continuous :

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Conditional Density :

let X and Y be 2 r.v. with joint pdf $f(x,y)$. The conditional density of X given $Y=y$ is defined as :

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} & f_Y(y) \neq 0 \\ 0 & f_Y(y) = 0 \end{cases}$$ (same as in discrete)

* If 2 r.v. have joint pdf, then they do've their own pdf.
* If y is continuous r.v.
  - $P\{Y=y\} = 0 \quad \forall \; y \in \mathbb{R}$
* Here, $\left(f_Y(y) \neq P\{Y=y\} \right)$

If $f_Y(y) > 0$, then

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\, dx = 1$$ (should be equal to 1)

integrating wrt x

$$\int_{-\infty}^{\infty} \frac{f(x,y)\, dx}{f_Y(y)} = \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} f(x,y)\, dx = \frac{f_Y(y)}{f_Y(y)} = \underline{\underline{1}}.$$

→ $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

If f_X is the pdf of r.v. X then for any 'Boral' subset A of $\mathbb{R}$

$$P(X \in A) = \int_A f_X(x)\, dx$$

Definition $P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\, dx$

$P(Y=y) = 0$ but if we can integrate, we'll get answer.

** "conditional density func"

* It allows to define conditional prob. even when the conditioned event has prob. 0.

Teacher's Signature.................

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Example: Let X & Y be two r.v. having the joint pdf :

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Compute $P\left(X \le \frac{2}{3} \,\middle|\, Y = \frac{3}{4}\right)$

If we try to use classical def^n, $P\left(Y = \frac{3}{4}\right) = 0$. & But, we don't use this

$$P\left(X \le \frac{2}{3} \,\middle|\, Y = \frac{3}{4}\right) = \int_{-\infty}^{2/3} f_{X|Y}\left(x \,\middle|\, \frac{3}{4}\right) dx$$

$\Downarrow$

$X \in \left(-\infty, \frac{2}{3}\right]$

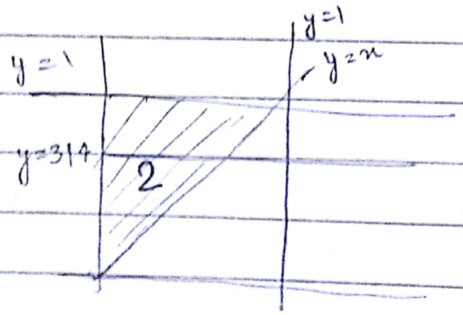
$\parallel$

A

$$f_{X|Y} = \begin{cases} \dfrac{f(x, 3/4)}{f_Y(3/4)} & \text{if } f_Y(3/4) \ne 0 \\ 0 & \text{if } f_Y\left(\frac{3}{4}\right) = 0 \end{cases}$$

$$f_Y\left(\frac{3}{4}\right) = \int_{-\infty}^{3/4} f\left(x, \frac{3}{4}\right) dx = \int_0^{3/4} 2 \cdot dx$$

$$= \frac{3}{2}.$$



$0 < x < y < 1$

$$f_{X|Y}\left(x \,\middle|\, 3/4\right) = \begin{cases} \dfrac{2}{3/2} = \dfrac{4}{3} & 0 < x < 3/4 \\ 0 & \text{Otherwise} \end{cases}$$

$$P\left(X \le \frac{2}{3} \mid Y = \frac{3}{4}\right) = \int_{0}^{2/3} \frac{4}{3}\, dx$$

$$= \boxed{\frac{8}{9}}$$

example: Let X & Y be independent continuous r.v. with pdf f_X & f_Y respectively. Determine the conditional density of $X+Y$ given X

Sol^n. If we can express the conditional distribution func^n of $X+Y$ given X as

$Z = X + Y$

$$F_{Z|X}(z|x) = P(Z \le z \mid X = x)$$

$$= \int_{-\infty}^{z} g(t, x)\, dt$$

conditional density of Z given $X = x$. (like in unconditional)

$$P(Z \le z \mid X = x) = P(X+Y \le z \mid X = x)$$

$$= P(x + Y \le z \mid X = x) \qquad [\because X = x]$$

$$= P(Y \le z - x \mid X = x)$$

Y & X are independent

$$= P(Y \le z - x)$$

$$= \int_{-\infty}^{z-x} f_Y(t)\, dt$$

~~fixed real no.~~

Let $t = s - x$

$$= \int_{-\infty}^{z} f_Y(s - x)\, ds$$

$$f_{Z|X}(s|x) = f_Y(s - x)$$

point is changing.

Other method : $f_{XY}(X+Y | X)$: find it

don't know it yet. Only know $f_X(x) \cdot f_Y(y)$

Example: Suppose X and Y are independent, identically distributed geometric (p) r.v. Find the ~~joint~~ conditional pmf of Y given $X+Y=n$ where $n \geq 2$

→ can either ~~find $f_{Y|X+Y}$~~

OR

We know :

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$P(X=x | Y=y)$

Here :-

$X = \{1, 2, 3, 4, \ldots\}$

$P(X=k) = p(1-p)^{k-1}$ $\quad k = 1, 2, \ldots$

$Z = X+Y$ $\quad R_Z = \{2, 3, 4, \ldots\}$

$P(Y=y | Z=n)$ → Y can take values $1, 2, \ldots, n-1$ ($X=1$ at least)

* $P(Y=y | Z=n) = 0$ if $y \geq n$

$y = \{1, 2, \ldots, n-1\}$

$$P(Y=y | Z=n) = \frac{P(Y=y, \overset{X+Y}{Z}=n)}{P(\underset{X+Y}{Z}=n)} = \frac{P(Y=y, \overset{X=n-y}{X+y=n})}{\sum_{k=1}^{n-1} P(X=k, Y=n-k)}$$

(Total Probability Thm)

X & Y are independent

$$\Rightarrow \frac{P(Y=y) \cdot P(X=n-y)}{\sum_{k=1}^{n-1} P(X=k) \cdot P(Y=n-k)}$$

$$= \frac{p(1-p)^{y-1} \cdot p(1-p)^{n-y-1}}{\sum_{k=1}^{n-1} p(1-p)^{k-1} p(1-p)^{n-k-1}}$$

$$= \frac{p^2 (1-p)^{n-2}}{p^2 \sum_{k=1}^{n-1} (1-p)^{n-2}} = \frac{(1-p)^{n-2}}{(n-1)(1-p)^{n-2}} = \boxed{\frac{1}{n-1}}$$

$$f_{Y|X+Y}(y|n) = \begin{cases} 1/n-1 & 1 \le y \le n-1 \\ 0 & \text{Otherwise } (y \ge n) \end{cases}$$

→ In original universe : y is Geometric r.v.
In conditional " : y is discrete uniform r.v.
(after modification)

Law of Total Probability : (Consequences of Total Probability Theorem)

Ω, $\{A_1, A_2, \dots, A_N\}$ is partition of Ω (can be uncountable also, i.e., $n \to \infty$)

Partition : pairwise disjoint

$A_i \cap A_j = \phi$ if $i \neq j$

$\bigcup_{i=1}^{N} A_i = \Omega$

Then, $P(B) = \sum_{i=1}^{N} P(B|A_i) \cdot P(A_i)$

Let Y be a discrete r.v. Then,

$\bigcup_{y \in R_Y} \{Y = y\} = \Omega$ → forms a partition (pairwise disjoint also)

$\{\{Y = y\}_{y \in R_Y}\}$: forms a partition

Total Probability theorem ⇒

$P(B) = \sum_{y \in R_Y} P(B|Y=y) \cdot P(Y=y)$

If Y has pdf f_Y, then

$$P(B) = \int_{-\infty}^{\infty} P(B \mid Y=y) \cdot f_Y(y)\, dy$$

$\downarrow$

won't use the usual defn as $P(Y=y)=0$

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Example: Let X & Y be 2 independent uniform $(0,1)$ random variables.

Find $P(X^3+Y>1)$

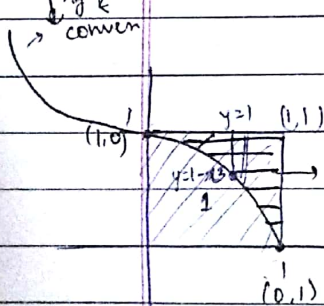
Solution: 3 possible solns :

$\boxed{(1)}$ we can find joint pdf :

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{Otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{Otherwise} \end{cases}$$

$$f(x,y) = \begin{cases} 1 & 0<x,y<1 \\ 0 & \text{Otherwise} \end{cases}$$



$\{X^3+Y>1\} = \{(X,Y) \in A\}$

where $A = \{(x,y) \in \mathbb{R}^2 \mid x^3+y>1\}$

$$P(X^3+Y>1) = \iint_A f(x,y)\, dx\, dy$$

Curve : $x^3+y=1 \rightarrow$

$\Rightarrow y = 1-x^3$

$f(x) = 1-x^3$

(1) domain : $\mathbb{R}$

(2) Range : $\mathbb{R}$

if $x<0$ then $x^3<0 \Rightarrow 1-x^3$: +ve axis

if $0 \le x \le 1$ then $x^3 \le 1 \Rightarrow 1-x^3 \geq 0$

if $x>0$ then $x^3>0 \Rightarrow 1-x^3<0$

$-2x^2 = 0$

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(3) $f'(x) = -3x^2 \le 0 \quad \forall \ x \in \mathbb{R}$

$\Rightarrow f(x) \downarrow$ on $\mathbb{R}$

(4) $-3x^2 = 0 \Rightarrow x = 0$: critical point

$\boxed{f''(x) = -6x} = 0 \Rightarrow$ Test ~~not~~ fails

(5) $+ \quad 0 \quad -$ $\quad f''(x) > 0$ if $x < 0$: convex

$f''(x) < 0$ if $x > 0$: concave

convex $\quad \Downarrow \quad$ concave

point of inflexion

$\Rightarrow$

(6) $x = 0 \quad f(x) = 1 \Rightarrow$ point $(0,1)$

$y = 0 \quad x^3 = 1 \Rightarrow x = 1 \quad (1,0)$

$$I = \int_0^1 \int_{1-x^3}^1 1 \cdot dy \, dx$$

$$= \int_0^1 x^3 \, dx = \boxed{\frac{1}{4}}$$

$\boxed{(2)}$ Let us condition wrt X. (Using Total Probability theorem)

$$P(X^3 + Y > 1) = \int_{-\infty}^{\infty} P(X^3 + Y > 1 \mid X = x) f_X(x) \, dx$$

$$= \int_0^1 P(\cancel{x^3} Y > 1 - x^3 \mid X = x) \, dx$$

X & Y are independent

$$I = \int_0^1 P(Y > 1 - x^3) \, dx$$

$\boxed{\text{we know } 1-x^3 < 1}$ we know pdf of Y,

$$P(Y > 1 - x^3) = \int_{1-x^3}^{1} 1 \cdot dy = \underline{\underline{x^3}}$$

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$$\Rightarrow I = \int_0^1 x^3\,dx = \frac{1}{4}$$

(3) Let us condition wrt y

$$P(X^3+Y>1) = \int_0^1 P(X^3+Y>1 \mid Y=y)\,dy$$

$$= \int_0^1 P(X^3>1-y \mid Y=y)\,dy$$

$$= \int_0^1 P(X^3>1-y)\,dy \qquad [\text{Independent}]$$

func^n of X, first have to get X.

$\cancel{P(X^3>1-y) = \int_{1-y}^1 1}$

$0<y<1 \qquad 0<1-y<1$

$$x^3 > 1-y \iff x > (1-y)^{1/3}$$

$$= \int_0^1 P(X > (1-y)^{1/3})\,dy$$

$$P(X>(1-y)^{1/3}) = \int_{(1-y)^{1/3}}^1 1.\,dx = 1-(1-y)^{1/3}$$

$$= \int_0^1 \left(1-(1-y)^{1/3}\right)dy$$

$$= \left[y + \frac{3(1-y)^{4/3}}{4}\right]_0^1$$

$$= \left[1 + \frac{3}{4}\{0-1\}\right] = \boxed{\frac{1}{4}}$$

* have to select wisely wrt to what I've to condition.

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Conditional Expectation : of X given Y=y

→ Let X and Y be discrete r.v. with conditional pmf $f_{X|Y}$. Then, the conditional expectation of X given Y=y is defined as :

$$E[X|Y=y] = \sum_{x \in R_X} x \, f_{X|Y}(x|y)$$ (talking about expec^n of X only)

provide $\sum_{x \in R_X} |x| \, f_{X|Y}(x|y) < \infty$

Example: Let X and Y be iid geometric (p) random variables. Find $E[Y|X+Y=n]$ where $n \geq 2$

Solution (found earlier in last class) $Z = X+Y$

$$f_{Y|Z}(y|n) = \begin{cases} 1/n-1 & 1 \leq y \leq n-1 \\ 0 & y \geq n \text{ (otherwise)} \end{cases}$$ → finite points ⇓ don't have to bother about absolute su

$$E[Y|Z] = \sum_{y=1}^{\infty} y \cdot f_{Y|Z}(y|z)$$

$$= \sum_{y=1}^{n-1} \frac{y}{n-1}$$

$$= \frac{1}{n-1} [n-1] \left[\frac{n}{2}\right] = \boxed{\frac{n}{2}}$$

→ Let X and Y be continuous r.v. with conditional pdf of X given Y=y. Then, the condional expectation of X given Y=y is defined as :

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, dx$$

provided $\int_{-\infty}^{\infty} |x| \, f_{X|Y}(x|y) \, dx < \infty$

Teacher's Signature.........

11 - 12:00 pm
Tomorrow

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Law of Total Expectation

Theorem: Let X and Y be discrete r.v. with joint pmf $f(x,y)$. If X has finite mean, then

$$E[X] = \sum_{y \in R_Y} E[X \mid Y=y]\, f_Y(y)$$

PROOF:

$$\text{RHS} = \sum_{y \in R_Y} E[X \mid Y=y]\, f_Y(y)$$

$$= \sum_{y \in R_Y} \sum_{x \in R_X} x\, f_{X|Y}(x|y) f_Y(y)$$

$$= \sum_{x \in R_X} x \sum_{y \in R_Y} \underbrace{f_{X|Y}(x|y)\, f_Y(y)}_{f(x,y)}$$

$$= \sum_{x \in R_X} x \sum_{y \in R_Y} f(x,y)$$

$$= \sum_{x \in R_X} x\, f_X(x) = E[X] = \text{LHS}$$

Hence Proved

24-03-18

Example Let X and Y be continuous r.v. with joint pdf

$$f(x,y) = \begin{cases} 6(y-x) & ; \quad 0 \le x \le y \le 1 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

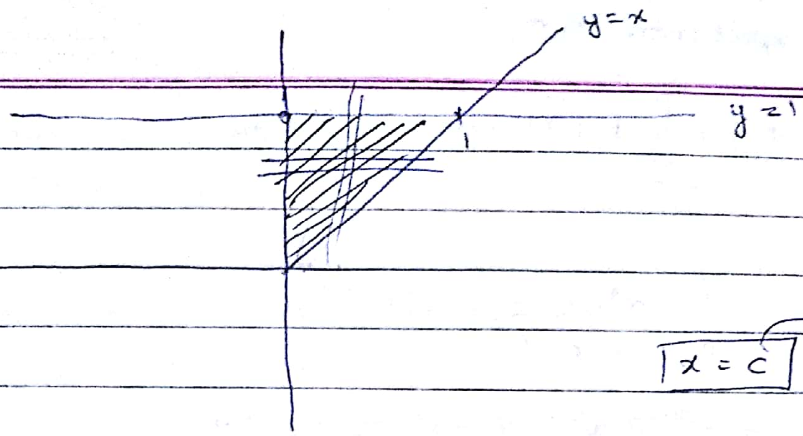
Find the $E[Y \mid X=x]$ and hence, calculate $E[Y]$

Solution: $E[Y \mid X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy$

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{f(x,y)}{f_X(x)} & , \quad f_X(x) > 0 \\ 0 & \quad \text{Otherwise} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

Teacher's Signature



→ real const.

$\boxed{x = c}$

If $x < 0 \Rightarrow f_X(x) = 0$

If $x > 1 \Rightarrow f_X(x) = 0$

$0 < x < 1$

$$f_X(x) = \int_x^1 6(y-x)\,dy = 6\left[\frac{y^2}{2} - xy\right]\Bigg|_x^1$$

$$\cancel{= 6\left[\frac{1}{2} - x - \frac{x^2}{2} + x\right] = 3(1-x^2)}$$

$$= 6\left[\frac{1}{2} - x - \frac{x^2}{2} + x^2\right] = 6\left[\frac{1}{2} - x + \frac{x^2}{2}\right]$$

$$= 6\left[\frac{1}{2} - x + \frac{x^2}{2}\right] = 3(x-1)^2$$

$$\rightarrow f_{Y|X}(y|x) = \begin{cases} \dfrac{6(y-x)}{3(x-1)^2}, & 0 \le x \le y < 1 \\ 0 & \text{Otherwise} \end{cases}$$

for $0 \le x < 1$

$$E[Y \mid X = x] = \int_x^1 \frac{2y(y-x)}{(x-1)^2}\,dy$$

$$= \frac{2}{(x-1)^2}\left[\frac{y^3}{3} - \frac{xy^2}{2}\right]\Bigg|_x^1$$

$$= \frac{2}{(x-1)^2}\left[\frac{1}{3} - \frac{x}{2} - \frac{x^3}{3} + \frac{x^3}{2}\right] = \frac{2}{(x-1)^2}\left[\frac{2 - 3x + x^3}{6}\right]$$

$$= \frac{x^2 + x - 2}{3(x-1)}$$

By Total Expectation Th^m,

$$E[Y] = \int_{-\infty}^{\infty} E[Y \mid X=x]\, f_X(x)\, dx$$

$$= \int_0^1 \frac{x^3+x-2}{\cancel{3(x-1)}} \cdot 3(x-1)^2\, dx$$

$$= \cancel{\int_0^1 (x^4 - x^3 + x^3 - 3x + 2)\,dx}$$

$$= \int_0^1 (x^3 - 3x + 2)\, dx = \frac{1}{4} - \frac{3}{2} + 2 = \boxed{\frac{3}{4}}$$

Th^m : Let X and Y be discrete r.v. with conditional ~~pdf~~ pmf of X given Y = y. If $g : \mathbb{R} \to \mathbb{R}$ is any func^n, then

$$E[g(X) \mid Y=y] = \sum_{x \in R_X} g(x)\, f_{X|Y}(x|y)$$

similar version for continuous case.

Covariance

→ for ~~dependence~~ measure dependence

The covariance of two random variables X & Y is defined as:

$$\mathrm{CoV}(X,Y) = E[\underbrace{(X - EX)}_{\text{mean of } X}\ \underbrace{(Y - EY)}_{\text{mean of } Y}]$$

↳ when CoV(X, Y) = 0, we say that X & Y are uncorrelated.

⇒ For large values of X & Y together

$X - EX > 0$

$Y - EY > 0$

$\mathrm{Cov}(X, Y) > 0$

⇒ Similarly if both take small values together ⇒
$Cov(X,Y) > 0$

→ If $Cov(X,Y) > 0$, we say that X & Y are +vely correlated.
→ If $Cov(X,Y) < 0$, we say that X & Y are -vely correlated

$Cov(X,Y) = E[XY - XEY - YEX + EXEY]$
Using linearity of expectation,
$$= E[XY] - E[X\underbrace{EY}_{\text{Real no.}}] - E[Y\underbrace{EX}_{\text{Real no.}}] + E[\underbrace{EXEY}_{\text{Real no.}}]$$

$$= E[XY] - EYEX - \cancel{EYEX} + \cancel{EXEY}$$

★★ $\boxed{Cov(X,Y) = E[XY] - EX\,EY}$

example : Suppose the joint pdf of X and Y is given as :

Find : $Cov(X,Y)$

X \ Y	-1	0	1		
-1	0	1/4	0	⇒ 1/4	$P(X=-1)$
0	1/4	0	1/4	⇒ 1/2	$P(X=0)$
1	0	1/4	0	⇒ 1/4	$P(X=1)$
	$P(Y=-1)$	$P(Y=0)$	$P(Y=1)$		

→ We need to find pmf of XY & then compute $E[XY]$
$R_{XY} = \{-1, 0, 1\}$

$P(XY=-1) = 0$ $P(XY=0) = 1$ $P(XY=1) = 0$

⇒ $E[XY] = XY \cdot P(XY=0) = 0 \times 1 = 0$

$EX = -1\left(\frac{1}{4}\right) + 0\left(\frac{1}{2}\right) + \frac{1}{4} = 0$

Similarly $EY = 0$

★ $\boxed{Cov(X,Y) = 0}$ ⇒ They are uncorrelated

Teacher's Signature..................

* If X and Y are independent.

$$E[XY] = EX\,EY$$

Hence

$\boxed{Cov(X,Y) = 0}$ → they are uncorrelated.

Converse is not true.

⇒ independent ⟹ covariance

covariance ⇏ independent

→ In last eg, we want to see whether X & Y are independent or not

$P(X=0, Y=0) = 0$

$P(X=0) = \frac{1}{2}$ $\quad P(Y=0) = \frac{1}{2}$

$P(X=0, Y=0) \neq P(X=0)\cdot P(Y=0)$ ⇒ X & Y are dependent.

→ **Proposition** : Let X, Y and Z be r.v. and $a, b \in \mathbb{R}$. Then,

(i) $Cov(X, X) = Var(X)$

(ii) $Cov(X,Y) = Cov(Y,X)$ (symmetric) (Just interchange X & Y)

(iii) $Cov(X, aY+b) = a\,Cov(X,Y)$

(iv) $Cov(X, Y+Z) = Cov(X,Y) + Cov(X,Z)$

Proof

(i) $Cov(X,X) = E[X^2] - EX\,EX = Var(X)$

(iii) $Cov(X, aY+b) = E[X\cdot(aY+b)] - E[X]\,E[aY+b]$

$= a\,E[XY] + b\,\cancel{E[X]} - a\,E[X]\,E[Y] - b\,\cancel{E[X]}$

$= a\,[E[XY] - E[X]\,E[Y]]$

$= a\,Cov(X,Y)$

(iv) $Cov(X, Y+Z) = E[X(Y+Z)] - E[X]\,E[Y+Z]$

$= E[XY + XZ] - E[X]\,[E[Y] + E[Z]]$

$= E[XY] + E[XZ] - EX\,EY - EX\,EZ$

$= Cov(X,Y) + Cov(X,Z)$

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Example: Let X & Y be two independent $N(0,1)$ random variables.

$$Z = 1 + X + XY^2 \quad , \quad W = 1 + X$$

Find $Cov(Z, W)$

$$Cov(X,Y) = 0 \Rightarrow E[XY] - EX\,EY = 0$$
$$\Rightarrow E[XY] = 0$$

~~not necessary~~

$$Cov(Z,W) = Cov(1 + X + XY^2, 1 + X)$$

(const. don't affect cov)

$$= cov(X + XY^2, X)$$
$$= Cov(X, X) + Cov(XY^2, X)$$
$$= Var(X) + E[X^2Y^2] - E[XY^2]\,E[X]$$
$$= 1 + E[X^2Y^2] \quad (X \text{ \& } Y \text{ are independent})$$
$$= 1 + E(X^2)\,E(Y^2)$$
$$= 1 + Var(X^2)\,Var(Y^2) \quad [Var(X^2) = E[X^2] - (E[X])^2]$$
$$= 1 + (1)(1) = 2$$

Example: Let X & Y be r.v. Then.

$$Var(X+Y) = ?$$
$$= Cov(X+Y, X+Y)$$
$$= Cov(X, X+Y) + Cov(Y, X+Y)$$
$$= Cov(X,X) + Cov(X,Y) + Cov(Y,X) + Cov(Y,Y)$$

★ $$\boxed{Var(X+Y) = Var(X) + 2\,Cov(X,Y) + Var(Y)}$$

26/03/18

→ $Cov(X,Y)$: It is a measure of linear relationship b/w X & Y in the sense that if

$X - EX$ & $Y - EY$ are of same sign $\Rightarrow Cov(X,Y) > 0$

$X - EX$ & $Y - EY$ are of opp. sign $\Rightarrow Cov(X,Y) < 0$

Teacher's Signature.................

Just tells about sign

→ Covariance just talks about sign, not about magnitude.

Reason: It depends on variability of both X & Y. To fix this problem, we've:

Correlation of two r.v.:

Let X & Y be two r.v. then the correlation b/w them is defines as:

$$\boxed{\rho(X,Y) = \frac{CoV(X,Y)}{\sqrt{Var(X)\,Var(Y)}}}$$ provided $Var(X) > 0$, $Var(Y) > 0$

Coefficient

(just divided ~~both~~ by variability)

$\boxed{-1 < \rho < 1}$

ρ : unitless.

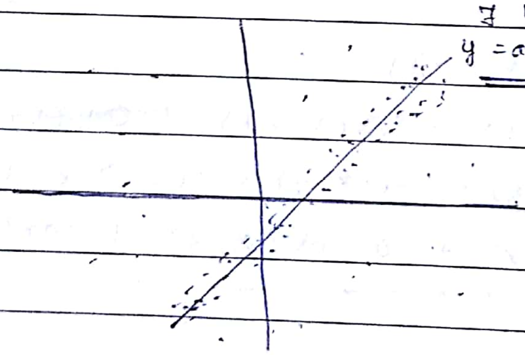
Theorem: The correlation b/w X & Y satisfies the foll.. properties:

(1) $|\rho(X,Y)| \le 1$

(2) $|\rho(X,Y)| = 1 \overset{iff}{\iff} \exists\, a, b \in \mathbb{R},\ a \neq 0$ s.t. $Y = aX + b$

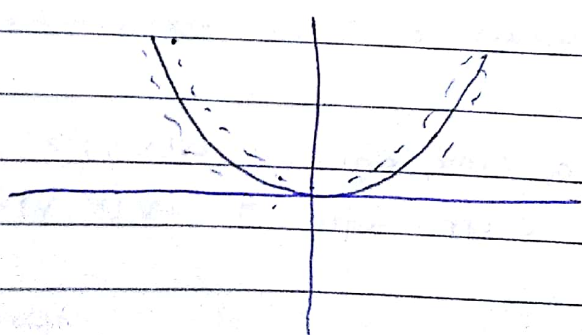
→ If $\rho(X,Y) = 1 \Rightarrow a > 0$ $(a>0 \Rightarrow cov > 0)$

→ If $\rho(X,Y) = -1 \Rightarrow a < 0$

→ 

∃ line $y = ax + b$ s.t. most of point lies near line ⇒ CoV exists

~~if no such line exists~~ ⇒ $CoV \approx 0$



⇒ Not linear ✗

⇒ $CoV \approx 0$.

(~~CoV reflects linear~~ relationship)

$CoV = 0$ ⇒ linear relationship done, some other " may exist

$\rightarrow$ let $\underline{EX^4 = 3}$ $\Rightarrow$ $\rho(X,Y) = \frac{b}{\sqrt{b^2+2c^2}} \leq 1$ (if $c \in \mathbb{R}$) $\left(\leq \frac{b}{\sqrt{2}c}\right) \Rightarrow$ linear rel^{n}ship

$X \sim N(0,1)$: $\boxed{\text{actually a normal r.v.}}$

$b = 0$
Corr $= 0$
but ~~cor~~ Y & X have strong rel^n.

Example: Suppose that a r.v. X satisfies : $EX = 0$, $EX^2 = 1$, $EX^3 = 0$, $Ex^4 = 8$

Let $Y = a + bX + cX^2$. Find the correlation coefficient. $\rho(X,Y)$

Solution:

$$\begin{aligned}
\text{Cov}(X,Y) &= \text{Cov}(X, a+bX+cX^2) \\
&= \text{Cov}(X, bX + cX^2) \\
&= \text{Cov}(X, bX) + \text{Cov}(X, cX^2) \\
&= b\,\text{Cov}(X,X) + c\,\text{Cov}(X, X^2) \\
&= b\,\text{Var}(X) + c\,\text{Cov}(X, X^2) \\
&= b[EX^2 - (EX)^2] + c\,E[(X - \overset{0}{EX})(X^2 - \overset{1}{EX^2})] \\
&= b(1) + c\,E[X^3 - X] \\
&= b + c[0 - 0] \\
&= \underline{b}.
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X) &= 1 \\
\text{Var}(Y) &= EY^2 - (EY)^2 = \text{Cov}(Y,Y) \\
&= \text{Cov}(bX+cX^2, bX+cX^2) \\
&= \text{Cov}(bX, bX) + \overset{0}{\cancel{\text{Cov}(bX, cX^2)}} + \overset{0}{\cancel{\text{Cov}(cX^2, bX)}} + \text{Cov}(cX^2, cX^2) \\
&= b^2\,\text{Var}(X) + c^2\,\text{Var}(X^2) \\
&= b^2(1) + c^2\,E[(X^2 - EX^2)^2] \\
&= b^2 \qquad [E(X^2)^2] - (EX^2)^2 = 0
\end{aligned}$$

$\rho(X,Y) = \frac{b}{\sqrt{b^2}} = \boxed{1} \Rightarrow$ * There doesn't exist r.v. X with above moments in this quest^n

$\Rightarrow$ From last theorem, we know $Y = aX + b$ ~~where~~ $\boxed{c=0}$
but here, $c \in \mathbb{R} \Rightarrow$ it is contradicting $\Rightarrow$ such X is not possible

Example: Let X and Y be 2 r.v.. Suppose $\text{Var}(X) = 4$ & $\text{Var}(Y) = 9$. If we know that r.v. $Z = 2X - Y$ and $W = X + Y$ are independent, then find the correlation coeff. $\rho(X,Y)$

Independent $\Rightarrow$ $\text{Cov}(Z, W) = 0$

$\Rightarrow \text{Cov}(2X - Y, X + Y) = 0$

Teacher's Signature.............

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* In this eg. Z & W both have variable X & Y. But we can't say they are dependent. For independence.

$$P(W)\,P(Z) = P(W \cap Z)$$

⇒ Just by looking, we can't say anything

$= 2\,Cov(X, X) - Cov(Y, X) + 2\,Cov(X, Y) - Cov(Y, Y)$

$= 2\,Var(X) + Cov(X, Y) - Cov(Y, Y) = 0$

$\Rightarrow \quad 2(4) + Cov(X, Y) - 9 = 0$

$\Rightarrow \quad Cov(X, Y) = 1$

$\Rightarrow \quad \rho(X, Y) = \dfrac{1}{\sqrt{36}} = \boxed{\dfrac{1}{6}}$

→ Real-valued random variable : $X : \Omega \longrightarrow \mathbb{R}$
(learnt till now)

→ Complex-valued r.v. :

$$Z : \Omega \longrightarrow \mathbb{C}$$

$$Z(\omega) = X(\omega) + i\,Y(\omega)$$

$$Z = X + iY$$

$\left.\begin{array}{l} X : \Omega \longrightarrow \mathbb{R} \\ Y : \Omega \longrightarrow \mathbb{R} \end{array}\right\}$ real valued random variables

→ $E[Z]$:

$E[Z] = E[X] + i\,E[Y]$ provided EX & EY exists

Characteristic Function :

For a r.v. X, its Ch. F. $\phi_X : \mathbb{R} \longrightarrow \mathbb{C}$ is defined as :

$$\phi_X(t) = E[e^{itX}] \,, \quad \forall\, t \in \mathbb{R}$$

↳ complex values funcⁿ

$= E[\cos(tX) + i\sin(tX)]$ ⇒ It exists for any value of random variable & any value of $t \in \mathbb{R}$

$|\cos tx| \le 1$ → bounded

⇒ $E[\cos tX] \le 1$

Eg. X is discrete ⇒ $E[\cos tX] = \sum_{x \in R_X} |\cos tx|\, f_X(x)$

$\le \sum_{x \in R_X} f_X(x) = 1$

→ If r.v. are is bounded, then $E[r.v.]$ is also bounded by the same constant

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Example: Let $X \sim \text{Bernoulli}(p)$. Ch. f. = ?

$\{P(X=0) = 1-p,\ P(X=1) = p\}$

$\phi_X(t) = E[e^{itX}] \quad \forall\ t \in \mathbb{R}$

discrete r.v. (since X : discrete r.v.)

$= e^{it0}(P(X=0)) + e^{it(1)} P(X=1)$

$$\boxed{\phi_X(t) = (1-p) + e^{it} p}$$

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Example: Let $X \sim N(0,1)$. $\phi_X(t) = ?$

$\phi_X(t) = E[e^{itx}] \quad \forall\ t \in \mathbb{R}$

$= E[\cos tX + i \sin tX]$

$= E[\cos tX] + i\, E[\sin tX]$

$= \int_{-\infty}^{\infty} \cos tx\, f_X(x)\, dx + i \int_{-\infty}^{\infty} \sin tx\, f_X(x)\, dx$

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

* If $\int_{-\infty}^{\infty} \sin(tx) f_X(x)\, dx < \infty$ convergent then

$\int_{-\infty}^{\infty} \sin(tx) \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \lim_{a\to\infty} \int_{-a}^{a} \underbrace{\sin tx}_{\text{odd}}\ \underbrace{e^{-x^2/2}}_{\text{even}}\, dx = \boxed{0}$

(principal value)

Converse is not true

Eg: $\int_{-\infty}^{\infty} \sin x\, dx$: diverges

but $\lim_{a\to\infty} \int_{-a}^{a} \sin x\, dx = 0$

→ $\int_{-\infty}^{\infty} \cos tx\, f_X(x)\, dx = \int_{-\infty}^{\infty}$

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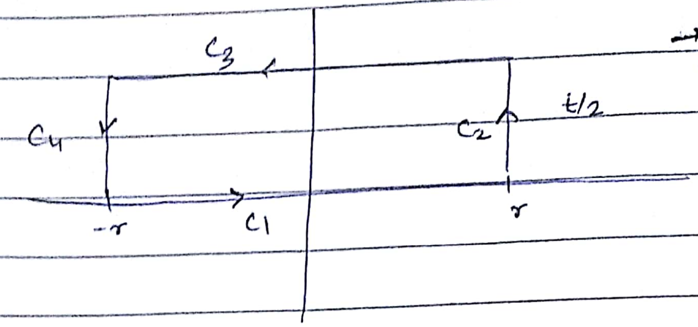
$\rightarrow \int_{-\infty}^{\infty} \cos tx \; e^{-x^2/2} \, dx = \oint e^{-t^2/2} \cdot \sqrt{\pi 2}$

(Complex Analysis)



$\rightarrow$ Contour Integral

$\oint_C e^{itz} \cdot e^{-x^2/2} \, dz = \int_{-r}^{r} e^{itx} e^{-\frac{x^2}{2}} dx + \int_{C_2} + \int_{C_3} + \int_{C_4}$
(C_1)

$\downarrow$

exponential func^n
$\Downarrow$
entire func^n

take $\lim_{r \to \infty}$

$\Rightarrow \boxed{\phi_X(t) = e^{-t^2/2}} \rightarrow$ Normal r.v.

Example : let X be a r.v. & $a, b \in \mathbb{R}$. Then,

$$\phi_{a+bX}(t) = E\left[e^{it(a+bX)}\right]$$
$$= E\left[e^{ita} * e^{itbX}\right]$$
$$= e^{iat} \, E\left[e^{itbX}\right] = e^{ita} \, \phi_X(bt)$$

Example : Let $X \sim N(\mu, \sigma^2)$ $\quad \sigma \neq 0$ (have σ in denominator in pdf of Normal)

Then we know $\sigma > 0$.

Define $Y = \frac{X - \mu}{\sigma}$. Then show that $Y \sim N(0,1)$

(② show it has mean 0 & $\sigma = 1$)

$F_Y(y) = P(Y \leq y)$
$= P\left(\frac{X - \mu}{\sigma} \leq y\right)$

$= P\left(X \le \sigma y + \mu\right)$

$$= \int_{-\infty}^{\sigma y+\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

putting $x = \sigma u + \mu$

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-u^2/2} \sigma \, du = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2} du}_{\text{cdf of std. Normal}}$$

$\Rightarrow Y \sim N(0,1)$

also, $\mu = 0$ & $\sigma = 1$ (from cdf)

Example : Let $Y \sim N(0,1)$. Define $X = \sigma Y + \mu$ where $\sigma > 0$ & $\mu \in \mathbb{R}$

$\phi_X(t) = e^{it\mu} \cdot \phi_Y(\sigma t)$ $\qquad \phi_Y(t) = e^{-t^2/2}$

$= e^{it\mu} \cdot e^{-\sigma^2 t^2/2}$

Example : Let X & Y be independent r.v. Show that

$$\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$$

Sol^n: $\phi_{X+Y}(t) = E\left[e^{it(X+Y)}\right]$

$= E\left[\underset{g(X)}{e^{itX}} \cdot \underset{g(Y)}{e^{itY}}\right]$ $\qquad X$ & Y : independent $\Rightarrow g(X)$ & $g(Y)$: "

$= E\left[e^{itX}\right] \cdot E\left[e^{itY}\right]$

$= \phi_X(t)\,\phi_Y(t)$ Hence Proved.

* More generally, if $X_1, X_2, \ldots, X_n$ are independent r.v, then

$$\boxed{\phi_{X_1+X_2+\cdots+X_n}(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{X_n}(t)}$$

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Example: let $X \sim B(n,p)$

1) $\phi_X(t) = E[e^{itX}] = \sum_{k=0}^{n} e^{itk} P(X=k)$

OR

2) Using last statement, note that X is sum of X_i where $X_i \sim$ Bernoulli (p) & $X_1, X_2, \ldots X_n$ are independent & identically distributed (have same parameters p)

$\phi_X(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t) \because$ all have same distribution $\Rightarrow$ Ch. Func^n will be same for all

$$= [\phi_{X_1}(t)]^n = [e^{it}p + (1-p)]^n$$

★★★ **Uniqueness Theorem**:

Let X_1 & X_2 be 2 r.v., s.t. $\phi_{X_1}(t) = \phi_{X_2}(t) \quad \forall \; t \in \mathbb{R}$, then X_1 & X_2 have the same probability distribution

Example let $X \sim B(n_1,p)$ & $Y \sim B(n_2,p)$ be 2 independent r.v. Show that $X+Y \sim B(n_1+n_2,p)$ — if had p_1 & p_2 $\Rightarrow$ can't be binomial (from Uniqueness) $(p_1 \neq p_2)$

1) Find pmf of X+Y and get desired expression

OR

2) $\phi_X(t) = (e^{it}p + (1-p))^{n_1}$

$\phi_Y(t) = (e^{it}p + (1-p))^{n_2}$

$\because$ X & Y are independent.

$$\phi_{X+Y}(t) = (e^{it}p + (1-p))^{n_1} \cdot (e^{it}p + (1-p))^{n_2}$$
$$= (e^{it}p + (1-p))^{n_1+n_2} \Rightarrow \text{char. func}^n \text{ of } B(n_1+n_2,p)$$

Using uniqueness th^m, X+Y has same pmf as $B(n_1+n_2,p)$ (since both have same char. func^n)

Example: Let $X \sim N(\mu_1, \sigma_1^2)$ & $Y \sim N(\mu_2, \sigma_2^2)$ be 2 independent r.v. Show that $X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

$\phi_X(t) = e^{it\mu_1} e^{-\sigma_1^2 t^2/2}$ $\phi_Y(t) = e^{it\mu_2} e^{-\sigma_2^2 t^2/2}$

$\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$

$= e^{it(\mu_1+\mu_2)} \cdot e^{-(\sigma_1^2+\sigma_2^2)t^2/2}$: char funⁿ of $N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

By uniqueness thⁿ, $X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

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→ Convex funcⁿ : line segment joining 2 pts. should lie above graph

Defⁿ: Let $f: I \to \mathbb{R}$ be a funcⁿ where I is an interval in $\mathbb{R}$. We say that f is convex if for any $x_1, x_2 \in I$ & $t \in (0,1)$, we 've

$$f(tx_1 + (1-t)x_2) \leq t f(x_1) + (1-t) f(x_2)$$

We say that f is concave if

$$f(tx_1 + (1-t)x_2) \geq t f(x_1) + (1-t) f(x_2)$$

⇒ f is convex $\Longleftrightarrow$ $-f$ is concave

1: Jensen's Inequality :

Let $f: I \to \mathbb{R}$ be convex funⁿ & X be a r.v. s.t. X & $f(X)$ have finite mean. (expectation)

Interval I contains range of X. (assumed) Then,

$$\boxed{f(EX) \leq E[f(x)]}$$

→ If f is concave then { f is convex ⇒ $-f$ is concave }

$$\boxed{f(EX) \geq E[f(x)]}$$

$\rightarrow$ $EX < \infty$ if $E|X| < \infty$

$EX = \sum_{x \in R_x} x f(x)$ given $\sum_{x \in R_x} |x| f_X(x) < \infty$

$f(x) = |x|$: convex func^n $\{f(x) = |x|\}$

$\Rightarrow$ $|EX| \leq E|X|$ $\{EX \leq |EX|\}$

$\Rightarrow$ $\boxed{EX \leq |EX| \leq E|X|}$

(keep in mind)

<u>Def^n</u> : Let ~~...~~ $r > 0$, $r \in \mathbb{R}$ & X be r.v. Then $E(X^r)$ is called <u>r^{th} - central moment of r.v. X</u> or <u>central moment of r.v. X of order r</u>.

central :- $E[(X-a)^r]$: r^{th} moment of X centred at a

where $\underline{a = 0}$.

$\hookrightarrow$ $E[|X|^r]$: r^{th} - absolute central moment of X.

<u>~~...~~</u> ★

<u>Example</u> : If the moment of order $q > 0$ exists for r.v. X, then show that moment of order p, where $0 < p < q$ exists

<u>Sol^n</u> : $f(x) = x^r$, $\forall x \in (0, \infty)$ & $\underline{r > 1}$

(determining convexity)

$f'(x) = r x^{r-1}$

$f''(x) = r(r-1) x^{r-2}$ $\underline{> 0}$ $\underline{(r > 1)}$

f is convex on $(0, \infty)$

$[E|X|]^r \leq E[|X|^r]$ (By Jensen's ~~eq~~ inequality)

$\Rightarrow$ $E|X| \leq (E[|X|^r])^{1/r}$ —— ①

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given : $E[X^q]$ exists

$\Rightarrow$ $E[|X|^q] < \infty$

let $p < q$ be given.

then, $\frac{q}{p} \geq 1$

so, take $r = q/p$ in ①

$\rightarrow$ $E|X| \leq \left(E(|X|^{q/p})\right)^{p/q}$ — ②

→ we need $E|X^p|$

In ②, replace $|X|$ with $|X|^p$. so,

$E|X^p| \leq \left(\underbrace{E(|X|^q)}_{\text{finite}}\right)^{p/q}$

$\Rightarrow$ $(E|X^q|)^{p/q} < \infty$ (finite)

$\Rightarrow$ $E|X^p| < \infty$ (finite)

$\Rightarrow$ moment of order p exists.

Example : let X be a r.v. s.t $EX = 10$. show that

$E[\ln\sqrt{x}] \leq \frac{1}{2}\ln 10$

↳ ~~can't find because~~ exact don't know pmf or pdf.

Using Jensen's :

$f(x) = \ln\sqrt{x} = \frac{1}{2}\ln x$ $x \in (0,\infty)$

$f'(x) = \frac{1}{2} \times \frac{1}{x}$ $f''(x) = \frac{-1}{2x^2} < 0$ $x \in (0,\infty)$

$\Rightarrow$ $f(x)$ is concave on $x \in (0,\infty)$

$\Rightarrow$ $f(EX) \geq E[f(x)]$

$\Rightarrow$ $\ln\sqrt{10} \geq E[\ln\sqrt{x}]$

$\Rightarrow$ $\frac{1}{2}\ln 10 \geq E[\ln\sqrt{x}]$

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2. Markov Inequality :

Let X be a non-negative r.v. with finite n^{th} moment

Then, $\forall\ \epsilon > 0$, we have

$$\boxed{P(X \geq \epsilon) = \frac{E(X^n)}{\epsilon^n}}$$

$\Rightarrow$ { Prob. that ϵ takes ↑ value is very small if n is small }

3. Chebyshev's Inequality :

Let X be a r.v. with finite mean μ & variance σ^2. Then,

$\forall\ \epsilon > 0$,

$$\boxed{P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}}$$

(X can be -ve : have mod)

variance $= \{EX^2 - (EX)^2$

exist ⇒ 2^{nd} moment exist (finite)

Proof : $|X-\mu| \geq 0$

Applying markov's Inequality,

$$P(|X-\mu| \geq \epsilon) \leq \frac{E(|X-\mu|^2)}{\epsilon^2}$$

$\because \sigma^2 = E\cdot X^2 - \mu^2$

$$\leq \frac{\sigma^2}{\epsilon^2}$$

spread of variability

if $\sigma^2 \downarrow \Rightarrow X$ won't take values far from mean with high prob.

Example : Let $X \sim B(n,p)$. Estimate $P(X \geq \alpha n)$ where $p < \alpha < 1$ using Markov Inequality for 1^{st} moment & chebyshev's Inequality. Compare both the estimates for $p = \frac{1}{4}$, $\alpha = \frac{3}{4}$

sol^n : Markov :

$$P(X \geq \alpha n) = \frac{EX}{\alpha n} \quad (n=1)$$

$EX = np$ (know)

$$\therefore P(X \geq \alpha n) = \frac{p}{\alpha} \quad \text{——} (1)$$

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Chebyshev :

$P(|X - np| \geq \alpha n)$ → we want $P(X \geq \alpha n)$

rewrite as

$P(X \geq \alpha n) = P(X - np \geq \alpha n - np)$

$\leq P(|X - np| \geq \alpha n - np)$

$\{ \because \{|Y| \geq a\} = \{Y \geq a\} \cup \{Y \leq -a\} \}$

$\leq \dfrac{Var(X)}{\alpha n - np} = \dfrac{p(1-p)}{(\alpha n - pn)^2}$ —— ②

$\alpha = 3/4$

① → $P(X \geq \frac{3}{4} n) = \frac{4}{3} \times \frac{1}{4} = \frac{1}{3}$

② → $P(X \geq \frac{3}{4} n) = \dfrac{\frac{1}{4} \times 3/4}{(\frac{1}{2})^2 n^2} = \dfrac{3}{4n^2}$

① → remains const ② → as $n \uparrow$, $P \downarrow$ ⇒ it controls ~~as $n \uparrow$~~

⇓

gives more accurate estimation ~~(more information)~~

Example: 102 → total students

$\mu = 14.5$ $\sigma = 4.46$

Ω = Student in A_2

$X : \Omega \to R$ uniform probability measure (mass associated with each student : 1/102)

$X(\omega)$ = mid-term of ω

For n=1 . Markov $P(X \geq 16) = ??$

$P(X \geq 16) \leq \dfrac{EX}{16} = 0.9025$

$P(\omega \in \Omega \mid X(\omega) \geq 16) \leq 0.90625$

⇒ No. of students who got marks > 16 is less than equal to $0.90625 \times 102 = 97.962 \approx 98$

(original data : 39)

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→ if you take $n = 2$ ⇒ get 91 : get much better estimation as $n \uparrow$