Intro to AI Homework 1

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February 23, 2024

1 Probability

1.1 Problem 1

1. Base Case:
$n = 2$

$$p(x_1, x_2) = p(x_2|x_1)p(x_1) = \prod_{i=1}^{2} p\left(x_i|x_1, \ldots, x_{i-1}\right)$$

2. Induction Step:

Suppose the following is true: $p(x_1, \ldots, x_k) = \prod_{i=1}^{k} p\left(x_i|x_1, \ldots, x_{i-1}\right)$
Now we must show that this holds true for

$p(x_1, ..., x_k) = p(a) = \prod_{i=1}^{k+1} p\left(x_i|x_1, \ldots, x_{i-1}\right)$ where $2 \leq k < n$.

Let's say $p(x_1, ..., x_k) = p(a) = \prod_{i=1}^{k} p\left(x_i|x_1, \ldots, x_{i-1}\right)$, and by using the product rule, $p(x_1, \ldots, x_{k+1}) = p(a, x_{k+1}) = p(x_{k+1}|a)p(a)$
$= p(x_{k+1}|x_1, \ldots, x_{i+1}) \prod_{i=1}^{k} p\left(x_i|x_1, \ldots, x_{i-1}\right) = \prod_{i=1}^{k+1} p\left(x_i|x_1, \ldots, x_{i-1}\right)$
Therefore we have proved the base case and induction step and proved that $p(x_1, \ldots, x_k) = \prod_{i=1}^{k} p\left(x_i|x_1, \ldots, x_{i-1}\right)$ implies $p(x_1, \ldots, x_{k+1}) = \prod_{i=1}^{k+1} p\left(x_i|x_1, \ldots, x_{i-1}\right)$.

1.2 Problem 2

If we apply the definition 2 on the equation below:

$$p(x, y|z) = \frac{p(x, y, z)}{p(z)} = \frac{p(x|y, z) \times p(y, z)}{p(z)} = p(x|y, z) \times \frac{p(y, z)}{p(z)} = p(x|y, z) \times p(y|z)$$

we have the following:

$$p(x|y, z) = p(x|y, z) \times p(y|z) = p(x|z) \times p(y|z)$$

Clearly, the result holds with definition 1 proving that both are indeed equivalent.

1.3 Problem 3

Statement 1: **False**
$X --> Z <-- Y$, X and Y, in this case, are unconditionally independent. Therefore, once Z is introduced, they become dependent.
Statement 2: **False**
$X <-- Z --> Y$, In contrast to above, X and Y are conditionally independent here. However, when Z is introduced, they become unconditionally dependent.

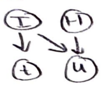
2 Bayesian Networks

2.1 Problem 4

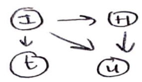
Please see below images for the work on problem 4

Problem 4

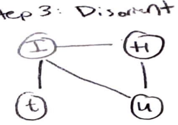
4.1) Step 1: Ancestral



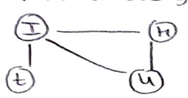
Step 2: Normalize



Step 3: Disorient

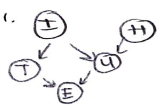


Step 4: Remove given

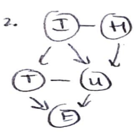


Observe path from t to u so not independent $t \not\perp u$

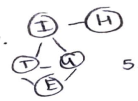
4.2) 1.



2.



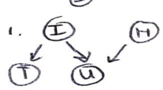
3.



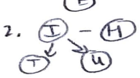
4. T − U

5. T and U are not conditionally independent b/c path from T to U.

4.3) 1.



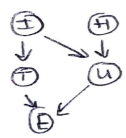
2.



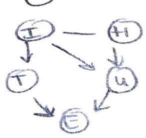
3.

4. T U Since there is no path from T to U they are conditionally independent

4.4) 1.



2.



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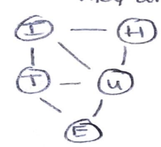
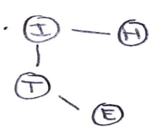


4.



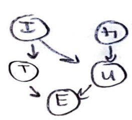
5. $E \not\perp H \mid U$ since path from E to H

4.5) Steps 1−3 above (4.4) are the same.

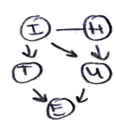
4.

5. $E \perp H \mid U, I, T$

4.6) 1.



2.



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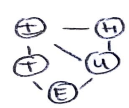
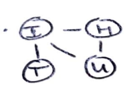
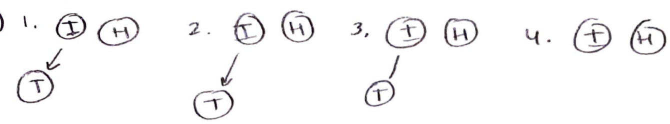


4.



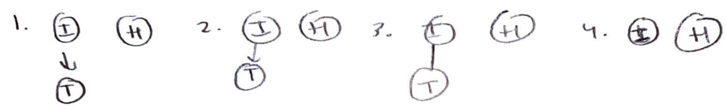
5. $I \not\perp H \mid E$

4.7) 1. 2. 3. 4.



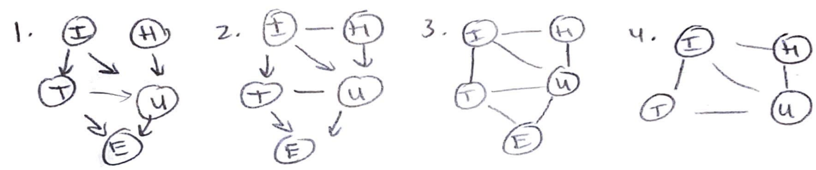
5. $I \perp H \mid T$

4.8) 1. 2. 3. 4.



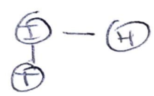
5. $T \perp H$

4.9) 1. 2. 3. 4.



5. $T \not\perp H \mid E$

4.10) Same steps 1-3 as (4.9) above. 4.



5. $T \not\perp H \mid E, U$

2.2 Problem 5

Since

$$P(+u|+e) = \frac{P(+u,+e)}{p(+e)}$$

Calculate $P(+u|+e)$:
$P(+u|+e) = \sum_{i,t,h}(+u,+e,i,t,h) \to \sum_h P(h) \sum_i P(i) \cdot P(+u|i,h) \cdot \sum_t P(t|i) \cdot P(+e|+u,t)$

$$f_2(h,i) = \sum_t P(t|i) \cdot P(+e|+u,t) ==> f_2(h,+i)$$
$$= P(+t|+i) \cdot P(+e|+u,+t) + P(-t|+i) \cdot P(+e|+u,-t) \to .86$$
$$f_2(h,-i) = P(+t|-i) \cdot P(+e|+u,+t) + P(-t|-i) \cdot P(+e|+u,-t) \to .8$$

In continuation,

$$f_1(h) = \sum_i P(i) \cdot P(+u|i,h) \cdot f_2(h,i)$$

$$f_1(+h) = P(+i) \cdot P(+u|+i,+h) \cdot f_2(+h,+i) + P(-i) \cdot P(+u|-i,+h) \cdot f_2(+h,-i) \to .6618$$

$$f_1(-h) = P(+i) \cdot P(+u|+i,-h) \cdot f_2(-h,+i) + P(-i) \cdot P(+u|-i,-h) \cdot f_2(-h,-i) \to .2046$$

Now we sum:
$$P(+u,+e) = \sum_h P(h) \cdot f_1(h)$$
$$= p(+h) \cdot f_1(+h) + p(-h) \cdot f_1(-h) \to .47892$$

Now we calculate: P(-u— +e)
$$P(-u|+e) = \sum_{i,t,h}(-u,+e,i,t,h)$$
$$= \sum_h P(h) \sum_i P(i) \cdot P(-u|i,h) \cdot \sum_t P(t|i) \cdot P(+e|-u,t)$$

$$f_2(h,i) = \sum_t P(t|i) \cdot P(+e|-u,t)$$

$f_2(h,+i) \to .46$
$f_2(h,-i \to .4)$

Therefore we can calculate $f_1 + h \&\& - h$ which are the following values after calculations:
$f_1(+h) = .0922$ and $f_1(-h) = .3334$

Now we calculate:
$$P(-u,+e) = \sum_h P(h) \cdot f_1(h)$$

$= P(+h) \cdot f_1(+h) + p(-h) \cdot f_1(-h) \rightarrow .18868$
Therefore,
$P(+e) = P(+u) \cdot P(+e|+u) + P(-u) \cdot P(+e|-u) \rightarrow \quad = P(+u, +e) + P(-u, +e) \rightarrow .6676$
Consequently, this proves that:

$$P(+u|+e) = \frac{P(+u, +e)}{P(+e)} = 0.7112$$