Optimal analysis of Best Fit bin packing ⋆

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Abstract. In early seventies it was shown that the *asymptotic* approximation ratio of BestFit bin packing is equal to 1.7. We prove that also the *absolute* approximation ratio for BestFit bin packing is exactly 1.7, improving the previous bound of 1.75. This means that if the optimum needs Opt bins, BestFit always uses at most $\lfloor 1.7 \cdot \mathrm{OPT} \rfloor$ bins. Furthermore we show *matching lower bounds* for all values of Opt, i.e., we give instances on which BestFit uses exactly $\lfloor 1.7 \cdot \mathrm{OPT} \rfloor$ bins. Thus we completely settle the worst-case complexity of BestFit bin packing after more than 40 years of its study.

1 Introduction

Bin packing is a classical combinatorial optimization problem in which we are given an instance consisting of a sequence of items with rational sizes between 0 and 1, and the goal is to pack these items into the smallest possible number of bins of unit size. BestFit algorithm packs each item into the most full bin where it fits, possibly opening a new bin if the item does not fit into any currently open bin. A closely related FirstFit algorithm packs each item into the first bin where it fits, again opening a new bin only if the item does not fit into any currently open bin.

Johnson's thesis [8] on bin packing together with Graham's work on scheduling [6, 7] belong to the early influential works that started and formed the whole area of approximation algorithms. The proof that the asymptotic approximation ratio of FirstFit and BestFit bin packing is 1.7 given by Ullman [14] and subsequent works by Garey et al. and Johnson et al. [5, 10] were among these first results on approximation algorithms.

We prove that also the *absolute* approximation ratio for BestFit bin packing is exactly 1.7, i.e., if the optimum needs Opt bins, BestFit always uses at most $\lfloor 1.7 \cdot \mathrm{Opt} \rfloor$ bins. This builds upon and substantially generalizes our previous upper bound for FirstFit from [3]. For the comparison of the techniques, see the beginning of Section 4. Furthermore we show *matching lower bounds* for *all* values of Opt, i.e., we give instances on which BestFit and FirstFit use exactly $\lfloor 1.7 \cdot \mathrm{Opt} \rfloor$ bins. This is also the first construction of an instance that has absolute approximation ratio exactly 1.7 for an arbitrarily large Opt.

Note that the upper bound for BestFit is indeed a generalization of the bound for FirstFit: The items in any instance can be reordered so that they arrive in the order of bins in the FirstFit packing. This changes neither FirstFit,

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nor the optimal packing. Thus it is sufficient to analyze FirstFit on such instances. On the other hand, on them BestFit behaves exactly as FirstFit, as there is always a single bin where the new item fits. Thus any lower bound for FirstFit applies immediately to BestFit and any upper bound for FirstFit is equivalent to a bound for BestFit for this very restricted subset of instances. To demonstrate that the extension of the absolute bound from FirstFit to BestFit is by far not automatic, we present a class of any-fit-type algorithms for which the asymptotic bound of 1.7 holds, but the absolute bound does not.

History and related work. The upper bound on BestFit (and FirstFit) was first shown by Ullman in 1971 [14]; he proved that for any instance, $\mathrm{BF}, \mathrm{FF} \leq 1.7 \cdot \mathrm{Opt} + 3$, where BF, FF and Opt denote the number of bins used by BestFit, FirstFit and the optimum, respectively. Still in seventies, the additive term was improved first in [5] to 2 and then in [4] to $\mathrm{BF} \leq \lceil 1.7 \cdot \mathrm{Opt} \rceil$; due to integrality of BF and Opt this is equivalent to $\mathrm{BF} \leq 1.7 \cdot \mathrm{Opt} + 0.9$. Recently the additive term of the asymptotic bound was improved for FirstFit to $\mathrm{FF} \leq 1.7 \cdot \mathrm{Opt} + 0.7$ in [16] and to $\mathrm{FF} \leq 1.7 \cdot \mathrm{Opt}$ in [3].

The absolute approximation ratio of FirstFit and BestFit was bounded by 1.75 by Simchi-Levy [13]. Recent improvements again apply only to FirstFit: after bounds of $12/7 \approx 1.7143$ by Xia and Tan [16] and Boyar et al. [1] and $101/59 \approx 1.7119$ by Németh [11], the tight bound of $\mathrm{FF} \leq 1.7 \cdot \mathrm{Opt}$ was given in our previous work [3].

For the lower bound, the early works give examples both for the asymptotic and absolute ratios. The example for the asymptotic bound gives $\mathrm{FF} = 17k$ whenever $\mathrm{Opt} = 10k + 1$, thus it shows that the asymptotic upper bound of 1.7 is tight, see [14, 5, 10]. For the absolute ratio, an example is given with $\mathrm{FF} = 17$ and $\mathrm{Opt} = 10$, i.e., an instance with approximation ratio exactly 1.7 [5, 10], but no such example was known for large Opt. In our previous work [3] we have given lower bound instances with $\mathrm{BF} = \mathrm{FF} = \lfloor 1.7 \cdot \mathrm{Opt} \rfloor$ whenever $\mathrm{Opt} \not\equiv 0, 3 \pmod{10}$.

We have mentioned only directly relevant previous work. Of course, there is much more work on bin packing, in particular there exist asymptotic approximation schemes for this problem, as well as many other algorithms. We refer to the survey [2] or to the recent excellent book [15].

Organization of the paper. The crucial technique of the upper bound is a combination of amortization and weight function analysis, following the scheme of our previous work [12, 3]. We present it first in Section 2 to give a simple proof of the asymptotic bound for BestFit and any-fit-type algorithms. We prove the lower bound in Section 3, as it illustrates well the issues that we need to deal with in the upper bound proof, which is then given in Section 4. We postpone most proofs to the appendix, but try to explain the main ideas behind them.

2 Notations and the simplified asymptotic bound

Let us fix an instance I with items $a_1, \ldots, a_n$ and denote the number of bins in the BestFit and optimal solutions by BF and Opt, respectively. We will often identify an item and its size. For a set of items A, let $s(A) = \sum_{a \in A} a$, i.e., the total size of items in A and also for a set of bins $\mathcal{A}$, let $s(\mathcal{A}) = \sum_{A\in\mathcal{A}} s(A)$. Furthermore, let $S = s(I)$ be the total size of all items of I. Obviously $S \leq \textsc{Opt}$.

We classify the items by their sizes: items $a \leq 1/6$ are **small**, items $a \in (1/6, 1/2]$ are **medium**, and items $a > 1/2$ are **huge**. A bin is called a **k-bin** or **k^+-bin**, if it contains exactly k items or at least k items, respectively, in the final packing. Furthermore, the **rank** of a bin is the number of medium and huge items in it. An item is called **k-item** if BF packs it into a k-bin.

The bins in the BF packing are ordered by the time they are opened (i.e., when the first item is packed into them). Expressions like "before", "after", "first bin", "last bin" refer to this ordering. At any time during the packing, the **level of a bin** is the total size of items currently packed in it, while by **size of a bin** we always mean its final level. A **level of an item** a denotes the level of the bin where a is packed, just before the packing of a.

The following properties of BestFit follow easily from its definition, see Appendix A.

Lemma 2.1. *At any moment, in the* BF *packing the following holds:*

(i) *The sum of levels of any two bins is greater than 1. In particular, there is at most one bin with level at most* $1/2$.

(ii) *Any item a with level at most* $1/2$ *(i.e., packed into the single bin with level at most* $1/2$*) does not fit into any bin open at the time of its arrival, except for the bin where the item a is packed.*

(iii) *If there are two bins B, B' with level at most* $2/3$*, in this order, then either B' contains a single item or the first item in B' is huge.*

To illustrate our technique, we now present a short proof of the asymptotic ratio 1.7 for BestFit. It uses the same weight function as the traditional analysis of BestFit. (In some variants the weight of an item is capped to be at most 1, which makes almost no difference in the analysis.) To use amortization, we split the weight of each item a into two parts, namely its bonus $\overline{w}(a)$ and its scaled size $\overline{\overline{w}}(a)$, defined as

$$\overline{w}(a) = \begin{cases} 0 & \text{if } a \leq \frac{1}{6}\,, \\ \frac{3}{5}(a - \frac{1}{6}) & \text{if } a \in \left(\frac{1}{6}, \frac{1}{3}\right)\,, \\ 0.1 & \text{if } a \in \left[\frac{1}{3}, \frac{1}{2}\right]\,, \\ 0.4 & \text{if } a > \frac{1}{2}\,. \end{cases}$$

For every item a we define $\overline{\overline{w}}(a) = \frac{6}{5}a$ and its weight is $w(a) = \overline{\overline{w}}(a) + \overline{w}(a)$. For a set of items B, $w(B) = \sum_{a\in B} w(a)$ denotes the total weight, similarly for $\overline{w}$ and $\overline{\overline{w}}$.

It is easy to observe that the weight of any bin B, i.e., of any set with $s(B) \leq 1$, is at most 1.7: The scaled size of B is at most 1.2, so we only need to check that $\overline{w}(B) \leq 0.5$. If B contains no huge item, there are at most 5 items with non-zero $\overline{w}(a)$ and $\overline{w}(a) \leq 0.1$ for each of them. Otherwise the huge item has bonus 0.4; there are at most two other medium items with non-zero bonus and it is easy to check that their total bonus is at most 0.1. This implies that the weight of the whole instance is at most $1.7 \cdot \textsc{Opt}$.

The key part is to show that, on average, the weight of each BF-bin is at least 1. Lemma 2.2 together with Lemma 2.1 implies that for almost all bins with two or more items, its scaled size plus the bonus of the *following* such bin is at least 1.

Lemma 2.2. *Let B be a bin such that $s(B) \geq 2/3$ and let c, c' be two items that do not fit into B, i.e., $c, c' > 1 - s(B)$. Then $\overline{\overline{w}}(B) + \overline{w}(c) + \overline{w}(c') \geq 1$.*

Proof. If $s(B) \geq 5/6$, then $\overline{\overline{w}}(B) \geq 1$ and we are done. Otherwise let $x = 5/6 - s(B)$. We have $0 < x \leq 1/6$ and thus $c, c' > 1/6 + x$ implies $\overline{w}(c), \overline{w}(c') > \frac{3}{5}x$. We get $\overline{\overline{w}}(B) + \overline{w}(c) + \overline{w}(c') > \frac{6}{5}(\frac{5}{6} - x) + \frac{3}{5}x + \frac{3}{5}x = 1$. □

Any BF-bin D with a huge item has $\overline{w}(D) \geq 0.4$ and $\frac{6}{5}s(D) > 0.6$, thus $w(D) > 1$.

For the amortization, consider all BF-bins B with two or more items, size $s(B) \geq 2/3$, and no huge item. For any such bin except for the last one choose C as the next bin with the same properties. Since C has no huge item, its first two items c, c' have level at most $1/2$ and by Lemma 2.1(ii) they do not fit into B. Lemma 2.2 implies $\overline{\overline{w}}(B) + \overline{w}(C) \geq \overline{\overline{w}}(B) + \overline{w}(c) + \overline{w}(c') \geq 1$.

Summing all these inequalities (note that each bin is used at most once as B and at most once as C) and $w(D) > 1$ for the bins with huge items we get $w(I) \geq \mathrm{BF} - 3$. The additive constant 3 comes from the fact that we are missing an inequality for at most three BF-bins: the last one from the amortization sequence, possibly one bin B with two or more items and $s(B) < 2/3$ (cf. Lemma 2.1(iii)) and possibly one bin B with a single item and $s(B) < 1/2$ (cf. Lemma 2.1(i)). Combining this with the previous bound on the total weight, we obtain $\mathrm{BF} - 3 \leq w(I) \leq 1.7 \cdot \textsc{Opt}$ and the asymptotic bound follows.

This simple proof holds for a wide class of any-fit-type algorithms: Call an algorithm a GAAF (generalized almost any fit) algorithm, if it uses the bin with level at most 1/2 only when the item does not fit into any previous bin (Lemma 2.1(i) implies that there is always at most one such bin). Our proof of the asymptotic ratio can be tightened so that the additive constant is smaller:

Theorem 2.3. *For any GAAF algorithm A and any instance of bin packing we have $A \leq \lfloor 1.7 \cdot \textsc{Opt} + 0.7 \rfloor \leq \lceil 1.7 \cdot \textsc{Opt} \rceil$.*

The proof is given in Appendix A, where we also give an example of a GAAF algorithm which does not satisfy the absolute bound of 1.7. The asymptotic bound for almost any fit (AAF) algorithms was proved in [8, 9], where the original AAF condition prohibits packing an item in the smallest bin if that bin is unique and the item does fit in some previous bin (but the restriction holds also if the smallest bin is larger than 1/2). Theorem 2.3 improves the additive term and generalizes the bound from AAF to the slightly less restrictive GAAF condition (although it seems that the original proof also uses only the GAAF condition).

3 Lower bound

The high level scheme of the lower bound for $\textsc{Opt} = 10k$ is this: For a tiny $\varepsilon > 0$, the instance consists of $\textsc{Opt}$ items of size approximately 1/6, followed by $\textsc{Opt}$ items of size approximately 1/3, followed by $\textsc{Opt}$ items of size $1/2 + \varepsilon$. The optimum packs in each bin one item from each group. BestFit packs the items of size about $1/6$ into $2k$ bins with 5 items, with the exception of the first and last of these bins that will have 6 and 4 items, respectively. The items of size about $1/3$ are packed in pairs. To guarantee this packing, the sizes of items differ from $1/6$ and $1/3$ in both directions by a small amount δ_i which is exponentially decreasing, but greater than ε for all i. This guarantees that only the item with the largest δ_i in a bin is relevant for its final size and this in turn enables us to order the items so that no additional later item fits into these bins.

Theorem 3.1. *For all values of* Opt*, there exists an instance I with* FF $=$ BF $= \lfloor 1.7 \cdot \text{Opt} \rfloor$.

4 Upper bound

At the high level, we follow the weight function argument from the simple proof in Section 2. As we have seen, the BF packing in the lower bound contains three types of bins that play different roles. To obtain the tight upper bound, we analyze them separately. For two of these types we can argue easily that the weight of each bin is at least 1: First, the bins with size at least $5/6$, called big bins below; these are the initial bins in the lower bound containing the items of size approximately $1/6$. Second, the 1-bins, called dedicated bins; these are the last bins with items $1/2 + \varepsilon$ in the lower bound. The remaining bins, common bins, are the middle bins of size approximately $2/3$ with items of size around $1/3$ (except for the first bin) in the lower bound. They are analyzed using the amortization lemma. This general scheme has several obstacles which we describe now, together with the intuition behind their solution.

Obstacle 1: There can be one dedicated bin with item $d_0 < 1/2$. We need to change its bonus to approximately 0.4, to guarantee a sufficient weight of this bin. This in turn possibly forces us to decrease the bonus of one huge item f_1 to 0.1, if d_0 and f_1 are in the same Opt-bin, so that the Opt-bin has weight at most 1.7.

Obstacle 2: The amortization lemma needs two items that do not fit into the previous bin. Unlike FirstFit, BestFit does not guarantee this, if the first item in a bin is huge. If this first item is not f_1, we can handle such bins, called huge-first bins, similarly as dedicated bins. If this happens for f_1, we need to argue quite carefully to find the additional bonus. This case, called the freaky case, is the most complicated part of our analysis.

Obstacle 3: Even on the instances similar to our lower bound, the amortization leaves us with the additive term of 0.1, because we cannot use the amortization on the last common bin, and its scaled weight is only about 0.8 if its size is around $2/3$. Here the parity of the items of size around $1/3$ comes into play: Typically they come in pairs in BF-bins, as in the lower bound, but for odd values of Opt one of them is missing or is in a FirstFit bin of 3 or more items. This allows us to remove the last 0.1 of the additive term, using the mechanism of an exceptional set, see Definition 4.9.

Obstacle 4: If the last common bin is smaller than $2/3$, the problem with amortizing it is even larger. Fortunately, then the previous common bins are larger than $2/3$ and have additional weight that can compensate for this, using a rather delicate argument, see Proposition 4.11.

4.1 Notations and preliminary lemmata

We classify the BF bins into four groups.

Any 1-bin D is a **dedicated bin**; $\mathcal{D}$ denotes the set of all dedicated bins and δ their number. If some dedicated bin has size at most $1/2$, denote the item in it d_0 and let $\Delta = 1/2 - d_0$; otherwise d_0 and Δ are undefined. Lemma 2.1(i) implies that there is at most one such item; also we shall see that in the tightest case Δ is close to 0.

If d_0 is defined, there may exist a (unique) huge item in its OPT-bin. In that case, denote it f_1 for the rest of the proof and leave f_1 undefined otherwise. Furthermore, if f_1 is the first item in a BF-bin, denote that bin F for the rest of the proof; otherwise let F undefined. Note that F cannot be a 1-bin as otherwise d_0 would fit there contradicting Lemma 2.1(i). Let f_2 be the second item in F.

If the first item of a 2^+-bin H is huge and $H \neq F$, we call H a **huge-first** bin; $\mathcal{H}$ denotes the set of all huge-first bins and η their number.

If a 2^+-bin B satisfies $s(B) \geq 5/6$ and it is not in $\mathcal{H}$, we call it a **big bin**; $\mathcal{B}$ denotes the set of all big bins and β their number.

Any remaining bin (i.e., any 2^+-bin with size less than $5/6$ and the first item small or medium, and also F if it is defined and not a big bin) is called a **common bin**; $\mathcal{C}$ denotes the set of all common bins, and γ their number.

An item is called an **H-item**, if it is d_o or a huge item different from f_1 (if defined). Note that each OPT-bin and each BF-bin contains at most one H-item.

The definitions imply that in every big or common bin different from F (if defined), the first item is small or medium. Then Lemma 2.1(ii) implies that the first two items of the bin do not fit into any previous bin.

Throughout the proof we distinguish two cases depending on the bin F.

If F is not defined, or it is a big bin, or f_2 does not fit into any previous common bin, then we call this the **regular case** and all the common bins, including F if it is defined and a common bin, are called **regular bins**.

If F is defined and it is a common bin, and f_2 would fit into some previous common bin at the time of its packing, fix one such bin G for the rest of the proof. We call this the **freaky case** and F the **freaky bin**. All the other common bins are called **regular bins**.

In both cases, denote the set of all regular bins by $\mathcal{R}$ and their number by ρ, furthermore number the regular bins $C_1, \ldots, C_\rho$, ordered by the time of their opening. In the freaky case, let g be the index of bin G in this ordering, i.e., let $C_g = G$. Note that $\rho = \gamma$ in the regular case and $\rho = \gamma - 1$ in the freaky case.

In the following lemma we significantly reduce the set of instances that we need to consider in our proof. In general, our goal is to reorder or remove the items in the sequence so that BESTFIT packs most items similarly as FIRSTFIT. For these transformations, we use two important properties of BESTFIT that follow from its definition. First, if we remove all the items from a BF-bin from the instance, the packing of the remaining items into the remaining bins does not change; often we use this so that we move the items to a later position in the instance and then this implies that the packing of the initial segment before the new position of the moved items does not change. Second, if two instances lead to the same configuration and we extend them by the same set of items, then the resulting configurations are also the same, where the configuration is the current multiset of levels of BF-bins. Note that this does not hold for FirstFit, as permuting the bins can change the subsequent packing, but the configuration is the same.

Lemma 4.1. *If there exists an instance with* BF $> 1.7 \cdot$ Opt, *then there exists such an instance I that in addition satisfies the following properties:*

(i) *All the* 1*-items form a final segment of the input instance.*

(ii) *If a* BF *bin B contains an item a such that for all other* BF *bins B' we have $a + s(B') > 1$ then B is an* 1*-bin.*

(iii) *In each* BF 2^+*-bin, the first two items are a_{j-1} and a_j for some j (i.e., they are adjacent in I). Furthermore, these two items are packed into different bins in* Opt.

(iv) *Suppose that for a* BF 3^+*-bin B, the first item in B is not huge, and no new bin is opened after opening B and before packing the third item into B. Then the first three items packed into B are a_{j-2}, a_{j-1} and a_j for some j (i.e., they are adjacent in I). Furthermore, these three items are packed into different bins in* Opt.

(v) *Suppose that a_j is the last item packed into a* BF *bin B. Then for all $j' > j$, we have $a_{j'} + s(B) > 1$ (i.e., no later items fit into B). Consequently, no later item has level $s(B)$ or larger in* BF *packing.*

For the rest of the proof we assume that our instance I satisfies the properties from Lemma 4.1. The following lemma states the consequences for the common bins: The medium items are packed as in FirstFit and the small items are restricted to only first few common bins.

Lemma 4.2. (i) *Any item $a_j > 1/6$ packed into a regular bin C_i has the property that at the time of its packing, a_j does not fit into any previous common bin.*

(ii) *If a small item a_j is packed into a common bin, then this is a common bin with the largest level at the time of packing a_j. Except for C_1 and F, any small item in a common bin has level at least $2/3$.*

(iii) *From the moment when there are two common bins with level at least $2/3$ on, no small item arrives. In particular, no small item is packed into a common bin opened later than C_2.*

(iv) *If $a_j \in C_2$ is small, some $a_k > 1/6$, $k > j$ (i.e., a_k is after a_j), is packed into C_1.*

In the next lemma we state some properties important for the freaky case. For the rest of the proof, let g_0 denote the item in bin G guaranteed by the next lemma. Note that the lemma implies that there are at least three items packed into G, as there are two other items in G when F opens.

Lemma 4.3. *In the freaky case, the* BF *packing satisfies the following:*

(i) *There exists an item g_0 that is packed into bin G such that g_0 arrives after f_2 and $s(F) + g_0 > 1$. Furthermore, $s(F) + s(G) > 1 + d_0$.*

(ii) *If the regular bins C_i and C_k are opened before F then $s(F) > 2/3$ and $s(C_i) + s(C_k) + s(F) > 2$.*

The following properties of BestFit are simple to show.

Lemma 4.4. *In the* BF *packing the following holds:*

(i) *The total size of any $k \geq 2$* BF*-bins is greater than $k/2$.*

(ii) *If d_0 is defined, then $s(\mathcal{H} \cup \mathcal{D}) \geq (\delta + \eta)/2 + (\delta + \eta - 2)\Delta$.*

(iii) *The total number of huge-first and dedicated bins is $\delta + \eta \leq \textsc{Opt}$.*

(iv) *Suppose that C is a regular bin of size $s(C) = 2/3 - 2x$ for some $x \geq 0$. For any bin B before C we have $s(B) > 2/3 + x$ and for any regular or big bin B after C we have $s(B) > 2/3 + 4x$.*

(v) *Suppose we have a set $\mathcal{A}$ of k common and big bins such that there are at least 3 common bins among them. Then $s(\mathcal{A}) > 2k/3$.*

Now we assume that the instance violates the absolute ratio 1.7 and derive some easy consequences that exclude some degenerate cases. Note that the values of $1.7 \cdot \textsc{Opt}$ are multiples of 0.1 and BF is an integer, thus $\mathrm{BF} > 1.7 \cdot \textsc{Opt}$ implies $\mathrm{BF} \geq 1.7 \cdot \textsc{Opt} + 0.1$. Typically we derive a contradiction with the lower bound $S \leq \textsc{Opt}$ on the optimum.

Lemma 4.5. *If* $\mathrm{BF} > 1.7 \cdot \textsc{Opt}$ *then the following holds:*

(i) $\textsc{Opt} \geq 7$.

(ii) *No common bin C has size $s(C) \leq 1/2$.*

(iii) *The total number of dedicated and huge-first bins is bounded by $\eta + \delta \geq 5$. If d_0 is not defined then there is no huge-first bin, i.e., $\eta = 0$.*

(iv) *The number of regular bins is at least $\rho \geq \textsc{Opt}/2 + 1 > 4$. If $\mathrm{BF} \geq 1.7 \cdot \textsc{Opt} + \tau/10$ for some integer $\tau \geq 1$ then $\rho > (\textsc{Opt} + \tau)/2$.*

4.2 The weight function, amortization, exceptional set

Now we give the modified and final definition of the weight function. The weight is modified only for d_0 and f_1 and their modified bonus is at least 0.1. Thus Lemma 2.2 still holds, as its proof uses at most 0.1 of bonus for each item.

Definition 4.6. *The weight function w, bonus $\overline{w}$ and scaled size $\overline{\overline{w}}$ are defined as follows:*

If d_0 is defined, we define $\overline{w}(d_0) = 0.4 - \frac{3}{5}\Delta$

If f_1 is defined, we define $\overline{w}(f_1) = 0.1$

For any other item a, we define
$$\overline{w}(a) = \begin{cases} 0 & \text{if } a \leq \frac{1}{6}, \\ \frac{3}{5}(a - \frac{1}{6}) & \text{if } c \in \left[\frac{1}{6}, \frac{1}{3}\right], \\ 0.1 & \text{if } a \in \left[\frac{1}{3}, \frac{1}{2}\right], \\ 0.4 & \text{if } a > \frac{1}{2}. \end{cases}$$

For every item a we define $\overline{\overline{w}}(a) = \frac{6}{5}a$ *and* $w(a) = \overline{\overline{w}}(a) + \overline{w}(a)$.

For a set of items A and a set of bins $\mathcal{A}$, let $w(A)$ and $w(\mathcal{A})$ denote the total weight of all items in A or $\mathcal{A}$; similarly for $\overline{\overline{w}}$ and $\overline{w}$. Furthermore, let $W = w(I)$ be the total weight of all items of the instance I.

Note that H-items are exactly the items with bonus greater than 0.1.

In the previous definition, the function $\overline{w}$ is continuous on the case boundaries, except for a jump at 0.4. Furthermore, if we have a set A of k items with size in $[\frac{1}{6}, \frac{1}{3}]$ and $d_0 \notin A$, then the definition implies that the bonus of A is exactly $\overline{w}(A) = \frac{3}{5}\left(s(A) - \frac{k}{6}\right)$. More generally, if A contains at least k items and no H-item, then we get an upper bound $\overline{w}(A) \le \frac{3}{5}\left(s(A) - \frac{k}{6}\right)$.

The analysis of OPT-bins and big, dedicated and huge-first BF-bins in the next two lemmata is easy. Some calculations are needed if d_0 is defined, as we need careful bounds on the lower order terms proportional to $\varDelta$.

Lemma 4.7. *For every optimal bin A its weight $w(A)$ can be bounded as follows:*
(i) *$w(A) \le 1.7$.*
(ii) *If A contains no H-item, then $w(A) \le 1.5$.*

Lemma 4.8. (i) *The total weight of the big bins is $w(\mathcal{B}) \ge \overline{\overline{w}}(\mathcal{B}) \ge \beta$.*
(ii) *If d_0 is undefined then total weight of the dedicated and huge-first bins is $w(\mathcal{D} \cup \mathcal{H}) \ge \delta + \eta$.*
(iii) *If d_0 is defined then the total weight of the huge-first and dedicated bins is*

$$w(\mathcal{D} \cup \mathcal{H}) \ge \delta + \eta + \frac{6}{5}(\delta + \eta - 2.5)\varDelta > \delta + \eta\,.$$

If in addition one of the huge-first bins has size at least $7/12$ then $w(\mathcal{D} \cup \mathcal{H}) > \delta + \eta + 0.1$.

The analysis of the common bins is significantly harder. Typically we prove that their weight is at least $\gamma - 0.2$ which easily implies that $\mathrm{BF} \le 1.7 \cdot \mathrm{OPT} + 0.1$. Due to the integrality of BF and OPT, this implies our main result whenever $\mathrm{OPT} \not\equiv 7 \pmod{10}$. To tighten the bound by the remaining 0.1 and to analyze the freaky case, we need to reserve the bonus of some of the items in the common bins instead of using it for amortization; this is possible if we still have two items in each regular bins whose bonus we can use. Now we define a notion of an exceptional set E, which contains these items with reserved bonus. In the freaky case, $g_0 \in E$, as its bonus is always needed to amortize for F. Other items are added to E only if $\mathrm{OPT} \equiv 7 \pmod{10}$, depending on various cases.

Definition 4.9. *A set of items E is called an* **exceptional set** *if*
(i) *for each $i = 2, \ldots, \rho$, the bin C_i contains at least two items $c, c' > \frac{1}{6}$ that are not in E;*
(ii) *if $\mathrm{OPT} \not\equiv 7 \pmod{10}$ then $E = \emptyset$ in the regular case and $E = \{g_0\}$ in the freaky case; and*
(iii) *if $\mathrm{OPT} \equiv 7 \pmod{10}$ then E has at most two items and $g_0 \in E$ in the freaky case.*

The next lemma modifies the amortization lemma for the presence of the exceptional set.

Lemma 4.10. (i) *Let $i = 2, \ldots, \rho$ and $s(C_{i-1}) \ge 2/3$. Then $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i \setminus E) \ge 1$.*
(ii) *In the freaky case, if $s(F) \ge 2/3$ then $\overline{\overline{w}}(F) + \overline{w}(f_1) + \overline{w}(g_0) \ge 1$.*

Proof. **(i):** Let $c, c' > \frac{1}{6}$ be two items in $C_i \setminus E$; their existence is guaranteed by the definition of the exceptional set. By Lemma 4.2(i), $c, c' > 1 - s(C_{i-1})$. The claim follows by Lemma 2.2 (which applies even to the modified weights, as we noted before).

(ii): Lemma 4.3(i) implies $g_0 > 1 - s(F)$. Trivially, $f_1 > 1/2 > 1 - s(F)$. Thus we can apply Lemma 2.2 with $c = g_0$ and $c' = f_1$ and the claim follows. $\square$

4.3 Analyzing the common bins

The following proposition is relatively straightforward if $s(C_\rho) \geq 2/3$, otherwise it needs a delicate argument, see Appendix D.

Proposition 4.11. *Let* $\textsc{Opt} \geq 8$, $\mathrm{BF} > 1.7 \cdot \textsc{Opt}$, *and* E *be an exceptional set. Then:*

(i) $w(\mathcal{R}) - \overline{w}(E) \geq \rho - 0.2$.
(ii) *If* C_ρ *has rank at least* 3 *then* $w(\mathcal{R}) - \overline{w}(E) \geq \rho$.
(iii) *In the freaky case, if* $E = \{g_0\}$, *and* $G = C_g \neq C_\rho$ *then we have* $w(\mathcal{R}) - \overline{w}(E) - \overline{\overline{w}}(C_g) - \overline{w}(C_{g+1}) \geq \rho - 1.2$.

We first show our upper bound with the additive term 0.1.

Proposition 4.12. *For any instance of bin packing with* $\textsc{Opt} \geq 8$, *we have* $W > \mathrm{BF} - 0.2$ *and* $W \leq 1.7 \cdot \textsc{Opt}$. *Thus also* $\mathrm{BF} \leq 1.7 \cdot \textsc{Opt} + 0.1$.

Proof. Suppose that $\mathrm{BF} > 1.7 \cdot \textsc{Opt}$. First we show that $w(\mathcal{C}) \geq \gamma - 0.2$, distinguishing three cases.

In the regular case we set $E = \emptyset$ and Proposition 4.11(i) gives $w(\mathcal{C}) \geq \gamma - 0.2$.

In the freaky case, if $s(F) \geq 2/3$, we set $E = \{g_0\}$, then sum Lemma 4.10(ii) and Proposition 4.11(i) to obtain $w(\mathcal{C}) = w(\mathcal{R}) - \overline{w}(E) + \overline{w}(g_0) + w(F) > \rho - 0.2 + 1 = \gamma - 0.2$.

In the freaky case, if $s(F) < 2/3$, then Lemma 4.3(ii) implies that F opens before C_2 and $G = C_1$. Each C_j, $j \geq 2$, contains two items larger than $1/3$, thus $w(C_j) > 1$. Finally, $f_1 < 2/3$, thus the level of C_1 when F opens is greater than $1/3$. Using Lemma 4.3(i) we have $s(F) + g_0 > 1$, thus also $g_0 > 1/3$ and $\overline{w}(g_0) \geq 0.1$. Thus $w(G) + w(F) \geq \overline{\overline{w}}(G) + \overline{\overline{w}}(F) + \overline{w}(g_0) + \overline{w}(f_1) \geq \frac{6}{5}(\frac{1}{3} + 1) + 0.1 + 0.1 = 1.8$. Summing this with $w(C_j) > 1$ for $j \geq 2$ we obtain $w(\mathcal{C}) > \gamma - 0.2$ as well.

Together with Lemma 4.8, $w(\mathcal{C}) > \gamma - 0.2$ implies $W = w(\mathcal{B}) + w(\mathcal{D}) + w(\mathcal{H}) + w(\mathcal{C}) > \beta + \eta + \delta + (\gamma - 0.2) = \mathrm{BF} - 0.2$. By Lemma 4.7(i) we have $W \leq 1.7 \cdot \textsc{Opt}$. Thus $\mathrm{BF} - 0.2 < W \leq 1.7 \cdot \textsc{Opt}$. Since BF and Opt are integers the theorem follows. □

Now after having proved $\mathrm{BF} \leq 1.7 \cdot \textsc{Opt} + 0.1$, we are going to prove our main result.

Theorem 4.13. *For any instance of bin packing we have* $\mathrm{BF} \leq 1.7 \cdot \textsc{Opt}$.

Proof. Suppose the theorem does not hold. Then Proposition 4.12 implies $\mathrm{BF} = 1.7 \cdot \textsc{Opt} + 0.1$ and integrality of Opt and BF then gives $\textsc{Opt} \equiv 7 \pmod{10}$, in particular Opt is odd.

In general, we try to save 0.1 in the analysis of the common bins, i.e., to prove $w(\mathcal{C}) > \gamma - 0.1$. In some of the subcases we need to use some additional weight of other bins and we then show $W > \mathrm{BF} - 0.1$. In both cases we then get $\mathrm{BF} - 0.1 < W \leq 1.7 \cdot \textsc{Opt}$ and the theorem follows by integrality of BF and Opt. In a few remaining cases we derive a contradiction directly.

The proof splits into three significantly different cases, $\textsc{Opt} = 7$, the regular case, and the freaky case. We start by two auxiliary lemmata. The first one enables the parity argument we mentioned before; it is needed in all the three cases. The second lemma excludes some easy cases.

Lemma 4.14. *Suppose that every* OPT*-bin contains an H-item. Then no* OPT*-bin contains two 2-items c_1 and c_2.*

Lemma 4.15. *Suppose that* $\text{OPT} \geq 8$ *and* $\text{BF} > 1.7 \cdot \text{OPT}$*. Then the following holds.*

(i) *For every* OPT*-bin A, $w(A) > 1.6$. Thus A contains an H-item.*
(ii) $\delta + \eta = \text{OPT}$.
(iii) *No huge-first bin is opened before C_ρ.*
(iv) *C_ρ has rank 2.*
(v) *Let B be a big or regular bin opened after C_i, $i = 1, \ldots, \rho - 1$. Then after opening of B, no $a_j > 1/6$ is packed into C_i.*
(vi) *No small item is packed into C_2.*

The general idea of the proof in the freaky case is that we try to find an item c different from f_1 such that the bonus of $\{g_0, c\}$ is sufficient and can be used to pay for the freaky bin F. If we find such c, we save the bonus 0.1 of f_1 and use it to tighten Proposition 4.11 by the necessary 0.1. We have three subcases.

Case 1: F opens after C_2. Thus F contains no small item by Lemma 4.2(iii); since f_1 is huge and $s(F) < 5/6$, it follows that F is a 2-bin containing only f_1 and f_2.

The intuition is that we use the bonus of f_2 instead of f_1 to pay for F. However, in general, the bonus of $\{g_0, f_2\}$ is not sufficient to pay for F, if F is smaller than C_g. In that case, the bonus of $\{g_0, f_2\}$ is sufficient to pay for G_g and we use the bonus of the next common bin, G_{g+1} to pay for F. A further complication is that the bonus of $\{g_0, f_2\}$ smaller than necessary by a term proportional to Δ; this is compensated by the dedicated and huge-first bins. The formal proof is in Appendix E.2.

Case 2: F opens before C_2 and there is some C_k which opens after F and has rank at least three. Let c be one of the medium items in C_k and set $E = \{g_0, c\}$. Then E is a valid exceptional set. Furthermore, c does not fit into F.

If $s(F) \geq 2/3$, we have $\overline{\overline{w}}(F) + \overline{w}(E) \geq 1$ by Lemma 2.2. Using Proposition 4.11(i) we have $w(\mathcal{C}) \geq (w(\mathcal{R}) - \overline{w}(E)) + (\overline{\overline{w}}(F) + \overline{w}(E)) + \overline{w}(f_1) \geq \rho - 0.2 + 1 + 0.1 = \gamma - 0.1$.

If $s(F) < 2/3$, we have $\overline{w}(E) = 0.2$ and by Lemma 4.4(iv), C_ρ is a 2-bin such that $s(C_\rho) + s(F) > 4/3$. Thus $\overline{w}(E) + w(F) + \overline{\overline{w}}(C_\rho) > 0.2 + 0.1 + 1.6 = 1.9$. Adding all the inequalities $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i \setminus E) \geq 1$, $i = 2, \ldots, \rho$ from Lemma 4.10(i), we get $w(\mathcal{C}) > \gamma - 0.1$.

Case 3: F opens before C_2 and each C_i, $i \geq 2$, has rank 2. Then all bins C_i, $i \geq 2$, are 2-bins and by Lemma 4.14, all items in these $\rho - 1$ bins are packed into different optimal bins. Thus there are at most $\text{OPT}/2$ such bins, and since OPT is odd (from $\text{OPT} \equiv 7 \pmod{10}$) we actually get $\rho \leq (\text{OPT} + 1)/2$. and thus $\gamma = \rho + 1 \leq \text{OPT}/2 + 3/2$.

Instead of using the weights, here we get a contradiction by bounding the size of all the bins as follows: Big bins have size at least 5/6; $s(C_1) + s(F) \geq 1 + d_0 = \frac{3}{2} - \Delta$ by Lemma 4.3(i), using also $G = C_1$; the remaining $\rho - 1$ common bins have average size at least 2/3 by Lemma 4.4(v) using also Lemma 4.5(iv);

finally the huge-first and dedicated bins are bounded by Lemma 4.4(ii). Since $\delta + \eta = \textsc{Opt}$ by Lemma 4.15(ii), the total size is

$$\begin{aligned}
S &> \frac{5}{6}(\mathrm{BF} - \gamma - \delta - \eta) + \frac{3}{2} - \Delta + \frac{2}{3}(\gamma - 2) + \frac{\delta + \eta}{2} + (\delta + \eta - 2)\Delta \\
&\geq \frac{5}{6}\mathrm{BF} - \frac{1}{6}\gamma - \frac{1}{3}(\delta + \eta) + \frac{1}{6} \\
&\geq \frac{5}{6}\left(\frac{17}{10}\textsc{Opt} + \frac{1}{10}\right) - \frac{1}{6}\left(\frac{\textsc{Opt}}{2} + \frac{3}{2}\right) - \frac{1}{3}\textsc{Opt} + \frac{1}{6} = \textsc{Opt}\,,\text{ a contradiction.}
\end{aligned}$$

This completes the proof of the freaky case; the regular case and case of $\textsc{Opt} = 7$ are proved in Appendix E. □

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A Lemma 2.1 and AnyFit algorithms

Proof (of Lemma 2.1). **(i):** The first item in any BF-bin does not fit in any open bin by the definition of BestFit, thus the sum of the levels of the two bins is greater than 1 already at the time when the second bin is opened.

(ii): Let $x \leq 1/2$ be the level of a, which is the level of its bin just before a is packed there. By (i), there is at most one bin with level at most $1/2$, thus at the time of packing of a all the other bins have level strictly greater than x. By the definition of BestFit, a does not fit into any of these bins.

(iii): If B' contains two items and the first one is not huge, then by (ii) the first two items in B' do not fit into B. Thus they are larger than $1/3$ and the level of B' is greater than $2/3$. □

Now we revisit the simple proof from Section 2. We save one of the three bins by noticing that we do not need to do amortization for bins that are after the bin of size smaller than $2/3$.

Proof (of Theorem 2.3). We observe that Lemma 2.1(i) and (iii) hold also for any GAAF algorithm A. Lemma 2.1(i) follows from the condition on opening a new bin, which remains the same as in BestFit. Next note that this and the GAAF condition together imply a weaker version of Lemma 2.1(ii), namely that any item with level at most $1/2$ does not fit into any previous bin. Now using this instead of Lemma 2.1(ii) in the proof of Lemma 2.1(iii) implies that Lemma 2.1(iii) also holds for A.

Lemma 2.2 holds for any GAAF algorithm as well, as it does not mention the algorithm at all.

Any A-bin (i.e., a bin of the GAAF algorithm) D with a huge item has $\overline{w}(D) \geq 0.4$ and $\frac{6}{5}s(D) > 0.6$, thus $w(D) > 1$. Similarly, any A-bin with two items larger than $1/3$ has $\overline{w}(D) \geq 0.2$ and $\frac{6}{5}s(D) > 0.8$, thus $w(D) > 1$.

For the amortization, consider all the A-bins B, with (i) two or more items, (ii) no huge item, and (iii) no pair of items both larger than $1/3$. For any such bin except for the last one choose C as the next bin with the same properties. Since C has no huge item, its first two items c, c' have level at most $1/2$ and by the GAAF condition they do not fit into B. Since C has no pair of items larger than $1/3$, we have $c \leq 1/3$ or $c' \leq 1/3$ and thus $s(B) \geq 2/3$. Lemma 2.2 now implies $\overline{\overline{w}}(B) + \overline{w}(C) \geq \overline{\overline{w}}(B) + \overline{w}(c) + \overline{w}(c') \geq 1$.

Let $\overline{C}$ be the last bin used in the amortization, if it exists, and $\overline{D}$ be the single bin with $s(\overline{D}) \leq 1/2$, if it exists.

If $\overline{C}$ and $\overline{D}$ both exist, we have $s(\overline{C}) + s(\overline{D}) > 1$ by Lemma 2.1(i) and thus $\overline{\overline{w}}(\overline{C}) + w(\overline{D}) > 1.2$. Summing this, all the amortization inequalities (note that each bin is used at most once as B or $\overline{C}$ and at most once as C) and $w(D) > 1$ for the bins with huge items or two items larger than $1/3$ we get $w(I) > A - 0.8$, where A denotes the number of A-bins. Combining this with the previous bound $w(I) \leq 1.7 \cdot \text{OPT}$ on the total weight, we obtain $A < w(I) + 0.8 \leq 1.7 \cdot \text{OPT} + 0.8$ and the theorem follows from the integrality of A and Opt.

If $\overline{C}$ exists but $\overline{D}$ does not, we have $s(\overline{C}) > 1/2$ and thus $\overline{\overline{w}}(\overline{C}) > 0.6$. Summing this and again both all the amortization inequalities and $w(D) > 1$ for the bins with huge items or two items larger than $1/3$ we get $w(I) \geq A - 0.4$ and the theorem follows again.

If $\overline{C}$ does not exist but $\overline{D}$ does, let C be an arbitrary bin other than $\overline{D}$ (if none exists, $A = 1 = \text{OPT}$ and the theorem is trivial). We have $s(C) + s(\overline{D}) > 1$ and thus $w(\overline{C}) + w(\overline{D}) > 1.2$. Summing this and $w(D) > 1$ for all the remaining bins, we get $w(I) > A - 0.8$ and the theorem follows as above.

Finally, if neither $\overline{C}$ nor $\overline{D}$ exists, we have $w(D) > 1$ for all the A-bins, thus $w(I) > A$ and the theorem follows as well. $\square$

Next we give an example showing that GAAF algorithms do not have an absolute approximation ratio 1.7. In particular, we give an instance with $\text{OPT} = 7$ and GAAF packing with 12 bins.

We first describe the GAAF packing; the input sequence contains items in the order of the bins, i.e., it starts by all the items from the first bin, then continues by items from the second bin, etc. The first bin contains 6 items of size 0.12, total of 0.72. The next three bins contain each 2 items of size 0.34; note that these do not fit into any previous bin. The fifth bin has items 0.52 and 0.01; the item 0.01 fits into the previous bins, but it is packed at a level larger than 0.5, so this satisfies the GAAF condition. The sixth bin contains a single item of size 0.48 and the remaining six bins contain each an item of size 0.53; again, these items do not fit into any previous bin.

OPT contains a bin with two items of sizes 0.52 and 0.48. The remaining 6 bins contain each three items of sizes 0.53, 0.34, and 0.12, total of 0.99; in addition one of them contains also the item 0.01. This packs all the items in the 7 bins and completes the example.

B Proof of the lower bound

Proof (of Theorem 3.1). We prove the theorem for $\text{OPT} = 10k$ and $\text{OPT} = 10k + 3$, $k \geq 1$. For the other values, the theorem follows from the results of [3] where we used another construction.

Let $\delta > 0$ be sufficiently small ($\delta = 1/50$ will be sufficient). Let $\delta_j = \delta/4^j$ and $\varepsilon < \delta_{10k+4}$.

The instance I contains the following items, reordered as described later: Items $b_j^+ = 1/6 + \delta_j$ and $c_j^- = 1/3 - \delta_j - \varepsilon$ for $j = 1, \ldots, \lfloor \text{OPT}/2 \rfloor$, items $b_j^- = 1/6 - \delta_j$ and $c_j^+ = 1/3 + \delta_j - \varepsilon$ for $j = 0, \ldots, \lceil \text{OPT}/2 \rceil - 1$, and OPT items of size $1/2 + \varepsilon$. Note the shifted indices in the two subsets of items; this is important for the construction.

The optimal packing uses bins $\{b_j^+, c_j^-, 1/2 + \varepsilon\}$, $j = 1, \ldots, \lfloor \text{OPT}/2 \rfloor$, and $\{b_j^-, c_j^+, 1/2 + \varepsilon\}$, $j = 0, \ldots, \lceil \text{OPT}/2 \rceil - 1$. All these bins have size exactly 1 and their number is OPT.

Now we describe the sequence of FF-bins for the case of $\text{OPT} = 10k$. The items are then issued in the order of FF-bins they are packed into, thus the BF and FF packings coincide. The first $2k$ bins $B_1, \ldots, B_{2k}$ contain all the b_j^+ and b_j^- items. Bin B_1 contains the 6 smallest items $b_0^-, b_1^-, \ldots, b_5^-$, bin B_{2k} contains the 4 largest items $b_1^+, b_2^+, b_3^+, b_4^+$. Each remaining bin B_i, $i = 2, \ldots, 2k - 1$ contains items b_{i+3}^+ and b_{i+4}^- (i.e., the largest and the smallest among the remaining ones of size about 1/6) and some other three items chosen arbitrarily from the remaining items b_j^+ and b_j^-. We need to verify that the first fit packing indeed behaves this way. Since δ is sufficiently small, the size of B_1 is close to 1 and no other item fits there. For $i = 2, \dots, 2k-1$, it is crucial that all items of size at most $1/6 - \delta_{i+3}$ are packed into previous bins $B_{i'}, i' < i$. First, this implies that $s(B_i) \geq b^+_{i+3} + b^-_{i+4} + 3b^-_{i+5} > 5/6 + \delta_{i+3}/2$. Second, this in turn implies that all items packed in later bins $B_{i'}$, $i' > i$ have size at least $1/6 - \delta_{i+5} > 1 - (5/6 + \delta_{i+3}/2) > 1 - s(B_i)$ and indeed they cannot be packed in B_i. The following part of FF packing contains $5k$ bins $C_1, \dots, C_{5k}$ and C_i contains items c^+_{i-1} and c^-_i. First note that no c^+_j or c^-_j item fits into $B_i, i < 2k$, as $s(B_i) > 5/6$. Also, $s(B_{2k}) > 2/3 + \delta_1 + \delta_2$, thus no c^+_j or c^-_j item fits into B_{2k}. Similarly as in the previous segment the fast decreasing δ_i and small ε yield $s(C_i) > 2/3 + 2\delta_i$, which guarantees that no later item c^+_{j-1} or c^-_j, $j > i$, fits there. Finally, the last segment of FF packing contains $10k$ bins with a single item $1/2 + \varepsilon$; all the bins have size more than $1/2$ so these items are packed separately. Altogether the FF packing contains $2k + 5k + 10k = 17k$ bins as needed.

It remains to describe the modification of the construction for $\textsc{Opt} = 10k+3$. The items and $\textsc{Opt}$ packing are already described; they are the same as in the instance for $\textsc{Opt} = 10k$ plus the new items $b^+_{5k+1}, c^-_{5k+1}, b^-_{5k}, b^-_{5k+1}, c^+_{5k}, c^+_{5k+1}$ and three items $1/2 + \varepsilon$. We pack items from the instance for $\textsc{Opt} = 10k$ as above, with the exception of b^+_{2k+3} which we replace by new b^-_{5k}. This creates no problems, as b^+_{2k+3} is among the arbitrarily assigned items for $\textsc{Opt} = 10k$, i.e., it is not the largest item in any B_i and its size is not relevant in any calculations. We pack the remaining items as follows: We create a bin $\overline{B} = \{b^+_{2k+3}, b^+_{5k+1}, b^-_{5k+1}, c^+_{5k+1}\}$ and insert it between B_{2k-1} and B_{2k}. None of these items fit in the previous bins, as the smallest one of them has size $1/6 - \delta_{5k+1}$ and $s(B_{2k-1}) > 5/6 + \delta_{2k+2}/2$. Furthermore, $s(\overline{B}) > 5/6$, thus no item from B_{2k} and later bins fits into $\overline{B}$. Next we add $C_{5k+1} = \{c^+_{5k}, c^-_{5k+1}\}$ after C_{5k}, following the pattern of bins C_i, and three bins with single items of size $1/2 + \varepsilon$. Thus $\text{FF} = 17k + 5$ and $\lfloor 1.7 \cdot \textsc{Opt} \rfloor = \lfloor 1.7(10k + 3) \rfloor = 17k + 5$ and we are done. $\square$

While we have only shown how to obtain an instance for $\textsc{Opt} = 10k + 3$ from the one for $\textsc{Opt} = 10k$, analogous construction can be used to modify the instance for any $\textsc{Opt}$ to one for $\textsc{Opt} + 3$ with additional 5 BF-bins. Thus we can get instances for any $\textsc{Opt}$ in a uniform way as follows: For any $k \geq 1$ and $i = 0, \dots 9$ use the instance for $\textsc{Opt} = 10k$ and repeat the transformation to construct an instance for $\textsc{Opt} = 10k + 3i$ with $\text{FF} = 17k + 15i = \lfloor 1.7 \cdot \textsc{Opt} \rfloor$. (The equality is easy to check for all residues.) Since the instances for $k \leq 2$ can be constructed trivially, we can get the lower bound instances also for the remaining small values of $\textsc{Opt}$.

C Omitted proofs of preliminary lemmata from Section 4

Proof (of Lemma 4.1). If some instance J does not satisfy the properties, we modify it into another instance I so that the number of BF bins stays the same and the number of $\textsc{Opt}$ bin does not increase. Most often, the constructed instances has fewer items, thus we can simply claim that the minimal counterexample satisfies the properties. In the remaining cases we reorder the items and decrease the overall number of violations of the conditions of the lemma.

Formally, it is easy to check that in each step, the following vector lexicographically decreases: "(the number of items; the number of 1-items violating (i); the number of bins violating (iii); the number of violating (iv))". Since all components are bounded by the number of items, this is sufficient to guarantee that we eventually reach an instance satisfying all the properties.

(i): If some 1-item a in bin B is followed by one or more 2^+-items in J, construct I by moving a to the end of J. Until a arrives, the sequence is the original sequence with all items in B removed, thus the BF packing on the other items does not change. When a arrives, it does not fit into any bin: In J, it did not fit into any bin fit before B, so it cannot if there in I either. No bin B' opened after B and before arrival of a can accommodate a, as in I the first item of B' did not fit into B with only a. So a opens a new bin. The number of items stays the same and the number of violations of (i) decreases.

(ii): If some such item a is in a 2^+-bin B, then remove all the items of B and put a at the end of the sequence. The condition $a_j + s(B') > 1$ guarantees that a_j opens a new bin. The number of items decreases.

(iii): If the first two items a' and a'' of B are not adjacent in J, move a' just before a'' in I. Check that BF behavior does not change, except for the permutation of the bins: Between the original position of a' and a'' the open bins are the same, except that the bin with only a' is missing. Since a' opened a new bin in J, it does not fit into any bin before B. Also bin B' opened after B in J can accommodate a', as in J the first item of B' did not fit into the bin with only a'. So a' opens a new bin in I and the configurations before packing a'' are identical in I and J. Thus a'' is packed in the bin of a' and the final configuration of I is the same as that of J.

If a' and a'' come from the same Opt-bin, we further modify I so that we replace a' and a'' by a single item $\bar{a}$ of size $a' + a''$. BestFit packs $\bar{a}$ into a new bin (since already a' did not fit in the currently open bins) and the configuration after packing $\bar{a}$ is the same as in the original packing after packing a''. Also, the number of bins in Opt does not change.

In this step, either the number of items decreases, or the number of violations of (iii) decreases, while the previous components of the vector do not increase.

(iv): Let a', a'', and a''' be the first three items in B. By (iii) we may assume that a' and a'' are not in the same Opt-bin.

If a'' and a''' are adjacent in J, let $I = J$ and proceed to the next paragraph. Otherwise construct I by moving a' and a'' just before a'''. By the assumption, the remaining items between a' and a''' in J are packed into bins opened before B, thus they are packed into the same bins in I. Since a' opened a new bin in J, it does not fit into any previous bin. Since $a' \leq 1/2$, Lemma 2.1(ii) for J implies that a'' also does not fit into any bin before B. In I, when a' arrives all the bins have the level equal to or greater than their level in J, thus a' and a'' do not fit into them, a' opens a new bin in which also a'' is packed. At this moment, just before the arrival of a''', the configurations are the same in J and I, thus the final configuration is also the same.

If a''' is in the same Opt-bin as a'', we further modify I so that we replace a'' and a''' by a single item $\bar{a}$ of size $a'' + a'''$. BestFit packs $\bar{a}$ into the bin of

a' (since already a'' did not fit into any other bin) and the configuration after this is the same as before after packing a'''.

If a''' is in the same OPT-bin as a', we further modify I so that we replace a' and a''' by a single item $\bar{a}$ of size $a'+a'''$. If $\bar{a}$ is huge, we put it after a'', otherwise before a''. Since $a' + a'' + a''' \leq 1$, as they are in the same bin in J, at most one of a'' and $\bar{a}$ is huge. By our choice of their order the first of these two items is not huge. Both a'' and $\bar{a}$ do not fit into any previous bin, as already a'' and a' did not fit. Thus they are packed together in a new bin and the configuration after packing them is the same as before after packing a'''.

In both cases, the resulting configuration is also the same as well as the number of OPT-bins. In this step, either the number of items decreased, or the number of violations of (iv) decreases, while the previous components of the vector do not increase.

(v): If this does not hold, we remove from the instance the last item $a_{j'}$ such that $j' > j$ and $a_{j'} \leq 1 - s(B)$. The level of $a_{j'}$ cannot be less than $s(B)$, as BESTFIT would pack it into B instead. Thus by removing $a_{j'}$, the level of its bin B' is still at least $s(B)$. Thus no remaining item after a_j can fit in B' and the BESTFIT packs the remaining items into the same bins as before.

In this step, the number of items decreased. □

Proof (of Lemma 4.2). **(i):** Since a common bin has size at most $5/6$, the level of a_j is less than $2/3$. By the definition of a regular bin, the first two items of C_i do not fit into any previous common bin, and one of them is smaller than $1/3$. Thus any previous common bin has level greater than $2/3$, thus by the definition of BESTFIT a_j would be packed into C_k if it would fit there.

(ii): The first part follows since any small item fits into any common bin. Once C_2 is open and receives its first two items, there is a common bin with level greater than $2/3$, so any subsequent small item must have level greater than $2/3$ as well.

(iii): Suppose that a_j is packed C_i and C_k is a bin different from C_i which has level at least $2/3$; C_k exists by the assumption of existence of two bins with level at least $2/3$. As the $s(C_k) < 5/6$ and the current level of C_k is at least $2/3$, no item larger than $1/6$ can be later packed there. As after packing of a_j the level of C_k is less than the level of C_i by (ii), no small item is packed into C_k either. But this means that a_j arrives after the last item was packed into C_k and fits there, which contradicts Lemma 4.1(v).

The second part follows, since after C_2 gets the first two items, one of C_1 and C_2 has level at least $2/3$, thus the level of any future small item is at least as large. However, if it would be packed in a common bin opened after C_2 at a level $2/3$ or larger, this would be the second bin of level at least $2/3$ at the time of packing the item, contradicting the first part.

(iv): The level of a_j is at least the level of C_1 at the time of packing a_j. By Lemma 4.1(v), another item must be later packed into C_1; the first such a_k cannot be small, as at the time of its packing C_2 is larger than C_1 and a small item would fit there. □

Proof (of Lemma 4.3). **(i):** Let x be the level of G at the arrival of f_2, the second item of bin F. We know that f_2 fits into G at the time of its packing, thus the level of f_2 is at least x. At the same time, by Lemma 4.1(v), G will receive another item, we denote g_0 the first such item. Since g_0 is packed into G and not F, which has higher level after packing f_2, by the definition of BESTFIT g_0 does not fit into F. To prove the second part, note that $f_1 + x > 1$, since f_1 was not put in G, and $f_1 + d_0 \leq 1$, as f_1 and d_0 are in the same OPT-bin. Thus $x > 1 - f_1 \geq d_0$ and together with $s(G) \geq x + g_0$ and the first part this implies the second part.

(ii): Since F opens after C_2, Lemma 4.2(iii) implies that $f_2 > 1/6$. Since f_1 is huge, $s(F) > 2/3$ follows. One of C_i and C_k has level at least $2/3$ at the time of arrival of f_2, suppose that this is C_i. Since $f_2 > 1/6$ and it is packed into a common bin, it has level less than $2/3$; it follows that $f_2 + s(C_i) > 1$ as otherwise it would be packed into C_i. Trivially, $f_1 + s(C_k) > 1$ and the claim follows by summing these two bounds. □

Proof (of Lemma 4.4). **(i) and (ii):** Use Lemma 2.1(i) for the two smallest bins and note that by Lemma 2.1(i) all the remaining bins have size greater than $1/2$ in (i), resp. greater than $1 - d_0 = 1/2 + \Delta$ in (ii).

(iii): The first items in the huge-first bins and the dedicated items except d_0 are strictly larger than a half, thus they are packed into different optimal bins. Also d_0 is packed into different optimal bin, as if the first item in some bin is huge and is packed with d_0, that bin was denoted F and excluded from the huge-first bins.

(iv): As a regular bin, C contains two items that do not fit into any previous bin and the size of one of them is at most $s(C)/2 = 1/3 - x$. Thus any B before C has size $s(B) > 1 - (1/3 - x) = 2/3 + x$. If B is regular or big and after C, the first two items do not fit into C and thus they have size greater than $1/3 + 2x$ and $s(B) > 2(1/3 + 2x) = 2/3 + 4x$.

(v): Among the common bins, there is at most one regular bin smaller than $2/3$ and also F may be smaller than $2/3$. Choose a set $\mathcal{A}$ of exactly three common bins containing the bins smaller than $2/3$. As all the other bins are larger than $2/3$, it is sufficient to prove that $s(\mathcal{A}) > 2$. If the last bin from $\mathcal{A}$ is regular, this follows from (iv). If the last bin is F, then this is exactly Lemma 4.3(ii). □

Proof (of Lemma 4.5). **(i):** Lemma 4.4(v) implies that if $\mathrm{BF} \geq 2 \cdot \mathrm{OPT}$ then $S > 4 \cdot \frac{1}{2} = \mathrm{OPT}$, a contradiction. Thus in the following we can assume that $\mathrm{BF} < 2 \cdot \mathrm{OPT}$.

For $\mathrm{OPT} \leq 3$, $\mathrm{BF} < 2 \cdot \mathrm{OPT}$ implies $\mathrm{BF} \leq 1.7 \cdot \mathrm{OPT}$ and we are done.

For $\mathrm{OPT} \in \{4, 5, 6\}$, $1.7 \cdot \mathrm{OPT} < \mathrm{BF} < 2 \cdot \mathrm{OPT}$ implies $\mathrm{BF} = 2\mathrm{OPT} - 1$; together with $\delta + \eta \leq \mathrm{OPT}$ we have also $\beta + \gamma \geq \mathrm{OPT} - 1$. If BF contains three bins with total size at least 2, Lemma 4.4(v) now implies that $S > 2 + (\mathrm{BF} - 3)/2 = \mathrm{OPT}$, a contradiction. So it is sufficient to find three such bins.

If $\beta + \rho \geq 3$. Take any three regular and big bins B_1, B_2, B_3; the last one of them starts by two items c, c' that do not fit into the previous bins by Lemma 2.1(ii), as the first item is not huge. Thus the $s(B_1) + c > 1$, $s(B_2) + c' > 1$, and $s(B_1) + s(B_2) + s(B_3) > 2$, a contradiction by the previous paragraph.

Since for $\mathrm{OPT} \geq 5$, $\beta + \gamma \geq \mathrm{OPT} - 1 \geq 4$ we have $\beta + \rho \geq 3$ and we are done. Also if $\gamma = 3$, Lemma 4.4(v) implies that the total size of the three common bins is at least 2 and we are done.

It remains to handle the case when $\mathrm{OPT} = 4$, F is defined, $\beta = 1$, $\gamma = 2$ and $\delta + \eta = 4$. Note that if F is defined, $\gamma > 2$ as also G is a common bin.

Using Lemma 4.3(i) we have $s(G) + s(F) > 1 + d_0 = 3/2 - \varDelta$. Combining with Lemma 4.4(ii) we have $S > 5/6 + 3/2 - \varDelta + 2 + 2\varDelta > 4 = \textsc{Opt}$, a contradiction.

(ii): For a contradiction, suppose that $s(C) \leq 1/2$ for some common bin. Then Lemma 4.4(iv) implies that any bin C' before C has $s(C') > 3/4$. Furthermore, any bin after C starts by a huge item not fitting in C and thus also in no other BF-bin. Thus d_0 is not defined and all the later bins are dedicated by Lemma 4.1(ii). By Lemma 4.4(i), the total size of C and all dedicated and huge-first bins is at least $(\delta + 1)/2$. Thus we obtain a contradiction by using $\textsc{Opt} \geq 7$ from (i) and $\delta \leq \textsc{Opt}$ from Lemma 4.4(iii) as follows:

$$\begin{aligned} S &> \frac{3}{4}(\mathrm{BF} - \delta - 1) + \frac{1}{2}(\delta + 1) = \frac{3}{4}\mathrm{BF} - \frac{1}{4}(\delta + 1) \\ &\geq \frac{3}{4}\left(\frac{17}{10}\textsc{Opt} + \frac{1}{10}\right) - \frac{1}{4}(\textsc{Opt} + 1) = \frac{41}{40}\textsc{Opt} - \frac{7}{40} \geq \textsc{Opt}\,. \end{aligned}$$

(iii): Suppose for a contradiction that $\delta + \eta \leq 4$. By Lemma 2.1(iii), there is at most one regular bin smaller than $2/3$; in addition also the freaky bin may be smaller than $2/3$. Thus we obtain

$$\begin{aligned} S &> \frac{2}{3}(\mathrm{BF} - \delta - \eta - 2) + \frac{1}{2}(\delta + \eta + 2) = \frac{2}{3}\mathrm{BF} - \frac{1}{6}(\delta + \eta + 2) \\ &\geq \frac{2}{3}\left(\frac{17}{10}\textsc{Opt} + \frac{1}{10}\right) - 1 = \frac{17}{15}\textsc{Opt} - \frac{14}{15} \geq \textsc{Opt}. \end{aligned}$$

If d_0 is not defined, by (ii) no BF-bin has size at most $1/2$, thus Lemma 4.1(ii) implies that all the huge items are in dedicated bins.

(iv): To obtain the first bound from the second one, use $\tau = 1$ and the integrality of $\textsc{Opt}$. Now consider the second claim and suppose for a contradiction that $\gamma \leq (\textsc{Opt} + \tau)/2$.

If $\gamma \leq 2$, we bound the size of all common, huge-first and big bins by Lemma 4.4(i) and obtain

$$\begin{aligned} S &> \frac{5}{6}(\mathrm{BF} - (\gamma + \delta + \eta)) + \frac{1}{2}(\gamma + \delta + \eta) = \frac{5}{6}\mathrm{BF} - \frac{1}{3}(\gamma + \delta + \eta) \\ &\geq \frac{5}{6}\left(\frac{17}{10}\textsc{Opt} + \frac{1}{10}\right) - \frac{1}{3}(\textsc{Opt} + 2) = \frac{13}{12}\textsc{Opt} - \frac{7}{12} \geq \textsc{Opt}\,, \end{aligned}$$

a contradiction. If $\gamma \geq 3$, we bound the size of all common bins by Lemma 4.4(v) and the size of huge-first and big bins by Lemma 4.4(i) and obtain

$$\begin{aligned} S &> \frac{5}{6}(\mathrm{BF} - (\gamma + \delta + \eta)) + \frac{2}{3}(\gamma - 2) + \frac{1}{2}(\delta + \eta) = \frac{5}{6}\mathrm{BF} - \frac{1}{6}\gamma - \frac{1}{3}(\delta + \eta) \\ &\geq \frac{5}{6}\left(\frac{17}{10}\textsc{Opt} + \frac{\tau}{10}\right) - \frac{\textsc{Opt} + \tau}{12} - \frac{1}{3}\textsc{Opt} = \textsc{Opt}\,, \end{aligned}$$

a contradiction. We have proved (iv) in the regular case, as $\rho = \gamma$, and also $\gamma \geq 5$ in the freaky case.

In the freaky case, to complete the proof, suppose for a contradiction that $\rho \leq (\textsc{Opt} + \tau)/2$. We bound $s(G) + s(F)$ by Lemma 4.3(i), the size of the remaining at least three common bins by Lemma 4.4(v) and the size of huge-first and big bins by Lemma 4.4(ii) and obtain (using $\delta+\eta \geq 3$ from (iii) in the first inequality)

$$\begin{aligned}
S &> \frac{5}{6}(\mathrm{BF} - (\delta+\eta+\rho+1)) + (1+d_0) + \frac{1}{3}(\rho-1) + \frac{1}{2}(\delta+\eta) + (\delta+\eta-2)\varDelta \\
&\geq \frac{5}{6}(\mathrm{BF}-2) - \frac{1}{6}(\rho-1) - \frac{1}{3}(\delta+\eta) + (1+d_0) + \varDelta \\
&\geq \frac{5}{6}\left(\frac{17}{10}\textsc{Opt} + \frac{\tau}{10} - 2\right) - \frac{\textsc{Opt}+\tau-2}{12} - \frac{1}{3}\textsc{Opt} + \frac{3}{2} = \textsc{Opt}\,,
\end{aligned}$$

and we obtain a contradiction as well. □

Proof (of Lemma 4.7). In all cases $\overline{\overline{w}}(A) \leq 1.2$, thus it remains to bound $\overline{w}(A)$. Recall that any Opt-bin contains at most one H-item and all other items have bonus at most 0.1.

Case 1: A contains no H-item. Either A contains at least 4 items with non-zero bonus, in which case their total bonus is at most $\overline{w}(A) \leq \frac{3}{5}(s(A) - \frac{4}{6}) \leq \frac{3}{5} \cdot \frac{2}{6} = 0.2$. Or else it contains at most 3 items with non-zero bonus and $\overline{w}(A) \leq 0.3$. In both subcases (ii) follows.

Case 2: A contains an H-item different from d_o. This item is huge and has bonus 0.4. If A has at most one additional item with non-zero bonus, its bonus is at most 0.1. Otherwise A has exactly two additional items with non-zero bonus, their total size is less than 1/2 and their bonus is at most $\frac{3}{5}(\frac{1}{2} - \frac{2}{6}) = 0.1$. In both subcases $\overline{w}(A) \leq 0.5$ and (i) follows.

Case 3: A contains d_0. By definition $\overline{w}(d_0) = 0.4 - \frac{3}{5}\varDelta$. If A has at most one additional item with non-zero bonus, its bonus is at most 0.1. If A has at least two additional items with non-zero bonus, their total size is less than $1/2 + \varDelta$ and their bonus is at most $\frac{3}{5}(\frac{1}{2}+\varDelta-\frac{2}{6}) = 0.1+\frac{3}{5}\varDelta$. In both subcases $\overline{w}(A) \leq 0.5$ and (i) follows. □

Proof (of Lemma 4.8). **(i):** For every big bin B, $w(B) \geq \overline{\overline{w}}(B) = \frac{6}{5}s(B) \geq \frac{6}{5} \cdot \frac{5}{6} = 1$.

(ii): If d_0 is undefined then for every dedicated bin D, $w(D) = \frac{6}{5}s(D) + 0.4 > \frac{6}{5} \cdot \frac{1}{2} + 0.4 = 1$. Moreover there is no huge-first bin by Lemma 4.5(iii), and the claim follows.

(iii): The bonus of any H-item except d_0 is exactly 2/5 and every huge-first or dedicated bin contains an H-item. Thus using Lemma 4.4(ii), $\overline{w}(d_0) = 0.4 - \frac{3}{5}\varDelta$ and $\delta+\eta \geq 5$, we get

$$\begin{aligned}
w(\mathcal{D} \cup \mathcal{H}) &\geq \frac{6}{5}\left(\frac{\delta+\eta}{2} + (\delta+\eta-2)\varDelta\right) + \frac{2}{5}(\delta+\eta) - \frac{3}{5}\varDelta \\
&= \delta+\eta + \frac{6}{5}(\delta+\eta-2.5)\varDelta > \delta+\eta.
\end{aligned}$$

If one of the huge-first bin has size 7/12 or larger, we get

$$\begin{aligned}
w(\mathcal{D} \cup \mathcal{H}) &\geq \frac{6}{5}\left(\frac{\delta+\eta-1}{2} + (\delta+\eta-3)\varDelta + \frac{7}{12}\right) + \frac{2}{5}(\delta+\eta) - \frac{3}{5}\varDelta \\
&= \delta+\eta+0.1 + \frac{6}{5}(\delta+\eta-3.5)\varDelta > \delta+\eta+0.1.
\end{aligned}$$

□

D Proof of Proposition 4.11

Let $\overline{C}_\rho$ denote the first two items in C_ρ and let $\overline{\mathcal{R}}$ denote the set of regular bins $\mathcal{R}$ with C_ρ replaced by $\overline{C}_\rho$. We distinguish several cases and prove that $w(\overline{\mathcal{R}}) - \overline{w}(E) \geq \rho - 0.2$, which implies Proposition 4.11(i). Proposition 4.11(ii) follows as well, since if C_ρ has rank at least three then $C_\rho \setminus \overline{C}_\rho$ contains some medium item c with $\overline{\overline{w}}(c) > 0.2$ thus $\overline{\overline{w}}(C_\rho) \geq \overline{\overline{w}}(\overline{C}_\rho) + 0.2$ and $w(\mathcal{R}) - \overline{w}(E) \geq 0.2 + w(\overline{\mathcal{R}}) - \overline{w}(E) \geq \rho$. Proposition 4.11(iii) is discussed in each case separately.

Case 1: Every regular bin and also $\overline{C}_\rho$ has size at least $2/3$. We apply Lemma 4.10(i) for every $i = 2, \ldots, \rho$. The scaled size of the last bin is at least $\overline{\overline{w}}(C_\rho) \geq \overline{\overline{w}}(\overline{C}_\rho) \geq \frac{6}{5} \cdot \frac{2}{3} = 0.8$. Summing all of these inequalities we obtain

$$w(\overline{\mathcal{R}}) - \overline{w}(E) \geq \overline{\overline{w}}(\overline{C}_\rho) + \sum_{i=2}^{\rho} (\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i \setminus E)) \geq 0.8 + (\rho - 1) = \rho - 0.2\,.$$

To obtain Proposition 4.11(iii) we omit from the summation the inequality $\overline{\overline{w}}(C_g) + \overline{w}(C_{g+1}) > 1$. Note that by the assumption $E = \{g_0\}$, E and C_{g+1} are disjoint.

Case 2: $s(C_k) = 2/3 - x$ for some $x > 0$ and $k < \rho$. Using Lemma 4.4(iv), each C_j, $j > k$, contains two items larger than $1/3 + x$; thus C_j has rank 2 and (i) C_j cannot contain an item from E and (ii) $C_j \neq G$, as $G = C_g$ has rank at least 3 if $g > 1$. It follows that $\overline{w}(C_j) = \overline{w}(C_j \setminus E) = 0.2$ and also $s(C_j) > 2/3 + 2x$ which implies $\sum_{i=k}^{\rho} s(C_i) > (\rho + 1 - k)\frac{2}{3}$. Combining these we have $\overline{\overline{w}}(C_k) + \sum_{j=k+1}^{\rho} w(C_j \setminus E) \geq (\rho + 1 - k) - 0.2$. Adding the last inequality and the inequalities $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i \setminus E) \geq 1$ from Lemma 4.10(i) for $i = 2, \ldots, k$, we get $w(\overline{\mathcal{R}}) - \overline{w}(E) \geq \rho - 0.2$.

To obtain Proposition 4.11(iii) if $g < k$, we again omit the inequality $\overline{\overline{w}}(C_g) + \overline{w}(C_{g+1}) > 1$ from the summation. We know that $g \leq k$ by (ii) above, so it remains to handle the case when $g = k$, i.e., $s(C_g) < 2/3$. But then $\overline{\overline{w}}(C_g) + \overline{w}(C_{g+1}) < 0.8 + 0.2 = 1$ and subtracting this from $w(\overline{\mathcal{R}}) - \overline{w}(E) \geq \rho - 0.2$ we obtain Proposition 4.11(iii) as well.

Case 3: $s(\overline{C}_\rho) < 2/3$. Fix $x > 0$ so that $s(\overline{C}_\rho) = \frac{2}{3} - 2x$. Lemma 4.5(ii) implies $s(\overline{C}_\rho) > 1/2$ and thus $x < 1/12$.

Since now the scaled size of $\overline{C}_\rho$ is less than 0.8, we need to compensate for this. This is indeed possible due to the fact that now Lemma 4.4(iv) implies that the regular bins C_i, $i = 2, \ldots, \rho - 1$ are larger than $2/3 + x$ and this allows us to improve the bounds of Lemma 4.10(i) by an amount proportional to x in the next stronger version of the amortization lemma. Note that common bins cannot have rank 5 or more, as a bin with size less than $5/6$ cannot contain 5 items larger than $1/6$.

Lemma D.1. *For $i = 2, \ldots, \rho - 1$ we have the following bounds:*
(i) *If the rank of C_i is 2 or 3, then $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 + \frac{3}{5}x$.*
(ii) *If the rank of C_i is 4, then $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 + \frac{3}{10}x$.*

Proof. Let y be such that $s(C_{i-1}) = \frac{5}{6} - y$. We have $\overline{\overline{w}}(C_{i-1}) = \frac{6}{5}(\frac{5}{6} - y) = 1 - \frac{6}{5}y$. Since C_{i-1} is a common bin, $y > 0$. On the other hand, by Lemma 4.4(iv) the size of C_{i-1} is greater than $\frac{2}{3} + x$ and thus also $y < \frac{1}{6} - x$. Lemma 4.2(i) implies that every item $c > \frac{1}{6}$ in C_i satisfies $c > \frac{1}{6} + y$.

We distinguish three cases. To see that they are exhaustive, note that if C_i has rank 2, it contains an item $c \geq 1/3$: Otherwise the two medium items in C_i reach level less than $2/3$ and Lemma 4.2(ii) implies no small item is packed into it, as $i \geq 2$; this contradicts $s(C_i) > 2/3$.

Case A: C_i has at least one item $c \geq 1/3$. There exist another item $c' > \frac{1}{6}$ in C_i and it satisfies $c' > \frac{1}{6} + y$. Thus

$$\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}y + 0.1 = 1.1 - \frac{3}{5}y \geq 1.1 - \frac{3}{5}\left(\frac{1}{6} - x\right) = 1 + \frac{3}{5}x\,.$$

Case B: C_i has rank 3 but no item with size at least $1/3$. We claim that one of the three medium items in C_i has size at least $\frac{1}{6} + x$. Then we get

$$\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}(y + y + x) = 1 + \frac{3}{5}x\,.$$

It remains to prove that one of the three medium items has size at least $1/6 + x$. First, the first three items packed into C_i are the medium items, as they are smaller than $1/3$ and no small item can be packed before the third medium item. If the total size of the three medium items in C_i is at least $3/4$ then one of them has size at least $1/4 \geq 1/6 + x$, using $x < 1/12$. Otherwise one of the medium items is smaller than $1/4$ and thus the level of C_{i-1} when the third item is assigned to C_i is greater than $3/4$. Thus C_i receives no small item at this point, either. By the assumption, $s(C_i) \geq 2/3 + x > 1/2 + 3x$, using $x < 1/12$ again, and one of the three items must have size at least $1/6 + x$.

Case C: C_i has rank 4 and all items are smaller than $1/3$. We claim C_i has two medium items with total size at least $\frac{1}{3} + \frac{x}{2}$. Then their total bonus is at least $\frac{3}{5} \cdot \frac{x}{2}$ and

$$\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i) \geq 1 - \frac{6}{5}y + \frac{3}{5}\left(y + y + \frac{x}{2}\right) = 1 + \frac{3}{10}x\,.$$

It remains to prove that two of the four medium items have total size at least $\frac{1}{3} + \frac{x}{2}$. Otherwise the total size of the four medium items is less than $\frac{2}{3} + x < 3/4$ and also each of them has size at most $\frac{1}{6} + \frac{x}{2} < 1/4$. As in the previous case, we show that the medium items are the first four items in C_i: At the time of opening of C_i, the level of C_{i-1} is greater than $3/4$, as otherwise the first item is larger than $1/4$. Thus no small items are packed into C_i, as the medium items do not reach the level $3/4$. Thus $s(C_i) < \frac{2}{3} + x$, a contradiction. $\square$

We cannot apply the previous lemma for all bins. We call a bin C_i exceptional if $i = \rho$, or if it contains an item from E, or if in the freaky case $E = \{g_0\}$ and $i = g + 1$. The remaining bins among $C_2, \ldots, C_{\rho-1}$ are **non-exceptional**; note that there are at least $\rho - 4$ non-exceptional bins. Let γ_k denote the number of rank k non-exceptional common bins and $\alpha = 2(\gamma_2 + \gamma_3) + \gamma_4$.

Lemma D.2. *Suppose that $s(\overline{C}_\rho) < 2/3$. The following holds:*
(i) *If $\alpha \geq 4$ then $w(\overline{\mathcal{R}}) - \overline{w}(E) \geq \rho - 0.3$.*
(ii) *If $\alpha \geq 8$ then Proposition 4.11 holds, in particular $w(\overline{\mathcal{R}}) - \overline{w}(E) \geq \rho - 0.2$.*

Proof. We apply Lemma 4.10 for any $i = 2, \ldots, \rho-1$ and such that C_i is disjoint with E. Otherwise, i.e., if C_i contains exceptional items and also for $i = \rho$ we apply Lemma 4.10(i). Summing all the resulting bounds on $\overline{\overline{w}}(C_{i-1}) + \overline{w}(C_i \setminus E)$ and $\overline{\overline{w}}(\overline{C}_\rho) = 0.8 - \frac{12}{5}x$ we obtain that the total weight of the regular bins is

$$w(\mathcal{R}) - \overline{w}(E) \geq \rho - 1 + (\gamma_2 + \gamma_3)\frac{3}{5}x + \gamma_4 \frac{3}{10}x + 0.8 - \frac{12}{5}x = \rho - 0.2 + \frac{(3\alpha - 24)x}{10}.$$

For $\alpha \geq 4$ we use $x < 1/12$, which gives $(3\alpha - 24)x \geq -12x \geq -1$ and (i) follows.

For $\alpha \geq 8$ we have $3\alpha \geq 24$ and $w(\overline{\mathcal{R}}) - \overline{w}(E) \geq \rho - 0.2$ follows. To obtain Proposition 4.11(iii) we omit from the summation the bound $\overline{\overline{w}}(C_g) + \overline{w}(C_{g+1}) \geq 1$. By the assumption of Proposition 4.11(iii), C_{g+1} exists and is exceptional but disjoint from E. ◻

For the rest of the proof of Proposition 4.11 we assume that BF $\geq 1.7 \cdot$ Opt $+ 0.1$. We distinguish several subcases of Case 3 and in each we either derive a contradiction or prove that $\alpha \geq 8$, which proves the proposition by Lemma D.2(ii).

Case 3.1: Opt ≥ 22. By Lemma 4.5(iv) we have $\rho > (\text{Opt} + \tau)/2 \geq 12$, i.e. $\rho \geq \gamma - 1 \geq 12$. Deducting C_1, C_ρ, and possibly 2 bins that contain items from E, at least 8 bins remain and $\alpha \geq \gamma_2 + \gamma_3 + \gamma_4 \geq 8$.

Case 3.2: $8 \leq$ Opt ≤ 21**,** Opt $\not\equiv 4 \pmod{10}$**, and** Opt $\not\equiv 7 \pmod{10}$**.** If Opt ≥ 10, then BF $\geq 1.7 \cdot$ Opt $+ 0.3$ and using Lemma 4.5(iv) with $\tau = 3$ we obtain $\rho > (\text{Opt} + \tau)/2 \geq 6.5$. If Opt $= 8, 9$, then even BF $\geq 1.7 \cdot$ Opt$+0.4$ and using Lemma 4.5(iv) with $\tau = 4$ we obtain $\rho > (\text{Opt} + \tau)/2 \geq 6$. Thus $\rho \geq 7$ in both cases. As Opt $\not\equiv 7 \pmod{10}$, E contains at most one item g_0, and we have $\alpha \geq \gamma_2 + \gamma_3 + \gamma_4 \geq 4$. Lemma D.2(i) implies $W > \beta + (\rho - 0.3) + \delta + \eta =$ BF $- 0.3 \geq 1.7 \cdot$ Opt, a contradiction with Lemma 4.7(i).

Case 3.3: Opt $= 14$ **and** BF ≥ 24**, or** Opt $= 17$ **and** BF ≥ 29**.** First we claim in each case separately that $\gamma_2 + \gamma_3 + \gamma_4 \geq 6$.

For Opt $= 14$, Lemma 4.5(iv) with $\tau = 2$ implies $\rho > (\text{Opt} + \tau)/2 = 8$, i.e. $\rho \geq 9$. We have at most 1 item in E, thus $\gamma_2 + \gamma_3 + \gamma_4 \geq 6$ follows.

For Opt $= 17$, Lemma 4.5(iv) implies $\rho > (\text{Opt} + 1)/2 = 9$, i.e. $\rho \geq 10$. We may have two regular bins intersecting E, but $\gamma_2 + \gamma_3 + \gamma_4 \geq 6$ follows again.

If $\gamma_2 + \gamma_3 \geq 2$ then $\alpha \geq 8$ and the proposition follows by Lemma D.2(ii). In the remaining case $\gamma_4 \geq 5$. We claim that these 5 common bins of rank 4 have total size at least 4: the items of the last bin do not fit in the previous ones, so always the size of one of the first bins and one of the items in the last bin is more than 1. We split the remaining bins into two parts. First, Opt bins that contain all dedicated and huge-first bins (using Lemma 4.4(iii)) and have total size at least Opt$/2$. Second, the remaining BF $-$ Opt $- 5$ bins that contain at least three common bins and only common and big bins; their average size is at least $2/3$ by Lemma 4.4(v).

For $\textsc{Opt} = 14$ we get $S > 4 + 7 + \frac{2}{3} \cdot 5 > \textsc{Opt}$, a contradiction.
For $\textsc{Opt} = 17$ we get $S > 4 + 17/2 + \frac{2}{3} \cdot 7 > \textsc{Opt}$, a contradiction.

E Proofs for Theorem 4.13

E.1 The lemmata

Proof (of Lemma 4.14). For a contradiction, assume we have such c_1 and c_2 and number them so that the BF-bin of c_1 is not after the BF-bin of c_2. Since the bin of c_1 is a 2-bin, the second item of this bin cannot be in the same $\textsc{Opt}$-bin by Lemma 4.1(iii). Thus c_2 is in a different BF-bin. Let c_3 be the other item in the BF-bin of c_1. Since the bin of c_1 and c_3 has only two items, they are adjacent by Lemma 4.1(iii). Thus c_2 arrives after c_3 and by Lemma 4.1(v) we have $c_1 + c_2 + c_3 > 1$. This implies that c_3 cannot be in the $\textsc{Opt}$-bin of c_1 and c_2. Every $\textsc{Opt}$-bin contains an H-item by the assumption; let d_1 be the H-item in the $\textsc{Opt}$-bin of c_1 and c_2 and d_3 the H-item in the $\textsc{Opt}$-bin of c_3. By Lemma 4.4(i), $d_1 + d_3 > 1$ and thus $c_1 + c_2 + c_3 + d_1 + d_3 > 2$. As all these items are in two $\textsc{Opt}$-bins, this is a contradiction. □

Proof (of Lemma 4.15). We prove most parts of the lemma by contradiction so that assuming that the assertion does not hold, we prove that $\mathrm{BF} \leq 1.7 \cdot \textsc{Opt}$, which contradicts the assumption of the lemma.

(i): If $w(A) \leq 1.6$ then by Lemma 4.7(i) for the remaining $\textsc{Opt}$-bins we have $W \leq 1.7 \cdot \textsc{Opt} - 0.1$. On the other hand, by Proposition 4.12, $W > \mathrm{BF} - 0.2$, thus $\mathrm{BF} < 1.7 \cdot \textsc{Opt} + 0.1$, a contradiction. If A contains only items with bonus at most 0.1 then we have even $w(A) \leq 1.5$ by Lemma 4.7(ii).

(ii): By Lemma 4.4(iii), $\delta + \eta \leq \textsc{Opt}$. If $\delta + \eta < \textsc{Opt}$, then one of the $\textsc{Opt}$-bins contains no H-item, contradicting (i).

(iii): If a huge-first bin H opens before C_ρ, then $s(H) > 7/12$, as otherwise C_ρ contains two items larger than 5/12, a contradiction with $s(C_\rho) < 5/6$. Now, by Lemma 4.8(iii), $w(\mathcal{D} \cup \mathcal{H}) > \delta + \eta + 0.1$. Thus $W > \mathrm{BF} - 0.1$ and $\mathrm{BF} \leq 1.7 \cdot \textsc{Opt}$, a contradiction.

(iv): If the rank of C_ρ is at least 3, Proposition 4.11(ii) gives $w(\mathcal{R}) - \overline{w}(E) \geq \rho$ instead of $w(\mathcal{R}) - \overline{w}(E) \geq \rho - 0.2$. We save 0.2 in the proof of Proposition 4.12 and obtain $\mathrm{BF} < 1.7 \cdot \textsc{Opt}$, a contradiction.

(v): Let y be the level of C_i when B opens. We have $y < 2/3$, as after packing $a_j > 1/6$ we still have $s(C_i) < 5/6$.

If B is big, it follows that its first item b has $\overline{w}(b) \geq 0.1$. Thus $w(\mathcal{B}) \geq \beta + 0.1$ and we obtain $W > \mathrm{BF} - 0.1$ and $\mathrm{BF} \leq 1.7 \cdot \textsc{Opt}$, a contradiction.

Otherwise B is common and we may assume $B = C_{i+1}$. The first two items of C_{i+1} are larger than 1/3, thus C_{i+1} is a 2-bin and $\overline{w}(C_{i+1}) = 0.2$. Also, $y > 7/12$, as otherwise the first two items in B are larger than 5/12 and B is not common. We have $s(C_i) > 7/12 + 1/6 = 3/4$ and $\overline{\overline{w}}(C_i) > 0.9$. Thus $\overline{\overline{w}}(C_i) + \overline{w}(C_{i+1}) > 1.1$, strengthening the amortization lemma by needed 0.1. We set $E = \emptyset$ in the regular case or $E = \{g_0\}$ in the freaky case. The bound $w(\mathcal{R}) - \overline{w}(E) \geq \rho - 0.1$ now follows as in Proposition 4.11: This is easy in Cases 1 and 2 in the proof; in Case 3 we treat C_{i+1} as an exceptional bin. Since E contains at most g_0, the argument works.

(vi): Follows immediately from (v) and Lemma 4.2(iv). □

E.2 The freaky case, Case 1 continued

Formally, we first claim that

$$\max(\overline{\overline{w}}(C_g), \overline{\overline{w}}(F)) + \overline{w}(g_0) + \overline{w}(f_2) \geq 1 - \frac{6}{5}\Delta\,. \tag{1}$$

Since $f_1 < 1 - d_0 = 1/2 + \Delta$ and g_0 does not fit into F, we have $g_0 + f_2 > 1/2 - \Delta$ and thus $\overline{w}(\{g_0, f_2\}) \geq 0.2 - \frac{3}{5}\Delta$. Furthermore, using Lemma 4.3(i) we get $\max(s(C_g), s(F)) \geq (s(C_g) + s(F))/2 > (1 + d_0)/2 = 3/4 - \Delta/2$ and thus $\max(\overline{\overline{w}}(C_g), \overline{\overline{w}}(F)) \geq \frac{6}{5}\left(\frac{3}{4} - \frac{\Delta}{2}\right) = 0.8 - \frac{3}{5}\Delta$. Summing the two bounds, (1) follows.

Next we claim that

$$\min(\overline{\overline{w}}(C_g), \overline{\overline{w}}(F)) + \overline{w}(C_{g+1}) > 1 - \frac{6}{5}\Delta\,. \tag{2}$$

Let c and c' be the first two items of C_{g+1}. We show that they do not fit in C_g or F. For C_g this is trivial and also for F if it opens before C_{g+1}. If F opens after C_{g+1}, then let x be the level of C_g when F opens. On one hand, $c, c' > 1 - x$, since C_{g+1} is already open. On the other hand $x < f_1 < s(F)$ by the definition of the freaky case. Thus c and c' do not fit into F.

If $\min(s(C_g), s(F)) \geq 2/3$, the claim follows by Lemma 2.2. We know that $s(F) > 2/3$ as it contains a huge item f_1 and a medium item f_2. It remains to handle the case $s(C_g) < 2/3$. Then $c, c' > 1/3$ and thus $\overline{w}(C_{g+1}) \geq 0.2$. Furthermore, when F is opened, the size of C_g is at least $1/2 - \Delta$ and later $g_o > 1/6$ is packed into C_g. Thus $s(C_g) > 2/3 - \Delta$, $\overline{\overline{w}}(C_g) > 0.8 - \frac{6}{5}\Delta$ and (2) follows.

Summing (1), (2) and $\overline{w}(f_1) = 0.1$ we get

$$\overline{\overline{w}}(C_g) + \overline{w}(C_{g+1}) + \overline{\overline{w}}(F) + \overline{w}(g_0) + \overline{w}(f_2) \geq 2.1 - 4\Delta\,. \tag{3}$$

Now we set $E = \{g_0\}$ which is a valid exceptional set, add Proposition 4.11(iii) to (3) and get

$$w(\mathcal{C}) \geq \gamma - 0.1 - 4\Delta$$

From Lemma 4.8(iii) and $\delta + \eta = \textsc{Opt} \geq 17$ we have $w(\mathcal{D} \cup \mathcal{H}) \geq \delta + \eta + 4\Delta$. Finally adding the last two inequalities and $w(\mathcal{B}) \geq \beta$ we get $W > \mathrm{BF} - 0.1$.

E.3 The regular case

Since Opt is odd, there exists an Opt-bin that contains no item from any 2-bin C_i, $i \geq 2$. Fix one such Opt-bin A. We define E as a set of all items in A with bonus in $(0, 0.1]$.

Lemma E.1. *E is an exceptional set and $w(A) \leq 1.6 + \overline{w}(E)$.*

Proof. Since A contains at most one item with bonus greater than 0.1 and this bonus is at most 0.4, we have $w(A) \leq \overline{\overline{w}}(A) + \overline{w}(A \setminus E) + \overline{w}(E) \leq 1.2 + 0.4 + \overline{w}(E)$ and the second part follows.

To prove that E is an exceptional set, we first claim that E contains at most two items. If A contains a huge item with bonus 0.4, then it indeed contains at most two other items with bonus. Otherwise A contains d_0. If A contains at least three other medium items, their total bonus is at most $\frac{3}{5}\Delta$, thus the bonus of A is at most 0.4 and $w(A) \leq 1.6$, contradicting Lemma 4.15(i). Thus E contains at most two items as well.

Observe that each C_i, $i \geq 2$, contains no small items by Lemma 4.2(iii) and Lemma 4.15(vi). Thus the rank of C_i is equal to the number of items in it.

Next we claim that no 3-bin C_i, $i \geq 2$, contains two items from E. Lemma 4.15(iv) implies $i < \rho$ and then Lemma 4.15(iii) together with Lemma 4.1(ii) implies that the next bin that opens after C_i is big or common. In that case, Lemma 4.15(v) implies that all three items in C_i are packed before the next bin opens. Then Lemma 4.1(iv) implies that the items from C_i are in different Opt-bins, thus C_i cannot contain both items of E, as they are both from A.

We conclude that each C_i, $i \geq 2$, contains at least two medium or huge items not in E. We have just proved it if C_i is a 3-bin, for a 4-bin it is trivial and 2-bins contain no item from E by the choice of A and E. This completes the proof that E is an exceptional set. □

Lemma E.1 for A and Lemma 4.7(i) for the other Opt-bins imply that $W \leq 1.7 \cdot \text{Opt} - 0.1 + \overline{w}(E)$. On the other hand, Proposition 4.11(i) together with Lemma 4.8 imply that $W = w(\mathcal{B}) + w(\mathcal{C}) + w(\mathcal{D} \cup \mathcal{H}) > \beta + (\gamma - 0.2 + \overline{w}(E)) + \delta + \eta = \text{BF} - 0.2 + \overline{w}(E)$. Combining the inequalities we get $\text{BF} < 1.7 \cdot \text{Opt} + 0.1$, a contradiction.

E.4 Opt = 7

Then BF ≥ 12. By Lemma 4.5(iv) there are at least 5 regular bins.

First we claim that BF $= 12$: Otherwise by Lemma 4.4(v) the size of the 5 regular bins is and by Lemma 4.4(i) the size of the remaining at least 8 bins is at least 4, thus $S > 5 \cdot \frac{2}{3} + 4 > 7$, a contradiction. Similarly we show that $\delta + \eta = 7$, as otherwise $S > 6 \cdot \frac{2}{3} + 6 \cdot \frac{1}{2} = 7$, a contradiction. By the same calculation, all the dedicated and huge-first bins have size less than 2/3, as otherwise we again have 6 bins with average size at least 2/3.

Since there are at least 5 regular bins, 7 dedicated and huge-first bins and 12 total bins, there are exactly 5 regular bins, no big bins or a freaky bin and we are in the regular case.

Next we claim that no two BF-bins have total size greater than or equal to 3/2. Otherwise there remain at least three regular BF-bins and $S > \frac{3}{2} + 3 \cdot \frac{2}{3} + 7 \cdot \frac{1}{2} = 7$, a contradiction.

Among the 5 common bins, there are at most three 2-bins, since by Lemma 4.14 no Opt-bin can contain two 2-items. Thus there are at least two 3^+-bins. Let C be the last 3^+-bin and B some common bin before it.

We claim that the first three items of C do not fit into B. First two items of C do not fit into B, since we are in the regular case and B and C are common bins. Thus also the level of B at the time of packing the first two items in C is greater than 1/2, as otherwise the first two items of C would be huge, which is impossible. Suppose that the third item fits into B. It must be packed into C because the level of C is greater, however then by Lemma 4.1(v) there must be another item a later packed into B, since at that point the level of B is less than the level of C, it must be the case that $a + s(C) > 1$. However, since the level of B is at least $1/2$ when a is packed there, we have $s(B) > 1/2 + a$ and $s(B) + s(C) \geq 1/2 + a + s(C) > 3/2$, a contradiction with the fact that no two bins have total size greater than $3/2$.

Then C contains three items larger than $1 - s(B)$ and $s(B) + s(C) \geq s(B) + 3(1 - s(B)) = 3 - 2 \cdot s(B)$. Since no two bins have total size $3/2$ or more, this implies $s(B) \geq 3/4$. Furthermore, this implies that there is a single bin before C: We proved that any bin B before C has size $s(B) \geq 3/4$ and if there would two bins B and B' before C, their total size would be at least $3/2$. I.e., it follows B is the first bin, C is the second bin and there are exactly three 2-bins.

We now claim that no bin is opened before the third item arrives in C. Suppose not, let C' be the first bin that opens after C and let y be the level of C when C' opens. If C' is a regular bin, then it is a 2-bin and a_j is packed after both items of C'. By Lemma 4.1(v) $a_j + s(C') > 1$. On the other hand, the level of C before C' opens must be greater than $1/2$, as otherwise the two items in C' are huge. It follows that $s(C) + s(C') > 3/2$, a contradiction. If C' is a huge-first bin, then at the time of opening C', the level of one of B and C is at least $2/3$, which is more than $s(C')$. Thus the second item in C' does not fit into one of the previous bins. The size of that bin plus $s(C')$ is then more than $3/2$, as the first item in C' is huge, a contradiction

We proved that the third item arrives in C before the next bin is opened. Thus we can use Lemma 4.1(iv) and conclude that the first three items in C are packed into different Opt-bins.

We claim that one of these three Opt-bins contains both a 2-item c and an H-item d with size $d > 1/2$: Each Opt-bin contains a H-item and there is at most one H-item of size at most $1/2$; furthermore, there is at most one Opt-bin not containing a 2-item, as there are six 2-items in the three 2-bins. Thus the condition excludes at most two Opt-bins. Fix c' to be an item from C packed with such a c and d in the same Opt-bin. Note that c and d are in later BF-bins than C, as B and C are the first bins and they are 3^+-bins.

We have $c' + c < 1/2$ as they are packed with $d > 1/2$ in an Opt-bin. On the other hand we claim that $s(C) - c' < 1/2$: otherwise we note that $c' > 1 - s(B)$, as c' was not packed in B and thus $s(B) + s(C) > s(B) + c' + 1/2 > s(B) + (1 - s(B)) + 1/2 = 3/2$, contradicting the first claim in the proof. Thus $s(C) + c = (s(C) - c') + (c' + c) < 1/2 + 1/2 = 1$ and c fits in C. However, c is one of the first two items packed in the 2-bin, thus in the regular case it cannot fit into a previous bin, but we have just shown that it fits into C. This is the final contradiction.