Von Neumann Stability Analysis for the 1D Heat Equation

Derivation of the stability condition for the explicit FTCS scheme

1. Continuous problem

Consider the one-dimensional heat (diffusion) equation

$$\frac{\partial u}{\partial t}(x,t) \;=\; \alpha\,\frac{\partial^2 u}{\partial x^2}(x,t), \qquad \alpha > 0, \tag{1}$$

on a spatial domain (e.g. $x \in \mathbb{R}$ or a finite interval with suitable boundary conditions).

2. Finite-difference discretization (FTCS)

Introduce a uniform grid

$$x_j = j\Delta x, \qquad t^n = n\Delta t,$$

and denote the numerical approximation by $u_j^n \approx u(x_j, t^n)$.

Use forward difference in time and central difference in space:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} \;=\; \alpha\,\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}.$$

Rearrange to the update (FTCS) form:

$$u_j^{n+1} \;=\; u_j^n \;+\; r\big(u_{j+1}^n - 2u_j^n + u_{j-1}^n\big), \qquad r := \frac{\alpha\,\Delta t}{(\Delta x)^2}. \tag{2}$$

3. Von Neumann (Fourier) stability ansatz

The von Neumann analysis studies growth of Fourier modes (or error components). Assume a single Fourier mode solution/error of the form

$$u_j^n \;=\; G^n\,e^{ikx_j}, \tag{3}$$

where $k \in \mathbb{R}$ is the wavenumber and G (generally complex) is the amplification factor per time step. Stability requires that the magnitude of every Fourier mode not grow:

$$|G| \le 1 \quad \text{for all admissible } k.$$

4. Substitute the Fourier mode into the scheme

Substitute (3) into (2). Compute the shifted terms:

$$u_{j\pm 1}^n = G^n e^{ikx_{j\pm 1}} = G^n e^{ikx_j} e^{\pm ik\Delta x}.$$

Plugging into (2):

$$G^{n+1} e^{ikx_j} = G^n e^{ikx_j} + r\Big(G^n e^{ikx_j}\big(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\big)\Big).$$

Divide both sides by $G^n e^{ikx_j} \neq 0$ to obtain the amplification factor:

$$\begin{aligned} G &= 1 + r\big(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\big) \\ &= 1 + r\big(2\cos(k\Delta x) - 2\big) \qquad (\text{since } e^{i\theta} + e^{-i\theta} = 2\cos\theta) \\ &= 1 - 2r\big(1 - \cos(k\Delta x)\big). \end{aligned}$$

Using the trigonometric identity $1 - \cos\theta = 2\sin^2(\frac{\theta}{2})$, we may write

$$\boxed{G(k) \;=\; 1 - 4r\,\sin^2\!\left(\frac{k\Delta x}{2}\right)}. \tag{4}$$

Note that $G(k) \in \mathbb{R}$ for all k (no imaginary part appears in this explicit FTCS for the heat equation).

5. Stability requirement

The stability condition is $|G(k)| \leq 1$ for every k. Since $G(k)$ in (4) is real, this reduces to

$$-1 \leq G(k) \leq 1 \quad \text{for all } k.$$

Observe

$$\min_k \sin^2\!\left(\frac{k\Delta x}{2}\right) = 0, \qquad \max_k \sin^2\!\left(\frac{k\Delta x}{2}\right) = 1.$$

Hence the maximum and minimum possible values of G (over k) are

$$G_{\max} = 1 \quad (\text{when } \sin^2(\tfrac{k\Delta x}{2}) = 0), \qquad G_{\min} = 1 - 4r \quad (\text{when } \sin^2(\tfrac{k\Delta x}{2}) = 1).$$

The most restrictive inequality is from the lower bound $G_{\min} \geq -1$:

$$1 - 4r \;\geq\; -1 \quad\Longrightarrow\quad 4r \;\leq\; 2 \quad\Longrightarrow\quad r \;\leq\; \frac{1}{2}.$$

Von Neumann Stability Condition

For the explicit FTCS scheme (2) applied to the 1D heat equation, a necessary and sufficient condition for linear stability under von Neumann analysis is

$$\boxed{\frac{\alpha\,\Delta t}{(\Delta x)^2} \;=\; r \;\leq\; \frac{1}{2}.}$$

6. Remarks and interpretation

- The FTCS scheme for the heat equation is therefore *conditionally stable*: stability requires a time-step restriction $\Delta t \leq (\Delta x)^2/(2\alpha)$.
- By contrast, fully implicit schemes (e.g., backward Euler) are unconditionally stable (von Neumann analysis yields $|G| \leq 1$ for all $r > 0$), and Crank–Nicolson is unconditionally stable in the sense that amplification factors satisfy $|G| \leq 1$ (but may be only marginally dissipative for some modes).