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THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
DEPARTMENT OF MATHEMATICS
MATH-221: PROBABILITY AND STATISTICS
END TERM

Maximum Time: 3 Hours | Date: 01/05/2018 | Maximum Marks: 50

Instruction: You should attempt all questions. Your writing should be legible and neat. Marks awarded are shown next to the question. **NO USE OF CALCULATORS.** Please make an index showing the question number and page number on the front page of your answer sheet in the following format, otherwise you may be penalized by the deduction of **4 marks**.

Question No.				
Page No.				

1. (a) Compute the probability that if 10 married couples are seated at random at a round table, then no wife sits next to her husband. **[03 Marks]**

   (b) Write down the probability density function f of a general normal random variable X. Prove that f is indeed a probability density function. Additionally, find expectation and variance of the random variable X. **[05 Marks]**

2. (a) If random variables X and Y are independent and uniformly distributed on $(0,1)$, then calculate the probability density function of $X+Y$. **[04 Marks]**

   (b) The joint probability density function of random variables X and Y is given as

   $$f(x,y) = \begin{cases} 6(1-x), & \text{if } 0 < y < x, 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

   Determine the probability density function of random variables X and Y. Are X and Y independent? Give reasons to your answers. **[04 Marks]**

3. (a) Let X and Y be random variables with the joint pmf $P\{X=i, Y=j\} = \frac{1}{N^2}, i,j = 1,2,\cdots,N$. Find the mean of the random variable $\min\{X,Y\}$. **[03 Marks]**

   (b) State the Jensen's Inequality. Hence, show that if the moment of order $q > 0$ exists for a random variable X, then moments of order p, where $0 < p < q$ exist. **[03 Marks]**

   (c) Let $X \sim B(n,p)$. Estimate $P(X \geq \alpha n)$, where $p < \alpha < 1$, using the Chebyshev's inequality. **[03 Marks]**

4. (a) Find the characteristic function of a Poisson random variable with the parameter $\lambda > 0$. State the uniqueness theorem. Hence, prove that sum of two independent Poisson random variable is a Poisson random variable. **[04 Marks]**

   (b) Let $X_1, X_2, \cdots, X_{25}$ be i.i.d. random variable with the following probability mass function

   $$f(t) = \begin{cases} 0.6 & \text{if } t = 1, \\ 0.4 & \text{if } t = -1, \\ 0 & \text{otherwise.} \end{cases}$$

   Let $Y = X_1 + X_2 + \cdots + X_{25}$. Using the central limit theorem compute approximate value of $P(4 \leq Y \leq 6)$. Use the standard normal table given in the end. **[04 Marks]**

5. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from the population with the following probability density function:

$$f(x) = \begin{cases} \theta\left(x - \frac{1}{2}\right) + 1, & 0 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

where, $\theta \in [-2, 2]$ is an unknown parameter. We define the estimator $\hat{\Theta}_n$ as $\hat{\Theta}_n = 12\overline{X} - 6$ to estimate θ, where $\overline{X}$ is the sample mean.

(i) Is $\hat{\Theta}_n$ an unbiased estimator of θ?

(ii) Find the mean squared error (MSE) of $\hat{\Theta}_n$.

(ii) Is $\hat{\Theta}_n$ a consistent estimator of θ?

[6 marks]

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample taken from a population with the following density function

$$f(x) = \begin{cases} e^{\lambda - x}, & x > \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Define $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$. Find $P\left(X_{(1)} > 3\lambda\right)$.

[3 marks]

6. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from a geometric distribution with the parameter p. Find the Maximum Likelihood Estimator (MLE) of p.

[4 marks]

(b) Define the pivotal quantity. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\theta, 9)$. Check the random variables $Q_1 = \overline{X} - \theta$ and $Q_2 = \dfrac{\sqrt{n}(\overline{X} - \theta)}{3}$ for pivotal quantity.

[4 marks]

TABLE 1 Values of the standard normal distribution function

x	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5363	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8204	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9430	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9648	.9656	.9664	.9671	.9678	.9686	.9693	.9700	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9762	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9874	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.	.9987	.9990	.9993	.9995	.9997	.9998	.9998	.9999	.9999	1.0000