An Analytical Approximation of Gravitational Waves

by

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Martin Project

Github Link:
https://github.com/devaoneil/AnalyticGravWave/blob/master/AnalyticGravWaveM40.ipynb

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Abstract

The goal of this project is to integrate pre-existing analytical models of gravitational waves into Python so that undergraduate students have access to a model gravitational waves that is straightforward and easy to understand. To facilitate this, we break the problem into two phases, the inspiral and the merger-ringdown. The inspiral phase is modeled using Post-Newtonian (PN) theory. The merger-ringdown phase utilizes an analytical model called the Implicit Rotating Source (IRS) that creates an analytical fit to data created by numerical relativity. To create the final waveform, we use two different matching techniques to combine the merger-ringdown and inspiral waveforms. Future research projects can use this template we have created to test the generated waveform with experimental data, create solutions for parameters such as non-zero eccentricity, and statistically determine the accuracy of the matching technique.

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1 Introduction

Gravitational waves were first predicted in 1916 by Albert Einstein in his general theory of relativity. They were not experimentally discovered until September 14, 2015 the Laser Interferometer Gravitational-Wave Observer, LIGO, detected gravitational waves emitted from a binary black holes (BBH) system. Gravitational waves are important to study and model because they provide a scope for looking at the universe that has never been seen before. Previously, all information about the universe that can be studied on Earth consisted of electromagnetic radiation and subatomic particles such as neutrinos (California Institute of Technology). Adding gravitational waves to this repertoire of lenses for research will provide new information about the universe.

To gain an understanding of these waves, we must discuss how they differ from better-known waves such as light waves and sound waves. While light waves are electric and magnetic fields propagating through spacetime and sound waves are vibrations propagating through a material, gravitational waves are ripples in spacetime that propagate through spacetime. Just like the familiar types of waves that have a noticeable presence in the world, gravitational waves also have notable effects. They create a cycle of pushing and pulling of spacetime to make it closer together or further apart as the wave propagates. This change in distance is the amplitude of the wave and can be parameterized using the strain, which is the change in length divided by the original length of the wave. The strain of a gravitational wave reveals the properties of that wave (Doyle and Erickson, 2020)

Solutions of Einstein's equations of general relativity have to be found in order to model gravitational waves. However, that is much more easily said than done because finding a solution to the differential equations requires knowledge of everything from the birth of the binary to the results after the binary merges. No exact solutions exist, but approximations can be made to find solutions. As an example, Schwarzchild spacetime, a solution to Einstein's equations, makes the assumption that the volume of space where the solution is valid is a vacuum (Guidry, 2019). By approximating solutions to Einstein's equations, we can discover properties of gravitational waves.

To do this, we first consult Post-Newtonian (PN) theory, which was developed by Einstein in order to help solve his equations of general relativity. In PN theory, a series of correction orders of powers $\frac{v}{c}$ is added to Newton's equation of gravity.

While useful, this approximation only works in weak gravitational fields with slow objects relative to c. As v approaches c, we have to resort to either numerical relativity or analytical models. Since this project is intended for undergraduate students, we have to use a model that is fairly easy to use. The generic Implicit Rotating Source (gIRS) model is simple and agrees with numerical relativity so it will work well for undergraduate students. The gIRS utilizes analytical fits to data created by numerical relativity. A model using these two techniques already exists and serves to be the basis for the rest of this project (Buskirk and Babiuc, 2018). In what follows, we will be mirroring the organization of their paper.

Additionally, we will be working in geometric units. Geometric units are used frequently in astrophysics to simplify calculations. The underlying assertion of geometric units is $G = c = 1$ where G is the gravitational constant and c is the speed of light. By setting those equal to 1, it allows us to perform dimensional analysis to measure time in units of meters, so that

$$c = 1 = 2.998 \times 10^8 m/s \Rightarrow 1s = 2.998 \times 10^8 m, 1m = 3.336 \times 10^9.$$

We can do the same thing with the gravitational constant to measure mass in units of distance:

$$G = 1 = 6.673 \times 10^{-11} m^3/kgs^2 \Rightarrow 1kg = 0.742 \times 10^{27} m = 0.742 \times 10^{-30} km.$$

By keeping these conversions in mind, we can describe mass, distance, and time in units of each other with simple calculations (Buskirk and Babiuc, 2018).

2 Inspiral

In an orbiting binary system of massive objects such as black holes or neutron stars, gravity as Newton modeled it would constantly pull the two objects towards each other, allowing the orbit to exist indefinitely. However, Einstein predicted that the two binary objects will gradually become closer to their center of mass. Since the system will gradually lose energy, something has to explain where the lost energy goes. An outward flux of gravitational waves propagating as ripples in spacetime produced by the orbit of two massive objects explains this energy loss. What this means is that a change in energy exists over a change in time that is equal to the flux

$$\frac{dE}{dt} = -\mathcal{F}. \tag{1}$$

where E is the energy of the binary and $\mathcal{F}$ is the emitted flux. Additionally, we can model the binary system as only being subject to gravitational forces, as follows:

$$F = ma = -\frac{GMm}{r^2}, \tag{2}$$

but since we are going to be working with PN approximations, one adds a correction order of $\frac{v}{c}^n$ displayed as an $n/2$ order of the PN expansion:

$$\frac{dv}{dt} = -G\frac{M}{r}[1 + \frac{1PN}{c^2} + \frac{1.5PN}{c^3} + \frac{2PN}{c^4} + ... \tag{3}$$

Current research in this PN expansion is able to compute up to the 3.5^{th} order. Using the PN-parameter $x = \frac{v^2}{c^2}$, the relationship between tangential velocity to angular frequency, $v = \omega r$ and Kepler's third law, $\omega^2 r^3 = GM$, we can develop the following relationship between the PN-parameter, x and the angular frequency of the binary system, ω

$$x = \frac{v^2}{c^2} = \frac{(GM\omega)^{2/3}}{c^2}. \tag{4}$$

By using the chain rule, we can rewrite equation (1) in terms of x instead of v

$$\frac{dx}{dt} = \frac{-\mathcal{F}}{dE/dx}. \tag{5}$$

This gives us a differential equation that by solving, we will be able to construct the PN-parameter x as a function of time, allowing us to discover the properties of the waveform. Several known solutions to this differential equation exists, but the one that agrees the most with numerical relativity is the Taylor T4 model (Hinder et al.)

$$\dot{x} = \dot{x}_{0PN}x^5 + \dot{x}_{1PN}x^6 + \dot{x}_{2PN}x^6 + \dot{x}_{3PN}x^8 + \dot{x}_{HT}. \tag{6}$$

While earlier it was mentioned that solutions only exist up to the 3.5^{th} order, $\dot{x}_{HT}$ represents the hereditary terms of the system including nonlinear PN terms up to order 6. These hereditary terms take into consideration the interactions between the gravitational waves and spacetime. Additionally, each PN term can be expressed as a power series of x:

$$\dot{x}\Big|^{6PN} = \frac{64}{5}\eta x^5\left(1 + \sum_{k=2}^{12} a_{\frac{k}{2}} x^{\frac{k}{2}}\right). \tag{7}$$

Solving this differential equation results in discovering the relationship between the PN-parameter, x, and time. However, in order to solve this, an initial value of x

and a final time bound of integration need to be found. The lower bound of x is determined based upon the smallest frequency that can currently be measured by experimental instruments. This threshold frequency for the Advanced LIGO detection band is $f_{low} = 10Hz$ due to interference from Earth's seismic activity (Buskirk and Babiuc, 2018). Using Kepler's third law, $v_0 = GM\omega_{low}^{1/3}$, and conversions between linear and angular frequency, we arrive at the following result:

$$x_{low} = \frac{{v_0}^2}{c} = \frac{GM\pi {f_{low}}^{2/3}}{c^3}. \tag{8}$$

To be able to find x, now we only need the final integration time. As mentioned previously, the PN approximation for the gravitational waveform only applies in weak fields and low speeds. The final integration time is determined by the last time that the PN approximation applies. To find this, we set up the differential equation seen in equation (7) in Python and solved using arbitrary final time values. The PN approximation stops modeling the waveform accurately when x approaches 1 and when the differential equation becomes stiff. By modifying the final time of the differential equation until it is no longer stiff, we are effectively able to determine the final integration time and solve the differential equation to find $x(t)$. Using equation (8) and geometric units, ω is written in terms of x as follows:

$$\omega = x^{3/2}. \tag{9}$$

After finding ω, we can notice that a change in phase over a change in time is equal to the angular frequency, or

$$\frac{d\phi}{dt} = \omega. \tag{10}$$

Solving this differential equation will result in the phase evolution of the inspiral. As Einstein predicted, the binary orbit will shrink over time, implying that the distance between binary objects, r, will also shrink. Using $v = \omega r$, Kepler's third law, $\omega^2 r^3 = GM$, and geometrical units, $r(t)$ becomes

$$r(t) = Mx(t)^{-1}. \tag{11}$$

In order to add more accuracy to this calculation, we can apply PN corrections to equation (11) up to order 3

$$r = M\big(r_{0PN}x^{-1} + r_{1PN} + r_{2PN}x + r_{3PN}x^2\big). \tag{12}$$

After finding the evolutions of phase and separation, we now know where the two binary objects are relative to each other. With this information, we can begin to model the gravitational waves. Gravitational waves are similar to light in that they have two separate states of polarization. However, unlike electromagnetic radiation, gravitational waves have polarizations that differ by 45 degrees. These two polarizations are known as plus-polarized and cross-polarized gravitational waves, denoted by h_+ and $h_\times$ respectively given by the following general formula:

$$h_+ = -\frac{M\eta}{r}\Bigg[(\cos^2\theta + 1)\Bigg[\Bigg(-\dot{r}^2 + r^2\dot{\Phi}^2 + \frac{M}{r}\Bigg)\cos 2\Phi + 2r\dot{r}\dot{\Phi}\sin 2\Phi\Bigg] + \Bigg(-\dot{r}^2 + r^2\dot{\Phi}^2 + \frac{M}{r}\Bigg)\sin^2\theta\Bigg], \quad (13)$$

$$h_\times = -2\frac{M\eta}{r}\cos\theta\Bigg[\Bigg(-\dot{r}^2 + r^2\dot{\Phi}^2 + \frac{M}{r}\Bigg)\sin 2\Phi - 2r\dot{r}\dot{\Phi}\cos 2\Phi\Bigg]. \quad (14)$$

These equations can be simplified by assuming the inclination angle, θ between the system and the observer is 0 (Tiwari et al., 2016)

$$h_+ = -\frac{M\eta}{r}\Bigg[\Bigg(-\dot{r}^2 + r^2\dot{\Phi}^2 + \frac{M}{r}\Bigg)\cos 2\Phi + 2r\dot{r}\dot{\Phi}\sin 2\Phi\Bigg], \quad (15)$$

$$h_\times = -2\frac{M\eta}{r}\Bigg[\Bigg(-\dot{r}^2 + r^2\dot{\Phi}^2 + \frac{M}{r}\Bigg)\sin 2\Phi - 2r\dot{r}\dot{\Phi}\cos 2\Phi\Bigg]. \quad (16)$$

The combined strain of the waveform can be found using (Huerta et al., 2017)

$$h_{ins}(t) = h_+(t) - ih_\times(t). \quad (17)$$

This equation can be further simplified by using Euler's formula to change sines and cosines into their exponential forms

$$h_{ins}(t) = A(t)e^{-2i\phi(t)}, \quad (18)$$

$$A = A_1 + iA_2,$$

$$A_1 = -2\frac{M\eta}{r}\Bigg(\dot{r}^2 + r^2\dot{\Phi}^2 + \frac{M}{r}\Bigg),$$

$$A_2 = -2\frac{M\eta}{r}\Bigg(2r\dot{r}\dot{\Phi}\Bigg).$$

The strain h_{ins} calculated above is identical, after rescaling by a constant, to that generated by the dominant spin-weighted spherical harmonic $(l, m, s) = (2, 2, -2)$ h_{22} (Hinder et al., 2017; Gopakumar and Iyer, 2002). This relationship can be seen by simply taking the results of equations (15,16) and using Euler's formula to exchange sines and cosines into their exponential forms

$$h_{22} = -4\frac{M\eta}{R}e^{-2i\Phi}\left(\frac{\pi}{5}\right)^{0.5}\left((r\dot{\Phi} + i\dot{r})^2 + \frac{M}{r}\right). \tag{19}$$

Figure 1 shows the strain constructed using the inspiral model consisting of both polarizations. The plus-polarized portion is out of phase and lags behind the cross-polarized portion.

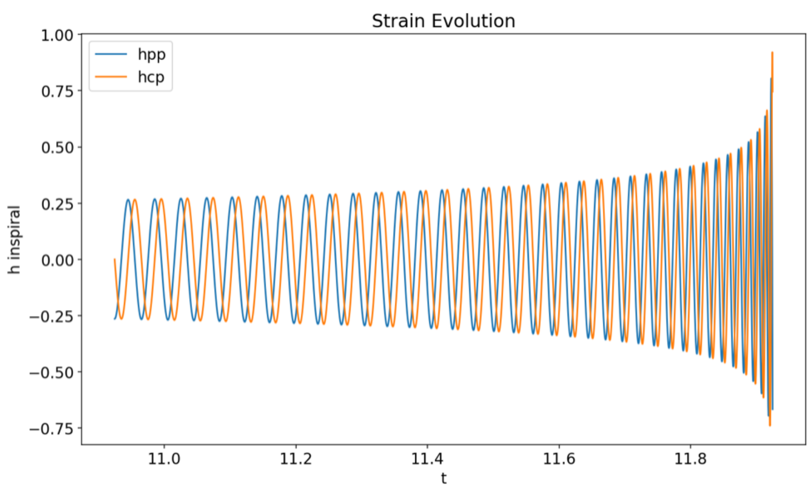

Figure 1: Inspiral Strain. where hpp is the real, plus-polarized portion of the strain and hcp is the imaginary, cross-polarized portion of the strain

3 Merger-Ringdown

The merger-ringdown starts at the innermost stable circular orbit (ISCO). Since the merger-ringdown phase is highly nonlinear, it has to be modeled by General Relativity, meaning that numerical solutions of Einstein's equations are needed to describe it. However, these simulations require strong computing power and are not easily accessible. Multiple different analytical models that fit to numerical relativity have been created to get around this barrier, one of which being the Implicit Rotating Source (IRS) model. It should be noted that the IRS makes the assumption that the waveform is circularly polarized, which is when the two components of strain, h_+ and $h_\times$, have the same amplitude, but different phases (Jones, 2013). Similar to what was done with the inspiral phase, we begin with an equation for strain:

$$h_{merger} = A(t)e^{i\Phi_{gIRS}(t)}, \tag{20}$$

where A is equal to (East et al., 2013)

$$A(t) = \frac{A_0}{\omega(t)}\left[\frac{\mid \hat{f} \mid}{1+\alpha(\hat{f}^2-\hat{f}^4)}\right]^{1/2}. \tag{21}$$

Notice how in order to find amplitude, frequency, $\hat{f}$, and angular orbital velocity, ω must be found. Both of these have been determined analytically (East et al., 2013)

$$\hat{f} = \frac{c}{2}\left(1+\frac{1}{\kappa}\right)^{1+\kappa}\left[1-(1+\frac{1}{\kappa}e^{-2t/b})^{-\kappa}\right], \tag{22}$$

$$\omega(t) = \omega_{QNM}(1-\hat{f}), \tag{23}$$

where ω_{QNM} is the least damped frequency of the quasi-normal modes given by (Huerta et al., 2017)

$$\omega_{QNM} = 1-0.63(1-\hat{s})^{0.3}. \tag{24}$$

The quantity $\hat{s}_{fin}$ is the final spin of the black hole given by (Huerta et al., 2017)

$$\hat{s}_{fin} = 2\sqrt{3}\eta - \frac{390}{79}\eta^2 + \frac{2379}{287}\eta^3 - \frac{4621}{276}\eta^4. \tag{25}$$

By using analytically determined functions of the symmetric mass ratio, η, we are able to find the frequency and angular orbital velocity. We can then take those results and plug them back into the amplitude and strain equations to create a waveform for the merger-ringdown phase. However, before we can do that, we need to be sure that the time axis is in units of time and the amplitudes and strains are unitless. Figure 2 shows the strain constructed using the gIRS model we utilized for the merger-ringdown phase.

The model we implement to create waveforms of gravitational waves emitted by binary black hole systems models the inspiral and merger-ringdown phases of the system. The ringdown phase is when the single black hole formed after the merger oscillates similarly to a struck bell (Giesler et al., 2019). Quasi-normal modes are the set of frequencies and damping times that correlate to a given black hole. The least damped or fundamental mode is the one that lasts the longest and can be used as described earlier in the gIRS model.

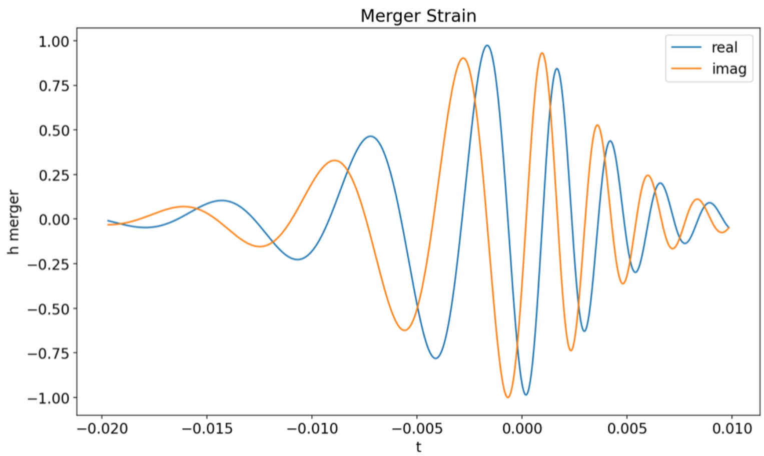

Figure 2: Merger Strain

4 Matching Techniques

In order to create a model for the full waveform of a gravitational wave, we have to match the inspiral waveform and the merger-ringdown waveform into one continuous piece. Our technique is largely influenced by (Buskirk and Babiuc, 2018), but has been made slightly more simple for undergraduates [1]. To do this, we first need to realize that since the amplitude of the merger-ringdown diminishes rapidly, it is only accurate for a small time period (Buskirk and Babiuc, 2018). This can be seen by analyzing the end of the inspiral amplitude graph created with a 6PN approximation. The amplitude rises exponentially but with the merger-ringdown, it is more of a gradual increase similar to a bell curve. This disagreement in the amplitude increase tells us that the inspiral is more accurate than the merger-ringdown away from the peak amplitude. With this in mind, we can now justify modeling the inspiral past the upper bound for x because the accuracy of the 6^{th}

[1]The model implemented by (Buskirk and Babiuc, 2018) uses four separate time parameters to match the waveform. The matching technique implemented here uses two parameters. While less formal, this technique is simple and can be done without user-inputted parameters in Python.

order PN approximation models the small time interval between ISCO and the light ring more accurately than the gIRS.

The first step in the matching process is to match the linear frequencies of the merger-ringdown and inspiral. To do this, we first shift the inspiral by $-t_F$ so that it ends at t = 0. From there, we shift the merger-ringdown so that the linear frequency of the merger-ringdown and inspiral match at $t = 0$. We determine that the amount of time the merger-ringdown was shifted to match the inspiral is a time parameter, τ. This parameter provides a general idea of how much the merger-ringdown strain needs to be shifted to match the inspiral strain. Additionally, by matching the frequency, the graphs of the inspiral and merger-ringdown strains are one step closer to a complete match since the time between peak strains is the same. From there, we just need to match the phase and amplitude.

In order to match the merger-ringdown strain with the inspiral strain, we notice that the h_+ strain is out of phase with the merger-ringdown strain, so we add a strain parameter, $\Phi = \pi$ to match them in phase. Next, we introduce a matching parameter, ξ, that is multiplied by τ and modified by user input until a best match is achieved. We define the best match as the full waveform consisting of the inspiral until $t = 0$ and the merger-ringdown from $t = 0$ to $t = t_f$ being as continuous as possible, determined by inspection. Ordinarily, we would also need to match the amplitude of the strain to obtain a perfect fit, however, the amplitudes of the merger-ringdown and inspiral strains are already matched. An additional matching technique exists in the python code that matches the inspiral strain and the merger-ringdown strain at $t = 0$. While not as adaptable, this second technique does avoid the need for the user to change ξ until it matches by inspection.

Figures 3 and 4 show the real and imaginary strain graphs before the match (above) and after the match (below) of the real portion and imaginary portion of the waveform. Figure 5 shows the completed waveforms we created for both the real (above) and imaginary (below) portions. The cut off point where the inspiral ends and the merger-ringdown begins was chosen to be $t = 0$ seconds because the inspiral was shifted to end at that time and it makes calculating the total time of the waveform easier.

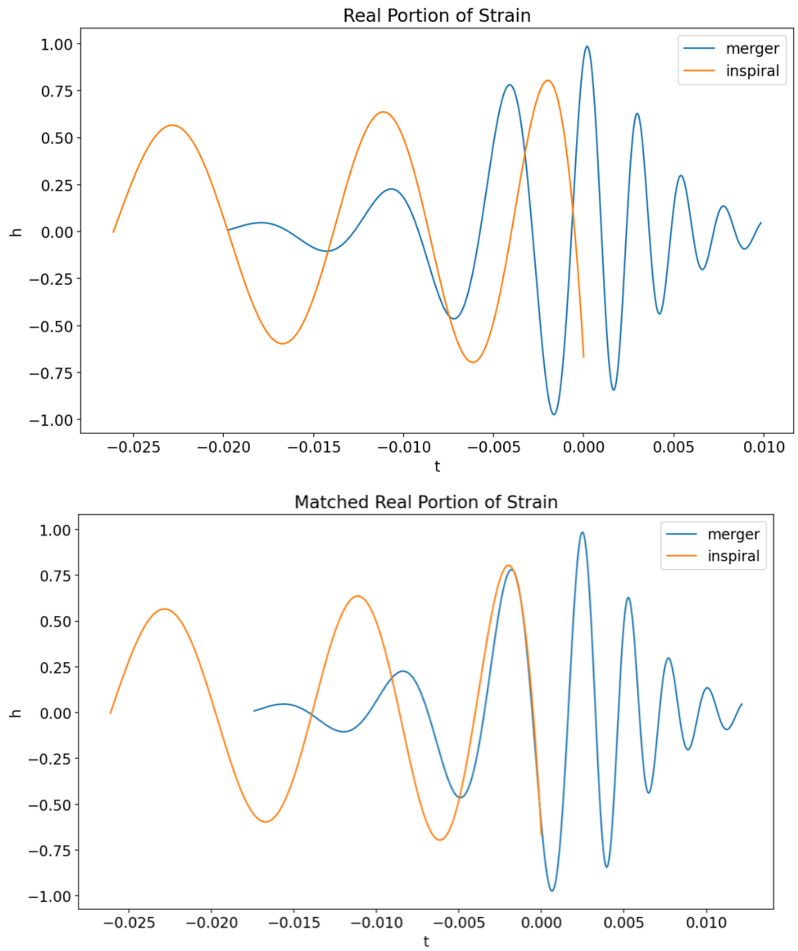

Figure 3: Real Propagation showing the waveform before (above) and after (below) matching

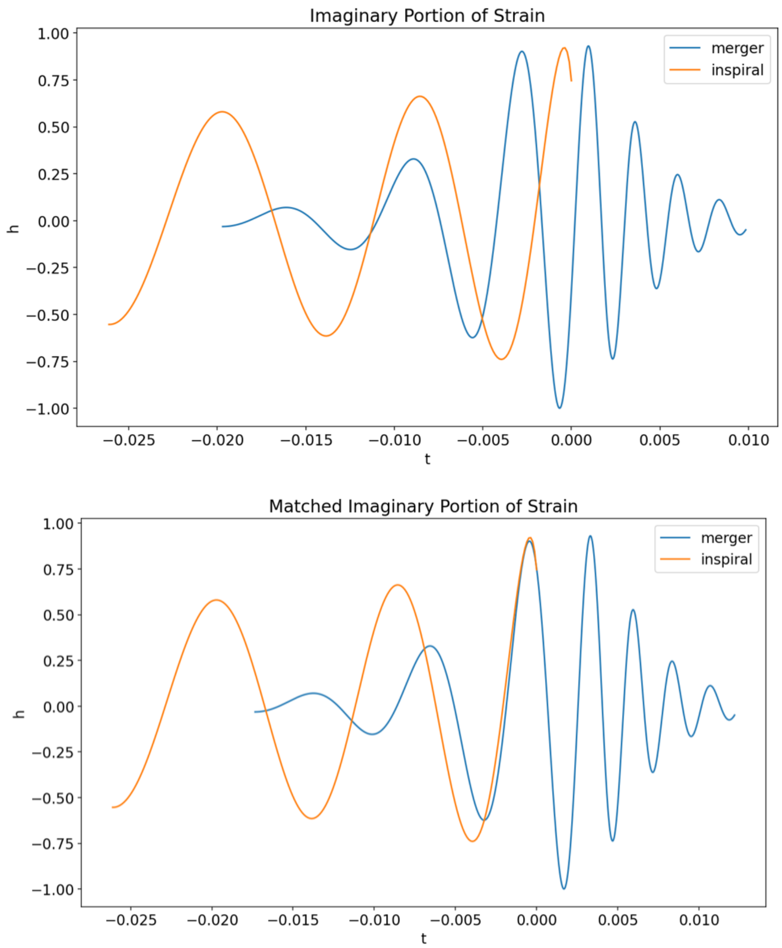

Figure 4: Imaginary Propagation showing the waveform before (above) and after (below) matching

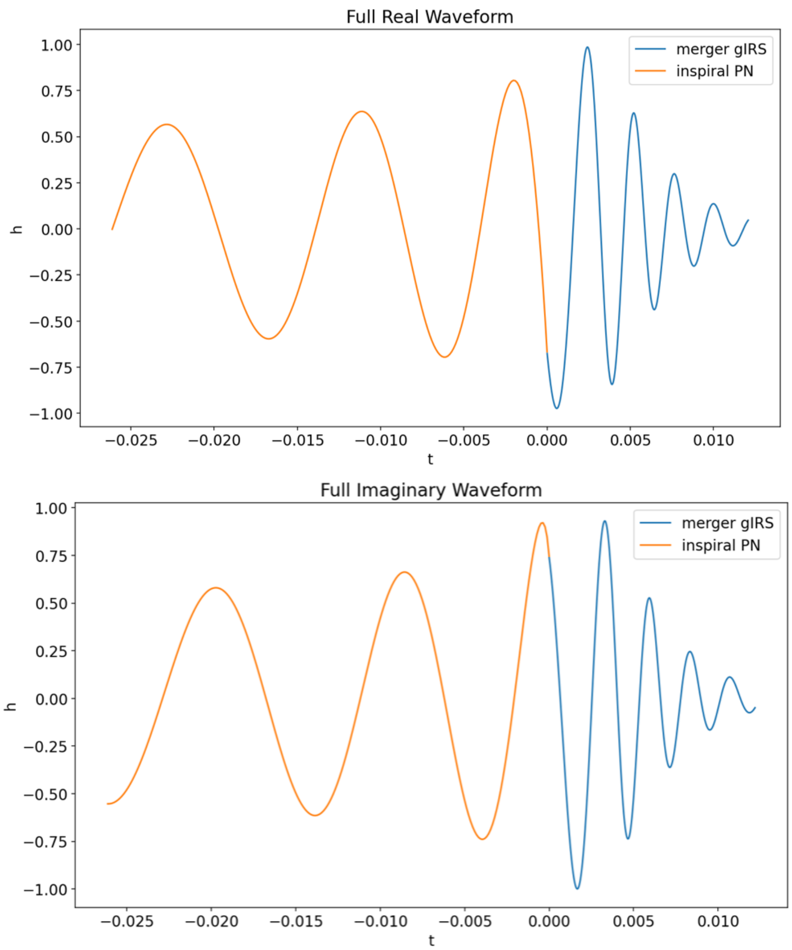

Figure 5: Full Waveforms

5 Conclusion

By following the procedures introduced in (Buskirk and Babiuc, 2018), we were able to successfully implement a model utillizing Post-Newtonian approximations and analytical fits in python to analytically calculate gravitational waves emitted through the inspiral and merger-ringdown phases of a binary black hole system in a manner that can be understood by undergraduates. While this model has yet to be tested with data collected experimentally and data produced through other analytical methods, the waveform generated shows high similarity when visually compared to other successful models (Buskirk and Babiuc, 2018; Huerta et al., 2017; East et al., 2013).

This project paves the way for numerous areas of research for undergraduate students such as exploring statistical analysis to measure how well the matching techniques work as well as how well they fit to experimentally determined data, creating a separate model for the ringdown phase, making a new, more optimized automated matching technique, and modifying the model to work for non-zero eccentricities and different mass ratios.

6 References

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A Appendix A - PN Coefficients

The coefficients used to calculate the PN-parameter x in equation (7) are below (Buskirk and Babiuc, 2018; Huerta et al., 2017).

$$a1 = \frac{-743}{336} - \frac{11}{4}\eta$$

$$a3/2 = 4\pi$$

$$a2 = \frac{34103}{18144} + \frac{13661}{2016}\eta + \frac{59}{18}\eta^2$$

$$a5/2 = \frac{-4159\pi}{672} - \frac{189\pi}{8}\eta$$

$a3 = \frac{-1712\gamma}{105} + \frac{16\pi^2}{3} + \frac{16447322263}{139708800} - \frac{856}{105}\log(16x(t)) + (-\frac{56198689}{217728} + \frac{451\pi^2}{48})\eta + \frac{541}{896}\eta^2 - \frac{5605}{2592}\eta^3$

$$a7/2 = (-\frac{4415}{4032} + \frac{358675}{6048}\eta + \frac{91945}{1512}\eta^2)\pi$$

$a4 = 170.799 - 742.551\eta + 370.173\eta^2 - 43.4703\eta^3 - 0.0249486\eta^4 + (14.143 - 150.692)\log(x(t))$

$$a9/2 = 1047.25 - 2280.56\eta + 923.756\eta^2 + 22.7462\eta^3 - 102.446\log(x(t))$$

$a5 = 714.739 - 1936.48\eta + 3058.95\eta^2 514.288\eta^3 + 29.5523\eta^4 - 0.185941\eta^5 + (-3.00846 + 1019.71\eta + 1146.13\eta^2)\log(x(t))$

$a11/2 = 3622.99 - 11498.7\eta + 12973.5\eta^2 - 1623\eta^3 + 25.5499\eta^4 + (83.1435 - 1893.65\eta)\log(x(t))$

$a6 = 11583.1 - 45878.3\eta + 33371.8\eta^2 - 7650.04\eta^3 + 648.748\eta^4 - 14.5589\eta^5 - 0.0925075\eta^6 + (-1155.61 + 7001.79\eta - 2135.6\eta^2 - 2411.92\eta^3)\log(x(t)) + 33.2307\log(x(t)^2)$

Below are the coefficients from equation (12) used for the PN expansion to add accuracy to the separation between binary objects, r (Memmesheimer et al., 2004).

$$r0PN = 1$$

$$r1PN = -1 + 0.333333\eta$$

$$r2PN = 4.75\eta + 0.111111\eta^2$$

B Appendix B - gIRS Coefficients

Below are the gIRS coefficients used to create an analytical model of the merger-ringdown waveform (Huerta et al., 2017)

$$Q = \frac{2}{(1 - \hat{s}_{fin})^0.45}$$

$$\alpha = \frac{1}{Q^2}(\frac{16313}{562} + \frac{21345}{124}\eta)$$

$$\kappa = \frac{713}{1056} - \frac{23}{193}\eta$$

$$b = \frac{16014}{979} - \frac{29132}{1343}\eta^2$$

$$c = \frac{206}{903} + \frac{180}{1141}\eta^{1/2} + \frac{424\eta^2}{1205\log(\eta)}$$