Classical Mechanics

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Calculation of the shortest length in a 2D Eucledian Space:

$$l = \int \sqrt{dx^2 dy^2}$$

$$l = \int \sqrt{1 + y'^2}$$

$$\delta l = \delta \int \sqrt{1 + y'^2}$$

$$l \to l + O(\delta y^2)$$

Energies:

Potential Energy

Energy associated with the body/particle due to virtue of its position or configuration.

Kinetic Energy

Energy associated with the body/particle due to virtue of its motion.

Generalized Coordinates:

Independent Coordinates

Action Priciple: Euler-Lagrange Equation

Action is defined by $S = \int_a^b \mathcal{L}(q, \dot{q}, t)dt$ where $\mathcal{L}$ is called the Lagrangian and is defined to be a function of q, $\dot{q}$, and t.

Suppose, if we assume Action be a line and if we introduce infintisimal variation in it, then the variation of the Action will only be Zero for extremum positions.

$$\delta S = \delta \int \mathcal{L}(q, \dot{(q)}, t)dt$$

Since, δ and $\int$ commute, the position of both can be interchanged.

$$\delta S = \int \delta \mathcal{L}(q, \dot{(q)}, t)dt$$

From the first principles,

$$\delta S = \int dt \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q} + \frac{\partial \mathcal{L}}{\partial t}\delta t)$$

Assuming, the δt is 0.

$$\delta S = \int dt \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q}$$

But, $\delta S = 0$

$$\therefore \int dt \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \ Lagr}{\partial \dot{q}}\delta \dot{q} = 0$$

$$\int dt \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial\ Lagr}{\partial \dot{q}}\delta \underset{=}{\frac{dq/dt}{}} 0$$

Since, δ and $\frac{d}{dt}$ commute,

$$\int dt \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial\ Lagr}{\partial \dot{q}}\frac{d}{dt}\delta q = 0$$

$$Using,\ \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d}{dt}\delta q = \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q) - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}})\delta q$$

$$\int dt \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q) - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}})\delta q = 0$$

For Integral from a to b, $\int_a^b dt\delta q = 0$, then

$$\int_a^b dt \frac{\partial \mathcal{L}}{\partial q}\delta q - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}})\delta q = 0$$

Diff both sides by $\frac{d}{dt}$,

$$\frac{\partial \mathcal{L}}{\partial q}\delta q - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}})\delta q = 0 \tag{1}$$

Obtained is nothing but the Euler-Lagrange Equation.
Suppose, $\mathcal{L}$ is not a functin of q(Generally, the potential term is zero),

$$\implies \frac{\partial \mathcal{L}}{\partial q} = 0$$

$$\therefore \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

$$\implies \frac{\partial \mathcal{L}}{\partial \dot{q}} = k\ \ where\ k\ is\ a\ constant$$

Suppose, $\mathcal{L}$ is not a function of $\dot{q}$(Generally, the Kinetic Energy term is Zero), then

$$\ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{\partial L}{\partial q} - \frac{d}{dt}(\dot{q}\frac{\partial L}{\partial \dot{q}}) = 0$$

Conditional Variation:

Suppose $I = \int_a^b F(y, y', x)dx$ with Constrint $J = \int_a^b G(y, y', x)dx$, then,

Consider, $K = I + \lambda J$

$$\therefore K = \int_a^b (F(y, y', x) + \lambda G(y, y', x))dx$$

Since, constrain (J) is a fixed function, then $\delta J = 0$.

$$\implies \delta K = \delta I$$

$$\mathcal{L} = F(y, y', x) + \lambda G(y, y', x)$$

where λ is underdetermined Lagrangian constant.

Conservation Laws and Symmetries:

Nother's Theorem For every symmetry, there always exists a conserved quantity.
If a $\mathcal{L}$ does not expcilitly depend on a coordinate $\rightarrow$ Conservation Law.

Translation Symmetry:

Suppose, a particle is at q_i with a velocity of $\dot{q}_i$ at t and has $\mathcal{L} = k{\dot{q}_i}^2$. Consider a coordinate transformation where $q_i^* = q_i + \delta$, where δ is an infintisimal variation, then

$$q_i^* = q_i + \delta$$
$$\dot{q_i^*} = \dot{q}_i$$
$$\therefore \delta\mathcal{L} = 0$$

By applying the Euler-Lagrange Eq.,

$$\frac{\partial L}{\partial \dot{q}_i} = 2k\dot{q}_i = 0$$
$$\implies \dot{P}_i = 0$$

$$P_i = Constant$$

Thus, Linear Momentum is conserved.

Rotational Symmetry:

Suppose, a particle is at x,y has a $\mathcal{L} = k(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2)$, where $k = \frac{1}{2}m$
Consider a rotation by δ angle, then

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tential

$$x' = x\cos\delta + y\sin\delta$$
$$y' = -x\sin\delta + y\cos\delta$$

Assuming, δ to be infintisimal, then

$$\sin\delta \sim \delta \; and \; \cos\delta \sim 1$$

$$\therefore x' = x + y\delta \; and \; y' = -x\delta + y$$

Hence, the variation in x is $\delta x = y\delta$ and variation in y is $\delta y = -x\delta$

$$\delta\dot{x} = y\delta \; and \; \delta\dot{y} = -x\delta$$

Therefore, the variation in $x^2 + y^2$is

$$\begin{aligned} \delta(x^2 + y^2) &= 2x\delta x + 2y\delta y \\ &= 2xy\delta - 2xy\delta \\ &= 0 \end{aligned} \tag{2}$$

Hence, the variation in V should also be 0.
For Variation in the Kinetic Term,

$$\begin{aligned} \delta(\dot{x^2} + \dot{y^2}) &= 2\dot{x}\delta\dot{x} + 2\dot{y}\delta\dot{y} \\ &= 2\dot{x}\dot{y}\delta - 2\dot{x}\dot{y}\delta \\ &= 0 \end{aligned} \tag{3}$$

Hence, the variation is Kinetic Term is also Zero.
Therefore, Variation in $\mathcal{L}$ is Zero.
Thus, for infinitisimal rotation of the system, the Lagrangian is Invarient.

Generalization:

Assuming, q_i and $\dot{q}_i$ as the generalized coordinates.
Suppose, a variation is brought in q,

$$\delta q = f_i(q)\delta$$

where δ is nothing but a small change, and is a constant.
Consider $\epsilon = \delta$ to reduce the clutter in notation.

It is to be noted that this infintisimal change can be stacked up to form a finite change, still these eqs will hold up, since, if $V(x, y)$ is the potential and $\delta V = \frac{\partial V}{\partial x}\delta x = 0, \implies \frac{\partial V}{\partial x}$ is 0 at all points.

Now,

$$\delta q = f_i(q)\epsilon \tag{4}$$

$$\delta \dot{q} = \frac{d}{dt}(f_i(q)\epsilon) \tag{5}$$

Then,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i + \frac{\partial\mathcal{L}}{\partial t}\delta t$$

Assuming, $\mathcal{L}$ is Independent of time. (Assuming time symmetry)

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i$$

Since, $\frac{\partial\mathcal{L}}{\partial\dot{q}}$ is nothing but the cannonical conjugate of Momentum and $\frac{\partial\mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}}$(1)

$$\delta\mathcal{L} = \dot{P}_i\delta q_i + P_i\delta\dot{q}_i \tag{6}$$

Using, (4) and (5)

$$\delta\mathcal{L} = \dot{P}_i f_i(q)\epsilon + P_i\frac{d}{dt}(f_i(q)\epsilon)$$

$$\delta\mathcal{L} = \epsilon\frac{d}{dt}(P_i f_i(q))$$

Assuming, there is a symmetry, then $\delta\mathcal{L} = 0$,

$$\implies \epsilon\frac{d}{dt}P_i f_i(q) = 0$$

$$\frac{d}{dt}P_i f_i(q) = 0$$

Therefore, the **conserved quantity** is,

$$Q = P_i f_i(q) \tag{7}$$

Time Symmetry:

When $\mathcal{L}$ doesn't expcilitly depend upon t, or $\mathcal{L} = \sum_i \mathcal{L}(q_i, \dot{q}_i)$). ($\sum_i$ is ommited in later text to reduce the clutter)

$$\frac{d\mathcal{L}}{dt} = \frac{\partial\mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\ddot{q}_i$$

But, since $P_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$ and $\dot{P}_i = \frac{\partial\mathcal{L}}{\partial q_i}$,

$$\frac{d\mathcal{L}}{dt} = \dot{P}_i q_i + P_i\ddot{q}_i$$

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}(P_i\dot{q}_i)$$

Therefore, the conserved quantity is,

$$\frac{d}{dt}(\mathcal{L} - P_i\dot{q}_i) = 0 \tag{8}$$

which is nothing but the total energy, called Hamiltonian.

$$-H = \mathcal{L} - P_i\dot{q}_i$$
$$\implies H = P_i\dot{q}_i - \mathcal{L} \tag{9}$$

Hamiltonian of a particle moving linearly in a potential field:

For such a particle, $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$, where $V(x)$ is the potential energy at position x.

$$\therefore H = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + V(x)$$
$$\implies H = \frac{1}{2}m\dot{x}^2 + V(x)$$

Assuming Time Dependence:

Suppose, the $\mathcal{L}$ depends upon time.

$$\therefore \frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$
$$\implies \frac{d\mathcal{L}}{dt} = \frac{d}{dt}(P_i\dot{q}_i) + \frac{\partial \mathcal{L}}{\partial t}$$
$$\frac{d\mathcal{L} - P_i\dot{q}_i}{dt} = \frac{\partial \mathcal{L}}{\partial t}$$
$$\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t} \tag{10}$$

Lagrangian in a Rotational Transformation:

Suppose, a particle is at x,y has a $\mathcal{L} = k(\dot{x}^2 + \dot{y}^2)$, where $k = \frac{1}{2}m$.
Consider a coordinate system, x', y', with the same origin but rotating with ω angular velocity.
Therefore, for the transformation eqs. will be,

$$x' = x\cos\omega t + y\sin\omega t \tag{11}$$
$$y' = -x\sin\omega t + y\cos\omega t \tag{12}$$

Using (11) and (12),

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\omega^2(x^2 + y^2) + m\omega(\dot{x}y - x\dot{y}) \tag{13}$$

Since, $\mathcal{L} = T - V$, the first term refers to the original $\mathcal{L}$ which is nothing but the Kinetic Energy, the second terms seems to be similar to Potential Energy and the third term corresponds to Correlios Force.

Hamiltonian:

Consider a $\mathcal{L} = \mathcal{L}(q, \dot{(q)}, t)$.
Then, then H is defined by,

$$H = \sum_i P_i\dot{q}_i - \mathcal{L} \tag{14}$$

For small variation in H,

$$\delta H = \frac{\partial H}{\partial P}\delta P + \frac{\partial H}{\partial q}\delta q \tag{15}$$
$$\delta H = \sum_i \delta P_i\dot{q}_i - \delta\mathcal{L}$$

$$\delta H = \sum_i \delta(P_i)\dot{q}_i + P_i\delta(\dot{q}_i) - \frac{\partial \mathcal{L}}{\partial q_i}\delta q_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta \dot{q}_i$$

Using (6), the $2nd$ and $4th$ turn cancels each other out,

$$\delta H = \sum_i \dot{q}_i \delta(P_i) - \dot{P}_i \delta(q_i)$$

Comparing this equation, with (15), (Dropping $\sum$ notation to increase clarity.)

$$\frac{\partial H}{\partial P} = \dot{q} \tag{16}$$

$$\frac{\partial H}{\partial q} = -\dot{P} \tag{17}$$

Note: The no. of Hamiltonian equations is double the Lagrangian equations.
The Hamiltonian eqs are $1st$ Order Differential Equations where as the Lagrangian eqs are $2nd$ Order.