in Carroll's heuristic, is equivalent to the heuristic studied here. Fisher [4] applied Carroll's heuristic to single-machine min tardiness problems with up to 50 jobs and found it produced solutions within 2.8% of optimal on average. Carroll's heuristic has also been used very successfully on real problems (Carroll [3]), a fact we believe heightens the interest of the analysis presented here.

Other worst-case results on scheduling heuristics include Graham's [8] pioneering paper on the parallel machine problem and the recent paper by Potts [11] on single-machine scheduling with release dates. In both these works the objective function is minimization of the time to complete all jobs, and because of this they are very different from the study reported here. For example, in Graham's work the starting time of jobs is irrelevant once a machine assignment has been made, and the proof of the worst-case result bears a strong resemblance to analyses of packing problems like bin packing. Because job start times play such a fundamental role in our problem, the worst-case proofs are totally different from packing type arguments.

1. Heuristic and worst-case examples

Jobs are scheduled sequentially by the heuristic, and the linear approximation of $p_i(\cdot)$ is specified at the time a job is considered for scheduling. Suppose several jobs have been scheduled for processing in the interval $[0, t]$ and we must choose one of the remaining jobs to start at time t. Then the approximation of $p_i(\cdot)$ is the linear function through the points $(t, p_i(t))$ and $(T, p_i(T))$. The job chosen maximizes $(p_i(t)-p_i(T))/t_i$. A formal statement of this heuristic is given below. This statement is written in the form of a computer program in that when a step uses a particular parameter, the value to be used is the last value that has been assigned.

Linearization Heuristic

1. Initialize $t=0$ and $U=\{1,\ldots,n\}$.
2. (a) Choose $i \in U$ such that

$$\frac{p_i(t)-p_i(T)}{t_i} = \max_{j\in U}\left[\frac{p_j(t)-p_j(T)}{t_j}\right].$$

If there are ties, choose the least index.

(b) Set

$$x_i = t,$$

$$t = t + t_i,$$

$$U = U - \{i\}.$$

3. If $U = \emptyset$, set $Z_H = \sum_{i=1}^{n} p_i(x_i)$ and STOP. Otherwise, go to 2.

We now give a family of examples for which Z_H/Z approaches $\frac{2}{3}$. In the examples n is even, $t_i = 1$ for all i and the profit functions are given by

$$p_i(t) = \begin{cases} \dfrac{n-i+1}{n}, & 0 \leq t < T, \quad i = 1, \ldots, \frac{1}{2}n, \\ 0, & t = T, \end{cases}$$

$$p_i(t) = 1 - \frac{t}{n}, \quad i = \tfrac{1}{2}n + 1, \ldots, n.$$

The heuristic can schedule jobs in the sequence $1, \ldots, n$ with $Z_H = \frac{1}{2}n + \frac{1}{2}$. An optimal order is $n, n-1, \ldots, 2, 1$ with $Z = \frac{3}{4}n + \frac{1}{2}$. Then $Z_H/Z = (2n+2)/(3n+2) \to \frac{2}{3}$ as $n \to \infty$. When n is odd, we can construct a worst-case example from the example given above by assuming that one job (say, job n) has $t_n = 0$ and $p_n(t) = 0$ for all t.

2. Bounds on heuristic performance

2.1. Characterization of worst-case profit functions

In this section we show that Z_H/Z is minimized by a particularly simple class of profit functions. We define three classes of piece-wise linear $p_j(\cdot)$, denoted by C, L_2 and L_3 (see Fig. 1). The set C includes all profit functions given by

$$p_j(t) = \begin{cases} a, & 0 \leq t < T, \\ 0, & t = T, \end{cases}$$

where a is an arbitrary nonnegative scalar. The set L_2 consists of all two-piece linear $p_j(\cdot)$ through the points $(0, a)$, (b, a) and $(T, 0)$ where $a > 0$ and $b \in (0, T)$ are arbitrary scalars. The set L_3 consists of all three-piece linear $p_j(\cdot)$ through the points $(0, a)$, (b, a), (c, d) and $(T, 0)$ where $T > c > b > 0$, $a > 0$ and $a > d > a(T-c)/(T-b)$.

Theorem 1. *An arbitrary instance of* (P) *can be converted to a problem with all* $p_j(\cdot) \in C \cup L_3$ *without increasing* Z_H/Z.

Proof. Let h_j and 0_j denote the start times of job j in the heuristic and an optimal solution respectively for the given instance of (P). If $0_j \geq h_j$, replace $p_j(\cdot)$ by the member of C with $a = p_j(h_j)$ (see Figure $2a$). If $0_j < h_j$, replace $p_j(\cdot)$ by the member of L_3 that goes through the points $(0, p_j(0_j))$, $(0_j, p_j(0_j))$, $(h_j, p_j(h_j))$ and $(T, 0)$ (see Figure $2b$). Let $\bar{p}_j(\cdot)$, $j = 1, \ldots, n$, denote the new profit functions and $\bar{Z}_H$ and $\bar{Z}$ the new heuristic and optimal values.

(i) The solution $h_1, \ldots, h_n$ can still be chosen by the heuristic for $\bar{p}_j(\cdot)$, $j = 1, \ldots, n$, since $\bar{p}_j(h_j) = p_j(h_j)$ and $\bar{p}_j(t) \leq p_j(h_j)$, $t \leq h_j$, for all j. This implies $\bar{Z}_H = Z_H$.

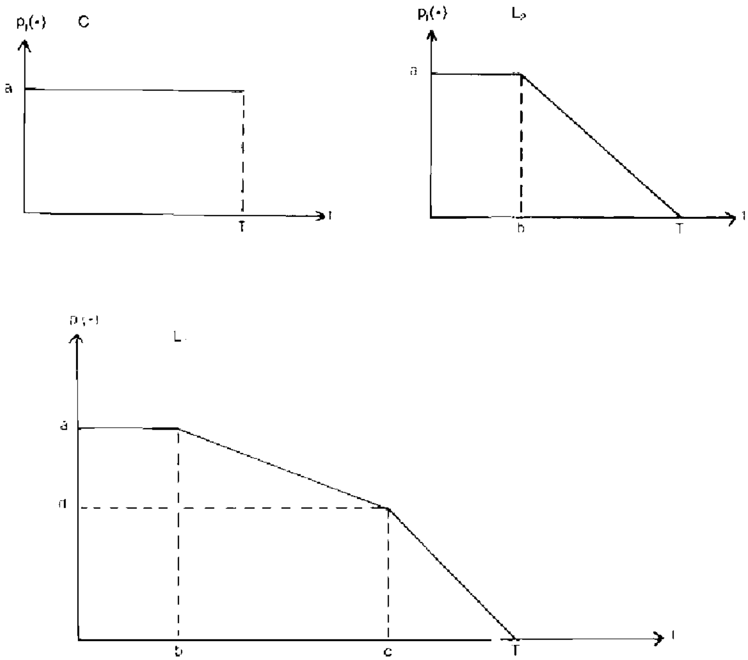

Fig. 1. Examples of C, L_2 and L_3.

(ii) The proof is completed by observing that

$$\bar{Z} \geqslant \sum_{j=1}^{n} \bar{p}_j(0_j) = \sum_{j=1}^{n} p_j(0_j) = Z.$$

2.2. Lower bound of $\frac{1}{2}$ on Z_H/Z

Although the bound of $\frac{1}{2}$ given here is weaker than the bound of $\frac{2}{3}$ stated in Section 2.3, the result is of interest because of the relative simplicity of its proof.

Let u_t denote a multiplier associated with the time interval $[t-1, t]$ for $t = 1, \ldots, T$. We use the Lagrangian relaxation from Fisher [4],

$$Z_D(u) = \sum_{t=1}^{T} u_t + \sum_{j=1}^{n} \max_{\substack{0 \leqslant x_j \leqslant T - t_j \\ x_j \text{ integral}}} \left(p_j(x_j) - \sum_{t=x_j+1}^{x_j+t_j} u_t \right), \tag{1}$$

and the Lagrangian dual

$$Z_D = \min_u Z_D(u). \tag{2}$$

It is well known (e.g., Fisher [6], p. 2) that $Z_D(u) \geqslant Z$ for any u.

Theorem 2. $Z_H \geqslant \frac{1}{2} Z_D$.

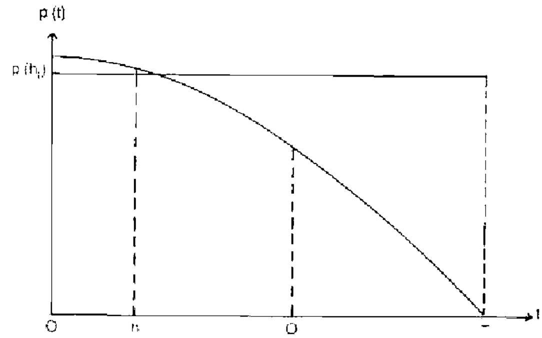

Fig. 2a. Converting $p_i(\cdot)$ for a member of C.

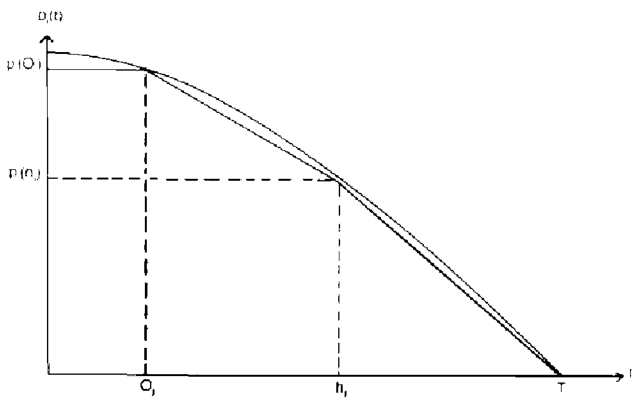

Fig. 2b. Converting $p_j(\cdot)$ for a member of L_3.

Proof. Let $h_1, \ldots, h_n$ denote the job start times in the heuristic solution. For $t = 1, \ldots, T$ let j_t denote the jobs in process in the heuristic solution during the interval $[t-1, t]$ and define

$$u_t(\mathrm{H}) = \frac{p_{j_t}(h_{j_t}) - p_{j_t}(T)}{t_{j_t}} \geq 0.$$

Note that $\sum_{t=1}^{T} u_t(\mathrm{H}) = Z_{\mathrm{H}} - \sum_{j=1}^{n} p_j(T) \leq Z_{\mathrm{H}}$.

Let

$$\bar{p}_i(t, u) = p_i(t) - \sum_{\tau=t+1}^{t+t_i} u_\tau.$$

We will show that $\bar{p}_i(t, u(\mathrm{H})) \leq p_i(h_i)$ for all i and t. Then using (1) and (2) we have $Z_{\mathrm{D}} \leq Z_{\mathrm{D}}(u(\mathrm{H})) \leq 2Z_{\mathrm{H}}$ and the theorem is proved.

For any i and $t \geq h_i$, $\bar{p}_i(t, u(\mathrm{H})) \leq p_i(h_i)$ follows from $p_i(\cdot)$ nonincreasing and $u(\mathrm{H}) \geq 0$. To analyze the case $t < h_i$, first note that if $\tau \leq t_i + t_i$ then $h_{j_\tau} \leq h_i$. The heuristic selection rule and $h_{j_\tau} \leq \tau \leq t + t_i$ then imply

$$u_\tau(H) = \frac{p_{j_\tau}(h_{j_\tau}) - p_{j_\tau}(T)}{t_{j_\tau}} \geq \frac{p_i(h_{j_\tau}) - p_i(T)}{t_i} \geq \frac{p_i(t + t_i) - p_i(T)}{t_i}$$

Then

$$\begin{aligned}
\bar{p}_i(t, u(H)) &= p_i(t) - \sum_{\tau=t+1}^{t+t_i} u_\tau(\mathrm{H}) \\
&\leq p_i(t) - \sum_{\tau=t+1}^{t+t_i} \frac{p_i(t+t_i) - p_i(T)}{t_i} \\
&= p_i(t) - p_i(t+t_i) - p_i(T) \\
&= p_i(h_i) + [p_i(t) - p_i(t+t_i)] - [p_i(h_i) - p_i(T)] \\
&\leq p_i(h_i),
\end{aligned}$$

where the last inequality follows from $t_i \leq h_i \leq T - h_i$ and $p_i(\cdot)$ concave nonincreasing.

2.3. *Lower bound of $\frac{2}{3}$ on Z_H/Z*

Fisher and Krieger [7] give a proof that $Z_H/Z \geq \frac{2}{3}$. This proof is based on an algorithm that takes an arbitrary instance of (P) and transforms it to a problem in which all $t_j = 1$, and all $p_j(\cdot) \in C \cup L_2$. We show that if the original instance of (P) satisfied $Z_H/Z < \frac{2}{3}$, then the transformed problem produced by our algorithm would also satisfy $Z_H/Z < \frac{2}{3}$. The proof is completed by showing that $Z_H/Z \geq \frac{2}{3}$ for any instance of (P) with all $t_j = 1$ and all $p_j(\cdot) \in C \cup L_2$. This proof is omitted here because it is long and complicated. Notational masochists can obtain a copy of the full proof from the authors.

3. Some arbitrarily bad linearizations

A natural question is whether an alternative method for linearizing profit functions might give better worst-case performance. This section exhibits a large class of plausible alternatives that are definitely not better.

When selecting a job to begin at time t, the alternative linear approximation is the line through the points $(t, p_i(t))$ and $(T - \Delta, p_i(T - \Delta))$, where Δ is a fixed positive scalar. The performance of this linear approximation is arbitrarily bad as shown by the example where $n = 2$, $t_1 = \varepsilon$, $t_2 = T - \varepsilon$, $p_1(\cdot)$ is the two piece linear function through the points $(0, 1)$, $(T - \Delta, 1)$ and $(T, 0)$ and $p_2(\cdot)$ is the linear function through the points $(0, \varepsilon T)$ and $(T, 0)$. With $\Delta > \varepsilon/(1 - \varepsilon^2)$ the optimal order is 1, 2 while the new linearization schedules the jobs 2, 1. This gives

$$Z = 1 + \varepsilon(T - \varepsilon), \quad Z_H = \varepsilon T + \varepsilon/\Delta \quad \text{and} \quad Z_H/Z \to 0 \text{ as } \varepsilon \to 0.$$

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