COSMOLOGY IN $f(Q)$ GRAVITY

Submitted in partial fulfillment for the award of the degree of

Master of Science

(FIVE YEARS INTEGRATED PROGRAMME)

in

PHYSICS

by

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June 2023

The important thing is to not stop questioning.

Albert Einstein

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APPROVAL

This is to certify that the work contained in this project report entitled **"Cosmology in $f(Q)$ gravity"** submitted by **Devansh Shukla (I18PH021)**, is approved for partial fulfillment for the award of the degree of **Five Years Integrated Master of Science** in **Department of Physics** at Sardar Vallabhbhai National Institute of Technology, Surat.

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CERTIFICATE

This is to certify that the dissertation entitled "**Cosmology in f(Q) gravity**", is a bona fide record of research work done by **Devansh Shukla** from Sardar Vallabhbhai National Institute of Technology, Surat. He carried out the study reported in this dissertation, independently under my supervision and guidance. I also certify that the subject matter of the dissertation has not formed the basis for the award of any degree or diploma of any university or institution.

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DECLARATION

I, **Devansh Shukla (I18PH021)**, hereby declare that, this report entitled **"Cosmology in $f(Q)$ gravity"** submitted to Sardar Vallabhbhai National Institute of Technology Surat towards the partial requirement of **Five Years Integrated Master of Science** in **Department of Physics**, is an original work carried out by me and has not formed the basis for the award of any degree or diploma, in this or any other institution or university before.

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Abstract

Albert Einstein's General Relativity has been quite successful in explaining multiple phenomena but it falls short in explaining the cosmic acceleration of the Universe, galaxy rotation curves and galaxy velocities without the inclusion of the mysterious "dark" matter. This provides a motivation to investigate other avenues such as modified gravity theories.

In this work, we will investigate in detail the motivations for modified gravity theories and look into the ideas for $f(\mathcal{Q})$ gravity. Furthermore, we will look into the cosmological perturbation theory and work towards developing the it and the Einstein's field equations for $f(\mathcal{Q})$ gravity. We will also develop a cosmological model using dust matter evolution in $f(\mathcal{Q})$ gravity.

Acknowledgement

I take an immense pleasure in acknowledging the kind help and guidance provided by **Prof. Dr. Salvatore Capozziello** and **Prof. Dr. Kamlesh Pathak** throughout the course of the project. Words cannot express my gratitude to my supervisors. I certainly wouldn't be able to complete this project without the help of them.

I would also like to thank my dog for all the entertainment and emotional support.

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Conventions

Unless explicitly said otherwise, throughout this thesis we use the following conventions:

For metric signature, connection, covariant derivative, curvature tensors and Lie derivative we follow the conventions of Misner, Thorne and Wheeler [1]. Explicitly, the metric signature is the "mostly plus" one

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \dots, +1) \tag{1}$$

so for example a space-like unit vector n has positive norm ($n_a n^a = +1$). In a metric manifold with metric g we will always use the unique symmetric connection compatible with the metric (Levi-Civita connection). The sign convention for the covariant derivative associated to the connection is

$$\nabla_\mu V^\nu = \partial_\mu + \Gamma^\nu{}_{\mu\lambda} V^\lambda \qquad \nabla_\mu V_\nu = \partial_\mu + \Gamma^\lambda{}_{\mu\nu} V_\lambda \tag{2}$$

and the Riemann curvature tensor is defined as:

$$\mathcal{R}^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\mu\lambda}\Gamma^\lambda{}_{\beta\nu} - \Gamma^\alpha{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\beta} \tag{3}$$

while the Ricci curvature tensor is defined as:

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^\alpha{}_{\mu\alpha\nu} \tag{4}$$

The sign convention for the Einstein equation is:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{5}$$

We use throughout the text the (Einstein) convention of implicit summation on repeated indices, and we use units of measure where the speed of light has unitary value $c = 1$. The reduced 4D Planck mass is defined as $M_p = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}$ GeV

Abbreviations

Throughout the work we use the following abbreviations:

EFE:	Einstein field equation
GR:	General Relativity
FRLW:	Friedmann-Lemaitre-Robertson-Walker
QFT:	Quantum field theory
SM:	Standard model of Particle Physics
DM:	Dark matter
CDM:	Cold Dark matter
AdS:	Anti-de Sitter
dS:	de Sitter

Chapter 1

Introduction

Our Universe is a stunning place for different physics objects and phenomena. It is teeming with planets, stars, galaxies, and every other type of energy and matter.

According to the big band theory, space and time first appeared together 13.783 billion years ago, at least based on what we currently comprehend.

The Earth was positioned at the centre of some of the oldest cosmological models of the world created by Greek and Indian philosophers. However, Nicolaus Copernicus gathered some observations which suggested that Sun is actually at the center of our Solar System and Sir Issac Newton actually used the Copernicus's work to develop the law of universal gravitation [2].

These universal laws were quite successful, however, discoveries in the early 20th century by Hubble suggested that the space has been expanding at increasing rate concluded from the red-shifted electromagnetic radiations from far away.

The Big Bang theory states that as the cosmos has expanded, the starting energy and matter have gotten less dense. The movement of galaxies has also revealed that there is far more matter in the universe than can be explained by visible things like stars, galaxies, nebula's, and ISM. The unseen matter is coined as the dark matter. From the observations like Planck 2018, there are strong but indirect evidences that suggests it exists but we have yet to discover it directly.

The universe's most widely accepted model is the CDM. It suggests that dark energy, which is responsible for accelerating the expansion of space, makes up approximately $69.2\% \pm 1.2\%$ of the mass and energy in the cosmos, and that dark matter makes up around $25.8\% \pm 1.1\%$. Therefore, the proportion of ordinary (or "baryonic") matter in the cosmos is only $4.84\% \pm 0.1\%$ [4].

The Lambda-CDM model has vast popularity among some physicists, however, others suggest that there may be a better plausible explanation for the manifestation of gravity in our universe like the modified gravity theories.

Modified gravity theories are alternatives to Einstein's General Relativity which attempts to describe the phenomenon of gravity in slightly different ways. The MOGs typically modified the Einstein-Hilbert action to include features like non-metricity, scalar fields, curvature etc.

1.1 Modified Gravity

1.2 Introduction

Although Einstein's General Relativity(GR) has been quite successful, it falls short in explaining the cosmic acceleration of the universe, galaxy rotation curves and galaxy velocities without the inclusion of the mysterious "dark" matter, which has to be about six times the baryonic matter to account for the disparity [5, 6]. These arguments provide sufficient motivation to investigate modified gravity theories.

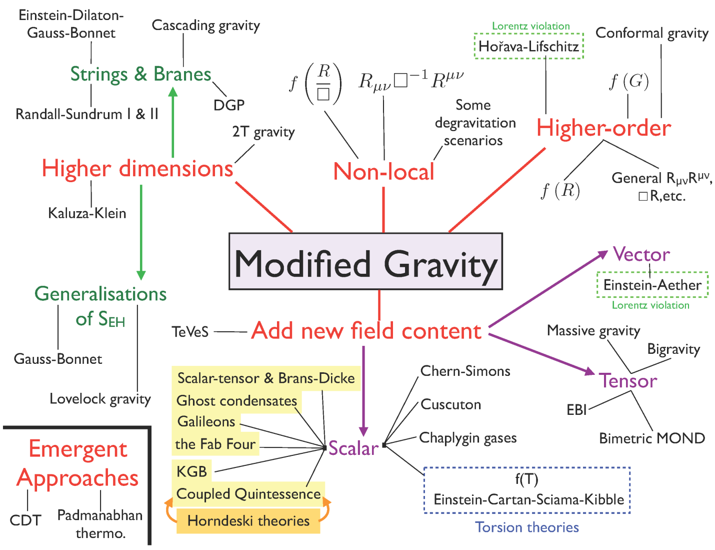

Figure 1.1. Illustration showing classification of modified gravity theory [1]

Figure 1.1 shows a lot of alternate theories of gravity along with there classification. In this work we will solely focus on the $f(Q)$ gravity under the Torsion theories umbrella.

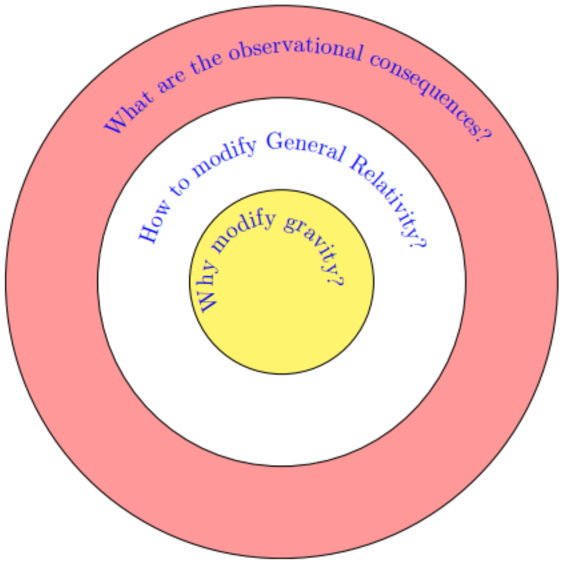

Figure 1.2. Illustration for developing modified gravity [2]

1.3 Incentive for modified theories of gravity

The standard model of cosmology called the ΛCDM, suggests that the universe is homogeneous and isotropic in the cosmological scales and is defined by the Friedmann–Lemaître–Robertson–Walker metric. It is a metric based on the exact solutions of the Einstein's field equations. The ΛCDM model makes the assumption that dark matter and dark energy are described by the cosmological constant Λ and that the gravity is explained by general relativity at cosmic scales. It is predicated that dark energy is in charge of the universe's accelerating expansion, whereas dark matter explains gravitational effects at cosmic scales. Despite its success in explaining wide range of cosmological observations, it still faces some challenges such as the nature of dark matter and the coincidence problem [2].

ΛCDM also faces some theoretical and observational challenges such as:

Hubble constant inconsistency: The computed value of the hubble's constant $\mathcal{H}_0$ using the FRLW metric differs significantly from the observations of the late universe. It shows a nearly $\sim 4.4\sigma$ variation [4], thus raising doubts about the validity of the model.

Flatness issue: ΛCDM model suggests that the universe should have zero curvature but the Planck Colloboration 2018 observations suggests that the universe is not flat but has a density parameter of $\Omega_k = 0.007 \pm 0.0019$ (+ve curvature).

Coincidence problem: Cosmological evidence implies that the dark energy density parameter's present value is $\Omega_\Lambda^{(0)} \sim 0.7$, which is on the same scale as the matter-energy density parameter's current value of $\Omega_m^{(0)} \sim 0.3$. Even though the energy density of matter (ρ_m) varies with time and that of dark energy (ρ_Λ) is constant, they are essentially the same order at the present, which looks strange and suggests that parameters in the early universe had to be adjusted [7].

Fine tuning problem: According to observations, the cosmological constant's present energy density is thought to be on the order of $\rho_\Lambda sim 10^{-47}; GeV$ [7–9]. This figure is in stark contrast to the vacuum energy density predicted by the quantum field theory, which has a value of $\rho_{vac} \sim 10^{74}\ GeV$ [10], assuming that the cosmological constant derives from a vacuum energy density. In other words, the value of ρ_Λ needs to be adjusted because it conflicts with the potential energy scales.

1.4 Recipe for modified gravity theories

In order to develop any theory of gravity, it must pass certain tests to consider it relevant. While some of these assessments are simple, others demand a close examination of the theory.

The theory must first be complete and capable of explaining gravity's behaviour at all conceivable scales and locations, including the scales of the neighbourhood playground and the Milky Way galaxy [11].

Secondly, the theory must be able to explain the experimental testing and observations. This includes tests such as gravitational redshift, deflection of light by gravity, precession of planetary orbits and shapiro delay. A theory of gravity may only be deemed credible if it passes these tests and is consistent with Newtonian and special relativity [11].

Furthermore, because we are using general relativity modifications, properties like Lorenz invariance and the equivalence principle are automatically inherited.

Chapter 2

Foundations of General Theory of Relativity

2.1 Introduction

General relativity is a gravitational theory developed by Albert Einstein during his golden years between 1907 and 1915. It claims that the observable gravitational effects involving masses are the product of spacetime wrapping. [12]

Sir Isaac Newton first proposed the theory of gravity, but it was unable to account for several events, such as the minute variations in Mercury's and other planets' orbits. Albert Einstein solved these minuscule anomalies via general relativity. In addition to anticipating novel effects like gravitational waves, gravitational lensing, and time dilation, it can explain the shortcomings of Newton's theory. It is the only current explanation of gravity that continues to make sense of observations.

While there are other relativistic theories that can explain gravity, Einstein's general relativity gives a fairly straightforward solution and has up to this point stayed consistent with the observational results.

2.2 Special to General Relativity

Initially, Albert Einstein published his theory of special relativty, which introduced a new framework to deal with effects of speed on mass, time and space. It included a coupling between mass and energy via the speed of light and stated that they are interchangeable. However, special relativty was more of an interpretation rather than a fully geometric theory of gravity like general relativity.

It is critical to first identify the properties that a relativistic theory of gravity must satisfy in order for it to be considered feasible. These include fundamental constraints such as free fall universality and space isotropy, as well as compatibility with a range of other measurements involving light propagation and massive body orbits.

2.3 Equivalence principle

The equivalence principles comes in two forms, **weak** and **strong** equivalence principle. The weak equivalence principle, which dates back to Galileo and Newton's time, stipulates that an object's "inertial mass" and "gravitational mass" are always equal to one other. As a result, the trajectories of particles in a gravitational field are constant regardless of their masses.

The strong equivalence principle, on the other hand, asserts that the laws of nature are the same in an analogous accelerating reference frame and a uniform static gravitational field. It implies that gravitational laws are independent of velocity and location.

The strong equivalence principle can be checked by looking for variations in Newton's gravitational constant G or, equivalently, variations in the masses of the fundamental particles over the course of the universe. G cannot have fluctuated by more than 10%, according to a number of independent constraints derived from Solar System orbits and Big Bang nucleosynthesis investigations [13].

2.4 Curvature

Einstein's brilliance was to propose that, while gravity manifests as a force, it is actually a product of the geometry of spacetime itself. He suggested that the matter wraps the spacetime. The sun, for example, encompasses spacetime, and it is the geometry to which the planets react rather than the sun itself. **This is a central tenent of the General theory of Relativity** [12].

This local curvature causes the geodesics to be "wonky" lines instead of the "straighest" possible lines. These predictions have been tested and they have been accurate to a very high degree [12]. This curvature can be described using tensors such as Riemannian curvature tensor, Ricci tensor, Ricci scalar, Weyl tensor etc.

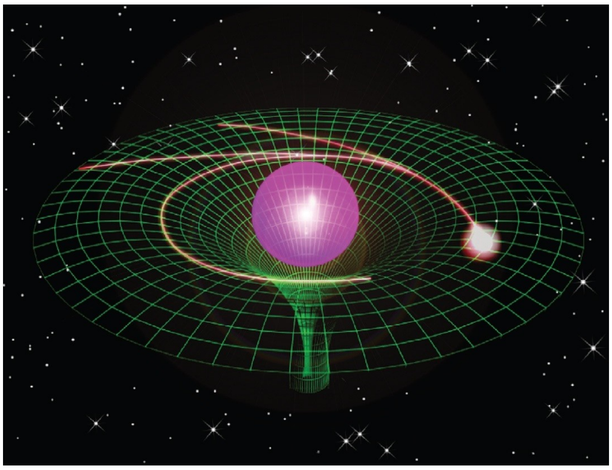

Figure 2.1. Artistic Representation for wrapping of the empty space around a massive star [3]

2.4.1 Covariant derivative

The partial derivate operator ∂_μ functions linearly on its arguments in Euclidean space and follows the Leibniz rule on tensor products [12]. However, when we move to four-dimensional space, the operator depends on the coordinate system being used, but all of this remains true in the more general scenario we would like to investigate. In order to accomplish the tasks of the partial derivative in curved spacetime in a manner independent of the coordinates, we define a **covariant** derivative operator ∇ [12].

It must satisfy two properties:

1. Linearity: $\nabla(A+B) = \nabla A + \nabla B$
2. Leibniz product rule: $\nabla(A \otimes B) = \nabla A \otimes B + A \otimes \nabla B$

In order to compute the derivative, we can write the new derivative as a linear combination of the bare partial derivative plus a correction term to make the overall result covariant because of the Leibniz rule.

Let us first compute the covariant derivative of a contravariant vector V^μ. The covariant derivative ∇_α is defined as the naked partial derivative of the vector plus a correction term given by the affine connection $(\Gamma^\lambda{}_{\alpha\beta})$ [14].

$$\nabla_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma^\mu{}_{\alpha\lambda} V^\lambda \tag{2.1}$$

In a similar manner, we can calculate the covariant derivative of a covariant vector V_ν, but in this instance, we use a new set of affine correction as the correction matrix to be $\widetilde{\Gamma}^{\alpha}_{\alpha\beta}$, because there is currently no requirement that the matrices representing contravariant and covariant must be identical.

$$\nabla_\alpha V_\mu = \partial_\alpha V_\mu + \widetilde{\Gamma}^{\lambda}_{\alpha\mu} V_\lambda \tag{2.2}$$

Trivially, the covariant derivative of a scalar must conform to the equations for covariant and contravariant vectors. Hence, to enforce this, we compute the derivative of a scalar $w = V^\mu V_\mu$:

$$\nabla_\alpha w = \nabla_\alpha (V^\mu V_\mu) \tag{2.3}$$

Utilizing the commutativity,

$$\nabla_\alpha w = \nabla_\alpha V^\mu V_\mu + V^\mu \nabla_\alpha V_\mu \tag{2.4}$$

$$\nabla_\alpha w = \left(\partial_\alpha V^\mu + \Gamma^{\mu}{}_{\alpha\lambda} V^\lambda\right) V_\mu + V^\mu \left(\partial_\alpha V_\mu + \widetilde{\Gamma}^{\lambda}_{\alpha\mu} V_\lambda\right) \tag{2.5}$$

In addition to the above properties, we would like our covariant derivative to reduce to partial derivative with scalar, considering scalars doesn't have any directions. Hence $\nabla_\alpha w = \partial_\alpha w$

$$\nabla_\alpha w = \partial_\alpha V^\mu V_\mu + V^\mu \partial_\alpha V_\mu \tag{2.6}$$

It is trivially straightforward to show that (2.5) and (2.5) are only equivalent when the two sets of affine connections have the same transformation properties but are negatives of each other.

Comparing (2.5) and (2.6) and modifying the dummy indices, we obtain:

$$\text{Loosely,} \quad \widetilde{\Gamma}^{\lambda}_{\mu\nu} = -\Gamma^{\lambda}{}_{\mu\nu} \tag{2.7}$$

Finally, we will compute the covariant derivative of the metric tensor $g_{\mu\nu}$ as:

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^{\lambda}{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^{\lambda}{}_{\alpha\nu} g_{\mu\lambda} \tag{2.8}$$

Now, we are at a cross-road, in order to develop the theory of gravitation, we can either take the vanilla torsion-free approach or newer curvature-free approach.

The Albert Einstein's general relativity uses the torsion free approach, which has two additional properties:

- Torsion-free:

$$T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{[\mu\nu]} = 0 \tag{2.9}$$

$$\implies \Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$$

- Metric-compatibility: $\nabla_\alpha g_{\mu\nu} = 0$

An affine connection is metric-compatible if the covariant derivative of the metric tensor is always zero. This characteristic places us in a set of unique coordinates known as the Minkowskian inertial reference frame or the Local inertial frame. In these special coordinates, we can eliminate the gravitational tidal forces that causes the background to become non-Euclidean, if we restrict ourselves to a sufficiently small region. Second, the torsion in our space-time geometry is defined by the commutator of the affine connections, which vanishes as a result of the assumed symmetry in the lower indices of the affine connection [12]. This implies a few admirable traits.

First, it's trivial to show that the covariant derivative of the inverse metric also vanishes:

$$\nabla_\alpha g^{\mu\nu} = 0 \tag{2.10}$$

Secondly, a covariant derivative that is compatible with metrics commutes with both increasing and lowering indices. So, for some vector field V^ρ:

$$g_{\mu\rho}\nabla_\alpha V^\rho = \nabla_\alpha (g_{\mu\rho}V^\rho) = \nabla_\alpha V_\mu \tag{2.11}$$

We now compute the metric compatibility equations for each of the three conceivable combinations in order to determine the special affine connections in our space-time:

$$\begin{aligned}
\nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\alpha\nu} g_{\mu\lambda} \\
\nabla_\mu g_{\nu\alpha} &= \partial_\mu g_{\nu\alpha} - \Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} - \Gamma^\lambda{}_{\mu\alpha} g_{\nu\lambda} \\
\nabla_\mu g_{\alpha\mu} &= \partial_\mu g_{\alpha\mu} - \Gamma^\lambda{}_{\mu\alpha} g_{\lambda\mu} - \Gamma^\lambda{}_{\mu\mu} g_{\alpha\lambda}
\end{aligned} \tag{2.12}$$

Solving (2.12) trivially, we obtain the expression for the christoffel symbols:

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{2.13}$$

This connection is particularly known as the **Levi-Civita connection** and sometimes called the **Riemannian connection** [12].

The second approach is to assume curvature-free non-torsion-free connections, this approach results into an additonal degree of freedom in the form of torsion tensor. The connections developed through this approach are called **Weitzenböck connection** and they form a basis of torsion based gravity models like $f(\mathcal{T})$ and $f(\mathcal{Q}, \mathcal{T})$.

2.5 Riemann curvature tensor

The Riemann curvature tensor which is computed using the affine connections assesses the space-time curvature. The idea behind this tensor is to quantify the space-time curvature. It is a fundamental tool in the study of the geometry of space-time [12].

It is defined as the commutator of the covariant derivatives. A Riemannian manifold is globally flat if and only if the Riemann curvature tensor vanishes.

The curvature tensor can be applied to any Riemannian, pseudo-Riemannian manifold, as well as to any other manifold having an affine connection.

The Riemannian tensor is defined as:

$$\mathcal{R}^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}{}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\beta} \tag{2.14}$$

Some properties of the Riemann curvature tensor:

- The Riemann tensor $\mathcal{R}^{\alpha}{}_{\beta\mu\nu}$ is antisymmetric in its last two indices

$$\mathcal{R}^{\alpha}{}_{\beta\mu\nu} = -\mathcal{R}^{\alpha}{}_{\beta\nu\mu} \tag{2.15}$$

- Contraction: $\mathcal{R}_{\alpha\beta\mu\nu} = g_{\alpha\lambda}\mathcal{R}^{\lambda}{}_{\beta\mu\nu}$:
- For a 4th rank covariant riemann tensor:

$$\mathcal{R}_{\alpha\beta\mu\nu} = -\mathcal{R}_{\mu\nu\alpha\beta} \tag{2.16}$$

2.6 Ricci tensor and Ricci scalar

The contravariant and a covariant indices of the Riemann tensor are frequently contracted to form the Ricci tensor:

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^{\lambda}{}_{\mu\lambda\nu} \tag{2.17}$$

whereas the Ricci tensor is form with the contraction of the Ricci tensor and the metric tensor:

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} \tag{2.18}$$

It is important to note that, in case the Ricci tensor and/or Ricci scalar vanishes, it doesn't necessarily means that the space is flat. Vanishing of the Riemann curvature tensor is the necessary and sufficient condition for a flat space-time.

2.7 Einstein Field Equations

The Einstein field equations (EFEs) in general theory of relativity link the geometry of spacetime to the distribution of energy and matter therein. They can be viewed as a collection of mathematical formulas that describe how energy-momentum impacts the curvature of spacetime.

EFEs are a group of tensor equations relating a set of symmetric 4x4 tensors. Considering a diagonal metric tensor, each of the symmetric 4x4 tensors have 10 independent components which are reduced to 6 by the four Bianchi identities [12].

The most general form of the Einstein field equations(EFE) is:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.19}$$

$$\text{with } G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci curvature tensor and $\mathcal{R}$ is the scalar curvature.

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} + \Lambda g^{\mu\nu} = \kappa T_{\mu\nu} \tag{2.20}$$

The expression on the right indicates the stress/energy-momentum content of spacetime, whereas the expression on the left shows the curvature of spacetime as determined by the metric.

2.8 Action Formalism for General Relativity

The field equation (2.20) can be obtained from the variation of an action, as is the case with most field theories. This is the Einstein-Hilbert action in the context of general relativity:

$$S_{EH} = \frac{1}{16\pi}\int d^4x\, \sqrt{-g}\mathcal{R} + \int d^4x\, \mathcal{L}_m \tag{2.21}$$

where $\mathcal{L}_m$ is the Lagrangian density of the matter fields, g is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{R}$ is the ricci scalar.

The corresponding energy-momentum tensor reads:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} \tag{2.22}$$

The Einstein field equations are obtained by varying the Einstein-Hilbert action (2.21) against the metric tensor metric tensor $g^{\mu\nu}$ [12].

Chapter 3

Cosmic dynamics

3.1 Introduction

Cosmology is a branch of astronomy which deals of figuring out the origin and evolution of the universe. More concretely, it is the study of large scale properties of the universe. [12]

The ΛCDM model, which is currently the recognised mainstream model of cosmology, postulates that the universe has three main constituents: the cosmological constant linked to dark energy, the speculated dark matter (CDM), and ordinary matter. This is the most straightforward model that adequately accounts for the following properties:

- the existence of cosmic microwave background
- the large-scale structure distribution in the universe
- cosmic acceleration: the accelerated expansion of the universe

But in order to create a model of the universe, we need more information, especially an understanding of how the universe might seem from various angles in addition to how it appears from our planet. Since we cannot actually accomplish that, we must make some assumptions. For example, it is reasonable to assume that we do not occupy a unique position in the universe (Copernican Principle), and that the universe would therefore appear to be isotropic (in the previously mentioned averaged sense) when viewed from every other point [12].

This presumption suggests that we can represent the visible cosmos as spatially homogeneous and isotropic on vast scales. Since it is impossible to explicitly demonstrate this assumption, it must be established a posteriori (after the fact) by comparing the model's predictions to the observations. Several distinct types of observations do, however, strongly support this assumption. [12]

To represent the dynamics of the universe as a whole, we largely rely on our understanding of physical processes on earth and in the solar system. In reality, it is natural to extrapolate the applicability of these concepts from the energy and length scales we can observe on and around our planet to arbitrary huge distances. Because we cannot assume that the system would remain consistent if the scales and complexity are modified, it may not have been the best course of action in hindsight, but it is still a legitimate assumption. We will thus assume that the only interaction that contributes to the large-scale structure of the cosmos is gravity. Therefore, we shall employ the general relativity [14, 15] framework proposed by Einstein to model the gravitational interactions.

In this approach, gravity is viewed as a geometrical influence, and the metric tensor $g_{\mu\nu}$ contains the geometry characteristics of the cosmos. The stress-energy tensor of matter fields $T_{\mu\nu}$ is the source term responsible for the universe's curvature and is determined using the Einstein's field equations, defined in (2.20):

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3.1}$$

3.2 The homogeneous and isotropic universe

The Cosmological Principle states that on a sufficiently large scale, i.e. on a scale of mega-parsecs, the spatial distribution of matter in the universe is homogeneous and isotropic. It is predicted that the forces act uniformly throughout the universe, thus avoiding any observable irregularities over the evolution of the universe [16].

The "Copernican principle," sometimes known as the assumption that the cosmos is essentially the same everywhere, is the foundation of modern cosmological models.

The Copernican principle is connected to isotropy and homogeneity, two further mathematically precise properties that a manifold may have. When it comes to a certain position in the room, the concept of isotropy states that the room appears to be the same from any aspect [12]. Homogeneity is the notion that the measure is constant over the entire region. Keep in mind that isotropy and homogeneity are not always correlated.

Simply stated, if a space is *isotropic* everywhere then it is *homogeneous*.

We will go forward assuming both homogeneity and isotropy since there is a all lot of observable evidence for isotropy and because the Copernican principle would have us believe that the cosmos is not centred on us and that observers elsewhere should therefore likewise experience isotropy. However, there is one issue, as Edwin Hubble pointed out through his observations that the universe is not static but is rather expanding. Hubble utilized the concept of cosmological redshift of light to figure out that the stars and galaxies are moving away from us at a velocity of $v = \mathcal{H}D$ [17], where $\mathcal{H}$ is the hubble constant and D is the distance to the star/galaxy.

Since we assume that the universe is homogeneous, isotropic, and expanding, we will use these concepts to build our cosmological model.

In general relativity, we can translate the homogeneous and isotropic with expanding spacetime by using a scale factor $a(t)$ as:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \tag{3.2}$$

Here t is the timelike coordinate, (x^1, x^2, x^3) are the spacelike coordinates, $a(t)$ is the scale factor and γ_{ij} is the maximally symmetric metric.

Using the form for maximally symmetric metric with (3.2) we obtain the Friedmann-Lemaître-Robertson-Walker(FRLW) metric, which we will discuss in detail in the next section.

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{3.3}$$

where $d\Omega^2 = (d\theta^2 + \sin^2(\theta)d\phi^2)$

3.2.1 The Friedmann-Lemaître-Robertson-Walker metric

The Friedmann-Lemaître-Robertson-Walker metric, which describes a homogeneous, isotropic, expanding (or otherwise, contracting) universe, is based on the exact solution of Einstein's field equations of general relativity [18].

The FLRW metric is applied on the assumption of homogeneous and isotropic space. It also takes into account the possibility that the spatial component of the metric may be time-dependent. The general measurement that complies with these specifications is [18]:

$$ds^2 = dr^2 + S_\kappa(r^2)d\Omega^2 \tag{3.4}$$

$$\text{where, } S_\kappa(r) = \begin{cases} R\sin\left(\frac{r}{R}\right), & \text{if } \kappa > 0 \\ r, & \text{if } \kappa = 0 \\ R\sinh\left(\frac{r}{R}\right), & \text{if } \kappa < 0 \end{cases} \tag{3.5}$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin(\theta)d\phi^2\right)\right] \tag{3.6}$$

Here κ defines the structure of the universe, flat($\kappa = 0$), spherical($\kappa = 1$) and hyperbolic ($\kappa = -1$)

3.3 Friedmann equations

The Friedmann equations are the dynamical equations of the universe which can be used to study the evolution of the scale factor $a(t)$ in time [19]. Assuming an isotropic and homogenous universe, we can utilize the FRLW metric to develop the dynamical equations [18].

The Friedmann equations are derived from the Einstein's field equations. They link the energy density, pressure and curvature of the universe to its expansion rate. We can understand the long-term evolution of the universe by solving the Friedmann equations [19].

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi}{3}\rho \tag{3.7}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \Lambda c^2 = -8\pi p \tag{3.8}$$

3.4 Hubble's law

The crucial discovery that stars and galaxies are moving away from Earth proportionate to their distance was discovered by Edwin Hubble in 1937. He used the redshift in apparent light of galaxies to calculate their velocities [12, 17].

Although the expanding nature of the universe was predicted by the Friedmann's equations, Hubble was the first to confirm it observationally. A few years later, using the galaxies' redshifts, he was also able to calculate a more precise value for the recession velocities.

Simply stated, the farther they are, the faster they are receding away. The receding velocity is computed using their redshift.

$$v = \mathcal{H}_0 D \tag{3.9}$$

Here $\mathcal{H}0$ is the proportionality constant called the **Hubble constant**. It can be interpreted as the relative rate of expansion of the universe with a value of $\sim 70\ kms^{-1}\mathrm{Mpc}^{-1}$.

3.5 Fluid and acceleration equations

The Friedmann equation determines $a(t)$ if we know $\mathcal{H}_0$ and the energy density $\epsilon(t)$ as a function of time. Through the *fluid equation* we can describe the evolution of energy density in an expanding universe [12, 18].

In an adiabatic, expanding/contracting universe dQ vanishes, hence, the 1st law of Thermodynamics reads:

$$dQ = dE + PdV \tag{3.10}$$

$$\dot{E} = -P\dot{V} \tag{3.11}$$

$$\dot{V}(t) = \frac{d}{dt}\frac{4\pi R^3 a^3}{3} = \frac{3\dot{a}}{a}V(t) \tag{3.12}$$

$$\dot{E} = V(t)\left(\frac{3\dot{a}}{a}\epsilon(t) + V(t)\,\dot{\epsilon}(t)\right) \tag{3.13}$$

Substituting (3.11), (3.12) into (3.13):

$$V(t)\left(\epsilon(t)\frac{3\dot{a}}{a} + \dot{\epsilon}(t)\right) = \frac{3\dot{a}}{a}V(t)P$$

$$\boxed{\dot{\epsilon}(t) + \frac{3\dot{a}}{a}\left(\epsilon(t) + P\right) = 0} \tag{3.14}$$

(3.14) is known as the fluid equation and it relates the evolution of energy density to the scale factor.

In order to obtain the acceleration equation, we differentiate the Friedmann equation (3.7) against time:

$$\partial_t\left(\frac{\dot{a}}{a}\right)^2 = \partial_t\frac{8\pi G}{3c^2}\epsilon(t) - \partial_t\frac{\kappa c^2}{R^2 a^2(t)} \tag{3.15}$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\epsilon + p\right)} \tag{3.16}$$

(3.16) is known as the acceleration equation.

3.6 Equation of State

The link between the two quantities, pressure (p) and energy density (ρ), is known as the equation of state. Almost all of the perfect fluids important to cosmology follow the straightforward *equation of state* [18]:

$$p = w\rho \tag{3.17}$$

Here w is a dimensionless quantity called the equation of state.

3.7 Cosmological redshift

When we take into consideration the effects that are described by the special theory of relativity, we can conclude that the cosmological redshift is the apparent change in the frequency of the light that is detected by the observer as a result of the relative motion between the source of light and the observer [12]:

In this section, we will try to develop a relationship between the redshift(z) and the scale factor of the universe $(a(t))$:

The general of the FRLW metric, disscussed in detail in 3.2.1, can be written as:

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega^2\right) \tag{3.18}$$

In cosmological redshift, we are interested in computing the photon trajectories which follows the null/radial geodesics, so the angular component and the space-time interval vanishes, using (3.18):

$$\int \frac{dt}{a(t)} = \pm \int dr \tag{3.19}$$

Suppose a wave of light is emitted at t_e while is observed at t_0 and considering only a single crest wave:

$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = \pm \int_0^r dr \tag{3.20}$$

Similarly, for a wave emitted at $t_e + \frac{\lambda_0}{c}$ and $t_0 + \frac{\lambda_0}{c}$, the travelled distance must be the same, hence:

$$\int_{t_e+\frac{\lambda_e}{c}}^{t_o+\frac{\lambda_o}{c}} \frac{dt}{a(t)} = \pm \int_0^r dr \tag{3.21}$$

Subtracting Eq.(3.20) from Eq.(3.21),

$$\int_{t_o}^{t_o+\frac{\lambda_o}{c}} \frac{dt}{a(t)} = \int_{t_e}^{t_e+\frac{\lambda_e}{c}} \frac{dt}{a(t)} \tag{3.22}$$

Assuming the scale-factor doesn't vary much during the integral, let the scale factor when the wave is emitted be $a(t_e)$ and when it is observed be $a(t_o)$:

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_o}{a(t_o)} \tag{3.23}$$

The redshift is defined as the radio of differenge of wavelenth observed and the wave-length emitted, $z = \dfrac{\lambda_o - \lambda_e}{\lambda_e}$,

$$\boxed{1 + z = \frac{a(t_o)}{a(t_e)} = \frac{1}{a(t_e)}} \tag{3.24}$$

3.8 Simple component universes

At certain epochs of cosmic history one component of matter-energy is overwhelmingly abundant and that component effectively drives the expansion rate of the universe and we can ignore other components. This makes our calculations dramatically easy to solve since we only have to worry about one energy-density at a time [12]. We can isolate the single component universe into one having curvature and the other spatially flat.

3.8.1 Curvature only

In this type of universe, we assume that the universe is empty, implies that the energy density vanishes. This causes the universe to expand/contract linearly, as predicted by the Friedmann equation [12, 18]:

$$\dot{a}^2 = -\frac{\kappa}{R^2} \tag{3.25}$$

In order to obtain a real solution, the universe must be hyperbolic in nature($\kappa = -1$):

$$\dot{a} = \pm\frac{1}{R} \tag{3.26}$$

$$a = \pm\frac{t}{t_0} \tag{3.27}$$

3.8.2 Spatially flat universe

In this type of universe, we assume that the universe is filled with some perfect fluid with equation-of-state parameter w and is spatially flat($\kappa = 0$).

The Friedmann equation reads:

$$\dot{a}^2 = \frac{8\pi}{3}\epsilon_0 a^{-1(1+3w)} \tag{3.28}$$

Integrating this we obtain:

$$a = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} + C' \tag{3.29}$$

For such a case, there can be three possible types:

Matter only

In this type of universe, we assume that the universe is filled with perfect fluid with equation-of-state parameter $w = 0$ and is spatially flat($\kappa = 0$)(3.29) [12, 18]:

$$a \propto t^{2/3} \tag{3.30}$$

Radiation only

In this type of universe, we assume that the universe is filled with perfect fluid with equation-of-state parameter $w = 1/3$ and is spatially flat($\kappa = 0$) (3.29) [12, 18]:

$$a \propto t^{1/2} \tag{3.31}$$

Lambda only

In this type of universe, we assume that the universe is filled with perfect fluid with equation-of-state parameter $w = -1$ and is spatially flat($\kappa = 0$). The result is a universe which will expand/contract exponentially.

$$a = e^{\mathcal{H}0(t-t_0)} \tag{3.32}$$

Chapter 4

f(Q) gravity

4.1 Introduction

The description of our understanding of gravity that has been most successful so far is General Relativity (GR). Its forecasts for the consequences of observation on solar system and cosmic scales are fairly accurate. However, other tensions, such the H_0 tension, have emerged in recent years, as discussed in 1.3.

Thus, it is beneficial to consider its alternate generalisations, such as $f(\mathcal{Q})$ gravity. In contrast to general relativity, it is based on a separate set of geometric postulates known as symmetric teleparallelism (ST) [20].

4.1.1 Motivation

$f(\mathcal{Q})$ gravity is a modification of GR that includes an extra scalar degree of freedom in the theory. This scalar degree of freedom could potentially explain the observed acceleration of the expansion of the universe without the need for dark energy. It has been shown to reduce the $\mathcal{H}_0$ stress by modifying how gravity behaves on big scales. However, it is still an active area of research and more work needs to be done to fully understand its implications.

F. Anagnostopoulos et al [21] suggested a unique $f(\mathcal{Q})$ model and tested it against observational data (SNIa, BAOs, CC, and RSD) in their work. They discovered that the models are stastically consistent for the CC + SNIa datasets, but that the $f(\mathcal{Q})$ gravity is marginally preferred over the ΛCDM model for the CC + SNIa + BAOs dataset. They found that the ΛCDM is somewhat preferred by the RSD + CC + SNIa data, but the two models are statistically comparable.

The current scenario easily satisfies the early universe requirements since the proposed $f(\mathcal{Q})$ tends to $\mathcal{Q}$ in the large redshift regime [21]. Thus, further studies in $f(\mathcal{Q})$ gravity is necessary.

4.1.2 Symmetric Teleparallelism (ST)

The main distinction between ST and GR is the role of the affine connection $\Gamma^{\alpha}{}_{\mu\nu}$ [20]. In general relativity, the affine connection is assumed to be torsionless and metric-compatible, which means its defined by the *Levi-Cevita* connection. The metric compatibility and torsionless is described in detail in 2.4.1. However, in ST, the metric-compatibility is dropped, instead one demands that the Riemann tensor($\mathcal{R}^{\alpha}{}_{\beta\mu\nu}$) must vanish. As long as the connection satisfies these postulates, they can be chosen arbitrarily, independent of the metric $g_{\mu\nu}$, with these constraints, the only non-trivial object left is the non-metricity tensor $\mathcal{Q}_{\alpha\mu\nu}$. This non-metricity scalar can be used to define a scalar $\mathcal{Q}$, similar to the GR's approach of defining a Ricci scalar from Riemann tensor [20, 22].

The non-metricity tensor is defined as:

$$\mathcal{Q}_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu} \tag{4.1}$$

with the modified Einstein-Hilbert action given as:

$$S = \int d^4x\sqrt{-g}\ \mathcal{Q} \tag{4.2}$$

ST's action provides a unique geometric interpretation of gravity. On generalising this approach even further by considering a priori arbitrary function $f(\mathcal{Q})$ in the action as [22]:

$$S = \int d^4x\sqrt{-g}\ f(\mathcal{Q}) \tag{4.3}$$

The generalised action obtained has degrees of freedom for both the metric and the connection. Basically, the $f(\mathcal{Q})$ is a modified gravity theory that proposes a symmetric teleparallel gravity in which the gravitational action is given by an arbitrarily chosen function f of the non-metricity scalar $\mathcal{Q}$.

4.2 Formalism

We being by constructing a non-metric affine connection in symmetric teleparallelism and define the non-metricity tensor [22]:

$$\mathcal{Q}_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu} \tag{4.4}$$

It is trivial that the non-metricity tensor is symmetric in the last two indices, the number of independent scalars that may be constructed from it is limited to five. It turns out that general relativity is recovered by a linear combination of only four of these contractions, which define the so-called non-metricity scalar. Trivially, a linear combination of the five separate components is an obvious starting point for building the action. The modified Einstein-Hilbert(2.21) action can be developed by adding terms that depend on the non-metricity scalar and its derivatives. However, there are a variety of ways to build such terms, and it is a recurring problem in $f(\mathcal{Q})$ gravity research to identify the one that yields the right physical predictions [20, 22].

In our approach, we will first take an arbitrary function $f(\mathcal{Q})$ to develop the field equations.

$$\mathcal{Q} - \frac{1}{4}\mathcal{Q}_{\alpha\beta\gamma}\mathcal{Q}^{\alpha\beta\gamma} + \frac{1}{2}\mathcal{Q}_{\alpha\beta\gamma}\mathcal{Q}^{\gamma\beta\alpha} + \frac{1}{4}\mathcal{Q}_{\alpha}\mathcal{Q}^{\alpha} - \frac{1}{2}\mathcal{Q}_{\alpha}\tilde{\mathcal{Q}}^{\alpha} \tag{4.5}$$

where $\mathcal{Q}_{\alpha} = g^{\mu\nu}\mathcal{Q}_{\alpha\mu\nu}$; $\quad \tilde{\mathcal{Q}}_{\alpha} = \mathcal{Q}^{\nu}{}_{\nu\alpha}$.

The affine connections can be represented as:

$$\Gamma^{\alpha}{}_{\mu\nu} = \mathring{\Gamma}^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu} \tag{4.6}$$

where $\mathring{\Gamma}^{\alpha}{}_{\mu\nu}$ is the Levi-Civita connection from the metric tensor $g_{\mu\nu}$ as represented in plain vanilla GR and $L^{\alpha}{}_{\mu\nu}$ is called the disformation tensor. It is represented by:

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(\mathcal{Q}^{\alpha}{}_{\mu\nu} - \mathcal{Q}_{\mu}{}^{\alpha}{}_{\nu} - \mathcal{Q}_{\nu}{}^{\alpha}{}_{\mu}\right) \tag{4.7}$$

As discussed earlier, the $f(\mathcal{Q})$ is constructed by assuming the curvature vanishes $\mathcal{R}^{\alpha}{}_{\beta\mu\nu} = 0$. It would mean that there exists a special coordinate system such that the affine connection $\Gamma^{\alpha}{}_{\mu\nu}$ vanishes. This gauge is called the coincident gauge, and causes the metric to be the only dynamical variable.

As we know that the Riemann curvature tensor is given by, discussed in detail in 2.5:

$$\mathcal{R}^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\beta} \tag{4.8}$$

Using the new form of affine connection, we can find that:

$$\mathcal{R}^{\alpha}{}_{\beta\mu\nu} = \mathring{\mathcal{R}}^{\alpha}{}_{\beta\mu\nu} + \mathring{\nabla}_{\mu}L^{\alpha}{}_{\nu\beta} - \mathring{\nabla}_{\nu}L^{\alpha}{}_{\mu\beta} + L^{\alpha}{}_{\mu\lambda}L^{\lambda}{}_{\nu\beta} - L^{\alpha}{}_{\nu\lambda}L^{\lambda}{}_{\mu\beta} \tag{4.9}$$

Hence the Ricci tensor and scalar reads:

$$\mathcal{R}_{\mu\nu} = \mathring{\mathcal{R}}_{\mu\nu} + \frac{1}{2}\mathring{\nabla}_{\nu}\mathcal{Q}_{\mu} + \mathring{\nabla}_{\lambda}L^{\lambda}{}_{\nu\mu} - \frac{1}{2}\mathcal{Q}_{\lambda}L^{\lambda}{}_{\nu\mu} - L^{\alpha}{}_{\nu\lambda}L^{\lambda}{}_{\alpha\mu} \tag{4.10}$$

$$\mathcal{R} = \mathring{\mathcal{R}} + \mathring{\nabla}_\lambda \mathcal{Q}^\lambda - \mathring{\nabla}_\lambda \tilde{\mathcal{Q}}^\lambda - \frac{1}{4}\mathcal{Q}_\lambda \mathcal{Q}^\lambda + \frac{1}{2}\mathcal{Q}_\lambda \tilde{\mathcal{Q}}^\lambda - L_{\alpha\nu\lambda}L^{\lambda\alpha\nu} \tag{4.11}$$

The field equations for $f(\mathcal{Q})$ gravity can be developed by varying the modified Einstein-Hilbert action against the metric tensor as:

$$S = \int d^4x \sqrt{-g}\left\{-\frac{1}{2\kappa^2}f(\mathcal{Q}) + \mathcal{L}_m\right\} \tag{4.12}$$

In the scenario when $f(\mathcal{Q}) = \mathcal{Q}$, we note that the aforementioned action is similar to the Einstein-Hilbert action up to a total derivative, which indicates that in this case, our adjustments will only affect the boundary terms and all physics effects will be recovered as is. When affine connections globally evaporate, Einstein's general relativity action is reinstated and the non-metricity tensor simply depends on the metric [20, 22].

In order to develop the gravitational field equations, the action obtained in (4.12) is varied against the metric as:

$$\begin{aligned}&\frac{2}{\sqrt{-g}}\nabla_\alpha\left\{\sqrt{-g}g_{\beta\nu}f_\mathcal{Q}\left[-\frac{1}{2}L^{\alpha\mu\beta} - \frac{1}{8}\left(g^{\alpha\mu}\mathcal{Q}^\beta + g^{\alpha\beta}\mathcal{Q}^\mu\right) + \frac{1}{4}g^{\mu\beta}\left(\mathcal{Q}^\alpha - \tilde{\mathcal{Q}}^\alpha\right)\right]\right\}\\ &+f_\mathcal{Q}\left[-\frac{1}{2}L^{\mu\alpha\beta} - \frac{1}{8}\left(g^{\mu\alpha}\mathcal{Q}^\beta + g^{\mu\beta}\mathcal{Q}^\alpha\right) + \frac{1}{4}g^{\alpha\beta}\left(\mathcal{Q}^\mu - \tilde{\mathcal{Q}}^\mu\right)\right]\mathcal{Q}_{\nu\alpha\beta} + \frac{1}{2}\delta^\mu{}_\nu f = T^\mu{}_\nu\end{aligned} \tag{4.13}$$

Here, $f_\mathcal{Q}$ represents $\partial_\mathcal{Q} f(\mathcal{Q})$

To further simplify it, we can use the bare general relativity's Ricci tensor ans scalar to develop the bare Einstein's tensor as:

$$\mathring{G}_{\mu\nu} = \mathring{\mathcal{R}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{\mathcal{R}} \tag{4.14}$$

Using (4.14) in (4.13) we can separate the contributions from general relativity and non-metricitiy scalar as:

$$\boxed{f'(\mathcal{Q})\mathring{G}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(f'(\mathcal{Q})\mathcal{Q} - f(\mathcal{Q})\right) + 2f''(\mathcal{Q})P^\alpha{}_{\mu\nu}\partial_\alpha\mathcal{Q} = 8\pi T_{\mu\nu}} \tag{4.15}$$

It is important to note that when $f(\mathcal{Q}) = \mathcal{Q}$, the (4.15) reduces to general relativity's action plus a boundary term (which gets integrated away). The boundary term is essential in ensuring that the additional degrees of freedom remains unphysical allowing us to retain our original understanding of physics.

4.3 Friedmann's equations

Using the FRLW metric [18]:

$$ds^2 = a^2\left(d\eta^2 - \left(dx^2 + dy^2 + dz^2\right)\right) \tag{4.16}$$

From the section above, we can see that the $0-0$th field equation will give us the Friedmann's equations:

$$f'(\mathcal{Q})\mathring{G}_{00} + \frac{1}{2}g_{00}\left(f'(\mathcal{Q})Q - f(\mathcal{Q})\right) + 2f''(\mathcal{Q})P^{\alpha}{}_{00}\,\partial_\alpha\mathcal{Q} = 8\pi T_{00} \tag{4.17}$$

Since the $\mathring{G}_{00} = 3\mathcal{H}^2 + 3K$ for Einstein's General Relativity, we obtain the Friedmann's equation for the $f(\mathcal{Q})$ gravity as:

$$3f'(\mathcal{Q})\left(\mathcal{H}^2 + K\right) + \frac{1}{2}a^2\left(f'(\mathcal{Q})Q - f(\mathcal{Q})\right) + 2f''(\mathcal{Q})P^{\alpha}{}_{00}\,\partial_\alpha\mathcal{Q} = 8\pi T_{00} \tag{4.18}$$

Here, $P^{\alpha}{}_{\mu\nu} = \frac{1}{4}\left(-2L^{\alpha}{}_{\mu\nu} + \left(\mathcal{Q}^\alpha - \tilde{\mathcal{Q}}^\alpha\right)g_{\mu\nu} - \frac{1}{2}\delta^{\alpha}{}_{\mu}\mathcal{Q}_\nu - \frac{1}{2}\delta^{\alpha}{}_{\nu}\mathcal{Q}_\mu\right)$ is called the super-potential tensor [22, 23].

In order to obtain the acceleration equation, we compute the $i-i$ form of the contracted EFEs:

$$f'(\mathcal{Q})\mathring{G}^{i}{}_{i} + \frac{1}{2}\delta^{i}{}_{i}\left(f'(\mathcal{Q})Q - f(\mathcal{Q})\right) + 2f''(\mathcal{Q})P^{\alpha}{}_{ii}\,g^{ii}\partial_\alpha\mathcal{Q} = 8\pi T^{i}_{i} \tag{4.19}$$

Since $\mathring{G}^{i}{}_{i} = -\mathcal{R} = 6\left(\dfrac{a'' + Ka}{a^3}\right)$ and $\delta^{i}{}_{i} = 4$;

$$f'(\mathcal{Q})\left(\frac{a'' + Ka}{a^3}\right) + \left(f'(\mathcal{Q})Q - f(\mathcal{Q})\right) + f''(\mathcal{Q})P^{\alpha}{}_{ii}\,g^{ii}\partial_\alpha\mathcal{Q} = \frac{4\pi}{3}\,T^{i}_{i} \tag{4.20}$$

In order to simply the obtained equations, we set $f(\mathcal{Q}) = Q + \Theta(\mathcal{Q})$, so $f'(\mathcal{Q}) = 1 + \Theta'(\mathcal{Q})$ and so on. The Friedmann and acceleration equation becomes [23]:

$$3\left(1 + \Theta'(\mathcal{Q})\right)\left(\mathcal{H}^2 + K\right) + \frac{1}{2}a^2\left(\Theta'(\mathcal{Q})\,\mathcal{Q} - \Theta(\mathcal{Q})\right) + 2\Theta''(\mathcal{Q})\,P^{\alpha}{}_{00}\,\partial_\alpha\mathcal{Q} = 8\pi T_{00} \tag{4.21}$$

$$\left(1 + \Theta'(\mathcal{Q})\right)\left(\frac{a'' + Ka}{a^3}\right) + \left(\Theta'(\mathcal{Q})\,Q - \Theta(\mathcal{Q})\right) + \Theta''(\mathcal{Q})\,P^{\alpha}{}_{ii}\,g^{ii}\partial_\alpha\mathcal{Q} = \frac{4\pi}{3}\,T^{i}_{i} \tag{4.22}$$

As a sake of consistency check, we can put $f(\mathcal{Q}) = \mathcal{Q}$ to make sure it reduces to our well-known Friedmann's equation. Following this, $f'(\mathcal{Q}) = \mathcal{Q}'$ and $f''(\mathcal{Q}) = 0$

$$\mathcal{H}^2 + K = \frac{8\pi}{3} T_{00} \tag{4.23}$$

$$\left(\frac{a'' + Ka}{a^3} \right) = \frac{4\pi}{3} T^i_{\ i} \tag{4.24}$$

Hence the Friedmann equations (4.23) matches (3.7); and (4.24) matched (3.8) thus satisfying the consistency check.

We can simply these equations by utilizing the FRLW metric (6.4) and by separating the contributions from the general relativity's and f(Q)'s part by setting $f(\mathcal{Q}) = Q + \Theta\left(\mathcal{Q}\right)$ while assuming the universe if filled with a perfect-fluid:

$$3\mathcal{H}^2 = \rho + \frac{\Theta\left(\mathcal{Q}\right)}{2} - \mathcal{Q}\Theta'\left(\mathcal{Q}\right) \tag{4.25}$$

$$\left(1 + 2\mathcal{Q}\Theta''\left(\mathcal{Q}\right) + \Theta'\left(\mathcal{Q}\right)\right)\mathcal{H}' + \frac{1}{4}\left(\mathcal{Q} + 2\mathcal{Q}\Theta'\left(\mathcal{Q}\right) - \Theta\left(\mathcal{Q}\right)\right) = -2p \tag{4.26}$$

Chapter 5

Cosmological Perturbations in f(Q)

5.1 Introduction

Modern cosmology is based on the hypothesis of linearized gravitational disturbances in an expanding cosmos, or cosmic perturbations. It is employed to compute the anticipated variations in the microwave background and to characterise the evolution of structure in the universe. [24]

Through various observational results, there is strong evidence that the cosmos was relatively homogeneous and isotropic at large scales in its early history. It is typically considered that the structures we currently see on the scales of galaxies were formed by modest early perturbations that resulted in gravitational instability. The linear gravitational perturbation theory initially uses the flat minkowskian metric $\eta_{\mu\nu}$ plus a perturbative metric to develop the growth of small inhomogenities [24–26].

The gravitational forces that cause minor perturbations to increase and eventually plant the seeds for the formation of stars, quasars, and other celestial bodies are calculated using general relativity. It only applies, though, when the cosmos is primarily homogeneous, as was the case during cosmic inflation. At cosmological scales, it is thought that the universe is still sufficiently homogeneous for the theory to be a good approximation [24].

In classical general relativity, perturbation theory is now studied from two different perspectives:

- *gauge-invariant perturbation theory*
- $1 + 3$ *covariant gauge-invariant perturbation theory*

The perturbation equation computation are done with the Python *pytearcat* package

[27].

5.2 Background model

Assuming the background to be a homogeneous, isotropic and spatially flat FRLW spacetime, whose metric is of the form, discussed in detail in 3.2.1:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2 \delta_{ij} dx^i dx^j \tag{5.1}$$

Let $\eta = t/a$ represent the conformal time:

$$ds^2 = a^2 \left(-d\eta^2 + a^2 \delta_{ij} dx^i dx^j\right) \tag{5.2}$$

5.3 Perturbations

In order to model a realistic universe we add the perturbations via $\delta g_{\mu\nu}$, hence the complete line element becomes [28]:

$$ds^2 = \left(g_{\mu\nu} + \delta g_{\mu\nu}\right) dx^\mu dx^\nu \tag{5.3}$$

Three different forms of perturbations are possible: scalar, vector, and tensor which are categorised according to how they are built with $\delta g_{\mu\nu}$ and transform when three-space coordinates are involved.

5.3.1 Tensor perturbation

We will first look at tensor perturbation, these are constructed using a symmetric traceless divergenless three-tensor h_{ij} satisfying [24]:

$$h^i_i = 0; \quad h^{|i}_{ij} = 0 \tag{5.4}$$

$$\delta g_{\mu\nu}{}^{(t)} = -a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix} \tag{5.5}$$

$$g_{\mu\nu} + \delta g_{\mu\nu}{}^{(t)} = a^2(\eta) \begin{pmatrix} -1 & 0 \\ 0 & (\gamma_{ij} + h_{ij}) \end{pmatrix} \tag{5.6}$$

Hence, the total line element becomes:

$$ds^2 = a^2(\eta)\left(-d\eta^2 + (\gamma_{ij} + h_{ij})\, dx^i dx^j\right) \tag{5.7}$$

The metric reads:

$$g_{00} = -a^2; \quad g_{i0} = g_{0i} = 0; \quad g_{ij} = a^2\left(\gamma_{ij} + h_{ij}\right) \tag{5.8}$$

Using the above metric while simplifying by using the Kronecker delta instead of γ_{ij} and perturbing the field equation (4.15) we obtain:

$$\gamma_{ij}'' + \left(2\mathcal{H} - (\log f'(\mathcal{Q}))'\right)\gamma_{ij}' + k^2\gamma_{ij} = 0 \tag{5.9}$$

5.3.2 Vector perturbations

These are constructed by considering two 3 vectors ensuring the two vector fields are divergenless, B_i. If these are not divergenless, then we can separate it into vector and gradient of a scalar, hence it would not be a pure vector perturbation [24].

$$B_i^{|i} = B_i^{|i} = 0$$

$$\delta g_{\mu\nu}{}^{(v)} = -a^2(\eta)\begin{pmatrix} 0 & -B_i \\ -B_i & F_{i|j} + F_{j|i} \end{pmatrix} \tag{5.10}$$

$$g_{\mu\nu} + \delta g_{\mu\nu}{}^{(v)} = a^2(\eta)\begin{pmatrix} -1 & B_i \\ B_i & \left(\gamma_{ij} + F_{i|j} + F_{j|i}\right) \end{pmatrix} \tag{5.11}$$

So, the total line element becomes:

$$ds^2 = a^2(\eta)\left\{d\eta^2 + 2S_i dx^i d\eta - \left(\gamma_{ij} + F_{i|j} + Fj|i\right) dx^i dx^j\right\} \tag{5.12}$$

By taking the vector part of the metric:

$$g_{00} = 0; \quad g_{0i} = B_i; \quad g_{i0} = 2\partial_i B_j + 2\partial_j B_i \tag{5.13}$$

By perturbing the field equation (4.15), we obtain:

$$k^2 f'(\mathcal{Q})\,(B - E') = 0; \quad k^2\left[f'(\mathcal{Q})\,(B - E')\right]' = 0 \tag{5.14}$$

5.3.3 Scalar perturbations

This type of perturbations results in growing inhomogenities which can lead to structure formation.

We can generate this via (1.) multiplying a scalar with γ_{ij}, (2.) covariant derivative of a scalar function, which reduces to ordinary derivative for $\mathcal{K}=0$ flat space-time [24].

$$\delta g_{\mu\nu}{}^{(s)} = a^2(\eta) \begin{pmatrix} 2\phi & -B_{|i} \\ -B_{|i} & 2\left(\psi\gamma_{ij} - E_{|ij}\right) \end{pmatrix} \tag{5.15}$$

$$g_{\mu\nu} + \delta g_{\mu\nu}{}^{(s)} = a^2(\eta) \begin{pmatrix} 1+2\phi & -B_{|i} \\ -B_{|i} & -\left[(1-2\psi)\,\gamma_{ij} + 2E_{|ij}\right] \end{pmatrix} \tag{5.16}$$

So, the total line element becomes:

$$ds^2 = a^2(\eta)\left\{(1+2\phi)\,d\eta^2 - 2B_{|i}dx^i d\eta - \left[(1-2\psi)\,\gamma_{ij} + 2E_{|ij}\right] dx^i dx^j\right\} \tag{5.17}$$

By perturbing the field equation (4.15) against the $0-0$ component of the metric tensor, we obtain the equation for scalar perturbations:

$$\frac{k^4\mathcal{Q}f'(\mathcal{Q})f''(\mathcal{Q})}{\mathcal{H}^2\left(2f'(\mathcal{Q}) + 3\mathcal{Q}f'(\mathcal{Q})\right)}\left(3\mathcal{H}E' + k^2E - 3\phi\right) = 0 \tag{5.18}$$

$$\frac{k^4\mathcal{Q}f'(\mathcal{Q})f''(\mathcal{Q})}{\mathcal{H}^2\left(2f'(\mathcal{Q}) + 3\mathcal{Q}f'(\mathcal{Q})\right)}\left(E'' + 18\mathcal{H}^2E' + k^2(3\mathcal{H}^2 - k^2)E - 9\mathcal{H}\phi' + 3(k^2 - 3\mathcal{H}^2)\phi\right) = 0 \tag{5.19}$$

Chapter 6

Cosmic dynamics in f(Q)

6.1 Cosmic Dynamics

We have computed the Friedmann's and acceleration equation in $f(\mathcal{Q})$ gravity in the previous section. In this section, we will assume the universe to be a perfect fluid and try to compute its cosmological parameters:

Assuming the universe to be a perfect fluid, with no viscosity and no energy flux at rest, the energy-momentum tensor can be characterized by [29]:

$$T_{\mu\nu} = diag(\rho, p, p, p) \tag{6.1}$$

$$T_{\mu\nu} = (\rho_m + p)\, u_\mu u_\nu + p g_{\mu\nu} \tag{6.2}$$

with space-positive metric signature.

Here, ρ and p represents the energy-density and pressure of the matter fluid, which for no-interactions will hold the continuity equation given by:

$$\dot{\rho} + 3\mathcal{H}\,(1 + w)\,\rho = 0 \tag{6.3}$$

where $w = p/\rho$ is the equation-of-state parameter.

6.1.1 Cosmological evolution model

The FRLW metric without the conformal time reads:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{6.4}$$

It is important to note that the non-metricity scalar obtained with this metric is $Q = 6\mathcal{H}^2$.

We have already computed the Friedmann's equation in (4.23), we will utilize that equation and simplify it using (6.2) and (6.4).

$$6f'(\mathcal{Q})\mathcal{H}^2 - \frac{1}{2}f(\mathcal{Q}) = 8\pi\rho \tag{6.5}$$

$$\left(12\mathcal{H}^2 f''(\mathcal{Q}) + f'(\mathcal{Q})\right)\dot{\mathcal{H}} = -4\pi\left(\rho + p\right) \tag{6.6}$$

In order to simply the above modified Friedmann's equation, we develop two additional parameters, p_Q and ρ_Q which represents the pressure and energy density for the non-metricity scalar $\mathcal{Q}$ as [29, 30]:

$$\rho_Q = \frac{3}{8\pi}\left[\mathcal{H}^2\left(1 - 2f'(\mathcal{Q})\right) + \frac{f(\mathcal{Q})}{6}\right] \tag{6.7}$$

$$p_Q = -\frac{1}{8\pi}\left[2\dot{\mathcal{H}}\left(1 - f'(\mathcal{Q})\right) + \frac{f(\mathcal{Q})}{2} + 3\mathcal{H}^2\left(1 - 8f''(\mathcal{Q})\dot{\mathcal{H}} - 2f'(\mathcal{Q})\right)\right] \tag{6.8}$$

Using (6.7) and (6.8) in the modified Friedmann's equations:

$$3\mathcal{H}^2 = 8\pi\left(\rho_m + \rho_Q\right) \tag{6.9}$$

$$3\mathcal{H}^2 + \dot{\mathcal{H}} = -8\pi\left(p_m + p_Q\right) \tag{6.10}$$

We now adopt a general form of $f(\mathcal{Q})$, which has the expression:

$$f(\mathcal{Q}) = \alpha + \beta Q^n \tag{6.11}$$

Here α, β and n represents three dimensionless free parameters.

Using the fact $f(\mathcal{Q}) = \alpha + \beta Q^n$, $f'(\mathcal{Q}) = \beta n Q^{n-1}$ and $f''(\mathcal{Q}) = \beta n(n-1)Q^{n-2}$. Substituting these values into the energy density and pressure equations for the non-metricitiy scalar:

$$\rho_Q = \frac{1}{16\pi}\left\{\alpha + \beta\left(-6^n\right)(2n-1)\mathcal{H}^{2n} + 6\mathcal{H}^2\right\} \tag{6.12}$$

$$p_Q = \frac{1}{48\pi\mathcal{H}^2}\left\{\beta 6^n(2n-1)\mathcal{H}^{2n}\left(2n\dot{\mathcal{H}} + 3\mathcal{H}^2\right) - 3\mathcal{H}^2\left(\alpha + 4\dot{\mathcal{H}} + 6\mathcal{H}^2\right)\right\} \tag{6.13}$$

Similarly, the equation-of-state parameter can be constructed as $w_Q = \frac{p_Q}{\rho_Q}$

$$w_q = \frac{\beta 6^n(2n-1)\mathcal{H}^{2n}\left(2n\dot{\mathcal{H}} + 3\mathcal{H}^2\right) - 3\mathcal{H}^2\left(\alpha + 4\dot{\mathcal{H}}\right) - 18\mathcal{H}^4}{3\mathcal{H}^2\left(\alpha + \beta\left(-6^n\right)(2n-1)\left(\mathcal{H}^2\right)^n + 6\mathcal{H}^2\right)} \tag{6.14}$$

Assuming the universe is only filled with dust matter $w_m = 0$, since $p_m = 0$ and $\rho_m = \rho_{m_0}/a^3$. The density parameter for matter and $\mathcal{Q}$ reads:

$$\Omega_m = \frac{8\pi}{3\mathcal{H}^2}\rho_m \tag{6.15}$$

$$\Omega_Q = \frac{8\pi}{3\mathcal{H}^2}\rho_Q \tag{6.16}$$

where Ω_m and Ω_Q represents ratio of the average density of matter and energy in the Universe to the critical density. Using the density parameters we can compute the hubble constant as [29]:

$$\mathcal{H} = \frac{\sqrt{\Omega_{m_0}}\mathcal{H}_0}{\sqrt{a^3\left(1-\Omega_Q\right)}} \tag{6.17}$$

And it's derivative as:

$$\dot{\mathcal{H}} = -\frac{\mathcal{H}}{2\left(1-\Omega_Q\right)}\left[3\mathcal{H}\left(1-\Omega_Q\right) - \partial_a\Omega_Q\right] \tag{6.18}$$

Transforming (6.18) to cosmological redshift parameter z defined as $a = \frac{1}{1+z}$:

$$\dot{\mathcal{H}} = -\frac{\mathcal{H}^2}{2\left(1-\Omega_Q\right)}\left[3\mathcal{H}\left(1-\Omega_Q\right) + (1+z)\,\Omega_Q'\right] \tag{6.19}$$

where prime reflects the derivative against the cosmological redshift.

6.1.2 Dust matter evolution

In this section, we seek to examine the cosmological evolution of the universe through the dynamical equations in $f(\mathcal{Q})$ gravity. We will use the equations developed in the last section to provide analytical solutions to the density parameter associated with the non-metricity scalar. We assume that the universe is filled with dust matter, hence $w_m = 0$ and $p_m = 0$ [30].

Using the continuity equation (6.3) for baryonic matter:

$$\rho_m = \frac{\rho_{m_0}}{a^3} \tag{6.20}$$

where ρ_{m_0} represents value at present scale factor $a_0 = 1$.

Since, the overall density parameter of the universe must be unity:

$$\Omega_m + \Omega_Q = \Omega_0 \tag{6.21}$$

Here $\Omega_0 = \rho/\rho_c$, ρ represents the total mass/energy density of the universe. To the accuracy of current cosmological observations and following our assuming of the flat universe, $\Omega_0 = 1$.

The density parameter can be recast into energy density using the critical density, $\rho_c = \dfrac{3\mathcal{H}^2}{8\pi G}$.

Following using (6.21) and (6.20):

$$\frac{\Omega_{m_0}}{a^3} + \frac{8\pi}{3\mathcal{H}^2}\rho_Q = 1 \tag{6.22}$$

Substituting (6.12) into the above equation:

$$\frac{\Omega_{m_0}}{a^3} + \frac{1}{6\mathcal{H}^2}\left\{\alpha + \beta\left(-6^n\right)(2n-1)\mathcal{H}^{2n} + 6\mathcal{H}^2\right\} = 1 \tag{6.23}$$

Substituting (6.17), the modified equation reads:

$$\frac{\Omega_{m_0}}{a^3} + \frac{a^3\left(1-\Omega_Q\right)}{6\Omega_{m_0}\mathcal{H}_0^2}\left\{\alpha - 6^n\beta(2n-1)\left(\frac{\sqrt{\Omega_{m_0}}\mathcal{H}_0}{\sqrt{a^3\left(1-\Omega_Q\right)}}\right)^{2n}\right\} = 0 \tag{6.24}$$

$$\frac{\Omega_{m_0}}{a^3} + \frac{a^3\left(1-\Omega_Q\right)}{6\Omega_{m_0}\mathcal{H}_0^2}\left\{\alpha - 6^n\beta(2n-1)\left(\Omega_{m_0}\mathcal{H}_0^2\right)^n\left(a^3\left(1-\Omega_Q\right)\right)^{-n}\right\} = 0 \tag{6.25}$$

Inserting the cosmological redshift in place of the scale factor, using $a = \dfrac{1}{1+z}$:

$$\frac{\Omega_{m_0}}{a^3} + \frac{(1+z)^{-3}\left(1-\Omega_Q\right)}{6\Omega_{m_0}\mathcal{H}_0^2}\left\{\alpha - 6^n\beta(2n-1)\left(\Omega_{m_0}\mathcal{H}_0^2\right)^n\left((1+z)^{-3}\left(1-\Omega_Q\right)\right)^{-n}\right\} = 0 \tag{6.26}$$

Simplifying the equation:

$$\frac{\Omega_{m_0}}{a^3} + \frac{(1+z)^{-3}\left(1-\Omega_Q\right)}{6\Omega_{m_0}\mathcal{H}_0^2}\alpha - 6^{(n-1)}\beta(2n-1)\left(\Omega_{m_0}\mathcal{H}_0^2\right)^{n-1}(1+z)^{3(n-1)}\left(1-\Omega_Q\right)^{1-n} = 0 \tag{6.27}$$

$$\Omega_m = -\frac{(1+z)^{-3}\left(1-\Omega_Q\right)}{6\Omega_{m_0}\mathcal{H}_0^2}\alpha - 6^{(n-1)}\beta(2n-1)\left(\Omega_{m_0}\mathcal{H}_0^2\right)^{n-1}(1+z)^{3(n-1)}\left(1-\Omega_Q\right)^{1-n} \tag{6.28}$$

As a boundary condition, our model must return to our vanilla general relativity when $f(\mathcal{Q}) = \mathcal{Q}$, i.e. $\alpha = 0, z = 0, n = 1$, which it does for (6.28). In such a model, there's no component from the non-metricity scalar and reduces to general relativity.

This equation can be simplified further numerically by computing relevant bionomial expansions. Computing the binomial expansion for $(1+z)^{3(n-1)}(1-\Omega_Q)^{1-n}$:

$$
\begin{aligned}
(1+z)^{3(n-1)}(1-\Omega_Q)^{1-n} = &\Big(2+(1-n)\Omega_Q+\frac{1}{2}(n-1)n\Omega_Q^2\\
&+\frac{1}{6}\left(n-n^3\right)\Omega_Q^3\Big)\\
&+3(n-1)z+\frac{3}{2}(n-1)(3n-4)z^2\\
&+\frac{1}{2}(n-1)(3n-5)(3n-4)z^3+O\left(z^4\right)+O\left(\Omega_Q^4\right)
\end{aligned}
\tag{6.29}
$$

Substituting (6.29) into (6.27):

$$
\begin{aligned}
&\frac{\Omega_{m_0}}{(1+z)^3}+\frac{(1+z)^{-3}(1-\Omega_Q)}{6\Omega_{m_0}\mathcal{H}_0^2}\alpha-6^{(n-1)}\beta(2n-1)\left(\Omega_{m_0}\mathcal{H}_0^2\right)^{n-1}\\
&\Big\{\Big(2+(1-n)\Omega_Q+\frac{1}{2}(n-1)n\Omega_Q^2+\frac{1}{6}\left(n-n^3\right)\Omega_Q^3\Big)\\
&+3(n-1)z+\frac{3}{2}(n-1)(3n-4)z^2\\
&+\frac{1}{2}(n-1)(3n-5)(3n-4)z^3+O\left(z^4\right)+O\left(\Omega_Q^4\right)\Big\}=0
\end{aligned}
\tag{6.30}
$$

$$
\begin{aligned}
\Omega_m = &-\frac{(1+z)^{-3}(1-\Omega_Q)}{6\Omega_{m_0}\mathcal{H}_0^2}\alpha-6^{(n-1)}\beta(2n-1)\left(\Omega_{m_0}\mathcal{H}_0^2\right)^{n-1}\\
&\Big(2+(1-n)\Omega_Q+\frac{1}{2}(n-1)n\Omega_Q^2+\frac{1}{6}\left(n-n^3\right)\Omega_Q^3\Big)\\
&+3(n-1)z+\frac{3}{2}(n-1)(3n-4)z^2\\
&+\frac{1}{2}(n-1)(3n-5)(3n-4)z^3+O\left(z^4\right)+O\left(\Omega_Q^4\right)=0
\end{aligned}
\tag{6.31}
$$

Applying the boundary condition $\alpha=0, n=1, z=0$, we can compute the value of $\beta=0.5$ to obtain proper density parameter for the matter.

The advantage of this model is that n can replace α in the equation above, leaving it with the same degrees of freedom as the ΛCDM model. We can compute the equation-of-state parameter w by using (6.12) and (6.13). Finally, we can compute another interesting called the deceleration parameter defined by (6.32). This parameter can be computed using the (6.17) and (6.19).

$$
q \equiv -\left(1+\frac{\dot{\mathcal{H}}}{\mathcal{H}^2}\right)
\tag{6.32}
$$

The gaps in general relativity have been filled by a number of modified gravity theories. However, the ΛCDM model has certain drawbacks while being quite effective at describing the rapid expansion. Hence, it makes sense to mimic the exact accelerated history via use of modified gravity theories like fQ

In summary, we have developed numerical and analytic equations for the density parameter associated with the non-metricity scalar $\mathcal{Q}$. In conclusion, modified gravity theories with the non-metricity scalar $\mathcal{Q}$ may provide exciting new perspectives for extending the study of our universe, but it is necessary to examine a variety of models to identify potentially effective substitutes.

Chapter 7

Conclusion and Future Prospects

$f(\mathcal{Q})$ gravity presents a uniquely situated theory which is able to explain the early constraints while also explaining the current scenario, as discussed in 4.1.1. It talks about how $f(\mathcal{Q})$ is so well suited as an alternative theory of gravity compared to other models like $f(\mathcal{R})$, $f(\mathcal{T})$ etc.

In this work we have successfully developed the Einstein's field equations in $f(\mathcal{Q})$ gravity 4.2. In addition to this, we have computed the tensor, vector and scalar cosmological perturbation equations in $f(\mathcal{Q})$ using the *Pytearcat* package [27].

Moreover, we have developed the Friedmann's equations 4.3, both trivial and acceleration one and have tried to use them to develop a cosmological evolution model. As an extension to his, we have developed density parameter and other factors for the dust matter evolution6.1.2 and have discussed it advantages in the two-component universe containing matter and $\mathcal{Q}$ non-metricity scalar.

In future, the work can be extended by developing a better multiple component universe containing matter, radiation, curvature and non-metricity scalar. In this case, the non-metricity will manifest the unexplained dark-energy/dark-matter in the universe. This model would be ideal in examining and comparing different cosmological parameters such as the density parameter, Hubble constant, etc, with the Planck collaboration model.

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Appendix A

Foundations of general relativitiy

A.1 Covariant derivative

Let us assume the covarient derivative of other sorts of tensors is given by,

$$\nabla_\mu w_\lambda = \partial_\mu w_\lambda + \tilde{\Gamma}^\sigma_{\mu\lambda} w_\sigma \tag{A.1}$$

Assumptions:

- covarient Derivative of a scaler is same as its oridinary derivative: $\nabla_\mu(\phi) = \partial_\mu \phi$
- contraction commutes: $\nabla_\mu T^\lambda_{\lambda\nu} = (\nabla T)_\mu{}^\lambda{}_{\lambda\nu}$

$$\therefore \quad \nabla_\mu(w_\lambda V^\lambda) = V^\lambda \partial_\mu(w_\lambda) + w_\lambda \partial_\mu(V^\lambda) + V^\lambda \tilde{\Gamma}^\sigma{}_{\mu\lambda} w_\sigma + w_\lambda \Gamma^\lambda{}_{\mu\rho} V^\rho \tag{A.2}$$

Since Equation A.2 is covarient derivative of a scaler, it should result into only partial derivative of the scaler,

$$\nabla_\mu(w_\lambda V^\lambda) = V^\lambda \partial_\mu(w_\lambda) + w_\lambda \partial_\mu(V^\lambda) \tag{A.3}$$

$$\therefore \quad V^\lambda \tilde{\Gamma}^\sigma{}_{\mu\lambda} w_\sigma + w_\lambda \Gamma^\lambda{}_{\mu\rho} V^\rho = 0 \tag{A.4}$$

$$V^\lambda \tilde{\Gamma}^\sigma{}_{\mu\lambda} w_\sigma = -w_\lambda \Gamma^\lambda{}_{\mu\rho} V^\rho \tag{A.5}$$

But, w_λ and V^λ are arbitary and but changing the dummy index,

$$\tilde{\Gamma}^\sigma{}_{\mu\lambda} = -\Gamma^\sigma{}_{\mu\lambda} \tag{A.6}$$

Hence from Equation A.1,

$$\nabla_\mu w_\lambda = \partial_\mu w_\lambda - \Gamma^\rho{}_{\mu\lambda} w_\rho \tag{A.7}$$

$$\nabla_\mu = \partial_\mu - \Gamma^\rho{}_{\mu\lambda} \tag{A.8}$$

A.2 Covarient derivative of the metric tensor

$$\nabla_p g_{mn} = \frac{\partial g_{mn}}{\partial y^p} - \Gamma^r_{pm} g_{rn} - \Gamma^r_{pn} g_{mr} \tag{A.9}$$

For a flat geometry, Equation A.9 is zero.

The christoffel symbols for a flat geoemtry are:

$$\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma} \left\{ \frac{\partial g_{\sigma\beta}}{\partial y^p} + \frac{\partial g_{\sigma\alpha}}{\partial y^m} - \frac{\partial g_{\alpha\beta}}{\partial y^\gamma} \right\}$$

RHS of Equation A.9

$$RHS = \frac{\partial g_{mn}}{\partial y^p} - \frac{1}{2} \left\{ g^{rd} \{ \frac{\partial g_{dm}}{\partial y^p} + \frac{\partial g_{dp}}{\partial y^m} - \frac{\partial g_{pm}}{\partial y^r} \} g_{rn} + g^{r'd'} \{ \frac{\partial g_{d'n}}{\partial y^p} + \frac{\partial g_{d'p}}{\partial y^n} - \frac{\partial g_{pn}}{\partial y^{r'}} \} g_{mr} \right\} \tag{A.10}$$

Since, the Geometry is flat, it implies that $d = r$ and $d' = r'$,

$$RHS = \frac{\partial g_{mn}}{\partial y^p} - \frac{1}{2} \left\{ \delta^r{}_n \{ \frac{\partial g_{dm}}{\partial y^p} + \frac{\partial g_{dp}}{\partial y^m} - \frac{\partial g_{pm}}{\partial y^r} \} + \delta^{r'}{}_m \{ \frac{\partial g_{d'n}}{\partial y^p} + \frac{\partial g_{d'p}}{\partial y^n} - \frac{\partial g_{pn}}{\partial y^{r'}} \} \right\} \tag{A.11}$$

From the properties of the metric tensor $g^{rr} g_{rn} = \delta^r{}_n$, Therefore, $r = n$ and $r' = m$,

$$RHS = \frac{\partial g_{mn}}{\partial y^p} - \frac{1}{2} \left\{ \frac{\partial g_{nm}}{\partial y^p} + \frac{\partial g_{np}}{\partial y^m} - \frac{\partial g_{pm}}{\partial y^n} + \frac{\partial g_{mn}}{\partial y^p} + \frac{\partial g_{mp}}{\partial y^n} - \frac{\partial g_{pn}}{\partial y^m} \right\} \tag{A.12}$$

From the properties of the metric tensor $g_{mn} = g_{nm}$,

Hence,

$$\nabla_p g_{mn} = RHS = 0$$

Appendix B

Classical perturbations

I Classical perturbations

B.1 Background model

Assuming the background to be a FRLW universe which represents a homogeneous and isotropic universe, which metric defined as:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2\delta_{ij}\left(1 + \frac{\mathcal{K}}{4}(x^2 + y^2 + z^2)\right)^{-2} dx^i dx^j \tag{B.1}$$

$$ds^2 = dt^2 - a^2\left(1 + \frac{\mathcal{K}}{4}(x^2 + y^2 + z^2)\right)^{-2}\left(dx^2 + dy^2 + dz^2\right) \tag{B.2}$$

Assuming conformal time, $d\eta = \dfrac{dt}{a}$.

$$ds^2 = a^2\left\{d\eta^2 - \left(1 + \frac{\mathcal{K}}{4}(x^2 + y^2 + z^2)\right)^{-2}\left(dx^2 + dy^2 + dz^2\right)\right\} \tag{B.3}$$

Let $Z(x, y, z) = \left(1 + \dfrac{\mathcal{K}}{4}(x^2 + y^2 + z^2)\right)^{-2}$ and $a' = \dfrac{da}{d\eta}$

$$ds^2 = a^2\left\{d\eta^2 - Z(x, y, z)\left(dx^2 + dy^2 + dz^2\right)\right\} \tag{B.4}$$

$$g_{\mu\nu} = diag(a^2, -a^2 Z(x, y, z), -a^2 Z(x, y, z), -a^2 Z(x, y, z)) \tag{B.5}$$

In order to obtain the dynamical equations, we have to compute the christoffel symbols, for which we assume a local inertial frame(LIF) in which the covariant derivate of the metric vanishes:

$$\nabla_\alpha g_{\mu\nu} = 0 \tag{B.6}$$

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{g^{\alpha\lambda}}{2}\left[\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right] \tag{B.7}$$

Since our background metric is diagonal, we can simplify:

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{g^{\alpha\alpha}}{2}\left[\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right] \tag{B.8}$$

For $\Gamma^{0}{}_{\mu\nu}$:

$$\Gamma^{0}{}_{\mu\nu} = \frac{g^{00}}{2}\left[\partial_\mu g_{0\nu} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu}\right] \tag{B.9}$$

$$\begin{aligned}
\Gamma^{0}{}_{00} &= \frac{1}{2a^2}\partial_0 g_{00} = \frac{1}{2a^2}\partial_0 a^2 = \frac{a'}{a} \\
\Gamma^{0}{}_{11} &= -\frac{1}{2a^2}\partial_0 g_{11} = \frac{1}{2a^2}\partial_0 Z(x,y,z)a^2 = \frac{a'}{a}Z(x,y,z) \\
\Gamma^{0}{}_{22} &= -\frac{1}{2a^2}\partial_0 g_{22} = \frac{1}{2a^2}\partial_0 Z(x,y,z)a^2 = \frac{a'}{a}Z(x,y,z) \\
\Gamma^{0}{}_{33} &= -\frac{1}{2a^2}\partial_0 g_{33} = \frac{1}{2a^2}\partial_0 Z(x,y,z)a^2 = \frac{a'}{a}Z(x,y,z)
\end{aligned} \tag{B.10}$$

For $\Gamma^{1}{}_{\mu\nu}$:

$$\Gamma^{1}{}_{\mu\nu} = \frac{g^{11}}{2}\left[\partial_\mu g_{1\nu} + \partial_\nu g_{\mu 1} - \partial_1 g_{\mu\nu}\right] \tag{B.11}$$

$$\begin{aligned}
\Gamma^{1}{}_{00} &= -\frac{1}{2g_{11}}\partial_1 g_{00} = -\frac{1}{2a^2 Z(x,y,z)}\partial_1 g_{00} = 0 \\
\Gamma^{1}{}_{11} &= \frac{1}{2g_{11}}\partial_1 g_{11} = \frac{1}{2a^2 Z(x,y,z)}\partial_1 Z(x,y,z)a^2 = \frac{-x\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}\left(x^2+y^2+z^2\right)\right)} \\
\Gamma^{1}{}_{22} &= -\frac{1}{2g_{11}}\partial_1 g_{22} = -\frac{1}{2a^2 Z(x,y,z)}\partial_1 Z(x,y,z)a^2 = \frac{x\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}\left(x^2+y^2+z^2\right)\right)} \\
\Gamma^{1}{}_{33} &= -\frac{1}{2g_{11}}\partial_1 g_{33} = -\frac{1}{2a^2 Z(x,y,z)}\partial_1 Z(x,y,z)a^2 = \frac{x\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}\left(x^2+y^2+z^2\right)\right)}
\end{aligned} \tag{B.12}$$

$$\begin{aligned}
\Gamma^{1}{}_{10} = \Gamma^{1}{}_{01} &= \frac{1}{2g_{11}}\partial_0 g_{11} = \frac{1}{2a^2}\partial_0 a^2 = \frac{a'}{a} \\
\Gamma^{1}{}_{12} = \Gamma^{1}{}_{21} &= \frac{1}{2g_{11}}\partial_2 g_{11} = \frac{1}{2Z(x,y,z)}\partial_2 Z(x,y,z) = \frac{-y\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}\left(x^2+y^2+z^2\right)\right)} \\
\Gamma^{1}{}_{13} = \Gamma^{1}{}_{31} &= \frac{1}{2g_{11}}\partial_3 g_{11} = \frac{1}{2Z(x,y,z)}\partial_3 Z(x,y,z) = \frac{-z\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}\left(x^2+y^2+z^2\right)\right)}
\end{aligned} \tag{B.13}$$

For ${\Gamma^2}_{\mu\nu}$:

$$ {\Gamma^2}_{\mu\nu} = \frac{g^{22}}{2}\left[\partial_\mu g_{2\nu} + \partial_\nu g_{\mu 2} - \partial_2 g_{\mu\nu}\right] \tag{B.14} $$

$$ \begin{aligned} {\Gamma^2}_{00} &= -\frac{1}{2g_{22}}\partial_2 g_{00} = -\frac{1}{2a^2 Z(x,y,z)}\partial_2 g_{00} = 0 \\ {\Gamma^2}_{11} &= -\frac{1}{2g_{22}}\partial_2 g_{11} = -\frac{1}{2a^2 Z(x,y,z)}\partial_2 Z(x,y,z)a^2 = \frac{y\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right)} \\ {\Gamma^2}_{22} &= \frac{1}{2g_{22}}\partial_2 g_{22} = \frac{1}{2a^2 Z(x,y,z)}\partial_2 Z(x,y,z)a^2 = \frac{-y\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right)} \\ {\Gamma^2}_{33} &= -\frac{1}{2g_{22}}\partial_2 g_{33} = -\frac{1}{2a^2 Z(x,y,z)}\partial_2 Z(x,y,z)a^2 = \frac{y\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right)} \end{aligned} \tag{B.15} $$

$$ \begin{aligned} {\Gamma^2}_{20} = {\Gamma^2}_{02} &= \frac{1}{2g_{22}}\partial_0 g_{22} = \frac{1}{2a^2}\partial_0 a^2 = \frac{a'}{a} \\ {\Gamma^2}_{21} = {\Gamma^2}_{12} &= \frac{1}{2g_{22}}\partial_1 g_{22} = \frac{1}{2Z(x,y,z)}\partial_1 Z(x,y,z) = \frac{-x\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right)} \\ {\Gamma^2}_{23} = {\Gamma^2}_{32} &= \frac{1}{2g_{22}}\partial_3 g_{22} = \frac{1}{2Z(x,y,z)}\partial_3 Z(x,y,z) = \frac{-z\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right)} \end{aligned} \tag{B.16} $$

For ${\Gamma^3}_{\mu\nu}$:

$$ {\Gamma^3}_{\mu\nu} = \frac{g^{33}}{2}\left[\partial_\mu g_{3\nu} + \partial_\nu g_{\mu 3} - \partial_3 g_{\mu\nu}\right] \tag{B.17} $$

$$ \begin{aligned} {\Gamma^3}_{00} &= -\frac{1}{2g_{33}}\partial_3 g_{00} = -\frac{1}{2a^2 Z(x,y,z)}\partial_3 g_{00} = 0 \\ {\Gamma^3}_{11} &= -\frac{1}{2g_{33}}\partial_3 g_{11} = -\frac{1}{2a^2 Z(x,y,z)}\partial_3 Z(x,y,z)a^2 = \frac{z\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right)} \\ {\Gamma^3}_{22} &= -\frac{1}{2g_{33}}\partial_3 g_{22} = -\frac{1}{2a^2 Z(x,y,z)}\partial_3 Z(x,y,z)a^2 = \frac{z\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right)} \\ {\Gamma^3}_{33} &= \frac{1}{2g_{33}}\partial_3 g_{33} = \frac{1}{2a^2 Z(x,y,z)}\partial_3 Z(x,y,z)a^2 = \frac{-z\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right)} \end{aligned} \tag{B.18} $$

$$\Gamma^3{}_{30} = \Gamma^3{}_{03} = \frac{1}{2g_{33}}\partial_0 g_{33} = \frac{1}{2a^2}\partial_0 a^2 = \frac{a'}{a}$$
$$\Gamma^3{}_{31} = \Gamma^3{}_{13} = \frac{1}{2g_{33}}\partial_1 g_{33} = \frac{1}{2Z(x,y,z)}\partial_1 Z(x,y,z) = \frac{-x\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}\left(x^2+y^2+z^2\right)\right)} \tag{B.19}$$
$$\Gamma^3{}_{32} = \Gamma^3{}_{23} = \frac{1}{2g_{33}}\partial_2 g_{33} = \frac{1}{2Z(x,y,z)}\partial_2 Z(x,y,z) = \frac{-y\mathcal{K}}{2\left(1+\frac{\mathcal{K}}{4}\left(x^2+y^2+z^2\right)\right)}$$

$$\boxed{\begin{aligned}
\Gamma^0{}_{00} = \Gamma^1{}_{10} = \Gamma^1{}_{01} = \Gamma^2{}_{20} = \Gamma^2{}_{02} = \Gamma^3{}_{30} = \Gamma^3{}_{03} &= \frac{a'}{a}\\
\Gamma^0{}_{11} = \Gamma^0{}_{22} = \Gamma^0{}_{33} &= \frac{a'}{a}\frac{1}{1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)}\\
\Gamma^1{}_{11} = \Gamma^2{}_{21} = \Gamma^2{}_{12} = \Gamma^2{}_{13} = \Gamma^3{}_{31} = -\Gamma^1{}_{22} = -\Gamma^1{}_{33} &= -\frac{1}{2}\frac{x\mathcal{K}}{1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)}\\
\Gamma^2{}_{22} = \Gamma^1{}_{12} = \Gamma^1{}_{21} = \Gamma^3{}_{23} = \Gamma^3{}_{32} = -\Gamma^2{}_{11} = -\Gamma^2{}_{33} &= -\frac{1}{2}\frac{y\mathcal{K}}{1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)}\\
\Gamma^3{}_{33} = \Gamma^2{}_{23} = \Gamma^2{}_{32} = \Gamma^1{}_{13} = \Gamma^1{}_{31} = -\Gamma^3{}_{11} = -\Gamma^3{}_{22} &= -\frac{1}{2}\frac{z\mathcal{K}}{1+\frac{\mathcal{K}}{4}(x^2+y^2+z^2)}
\end{aligned}} \tag{B.20}$$

We can use these Christoffel connections to compute the Riemann's curvature tensor and subsequencty the Ricci tensor and Ricci scalar as:

$$\mathcal{R}^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} + \Gamma^\alpha{}_{\mu\lambda}\Gamma^\lambda{}_{\beta\nu} - (\mu \leftrightarrow \nu) \tag{B.21}$$

$$\mathcal{R}_{\beta\nu} = \mathcal{R}^\mu{}_{\beta\mu\nu}; \quad \mathcal{R} = g^{\beta\nu}\mathcal{R}_{\beta\nu} \tag{B.22}$$

$$\mathcal{R}_{\beta\nu} = \partial_\mu \Gamma^\mu{}_{\beta\nu} + \Gamma^\mu{}_{\mu\lambda}\Gamma^\lambda{}_{\beta\nu} - \partial_\nu \Gamma^\mu{}_{\mu\beta} - \Gamma^\mu{}_{\nu\lambda}\Gamma^\lambda{}_{\beta\mu} \tag{B.23}$$

$$\boxed{\begin{aligned}
\mathcal{R}_{00} &= \partial_\mu \Gamma^\mu{}_{00} + \Gamma^\mu{}_{\mu\lambda}\Gamma^\lambda{}_{00} - \partial_0 \Gamma^\mu{}_{\mu 0} - \Gamma^\mu{}_{0\lambda}\Gamma^\lambda{}_{0\mu} = \frac{3\left(a'^2 - aa''\right)}{a^2} \\
\mathcal{R}_{11} &= \partial_\mu \Gamma^\mu{}_{11} + \Gamma^\mu{}_{\mu\lambda}\Gamma^\lambda{}_{11} - \partial_1 \Gamma^\mu{}_{\mu 1} - \Gamma^\mu{}_{1\lambda}\Gamma^\lambda{}_{1\mu} = \frac{(aa'' + a'^2 + 2\mathcal{K}a^2)}{a^2\left[1 + \frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right]^2} \\
\mathcal{R}_{22} &= \partial_\mu \Gamma^\mu{}_{22} + \Gamma^\mu{}_{\mu\lambda}\Gamma^\lambda{}_{22} - \partial_2 \Gamma^\mu{}_{\mu 2} - \Gamma^\mu{}_{2\lambda}\Gamma^\lambda{}_{2\mu} = \frac{(aa'' + a'^2 + 2\mathcal{K}a^2)}{a^2\left[1 + \frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right]^2} \\
\mathcal{R}_{33} &= \partial_\mu \Gamma^\mu{}_{33} + \Gamma^\mu{}_{\mu\lambda}\Gamma^\lambda{}_{33} - \partial_3 \Gamma^\mu{}_{\mu 3} - \Gamma^\mu{}_{3\lambda}\Gamma^\lambda{}_{3\mu} = \frac{(aa'' + a'^2 + 2\mathcal{K}a^2)}{a^2\left[1 + \frac{\mathcal{K}}{4}(x^2+y^2+z^2)\right]^2}
\end{aligned}} \tag{B.24}$$

Plugging (B.24) into $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ allows us to compute the ricci scalar, considering the tensor is diagonal makes our job easier:

$$\mathcal{R} = g^{00}\mathcal{R}_{00} + g^{11}\mathcal{R}_{11} + g^{22}\mathcal{R}_{22} + g^{33}\mathcal{R}_{33} \tag{B.25}$$

$$\boxed{\mathcal{R} = -\frac{6\left(a'' + \mathcal{K}a\right)}{a^3}} \tag{B.26}$$

Plugging these into the field equation $\mathcal{R}_{\mu\nu} - g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu}$ would give us the dynamical equations; The 0-0 equation reduces to the Friedmann's equation:

$$G_{00} = \mathcal{R}_{00} - g_{00}\mathcal{R} = 8\pi G T_{00} \tag{B.27}$$

Giving us the Friedmann's equation:

$$\boxed{\left(\frac{a'}{a}\right)^2 + \mathcal{K} = \frac{8\pi G}{3} T_{00}} \tag{B.28}$$

The same result can be obtained using the contracted form of the EFE:

$$g^{\mu\lambda}\left[\mathcal{R}_{\lambda\nu} - \frac{1}{2}g_{\lambda\nu}R = 8\pi G T_{\lambda\nu}\right]$$

$$\mathcal{R}^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = 8\pi G T^\mu{}_\nu \tag{B.29}$$

$0 - 0$th equation giving us

$$\frac{1}{a^2}\frac{3}{a^2}\left(a'^2 + aa''\right) + \frac{1}{2}\frac{6\left(a'' + \mathcal{K}a\right)}{a^3} = 8\pi G T^0{}_0$$

$$\boxed{a'^2 + \mathcal{K}a^2 = \frac{8\pi G}{3} T^0{}_0\, a^4} \tag{B.30}$$

and the $i-i$th giving us the accleration of scale factor equation: $(\delta^{\mu}{}_{\mu}=4)$

$$\mathcal{R}=-8\pi GT$$

$$\boxed{a''+\mathcal{K}a=\frac{4\pi G}{3}Ta^3} \tag{B.31}$$

In order to obtain the continuity equation, we differentiate (B.30) and subtract $2a'$ times (B.31):

$$\begin{aligned}
2a'a''+2\mathcal{K}aa' &= \frac{8\pi G}{3}T^0{}_0\left(4a^3a'\right)+\frac{8\pi G}{3}a^4dT^0{}_0 \\
-2a'a''-2\mathcal{K}aa' &= -\frac{4\pi G}{3}T\left(2a'a^3\right)
\end{aligned}$$

$$\boxed{dT^0{}_0=-(4T^0{}_0-T)\frac{a'}{a}=-(4T^0{}_0-T)d\ln a} \tag{B.32}$$

B.2 Perturbations

In order to model a realistic universe we add the perturbations, full line element becomes:

$$ds^2=\left(g_{\mu\nu}+\delta g_{\mu\nu}\right)dx^{\mu}dx^{\nu} \tag{B.33}$$

The perturbations can be of three types: Scalar, Vector and Tensor. This classification refers to the way in which fields from which $\delta g_{\mu\nu}$ are constructed and transform under three-space coordinate transformations.

B.2.1 Scalar perturbations

This type of perturbations results in growing inhomogenities which can lead to structure formation.

We can generate this via (1.) multiplying a scalar with γ_{ij}, (2.) covariant derivate of a scalar function, which reduces to ordinary derivative for $\mathcal{K}=0$ flat spacetime.

$$\delta g_{\mu\nu}{}^{(s)}=a^2(\eta)\begin{pmatrix} 2\phi & -B_{|i} \\ -B_{|i} & 2\left(\psi\gamma_{ij}-E_{|ij}\right)\end{pmatrix} \tag{B.34}$$

$$g_{\mu\nu}+\delta g_{\mu\nu}{}^{(s)}=a^2(\eta)\begin{pmatrix} 1+2\phi & -B_{|i} \\ -B_{|i} & -\left[(1-2\psi)\gamma_{ij}+2E_{|ij}\right]\end{pmatrix} \tag{B.35}$$

So, the total line element becomes:

$$ds^2 = a^2(\eta)\left\{(1+2\phi)\,d\eta^2 - 2B_{|i}dx^i d\eta - \Big[(1-2\psi)\,\gamma_{ij} + 2E_{|ij}\Big]\,dx^i dx^j\right\} \tag{B.36}$$

B.2.2 Vector perturbations

These are constructed by considering two 3 vectors and two constraints ensuring the two vector fields are divergenless, S_i and F_i. If these are not divergenless, then we can seperate it into vector and gradiant of a scalar, hence it would not be a pure vector perturbation.

$$S_i^{|i} = F_i^{|i} = 0$$

$$\delta g_{\mu\nu}{}^{(v)} = -a^2(\eta)\begin{pmatrix} 0 & -S_i \\ -S_i & F_{i|j} + F_{j|i} \end{pmatrix} \tag{B.37}$$

$$g_{\mu\nu} + \delta g_{\mu\nu}{}^{(v)} = a^2(\eta)\begin{pmatrix} 1 & S_i \\ S_i & -\left(\gamma_{ij} + F_{i|j} + F_{j|i}\right) \end{pmatrix} \tag{B.38}$$

So, the total line element becomes:

$$ds^2 = a^2(\eta)\left\{d\eta^2 + 2S_i dx^i d\eta - \left(\gamma_{ij} + F_{i|j} + Fj|i\right)dx^i dx^j\right\} \tag{B.39}$$

B.2.3 Tensor perturbation

These are constructed using a symmetric traceless three-tensor h_{ij} satisfying:

$$h_i^i = 0; \quad h_{ij}^{|i} = 0 \tag{B.40}$$

$$\delta g_{\mu\nu}{}^{(t)} = -a^2(\eta)\begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix} \tag{B.41}$$

$$g_{\mu\nu} + \delta g_{\mu\nu}{}^{(t)} = a^2(\eta)\begin{pmatrix} 1 & 0 \\ 0 & -\left(\gamma_{ij} + h_{ij}\right) \end{pmatrix} \tag{B.42}$$

$$ds^2 = a^2(\eta)\left(d\eta^2 - \left(\gamma_{ij} + h_{ij}\right)dx^i dx^j\right) \tag{B.43}$$

B.3 Gauge transformations

Consider an infitisimal transformation, described by $\xi^\alpha = (\xi^0, \xi^i)$:

$$x^\alpha \to \widetilde{x}^\alpha = x^\alpha + \xi^\alpha \tag{B.44}$$

$$x^\alpha = \widetilde{x}^\alpha - \xi^\alpha$$

so the transformation law is:

$$X^\alpha_{\tilde{\mu}} = \frac{dx^\alpha}{d\widetilde{x}^\mu} = \delta^\alpha_\mu - \xi^\alpha_{,\mu}$$

In order to obtain the transformation eq for the metric, we generalize it using a type (0, 2) 4-tensor field:

$$\widetilde{B}_{\mu\nu} = X^\alpha_{\tilde{\mu}} X^\alpha_{\tilde{\mu}} \left(B_{\alpha\beta} - B_{\alpha\beta,\lambda}\xi^\lambda \right) \tag{B.45}$$

$$\widetilde{B}_{\mu\nu} = \left(\delta^\alpha_\mu - \xi^\alpha_{,\mu}\right)\left(\delta^\beta_\nu - \xi^\beta_{,\nu}\right)\left(B_{\alpha_\beta} - B_{\alpha\beta,\lambda}\xi^\lambda\right)$$

Ignoring the 2nd order perturbations:

$$\widetilde{B}_{\mu\nu} = B_{\mu\nu} - \xi^\alpha_{,\mu} B_{\alpha\nu} - \xi^\beta_{,\nu} B_{\mu\beta} - B_{\mu\nu,\alpha}\xi^\alpha \tag{B.46}$$

Following this equation, we can compute the linear perturbations for our metric:

$$\delta\widetilde{g}_{\mu\nu} = \delta g_{\mu\nu} - \xi^\alpha_{,\mu}\, g_{\alpha\nu} - \xi^\beta_{,\nu}\, g_{\mu\beta} - g_{\mu\nu,\alpha}\, \xi^\alpha \tag{B.47}$$

New functions $\widetilde{\phi}, \widetilde{\psi}, \widetilde{B}, \widetilde{E}$ can be calculated as:

$$\begin{aligned} \delta\widetilde{g}_{00} = 2a^2\widetilde{\phi} &= \delta g_{00} - \xi^\alpha_{,0}\, g_{\alpha 0} - \xi^\beta_{,0}\, g_{0\beta} - g_{00,\alpha}\, \xi^\alpha \\ 2a^2\widetilde{\phi} &= 2a^2\phi - 2a^2\xi^{0'} - (2aa')\xi^0 \\ \widetilde{\phi} &= \phi - \frac{a'}{a}\xi^0 - \xi^{0'} \end{aligned} \tag{B.48}$$

$$\begin{aligned} \delta\widetilde{g}_{0i} = -a^2\widetilde{B}_{|i} &= -a^2 B_{|i} - \xi^\alpha_{,0}\, g_{\alpha i} - \xi^\beta_{,i}\, g_{0\beta} - g_{0i,\alpha}\, \xi^\alpha \\ -a^2\widetilde{B}_{|i} &= -a^2 B_{|i} - \xi^i_{,0}\, g_{ii} - \xi^0_{,i}\, g_{00} \\ \widetilde{B}_{|i} &= B_{|i} + \xi^i_{,0} + \xi^0_{,i} \end{aligned} \tag{B.49}$$

$$
\begin{aligned}
\delta\tilde{g}_{ij} = 2a^2(\tilde{\psi}\gamma_{ij} - \tilde{E}_{|ij}) &= \delta g_{ij} - \xi^{\alpha}_{,i}\, g_{\alpha j} - \xi^{\beta}_{,j}\, g_{i\beta} - g_{ij,\alpha}\, \xi^{\alpha} \\
2a^2(\tilde{\psi}\gamma_{ij} - \tilde{E}_{|ij}) &= 2a^2(\psi\gamma_{ij} - E_{|ij}) - \xi^{\alpha}_{,i}\, g_{\alpha j} - \xi^{\beta}_{,j}\, g_{i\beta} - g_{ij,\alpha}\, \xi^{\alpha} \\
2a^2(\tilde{\psi}\gamma_{ij} - \tilde{E}_{|ij}) &= 2a^2(\psi\gamma_{ij} - E_{|ij}) - \xi^{j}_{,i}\, g_{jj} - \xi^{i}_{,j}\, g_{ii} - g_{ij,\alpha}\, \xi^{\alpha} \\
&\text{comparing the traceless part} \\
\tilde{\psi} = \psi + \frac{a'}{a}\xi^0, &\quad \tilde{E} = E - \xi
\end{aligned}
\tag{B.50}
$$

B.3.1 Synchronous gauage

It is defined by $\tilde{\phi} = \tilde{B} = 0$

Using $\tilde{\phi} = 0$ and solving the ODE:

$$
\begin{aligned}
\xi^{0\prime} + \frac{a'}{a}\xi^0 &= \phi \\
\xi^0 &= e^{-\int d\eta \frac{a'}{a}} \int d\eta\, e^{\int d\eta \frac{a'}{a}} \phi \\
\xi^0 &= a^{-1} \int d\eta\, a\phi
\end{aligned}
\tag{B.51}
$$

Using $\tilde{B} = 0$ and solving the ODE:

$$
\begin{aligned}
\xi' &= B + \xi^0 \\
\xi &= \int d\eta \left(B + a^{-1} \int d\eta\, a\phi \right)
\end{aligned}
\tag{B.52}
$$

Using (B.51) and (B.52):

$$
\eta \to \eta_s = \eta + a^{-1} \int d\eta\, a\phi, \qquad x^i \to x^i_s = x_i + \gamma^{ij} \left(\int d\eta \left(B + a^{-1} \int d\eta\, a\phi \right) \right)_{|j}
\tag{B.53}
$$

However, due to the integration constants, the synchronous-gauge has residual coordinate freedom which leads to appearance of unphysical gauge modes.

B.3.2 Longitudinal gauge

It is defined by $B = E = 0$. Since, the conditions uniquely fixes ξ and ξ^0, there are no residual modes.

$$
\tilde{E} = E - \xi = 0 \implies \xi = E
\tag{B.54}
$$

$$\begin{aligned} \xi_0 &= \xi' - B \\ \xi_0 &= E' - B \end{aligned} \tag{B.55}$$

$$\eta \to \eta_l = \eta + (E' - B), \qquad x^i \to x^i_l = x_i + \gamma^{ij} E_{|j} \tag{B.56}$$

The metric variables transform as:

$$\begin{aligned} \phi \to \phi_l &= \phi + a^{-1}\left\{a(B - E')\right\}' = \Phi \\ \psi \to \psi_l &= \psi - \frac{a'}{a}(B - E') = \Psi \\ B \to B_l &= 0 \\ E \to E_l &= 0 \end{aligned} \tag{B.57}$$

With the metric defined by:

$$ds^2 = a^2(\eta)\left((1 + 2\Phi)d\eta^2 - (1 - 2\Psi)dx^i dx^j\right) \tag{B.58}$$

If the stress-energy tensor has diagonal spatial parts, i.e. $\delta T^i{}_j \sim \delta^i_j$, it follows that $\phi_1 = \psi_1$ or $\Phi = \Psi$, leaving only one free metric perturbation variable, which is a generalization of the Newtonian gravitational potential ϕ. That's why this gauge is called "conformal-Newtonian" gauge.

The gauge invariant Φ and Ψ represents the amplitudes of metric perturbations in conformal-Newtonian gauge.

B.4 General form of the equations for cosmological perturbations

Using the EFE as defined in our background metric (B.1):

$$G^\mu{}_\nu = \mathcal{R}^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu \mathcal{R} = 8\pi G T^\mu{}_\nu$$

From (B.30)

$${}^{(0)}G^0{}_0 = \frac{3}{a^2}\left(\mathcal{H}^2 + \mathcal{K}\right), \quad {}^{(0)}G^0{}_i = 0, \quad {}^{(0)}G^i{}_j = \delta^i{}_j a^{-2}\left(\mathcal{H}' + \mathcal{H}^2 + \mathcal{K}\right) \tag{B.59}$$

$$
\begin{aligned}
{}^{(0)}G^{i}{}_{j} &= g^{ik}\mathcal{R}_{kj} - \frac{1}{2}\delta^{i}{}_{j}\mathcal{R} \\
&= -\delta^{i}{}_{j}a^{-4}\left(aa'' + a'^{2} + 2\mathcal{K}a^{2}\right) - \frac{1}{2}\delta^{i}{}_{j}\mathcal{R} \\
&= \delta^{i}{}_{j}a^{-4}\left\{-\left(aa'' + a'^{2} + 2\mathcal{K}a^{2}\right) + 3\left(aa'' + 3\mathcal{K}a^{2}\right)\right\} \\
&= \delta^{i}{}_{j}a^{-4}\left\{2aa'' - a'^{2} + \mathcal{K}a^{2}\right\} \\
&= \delta^{i}{}_{j}a^{-2}\left(\mathcal{H}' + \mathcal{H}^{2} + \mathcal{K}\right)
\end{aligned}
\tag{B.60}
$$

Here, $\mathcal{H} = a'/a$ and $a' = \dfrac{da}{d\eta}$

The form of ${}^{(0)}G^{\mu}{}_{\nu}$ using physical time can be obtained by inserting the change of variables $t = t(\eta)$;

$$
{}^{(0)}G^{\mu}{}_{\nu} = 8\pi G^{(0)}T^{\mu}{}_{\nu}
$$

The background energy-momentum tensor, represented by ${}^{(0)}T^{\mu}{}_{\nu}$, must be diagonal and follow following symmetries:

$$
{}^{(0)}T^{i}_{\ 0} ={}^{(0)} T^{0}_{\ i} = 0, \quad {}^{(0)}T^{i}_{\ j} \propto \delta^{i}_{j}
$$

For a metric with small perturbations:

$$
G^{\mu}{}_{\nu} ={}^{(0)} G^{\mu}{}_{\nu} + \delta G^{\mu}{}_{\nu} + ...
$$

For small linearized perturbations:

$$
\delta G^{\mu}{}_{\nu} = 8\pi G \delta T^{\mu}{}_{\nu} \tag{B.61}
$$

Considering scalar type metric perturbations with line element defined by:

$$
ds^{2} = a^{2}(\eta)\left\{(1+2\phi)\, d\eta^{2} - 2B_{|i}dx^{i}d\eta - \left[(1-2\psi)\,\gamma_{ij} + 2E_{|ij}\right]dx^{i}dx^{j}\right\} \tag{B.62}
$$

Computing christoffel symbols:

$$
\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2g_{\alpha\lambda}}\left(\partial_{\mu}g_{\lambda\nu} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu}\right) \tag{B.63}
$$

For $\Gamma^{\alpha}{}_{00}$:

$$
\Gamma^{\alpha}{}_{00} = \frac{1}{2g_{00}}\left(\left(\partial_{0}g_{\lambda 0} + \partial_{0}g_{0\lambda} - \partial_{\lambda}g_{00}\right)\right)
$$

$$\Gamma^0{}_{00} = \frac{1}{2g_{\lambda 0}}\left(\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00}\right) = \frac{1}{2a^2(1+2\phi)}\left(\partial_0 a^2(1+2\phi)\right) + \frac{1}{2a^2 B_{|i}}\left(2\partial_0(a^2 B_{|i}) + \partial_i(a^2(1+$$
$$\Gamma^i{}_{00} = \frac{1}{2a^2(1+2\phi)}\partial_0\left(a^2(1+2\phi)\right) + \frac{1}{2a^2\left((1-2\psi)\gamma_{ij} + 2E_{|ij}\right)}\left(2\partial_0(a^2 B_{|j}) + \partial_j(a^2(1+2\phi))\right)$$

For $\Gamma^\alpha{}_{0i}$:

$$\begin{aligned}\Gamma^0{}_{0i} &= \frac{1}{(1+2\phi)}\partial_i\phi + \frac{1}{2a^2 B_{|j}}\left(\partial_0\left(a^2(1-2\psi)\gamma_{ij} + 2a^2 E_{|ij}\right) + \partial_i(2a^2 B_{|j}) - \partial_j(2a^2 B_{|i})\right)\\ \Gamma^j{}_{0i} &= -\frac{1}{B_{|j}}\partial_i\phi + \frac{1}{2a^2((1-2\psi)\gamma_{jk}) + 2E_{|jk}}\left\{\partial_0\left(a^2(1-2\psi)\gamma_{ki} + 2E_{|ki}\right) + \partial_i B_{|k}a^2 - a^2\partial_k B_{|i}\right\}\end{aligned} \tag{B.65}$$

Computing the Ricci Scalar and putting into the field equation, we obtain:

The perturbed Einstein equations are:

$$\begin{aligned}\delta G^0{}_0 &= 2a^{-2}\left\{-3\mathcal{H}\left(\mathcal{H}\phi + \psi'\right) + \nabla^2\left(\psi - \mathcal{H}(B-E')\right) + 3\mathcal{K}\psi\right\} = 8\pi G\delta T^0{}_0\\ \delta G^0{}_i &= 2a^{-2}\left\{\mathcal{H}\phi + \psi' - \mathcal{K}(B-E')\right\}_{|i} = 8\pi G\delta T^0{}_i\\ \delta G^i{}_j &= -2a^{-2}\left\{\left((2\mathcal{H}' + \mathcal{H}^2)\phi + \mathcal{H}\phi' + \psi'' + 2\mathcal{H}\psi' - \mathcal{H}\psi + \frac{1}{2}\nabla^2 D\right)\delta^i_j - \frac{1}{2}D_{|ij}\right\} = 8\pi G T^i{}_j\end{aligned} \tag{B.66}$$

where, $D = (\phi - \psi) + 2\mathcal{H}(B-E') + (B-E')'$

For perturbation in a (1, 1) type tensor:

$$\begin{aligned}\widetilde{A}^\mu_\nu &= X^{\tilde{\mu}}_\alpha X^\beta_{\tilde{\nu}} A^\alpha_\beta\\ \widetilde{A}^\mu_\nu &= \left(\delta^\mu_\alpha + \xi^\mu_{,\alpha}\right)\left(\delta^\beta_\nu - \xi^\beta_{,\nu}\right)\left(A^\alpha_\beta - A^\alpha_{\beta,\lambda}\xi^\lambda\right)\\ \widetilde{A}^\mu_\nu &= A^\mu_\nu + A^\alpha_\nu\xi^\mu_{,\alpha} - A^\mu_\beta\xi^\beta_{,\nu} - A^\mu_{\nu,\lambda}\xi^\lambda\end{aligned} \tag{B.67}$$

Using this equation to compute the transformation for the Einstein tensor:

$$\delta\widetilde{G^0}_0 = \delta G^0{}_0 + G^\alpha{}_0\xi^0_{,\alpha} - G^0{}_\beta\xi^\beta_{,0} - {}^{(0)}G^0{}_{0,\lambda}\xi^\lambda \tag{B.68}$$

Since, the Einstein tensor is diagonal we can simplify:

$$\delta\widetilde{G^0}_0 = \delta G^0{}_0 - {}^{(0)}G^0{}_{0,0}\xi^0 \tag{B.69}$$

Under the transformation, $\eta \to \tilde{\eta} = \eta + \xi^0$; $x^i \to \tilde{x}^i = x^i + \gamma^{ij}\xi_{ij}(\eta, x)$ and $\widetilde{\delta Q} - \delta Q = \mathcal{L}_\xi Q$

$$\delta G^0{}_0 \to \delta G^0{}_0 - ({}^{(0)}G^0{}_0)'\xi^0\,, \quad \delta G^0{}_i \to \delta G^0{}_i - \left({}^{(0)}G^0{}_0 - \frac{1}{3}{}^{(0)}G^k{}_k\right)\xi^0_{|i} \tag{B.70}$$

$$\delta G^i{}_j \to \delta G^i{}_j - \left({}^{(0)}G^i{}_j\right)'\xi^0 \tag{B.71}$$

By substituting (B.66) in (B.70) and (B.71) while utilizing the gauage-invariant variables Φ and Ψ:

$$\Phi = \phi + \frac{1}{a}\left[(B - E')\,a\right]' \ , \quad \Psi = \psi - \frac{a'}{a}(B - E') \tag{B.72}$$

$$\begin{aligned}
\delta G^0{}_0 &= 2a^{-2}\left\{-3\mathcal{H}\left(\mathcal{H}\phi + \psi'\right) + \nabla^2\left(\psi - \mathcal{H}(B - E')\right) + 3\mathcal{K}\psi\right\}\\
&= 2a^{-2}\left\{-3\mathcal{H}\left(\mathcal{H}\left(\Phi - \frac{1}{a}\left[(B - E')\,a\right]'\right) + \left(\Psi + \frac{a'}{a}(B - E')\right)'\right)\right.\\
&\left.+\nabla^2\left(\left(\Psi + \frac{a'}{a}(B - E')\right) - \mathcal{H}(B - E')\right) + 3\mathcal{K}\left(\Psi + \frac{a'}{a}(B - E')\right)\right\}\\
&= 2a^{-2}\left\{-3\mathcal{H}\left(\mathcal{H}\Phi + \Psi'\right) + \nabla^2\Psi + 3\mathcal{K}\Psi + 3\mathcal{H}\left[-\mathcal{H}' + \mathcal{H}^2 + \mathcal{K}\right](B - E')\right\}\\
\delta G^0{}_i &= 2a^{-2}\left\{\mathcal{H}\phi + \psi' - \mathcal{K}(B - E')\right\}_{|i}\\
&= 2a^{-2}\left\{\mathcal{H}\left(\Phi - \frac{1}{a}\left[(B - E')\,a\right]'\right) + \left(\Psi + \frac{a'}{a}(B - E')\right)' - \mathcal{K}(B - E')\right\}_{|i}\\
&= 2a^{-2}\left\{\mathcal{H}\Phi + \Psi' + \left(\mathcal{H}' - \mathcal{H}^2 - \mathcal{K}\right)(B - E')\right\}_{|i}\\
\delta G^i{}_j &= -2a^{-2}\left\{\left((2\mathcal{H}' + \mathcal{H}^2)\phi + \mathcal{H}\phi' + \psi'' + 2\mathcal{H}\psi' - \mathcal{H}\psi + \frac{1}{2}\nabla^2 D\right)\delta^i_j - \frac{1}{2}D_{|ij}\right\}\\
&= -2a^{-2}\left\{\left((2\mathcal{H}' + \mathcal{H}^2)\left(\Phi - \frac{1}{a}\left[(B - E')\,a\right]'\right) + \mathcal{H}\left(\Phi - \frac{1}{a}\left[(B - E')\,a\right]'\right)' + \left(\Psi + \frac{a'}{a}(B - E'\right)\right.\right.\\
&\left.+2\mathcal{H}\left(\Psi + \frac{a'}{a}(B - E')\right)' - \mathcal{H}\left(\Psi + \frac{a'}{a}(B - E')\right) + \frac{1}{2}\nabla^2(\Phi - \Psi)\right)\delta^i_j - \frac{1}{2}(\Phi - \Psi)_{|ij}\Big\}\\
&= -2a^{-2}\left\{\left[\left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' - \mathcal{K}\Psi + \frac{1}{2}\nabla^2 D\right]\delta^i{}_j\right.\\
&\left.\left(\mathcal{H}'' - \mathcal{H}\mathcal{H}' - \mathcal{H}^3 - \mathcal{K}\mathcal{H}\right)(B - E')\,\delta^i{}_j - \frac{1}{2}\gamma^{ij}(\Phi - \Psi)_{kj}\right\}
\end{aligned}$$

As we learned in previous sections, the gauge-invariant variables $\delta G^{(gi)}{}^\alpha_\beta$ and $\delta T^{(gi)}{}^\alpha_\beta$ as:

$$\begin{aligned}
&\delta G^{(\mathrm{gi})0}_0 = \delta G^0_0 + \left({}^{(0)}G^0_0\right)'(B - E')\,, \quad \delta G^{(\mathrm{gi})0}_i = \delta G^0_i + \left({}^{(0)}G^0_0 - \frac{1}{3}\,{}^{(0)}G^k_k\right)(B - E')_{|i}\,,\\
&\delta G^{(\mathrm{gi})i}_j = \delta G^i_j + \left({}^{(0)}G^i_j\right)'(B - E')\,,
\end{aligned} \tag{B.74}$$

$$\begin{aligned}
&\delta T^{(\mathrm{gi})0}_0 = \delta T^0_0 + \left({}^{(0)}T^0_0\right)'(B - E')\,, \quad \delta T^{(\mathrm{gi})0}_i = \delta T^0_i + \left({}^{(0)}T^0_0 - \frac{1}{3}\,{}^{(0)}T^k_k\right)(B - E')_{i}\\
&\delta T^{(\mathrm{gi})i}_j = \delta T^i_j + \left({}^{(0)}T^i_j\right)'(B - E')
\end{aligned} \tag{B.75}$$

$$\delta G^{(gi)}{}^\mu_\nu = 8\pi G\,\delta T^{(gi)}{}^\mu_\nu \tag{B.76}$$

Using (B.73), (B.74) and (B.76), we can see that B and E cancel on the LHS:

$$\begin{aligned} -3\mathcal{H}\left(\mathcal{H}\Phi+\Psi'\right)+\nabla^2\Psi+3\mathcal{K}\Psi &= 4\pi G a^2 \delta T_0^{(gi)0} \\ \left(\mathcal{H}\Phi+\Psi'\right)_{,i} &= 4\pi G a^2 \delta T_i^{(gi)0} \\ \left[\left(2\mathcal{H}'+\mathcal{H}^2\right)\Phi+\mathcal{H}\Phi'+\Psi''+2\mathcal{H}\Psi'-\mathcal{K}\Psi+\frac{1}{2}\nabla^2 D\right]\delta^i_j-\frac{1}{2}\gamma^{ik}D_{|kj} &= -4\pi G a^2 \delta T_j^{(gi)i} \end{aligned} \tag{B.77}$$

where $D=\Phi-\Psi$

B.5 Hydrodynamical Perturbation

In this chapter, we'll study about the growth rates of cosmological perturbaition in models with conventional Hydrodynamical matter.

Firstly, we'll consider the energy-momentum tensor to be adiabatic and entropy perturbation evolves.

B.5.1 Energy-momentum tensor

Focusing solely on scalar perturbations, we can express the first-order perturbations as:

$$\delta T^{\mu}{}_{\nu} = \begin{pmatrix} \delta\epsilon & -(\epsilon_0 p_0)a^{-1}V_{,i} \\ (\epsilon_0 p_0)a^{-1}V_{,i} & -\delta p\delta_{ij}+\sigma_{ij} \end{pmatrix} \tag{B.78}$$

Here we have taken the most general form of first order perturbation. The $\delta\epsilon$ and δp represents perturbed energy density and pressure, V is the potential and σ is the anisotropic stress.

But for a relatvistic perfect fluid the energy-momentum tensor can be simplifed since the anisotropic stress vanishes:

$$T^{\alpha}_{\beta} = (\epsilon+p)u^{\alpha}u_{\beta} - p\delta^{\alpha}_{\beta} \tag{B.79}$$

Varying the pressure $p(\epsilon, S)$ reads:

$$\delta p = \frac{\partial p}{\partial \epsilon}|_S\,\delta\epsilon + \frac{\partial p}{\partial S}|_{\epsilon}\,\delta S \tag{B.80}$$

$$\delta p = c_s^2\,\delta\epsilon + \tau\,\delta S \tag{B.81}$$

For Hydrodynamical matter, c_s is interpreted as the sound velocity.

In a single component ideal gas universe there are no entropy components. But since there are at-least two components: plasma and radiation, hence entropy perturbations may be important.

At late times, when the temperature T is low compated to the masses of baryon, the baryon pressure can be neglected ($p_m = 0$), therefore the total pressure reads:

$$p = p_r = \frac{1}{3}\epsilon_r \tag{B.82}$$

$$\text{Hence, } \delta p = \frac{1}{3}\delta\epsilon_r \tag{B.83}$$

The entropy per baryon is proportional to T^3/n_b where n_b is the number density of baryons, T is the temperature and $\epsilon_r \propto T^4$.

Since, $\epsilon_r \propto T^4 \implies \delta\epsilon_r = 4T^3\delta T$ and $\epsilon_m \propto n_b$

$$\begin{aligned}
S &\propto T^3/n_b \\
\delta S &= 3\frac{T^2}{n_b}\,\delta T - \frac{T^3}{n_b^2}\,\delta n_b \\
\delta S &= \frac{3}{4}\frac{1}{Tn_b}\,\delta\epsilon_r - \frac{T^3}{n_b^2}\,\delta n_b \\
\delta S &= \frac{3}{4}\frac{T^3}{n_b}\,\frac{\delta\epsilon_r}{\epsilon_r} - \frac{T^3}{n_b}\,\frac{\delta n_b}{n_b} \\
\frac{\delta S}{S} &= \frac{3}{4}\frac{\delta\epsilon_r}{\epsilon_r} - \frac{\delta\epsilon_m}{\epsilon_m}
\end{aligned} \tag{B.84}$$

Since $\delta\epsilon = \delta\epsilon_r + \delta\epsilon_m$

$$\frac{\delta S}{S} = \frac{3}{4}\frac{\delta\epsilon_r}{\epsilon_r} - \frac{\delta\epsilon_m}{\epsilon_m} \tag{B.85}$$

$$\delta\epsilon_m = \epsilon_m\left(\frac{3}{4}\frac{\delta\epsilon_r}{\epsilon_r} - \frac{\delta S}{S}\right) \tag{B.86}$$

$$\delta\epsilon = \delta\epsilon_r + \epsilon_m\left(\frac{3}{4}\frac{\delta\epsilon_r}{\epsilon_r} - \frac{\delta S}{S}\right)$$

$$\delta\epsilon_r = \left(1 + \frac{3}{4}\frac{\epsilon_m}{\epsilon_r}\right)^{-1}\delta\epsilon + \left(1 + \frac{3}{4}\frac{\epsilon_m}{\epsilon_r}\right)^{-1}\epsilon_m\frac{\delta S}{S} \tag{B.87}$$

Since $\delta p = \frac{1}{3}\delta\epsilon_r$

$$\boxed{\delta p = \frac{1}{3}\left(1 + \frac{3}{4}\frac{\epsilon_m}{\epsilon_r}\right)^{-1}\delta\epsilon + \frac{1}{3}\left(1 + \frac{3}{4}\frac{\epsilon_m}{\epsilon_r}\right)^{-1}\epsilon_m\frac{\delta S}{S}} \tag{B.88}$$

Comparing (B.88) and (B.81)

$$c_s^2 = \frac{1}{3}\left(1 + \frac{3}{4}\frac{\epsilon_m}{\epsilon_r}\right)^{-1} \quad , \quad \tau = \frac{c_s^2 \epsilon_m}{S} \tag{B.89}$$

When this is applied to the early universe, this model describes a smooth transition from the radiation dominated epoch($\epsilon_r >> \epsilon_m \implies c_s^2 = 1/3$) to the matter dominated epoch ($\epsilon_m >> \epsilon_m \implies c_s^2 = 0$).

Following (B.78) we can express the first-order perturbations as:

$$\delta T_0^0 = \delta\epsilon, \quad \delta T_i^0 = (\epsilon_0 + p_0)\, a^{-1}\delta u_i, \quad \delta T_j^i = -\delta p \delta_j^i \tag{B.90}$$

For gauge-invariant perturbations:

The gauge-invariant energy density and pressure perturbations $\delta\epsilon^{(gi)}$ and $\delta p^{(gi)}$ are defined in the same way as a general four-scalar:

$$\delta\epsilon^{(gi)} = \delta\epsilon + \epsilon_0' \left(B - E'\right), \quad \delta p^{(gi)} = \delta p + p_0' \left(B - E'\right) \tag{B.91}$$

With the gauge invariant three velocity obtained by:

$$\delta u^{(gi)}{}_i = \delta u_i + a\left(B - E'\right)_{|i} \tag{B.92}$$

Since ϵ and p are not time-dependent then (B.91) should be gauge-invariant:

$$\delta T^{(gi)}{}_0^0 = \delta\epsilon^{(gi)}, \quad \delta T^{(gi)}{}_i^0 = (\epsilon_0 + p_0)\, a^{-1}\delta u_i^{(gi)}, \quad \delta T^{(gi)}{}_j^i = -\delta p^{(gi)}\delta_j^i \tag{B.93}$$

The physical meaning of $\delta\epsilon^{(gi)}$, $\delta p^{(gi)}$ and $\delta u^{(gi)}{}_i$ is that they coincide with the perturbations of energy density, pressure and velocity in the longitudinal gauge. Using (B.97)

$$\delta\varepsilon^{(\mathrm{gi})}/\varepsilon_0 = 2\left[3\left(\mathcal{H}^2 + \mathcal{K}\right)\right]^{-1}\left[\nabla^2\Phi - 3\mathcal{H}\Phi' - 3\left(\mathcal{H}^2 - \mathcal{K}\right)\Phi\right] \tag{B.94}$$

$$\delta u_i^{(\mathrm{gi})} = -a^{-2}\left(\mathcal{H}^2 - \mathcal{H}' + \mathcal{K}\right)^{-1}(a\Phi)'_{,i} \tag{B.95}$$

B.5.2 Equation of motion

Using (B.77) we can write the gauge-invariant EOMs:

$$\begin{aligned} -3\mathcal{H}\left(\mathcal{H}\Phi + \Psi'\right) + \nabla^2\Psi + 3\mathcal{H}\Psi &= 4\pi G a^2 \delta\epsilon^{(gi)} \\ \left(\mathcal{H}\Phi + \Psi'\right)_{,i} &= 4\pi G a \left(\epsilon_0 + p_0\right)\delta u^{(gi)} \\ \left[\left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' - \mathcal{K}\Psi + \frac{1}{2}\nabla^2 D\right]\delta_j^i - \frac{1}{2}\gamma^{ik}D_{|kj} &= 4\pi G a^2 \delta p^{(gi)}\delta_j^i \end{aligned} \tag{B.96}$$

where as earlier $D = \Phi - \Psi$

Since the energy-momentum tensor doesn't have any nondiagonal space-space terms, we can simply these equations. It follows from the $i \neq j$ equation that $\Phi = \Psi$.

In a flat universe, there is a sompler way to reach this conclusion, we go to momentum space. The $i \neq j$ equation must hold for each mode separately, and the only way for this to happen is if all fourier coefficients vanish, so:

$$\begin{aligned} \nabla^2\Phi - 3\mathcal{H}\Phi' - 3\left(\mathcal{H}^2 - \mathcal{K}\right)\Phi &= 4\pi G a^2 \delta\varepsilon^{(\mathrm{gi})} \\ (a\Phi)'_{,i} &= 4\pi G a^2 \left(\varepsilon_0 + p_0\right)\delta u_i^{(\mathrm{gi})} \\ \Phi'' + 3\mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2 - \mathcal{H}\right)\Phi &= 4\pi G a^2 \delta p^{(\mathrm{gi})} \end{aligned} \tag{B.97}$$

The first equation in (B.97) is the usual Poisson's equation for the gravitational potential with some perturbation. This supports the interpretation that Φ as the relatvistic generaltization of the Newtonian gravitational potential ϕ. It generalizes the potential by taking into account the scale-factor/expansion of the universe.

Comparing gauge-invariant and synchronous-gauge equations. To obtain these equations, we start from the general gauge-dependent Einstein equation and set $\phi = B = 0$:

$$\begin{aligned} &\nabla^2\left(\psi_{\mathrm{s}} + \mathcal{H}E_{\mathrm{s}}'\right) - 3\mathcal{H}\psi_{\mathrm{s}}' = 4\pi G a^2 \delta\varepsilon_{\mathrm{s}} \\ &\psi'_{\mathrm{s},i} = 4\pi G a^3 \left(\varepsilon_0 + p_0\right)\delta u_{\mathrm{s}i} \\ &\psi_{\mathrm{s}}'' + 2\mathcal{H}\psi_{\mathrm{s}}' + \frac{1}{2}\left(\nabla^2 D_{\mathrm{s}} - D_{\mathrm{s}|ii}\right) = 4\pi G \delta p_{\mathrm{s}} \\ &D_{\mathrm{s}|ij} = 0, \quad i \neq j \end{aligned} \tag{B.98}$$

where the subscript s represents the synchronous gauge and $D_s = -\psi_s - E_s'' - 2\mathcal{H}E_s'$

Unfortunately, (B.98) contains third derivatives of E_s, which is hard to solve. This mathematical problem is a reflection of the fac that synchronous gauge has a residual gauge freedom. Thus, the synchronous gauge equations are manifestly more complicated than the gauge-invariant onces for our case. This gives additional motiviation for gauge-invariant approach besides its intrinsic advantages related to the physical interpretation.

Combining first and third equations of (B.97) gives:

$$\boxed{\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' - c_s^2\nabla^2\Phi + \left[2\mathcal{H}' + (1 + 3c_s^2)(\mathcal{H}^2 - \mathcal{K})\right]\Phi = 4\pi G a^2 \tau \delta S} \tag{B.99}$$

Recasting the above equation by introducing a gauge-invariant function:

$$\zeta = \frac{2}{3}\frac{\left(H^{-1}\dot{\Phi} + \Phi\right)}{1 + w} + \Phi \tag{B.100}$$

$$\Phi = 4\pi G\left(\epsilon_0 + p_0\right)^{1/2} u = (4\pi G)^{1/2}\left[\left(\mathcal{H}^2 - \mathcal{H}' + \mathcal{K}\right)/a^2\right]^{1/2} u \tag{B.101}$$

After some calculations, the equation of motion for u can be obtained as:

$$\boxed{u'' - c_s^2\nabla^2 u - \left(\theta''/\theta\right) u = \mathcal{N}} \tag{B.102}$$

$$\theta = (\mathcal{H}/a)\left[\frac{2}{3}\left(\mathcal{H}^2 - \mathcal{H}' + \mathcal{K}\right)\right]^{-1/2} = \frac{1}{a}\left(\frac{\varepsilon_0}{\varepsilon_0 + p_0}\right)^{1/2}\left(1 - \frac{3\mathcal{K}}{8\pi G a^2 \varepsilon_0}\right)^{1/2} \tag{B.103}$$

$$\mathcal{N} = (4\pi G)^{1/2} a^3\left(\mathcal{H}^2 - \mathcal{H}' + \mathcal{K}\right)^{-1/2}\tau\delta S = a^2\left(\varepsilon_0 + p_0\right)^{-1/2}\tau\delta S \tag{B.104}$$

The (B.99) and (B.102) are the main result from this section.

B.5.3 Adiabatic perturbations

In these type of perturbations, the source term vanishes, leaving (B.99) and (B.102) homogeneous.

Considering only cold matter-dominated universe($p = 0$)

$$a(\eta) = a_m \begin{cases} \cosh\eta - 1, & \mathcal{K} = -1, \\ \eta^2/2, & \mathcal{K} = 0, \\ 1 - \cosh\eta, & \mathcal{K} = 1. \end{cases} \tag{B.105}$$

As we know, for a flat universe $a(t) \sim t^{2/3}$, $p = 0$ and $c_s^2 = 0$, (B.102) becomes:

$$u'' - (\theta''/\theta)u = 0 \tag{B.106}$$

Solving this reveals:

$$u(x,\eta) = C_1'(x)\theta(\eta) + C_2'(x)\theta(\eta)\int\frac{d\eta'}{\theta^2} \tag{B.107}$$

$$\Phi = (4\pi G)^{1/2}\left[\left(\mathcal{H}^2 - \mathcal{H}'\right)/a^2\right]^{1/2} u$$

$$\Phi = (4\pi G)^{1/2}\left[\left(\mathcal{H}^2 - \mathcal{H}'\right)/a^2\right]^{1/2}\left[C_1'(x)\theta(\eta) + C_2'(x)\theta(\eta)\int\frac{d\eta'}{\theta^2}\right]$$

$$\Phi = (4\pi G)^{1/2}\left[\left(\mathcal{H}^2 - \mathcal{H}'\right)/a^2\right]^{1/2}\left[C_1'(x)\theta(\eta) + C_2'(x)\theta(\eta)\int\frac{d\eta'}{\theta^2}\right]$$

where $\theta = (\mathcal{H}/a)\left[\frac{2}{3}\left(\mathcal{H}^2 - \mathcal{H}'\right)\right]^{-1/2}$

$$\Phi = (4\pi G)^{1/2}\left[\left(\mathcal{H}^2 - \mathcal{H}'\right)/a^2\right]^{1/2}\frac{\mathcal{H}}{a}\left[\frac{2}{3}\left(\mathcal{H}^2 - \mathcal{H}'\right)\right]^{-1/2}\left[C_1'(x) + C_2'(x)\int\frac{d\eta'}{\theta^2}\right]$$

$$\Phi(x,\eta) = C_1(x) + C_2(x)\eta^{-5} \tag{B.108}$$

Using (B.94) and (B.95):

$$\delta\epsilon^{(gi)}/\epsilon_0 = \frac{2}{3\mathcal{H}^2}\left[\nabla^2\Phi - 3\mathcal{H}\Phi' - \mathcal{H}^2\Phi\right] \tag{B.109}$$

$$\delta u_i^{(gi)} = \frac{1}{a_m}\left(\frac{2C_{2,i}}{\eta^2} - \frac{4}{3\eta}C_{1,i}\right) \tag{B.110}$$

The gravitational potential Φ remains constant. In length, smaller the hubble radius, the energy-density perturbations increases as η^2 or $t^{2/3}$, whereas for scales larger than the hubble radius, the energy-density perturbations remains constant.

Devansh

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