

Problem Set 3 - Solutions

1. [HMU], Exercise 2.3.2. Convert to a DFA the following NFA:

NFA:

State	0	1
$\rightarrow p$	$\{q, s\}$	$\{q\}$
$*q$	$\{r\}$	$\{q, r\}$
r	$\{s\}$	$\{p\}$
$*s$	ϕ	$\{p\}$

DFA:

State	0	1
$\rightarrow \{p\}$	$\{qs\}$	$\{q\}$
$*\{q\}$	$\{r\}$	$\{qr\}$
$\{r\}$	$\{s\}$	$\{p\}$
$*\{s\}$	ϕ	$\{p\}$
ϕ	ϕ	ϕ
$*\{qs\}$	$\{r\}$	$\{pqr\}$
$*\{qr\}$	$\{rs\}$	$\{pqr\}$
$*\{rs\}$	$\{s\}$	$\{p\}$
$*\{pqr\}$	$\{qrs\}$	$\{pqr\}$
$*\{qrs\}$	$\{rs\}$	$\{pqr\}$

2. [HMU], Exercise 2.3.4. Give nondeterministic finite automata to accept the following languages. Try to take advantage of nondeterminism as much as possible.

- (a) The set of strings over alphabet $\{0,1,\dots,9\}$ such that the final digit has appeared before.

Define an NFA $N_a = (Q, \Sigma, \delta, q_s, F)$, where

- $Q = \{q_s, q_f, q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$;
- $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$;
- $F = \{q_f\}$;
- δ as shown in the following transition table:

State	0	1	2	3	4	5	6	7	8	9
$\rightarrow q_s$	$\{q_s, q_0\}$	$\{q_s, q_1\}$	$\{q_s, q_2\}$	$\{q_s, q_3\}$	$\{q_s, q_4\}$	$\{q_s, q_5\}$	$\{q_s, q_6\}$	$\{q_s, q_7\}$	$\{q_s, q_8\}$	$\{q_s, q_9\}$
q_0	$\{q_f, q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$
q_1	$\{q_1\}$	$\{q_f, q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$
q_2	$\{q_2\}$	$\{q_2\}$	$\{q_f, q_2\}$	$\{q_2, 0\}$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$
q_3	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	$\{q_f, q_3\}$	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$
q_4	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$	$\{q_f, q_4\}$	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$
q_5	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$	$\{q_f, q_5\}$	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$
q_6	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$	$\{q_f, q_6\}$	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$
q_7	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_f, q_7\}$	$\{q_7\}$	$\{q_7\}$
q_8	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_f, q_8\}$	$\{q_8\}$
q_9	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_f, q_9\}$
$*q_f$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

(b) The set of strings over alphabet $\{0,1,\dots,9\}$ such that the final digit has *not* appeared before.

Define an NFA $N_b = (Q, \Sigma, \delta, q_s, F)$, where

- $Q = \{q_s, q_f, q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$;
- $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$;
- $F = \{q_f\}$;
- δ as shown in the following transition table ($Q_s^i = Q - \{q_s, q_i\}$):

State	0	1	2	3	4	5	6	7	8	9
$\rightarrow q_s$	Q_s^0	Q_s^1	Q_s^2	Q_s^3	Q_s^4	Q_s^5	Q_s^6	Q_s^7	Q_s^8	Q_s^9
q_0	$\{q_f\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$	$\{q_0\}$
q_1	$\{q_1\}$	$\{q_f\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$
q_2	$\{q_2\}$	$\{q_2\}$	$\{q_f\}$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$
q_3	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	$\{q_f\}$	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$
q_4	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$	$\{q_f\}$	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$
q_5	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$	$\{q_f\}$	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$
q_6	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$	$\{q_f\}$	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$
q_7	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$	$\{q_f\}$	$\{q_7\}$	$\{q_7\}$
q_8	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$	$\{q_f\}$	$\{q_8\}$
q_9	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$	$\{q_f\}$
$*q_f$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

(c) The set of strings of 0's and 1's such that there are two 0's separated by a number of positions that is a multiple of 4. Note that 0 is an allowable multiple of 4.

Define an NFA $N_c = (Q, \Sigma, \delta, q_s, F)$, where

- $Q = \{q_s, q_f, q_0, q_1, q_2, q_3\}$;
- $\Sigma = \{0, 1\}$;
- $F = \{q_f\}$;
- δ as shown in the following transition table:

State	0	1
$\rightarrow q_s$	$\{q_s, q_0\}$	$\{q_s\}$
q_0	$\{q_f\}$	$\{q_1\}$
q_1	$\{q_2\}$	$\{q_2\}$
q_2	$\{q_3\}$	$\{q_3\}$
q_3	$\{q_0\}$	$\{q_0\}$
$*q_f$	$\{q_f\}$	$\{q_f\}$

3. [HMU], Exercise 2.5.3. Design ϵ -NFA's for the following languages. Try to use ϵ transitions to simplify your design.

- (a) The set of strings consisting of zero or more a's followed by zero or more b's, followed by zero or more c's.

Define an ϵ -NFA $N_a = (Q, \Sigma, \delta, q_s, F)$, where

- $Q = \{q_s, q_a, q_b, q_c\}$;
- $\Sigma = \{a, b, c\}$;
- $F = \{q_c\}$;
- δ as shown in the following transition table:

State	ϵ	a	b	c
$\rightarrow q_s$	$\{q_a\}$	ϕ	ϕ	ϕ
q_a	$\{q_b\}$	$\{q_a\}$	ϕ	ϕ
q_b	$\{q_c\}$	ϕ	$\{q_b\}$	ϕ
$*q_c$	ϕ	ϕ	ϕ	$\{q_c\}$

- (b) The set of strings that consist of either 01 repeated one or more times or 010 repeated one or more times.

Define an ϵ -NFA $N_b = (Q, \Sigma, \delta, q_s, F)$, where

- $Q = \{q_s, q_1, q_2, q_3, q_4, q_5, q_6, q_7\}$;
- $\Sigma = \{0, 1\}$;
- $F = \{q_3, q_7\}$;
- δ as shown in the following transition table:

State	ϵ	0	1
$\rightarrow q_s$	$\{q_1, q_4\}$	ϕ	ϕ
q_1	ϕ	$\{q_2\}$	ϕ
q_2	ϕ	ϕ	$\{q_3\}$
$*q_3$	$\{q_1\}$	ϕ	ϕ
q_4	ϕ	$\{q_5\}$	ϕ
q_5	ϕ	ϕ	$\{q_6\}$
q_6	ϕ	$\{q_7\}$	ϕ
$*q_7$	$\{q_4\}$	ϕ	ϕ

- (c) The set of strings of 0's and 1's such that at least one of the last ten positions is a 1.

Define an ϵ -NFA $N_c = (Q, \Sigma, \delta, q_s, F)$, where

- $Q = \{q_s, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}\}$;
- $\Sigma = \{0, 1\}$;
- $F = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}\}$;
- δ as shown in the following transition table:

State	ϵ	0	1
$\rightarrow q_s$	ϕ	$\{q_s\}$	$\{q_s, q_1\}$
$*q_1$	$\{q_2\}$	$\{q_2\}$	$\{q_2\}$
$*q_2$	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$
$*q_3$	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$
$*q_4$	$\{q_5\}$	$\{q_5\}$	$\{q_5\}$
$*q_5$	$\{q_6\}$	$\{q_6\}$	$\{q_6\}$
$*q_6$	$\{q_7\}$	$\{q_7\}$	$\{q_7\}$
$*q_7$	$\{q_8\}$	$\{q_8\}$	$\{q_8\}$
$*q_8$	$\{q_9\}$	$\{q_9\}$	$\{q_9\}$
$*q_9$	$\{q_{10}\}$	$\{q_{10}\}$	$\{q_{10}\}$
$*q_{10}$	ϕ	ϕ	ϕ

4. It was mentioned in class that if L_1 and L_2 are regular languages, then $L_1 \cup L_2$ is regular. A proof was suggested along the following lines. Let $L_1 = L(M_1)$ and $L_2 = L(M_2)$ for two DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_1^0, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2^0, F_2)$. Construct a new ϵ -NFA as follows: $M = (\{q_0\} \cup Q_1 \cup Q_2, \Sigma, \delta, q_0, F_1 \cup F_2)$, where

$$\delta(q, a) = \begin{cases} \phi & \text{if } q \neq q_0 \text{ and } a = \epsilon, \\ \{\delta_1(q, a)\} & \text{if } q \in Q_1 \text{ and } a \neq \epsilon, \\ \{\delta_2(q, a)\} & \text{if } q \in Q_2 \text{ and } a \neq \epsilon, \\ \{q_0^1, q_0^2\} & \text{if } q = q_0 \text{ and } a = \epsilon, \\ \phi & \text{if } q = q_0 \text{ and } a \neq \epsilon. \end{cases} \quad (1)$$

Observe that $\hat{\delta}(q_0, \epsilon) = \{q_0, q_0^1, q_0^2\}$. Prove by induction that $\hat{\delta}(q_0, w) = \{\hat{\delta}_1(q_0^1, w), \hat{\delta}_2(q_0^2, w)\}$ for any $w \neq \epsilon$. Conclude that this machine indeed accepts the union of the languages L_1 and L_2 .

Proof:

The proof is in two parts: the basis and the inductive step; we prove each in turn.

BASIS: If $|w| = 1$,

$$\begin{aligned} \hat{\delta}(q_0, w) &= \bigcup_{p \in \hat{\delta}(q_0, \epsilon)} \delta(p, w) \\ &= \delta(q_0, w) \bigcup \delta(q_0^1, w) \bigcup \delta(q_0^2, w) \text{ (By the above observation)} \\ &= \phi \bigcup \{\delta_1(q_0^1, w)\} \bigcup \{\delta_2(q_0^2, w)\} \text{ (By def. of } \delta) \\ &= \{\delta_1(q_0^1, w), \delta_2(q_0^2, w)\} \\ &= \{\hat{\delta}_1(q_0^1, w), \hat{\delta}_2(q_0^2, w)\} \end{aligned}$$

Thus, it is true when $|w| = 1$.

INDUCTION: Now, suppose it is true for $|\tilde{w}| \leq n$. We need to show that it is true for $|w| = n + 1$. Let $w = xa, |x| = n, a \in \Sigma$. We have:

$$\begin{aligned} \hat{\delta}(q_0, w) &= \delta(\hat{\delta}(q_0, x), a) \\ &= \delta(\hat{\delta}_1(q_0^1, x), a) \bigcup \delta(\hat{\delta}_2(q_0^2, x), a) \text{ (By IH)} \\ &= \{\delta_1(\hat{\delta}_1(q_0^1, x), a)\} \bigcup \{\delta_2(\hat{\delta}_2(q_0^2, x), a)\} \\ &\quad (\delta_1(q_0^1, x) \in Q_1, \delta_2(q_0^2, x) \in Q_2) \\ &= \{\hat{\delta}_1(q_0^1, xa), \hat{\delta}_2(q_0^2, xa)\} \text{ (By def. of } \hat{\delta}) \\ &= \{\hat{\delta}_1(q_0^1, w), \hat{\delta}_2(q_0^2, w)\} \end{aligned}$$

Based on the basis and the inductive step, we know that it is true for any $w \neq \epsilon$.

If $w \in L_1 \cup L_2$, $\hat{\delta}_1(q_0^1, w) \cap F_1 \neq \phi$ or $\hat{\delta}_2(q_0^2, w) \cap F_2 \neq \phi$. Hence, $\hat{\delta}(q_0, w) \cap (F_1 \cup F_2) \neq \phi$, which means w is recognized by this new machine. On the other hand, if $w \notin L_1 \cup L_2$, $\hat{\delta}(q_0, w) \cap (F_1 \cup F_2) = \phi$. So w is rejected by this machine. We conclude that this machine accepts the union of the two languages.

5. Prove that any finite language is regular. (Hint: Start by showing that any language consisting of a single string is regular.)

Proof:

BASIS: $\phi, \{\epsilon\}$ are regular. For any language consisting of a single string $w \neq \epsilon$, we can construct a DFA to recognize it. Given $w = w_1 \dots w_n$, define a DFA $M_d = (Q, \Sigma, \delta, q_0, \{q_n\})$, where:

- $Q = \{q_0, q_1, q_2, \dots, q_n, q_d\};$
-

$$\delta(q, a) = \begin{cases} q_d & \text{if } q = q_{i-1} \text{ and } a \neq w_i, \\ q_i & \text{if } q = q_{i-1} \text{ and } a = w_i, \\ q_d & \text{if } q \in q_d, q_n \text{ and } a \neq \epsilon, \end{cases} \quad (2)$$

Given M_d , we can check that $\hat{\delta}(q_0, w) = q_n$. For any other string $v \neq w$, assume they are different at position i ($v_i \neq w_i$), $\hat{\delta}(q_0, v) = q_d$ (if $v_i \neq \epsilon$), $\hat{\delta}(q_0, v) = q_i$ (if $|v| = i - 1$). In both cases, v is rejected. Therefore, M_d recognize this language. Such a language is regular.

INDUCTION: Now, assume that any finite language L with $|L| \leq n$ is regular. We need to show that any finite language L' consisting of $n + 1$ strings is regular too. Randomly, we pick a string from L' , say w . We put w in a set, which corresponds the language $\{w\}$, and $L' - \{w\}$ will be another finite language, denoted by L_1 . We have that $L' = L_1 \cup \{w\}$. Since both L_1 and $\{w\}$ are finite, they are regular (by IH). Based on the conclusion given in the previous question, we know that L' is also regular.

Therefore, any finite language is regular.

6. For a language L , let $L^R = \{w_n \dots w_1 | w_1 \dots w_n \in L\}$ be the reverse of L ; it consists of all strings obtained by reversing a string in L . Show that if L is regular, then L^R is regular. (Hint: there are two natural approaches to this problem: (a.) given a DFA for L , construct an NFA for L^R and prove that, indeed, the NFA accepts the right language, or (b.) given a regular expression for L , construct a regular expression for L^R and prove that this denotes the correct language.)

Proof:

If L is regular, there is a DFA $M_L = (Q, \Sigma, \delta, q_0, F)$. Given M_L , we construct an NFA $N^R = (Q \cup q_0^r, \Sigma, \delta_r, q_0^r, q_0)$, where:

$$\delta_r(q, a) = \begin{cases} F & \text{if } q = q_0^r \text{ and } a = \epsilon, \\ \{p | \delta(p, a) = q\} & \text{if } q \neq q_0^r \text{ and } a \neq \epsilon, \\ \phi & \text{if } q = q_0^r \text{ and } a \neq \epsilon, \end{cases} \quad (3)$$

Claim 1 For any w , $\hat{\delta}(q_0, w) = p \Leftrightarrow q_0 \in \hat{\delta}_r(p, w_r)$, where $p \in Q$ and w_r is obtained by reversing w .

BASIS: If $w = \epsilon$, $w_r = \epsilon$, $\hat{\delta}(q_0, \epsilon) = q_0$ and $q_0 \in \hat{\delta}_r(q_0, \epsilon)$.

INDUCTION: Assume that for any w ($|w| \leq n$), it is true. Consider v ($|v| = n + 1$). Let $v = xa$ ($|x| = n$). By IH, there is p , $\hat{\delta}(q_0, x) = p$ and $q_0 \in \hat{\delta}_r(p, x_r)$, where x_r is obtained by reversing x . Now, suppose $\delta(p, a) = q$ ($q \in Q$), by definition of δ_r , we have $p \in \delta_r(q, a)$. Then, $\hat{\delta}_r(q, ax_r) = \hat{\delta}_r(\delta_r(q, a), x_r) \Rightarrow \hat{\delta}_r(p, x_r) \subseteq \hat{\delta}_r(q, ax_r) \Rightarrow q_0 \in \hat{\delta}_r(p, x_r) (\subseteq \hat{\delta}_r(q, ax_r)) \Rightarrow q_0 \in \hat{\delta}_r(q, ax_r)$. The other direction is similar.

Therefore, the claim is true.

Now, if $w \in L$, $\hat{\delta}(q_0, w) \cap F \neq \phi$. Let $\hat{\delta}(q_0, w) = q_f (\in F)$. By our claim, we have $q_0 \in \hat{\delta}_r(q_f, w_r)$. Since $\delta_r(q_0^r, \epsilon) = F$, $q_0 \in \hat{\delta}_r(q_0^r, w_r)$ and w_r is accepted by N^R . If $w_r \in L^R$, $q_0 \in \hat{\delta}_r(q_0^r, w_r)$. By tracing the path of N^R , from q_0^r , an ϵ transition brings N^R into a state q_f in Q ($q_f \in F$). Hence, $q_0 \in \hat{\delta}_r(q_f, w_r)$. By our claim,

$\hat{\delta}(q_0, w) = q_f$, which means $w \in L$. Therefore N^R accepts the right language L^R . We can conclude that L^R is a regular language.