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All of Physics 134A first semester
Quantum Mechanics

Correspondence Principle:

Quantum Mechanics → Classical Mechanics

$\hbar \to 0$, $n \to \infty$

Superposition

$\psi_1(\vec{r},t)$, $\psi_2(\vec{x},t)$ satisfy Schrodinger's Equation, so does

$$\psi_1(\vec{x},t) + \psi_2(\vec{x},t) + \dots$$

Uncertainty Principle

$$\Delta x \Delta p \le \frac{\hbar}{2}$$

$(\Delta x)^2 = \langle (\hat{x} - \langle \hat{x} \rangle)^2 \rangle$ is the variance — Δx standard deviation

since $\Delta E = \frac{\Delta p^2}{2m} = \frac{p}{m}\Delta p$ $\Delta p \sim \Delta E$

$$\Delta t = \frac{\Delta x}{v} = \frac{m}{p}\Delta x \ge \frac{m}{p}\frac{\hbar}{2\Delta E} \Longrightarrow \Delta E \Delta t \ge \frac{\hbar}{2}$$

Schrodinger Equation Operators

$$\hat{x} = x \qquad \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x} \longrightarrow \hat{E} = i\hbar\frac{\partial}{\partial t}$$

Schrodinger Equations

time-dependent: $$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

time-independent: $$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x)$$

Position Probability Density

$$P(x,t) = \psi(x,t)\,\psi^*(x,t) = |\psi(x,t)|^2$$

$$dP(x,t) = |\psi(x,t)|^2\,dx$$

Normalization Condition

$$\int |\psi(x,t)|^2\,dx = 1, \qquad \frac{d}{dt}\int |\psi(x,t)|^2\,dx = 0$$

Expectation values of operators.

$$\langle \hat{O} \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{O}\,\psi(x,t)\,dx$$

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} x\,|\psi(x,t)|^2\,dx$$

$$\langle \hat{p} \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi(x,t)\,dx$$

Stationary States

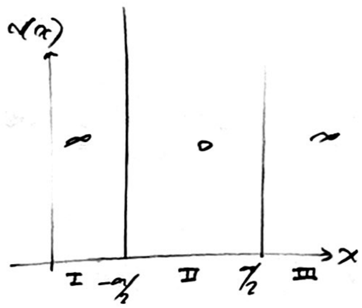
$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi$$

probability density independent of time $\frac{\partial}{\partial t}|\psi(x,t)|^2=0$

Stationary state solutions form orthonormal basis $\int_{-\infty}^{\infty}\psi_j^*(x)\psi_i(x)\,dx=\delta_{ij}$

$$\Psi(x,t)=\sum_{n}^{\infty}c_i\psi(x), \qquad c_i=\int_{-\infty}^{\infty}\psi_i^*\Psi(x)\,dx$$

The infinite square well



$$\psi_I=\psi_{III}=0$$

II.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi=E\psi \longrightarrow \psi=A\sin(kx)+B\cos(kx), \qquad k=\frac{\sqrt{2mE}}{\hbar}$$

Boundary Conditions

$$\psi_{II}\left(-\frac{a}{2}\right)=0 \qquad -A\sin\left(\frac{ka}{2}\right)+B\cos\left(\frac{ka}{2}\right)=0$$

$$\psi_{II}\left(\frac{a}{2}\right)=0 \qquad A\sin\left(\frac{ka}{2}\right)+B\cos\left(\frac{ka}{2}\right)=0$$

$2B\cos\left(\frac{ka}{2}\right)=0 \longrightarrow \frac{ka}{2}=n\frac{\pi}{2}, \Rightarrow k=\frac{n\pi}{a}, \quad n=1,3,5,\dots$ Even soln

$-2\sin\left(\frac{ka}{2}\right)=0 \longrightarrow \frac{ka}{2}=n\pi, \Rightarrow k=\frac{n\pi}{a}, \quad n=2,4,\dots$ Odd soln

Therefore,

$$\psi(x) = \begin{cases} A\sin(kx) & k = \frac{n\pi}{a}, \quad n = 0, 2, 4, \dots \\ B\cos(kx) & k = \frac{n\pi}{a}, \quad n = 1, 3, 5, \dots \end{cases}$$

$\hat{p}\,x$

$\hat{x}$

Normalization Condition

$$\int_{-\infty}^{\infty} A^2 \sin^2(kx) = \int_{-a}^{a} B^2 \cos^2(kx)\,dx = \frac{a}{2} \Longrightarrow A, B = \sqrt{\frac{2}{a}}$$

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin(kx), & k = \frac{n\pi}{a}, \quad n = 0, 2, 4 \dots \\ \sqrt{\frac{2}{a}}\cos(kx), & k = \frac{n\pi}{a}, \quad n = 1, 3, 5, \dots \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \Longrightarrow E = \frac{\hbar^2 k}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$ Energy Eigenvalues.

Harmonic Oscillator

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2, \quad \omega = \sqrt{\frac{k}{m}}$$

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}m\omega^2 x^2 \psi = E\psi \rightarrow H = \frac{1}{2}\left[\frac{-\hbar^2}{m}\frac{d^2}{dx^2} + m\omega^2 x^2\right]$$

$$H = \frac{1}{2m}\left[\hat{p}^2 + (m\omega x)^2\right] \qquad \text{let} \quad \hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x)$$

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega x)$$

$$\hat{a}_+\hat{a}_+ = \frac{1}{2\hbar m\omega}\left(\hat{p} + (m\omega x)^2 + i\hat{p}\,m\omega x - m\omega x i\hat{p}\right)$$

$$\hat{a}_- \hat{a}_+ = \frac{1}{2\hbar m\omega}\left(\hat{p} + (m\omega x)^2 - im\omega\left(\hat{x}\hat{p}x - \hat{p}\hat{x}\right)\right)$$

$$= \frac{1}{2\hbar m\omega}\left(\hat{p} + (m\omega x)^2 - im\omega[\hat{x}, \hat{p}]\right)$$

$[\hat{x}, \hat{p}] = i\hbar$

$$= \frac{1}{2\hbar m\omega}\left(\hat{p} + (m\omega x)^2 + m\omega\hbar\right) = \frac{1}{\hbar\omega}H + \frac{1}{2} \longrightarrow H = \left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\hbar\omega$$

Therefore,

$$\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\psi = E\psi \quad (1) \longrightarrow E = \frac{\hbar\omega}{2}$$

Claim:

if ψ solves (1) w/ Energy E, $\hat{a}_+\psi$ solves it w/ $(E + \hbar\omega)$.

Knowing that $\hat{a}_-\psi_0 = 0 \Longrightarrow \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega x)\psi_0 = 0$

$$(i\hat{p} + m\omega x)\psi_0 = 0 \longrightarrow \hbar\frac{\partial}{\partial x}\psi_0 = -m\omega x\psi_0$$

Therefore

$$\frac{\partial}{\partial x}\psi_0 = \frac{-m\omega x}{\hbar}\psi_0 \longrightarrow \frac{1}{\psi_0}d\psi_0 = \frac{-m\omega x}{\hbar}dx$$

$$\psi_0 = Ae^{-\frac{m\omega x^2}{2\hbar}}, \quad \int_{-\infty}^{\infty} e^{-\frac{m\omega x^2}{2\hbar}}dx = \sqrt{\frac{2\hbar\pi}{m\omega}}, \quad A = \sqrt{\frac{m\omega}{2\hbar\pi}}$$

$$\psi_0 = \sqrt{\frac{m\omega}{2\hbar\pi}}e^{-\frac{m\omega x^2}{2\hbar}}, \quad \psi_n = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\psi_0, \quad E_n = \frac{\hbar\omega}{2} + \hbar\omega n$$

$$= \hbar\omega\left(n + \frac{1}{2}\right)$$

The free particle

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \longrightarrow \psi = Ae^{ikx} + Be^{-ikx} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\Psi_k(x,t) = Ae^{ik\left(x - \frac{iEt}{\hbar}\right)} + Be^{-ik\left(x - \frac{iEt}{\hbar}\right)} \qquad E = \frac{\hbar^2 k}{2m}$$

$$= Ae^{ik\left(x - \frac{\hbar k t}{2m}\right)} + Be^{ik\left(x - \frac{\hbar k t}{2m}\right)}$$

$$= Ae^{i\left(kx - \frac{\hbar k^2 t}{2m}\right)} \qquad k = \frac{\pm\sqrt{2mE}}{\hbar}$$

$k > 0$ right $\rightarrow$

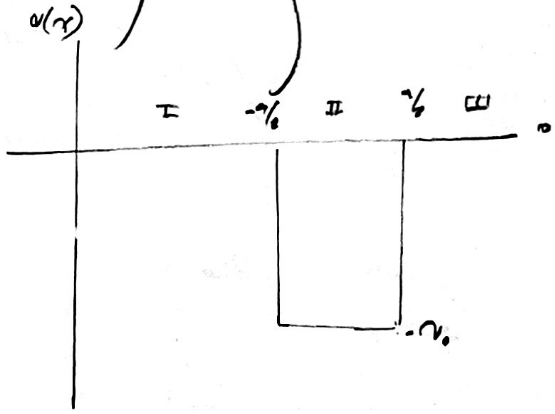
$k < 0$ left $\leftarrow$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k) e^{-i\left(kx - \frac{\hbar k^2 t}{2m}\right)} dk$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,t) e^{-i\left(kx - \frac{\hbar k^2 t}{2m}\right)} \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}$$

$$E = \frac{\hbar^2 k^2}{2m}$$

finite square well



Even solution

$$\psi(x) = \begin{cases} Ce^{\frac{\kappa a}{2}} \cos\left(\frac{ka}{2}\right) e^{\kappa x} \\ C\cos(kx) \\ Ce^{\frac{\kappa a}{2}} \cos\left(\frac{ka}{2}\right) e^{-\kappa x} \end{cases}$$

$$k \tan\left(\frac{ka}{2}\right) = \kappa$$

Odd solution

$$\psi(x) = \begin{cases} -De^{\frac{\kappa a}{2}} \sin\left(\frac{ka}{2}\right) e^{\kappa x} \\ D\sin(ka) \\ De^{\frac{\kappa a}{2}} \sin\left(\frac{ka}{2}\right) e^{-\kappa x} \end{cases}$$

$$k \cot\left(\frac{ka}{2}\right) = -\kappa$$

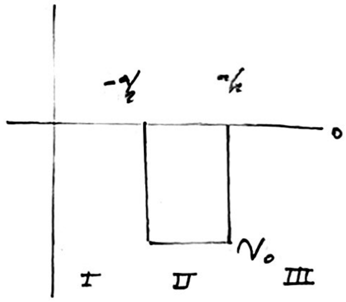
letting $z = \frac{ka}{2}$, $z_0 = \frac{a}{2\hbar}\sqrt{2mV_0} \Longrightarrow \frac{\kappa a}{2} = \sqrt{z_0^2 - z^2}$

Even: $\tan(z) = \sqrt{\frac{z_0^2}{z^2} - 1}$

Odd: $\cot(z) = -\sqrt{\frac{z_0^2}{z^2} - 1}$

if $(n+1)^2 \pi^2 \frac{\hbar^2}{2m} > a^2 V_0 > n^2 \pi^2 \frac{\hbar^2}{2m}$, $n = 0, 1, 2$

There are $n+1$ bound states.



E<0 Bound States

I:

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \longrightarrow \frac{d^2}{dx^2}\psi = \frac{-2mE}{\hbar^2}\psi$$

$$\psi_I = \cancel{A\sin^{-\kappa x}} + Be^{\kappa x} = Be^{\kappa x} \qquad \kappa = \frac{\sqrt{-2mE}}{\hbar^2}$$

II: E<0

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi - V_0\psi = E\psi \longrightarrow \frac{d^2}{dx^2}\psi = -(E+V_0)\psi \cdot \frac{2m}{\hbar^2}$$

$$\psi_{II} = C\sin(lx) + D\cos(lx) \qquad l = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

III

$$\psi_{III} = Fe^{-\kappa x}$$

$$\psi = \begin{cases} Be^{-\kappa x} & x < -a/2 \\ D\sin(lx) + C\cos(lx) & -a/2 < x < a/2 \\ Ae^{-\kappa x} & x > a/2 \end{cases}$$

Even soln: $D\cos(lx)$ $\quad -\frac{a}{2} < x < \frac{a}{2}$

Boundary Conditions

$$A = Ce^{\frac{\kappa a}{2}}\cos\left(\frac{la}{2}\right) \qquad l\tan\left(\frac{la}{2}\right) = \kappa$$

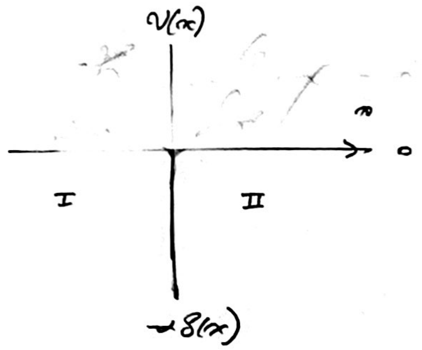
$$\longrightarrow$$

$$B = Ce^{\frac{\kappa a}{2}}\cos\left(\frac{la}{2}\right)$$

Delta function potential

$V(x) = -\alpha\delta(x)$

$\underline{E<0}$



I:

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \longrightarrow \frac{d^2}{dx^2}\psi = \frac{-2mE}{\hbar^2}\psi$$

$$k = \frac{\sqrt{-2mE}}{\hbar}$$

$x<0$: $\psi_I = Ae^{kx} + \cancel{Be^{-kx}}^{\,0}$

$x>0$: $\psi_{II} = \cancel{Ce^{kx}}^{\,0} + De^{-kx}$

$$\psi(x) = \begin{cases} Ae^{kx} & x<0 \\ De^{-kx} & x>0 \end{cases}$$

Boundary Conditions

$$Ae^{k(0)} = De^{-k(0)} \longrightarrow A = D \qquad \psi = Ae^{kx},\ Ae^{-kx} = Ae^{-k|x|}$$

$$\frac{-\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2}{dx^2}\psi + \int_{-\epsilon}^{\epsilon}\alpha\delta(x)\psi = \cancel{\int_{-\epsilon}^{\epsilon}E\psi}^{\,0} \longrightarrow \frac{-\hbar^2}{2m}\left(\frac{d}{dx}\psi\Big|_{\epsilon} - \frac{d}{dx}\psi\Big|_{-\epsilon}\right) = \alpha\psi(0)$$

Therefore

$$\left.\frac{d\psi}{dx}\right|_{\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon} = \frac{-2m}{\hbar^2}\alpha\psi(0) = -\frac{2m}{\hbar^2}\alpha A$$

$$\left. -kAe^{-k\epsilon} - kAe^{k\epsilon}\right|_{\epsilon\to 0} = \frac{-2m}{\hbar^2}\alpha A$$

$$-\cancel{2}kA = \frac{-\cancel{2}m}{\hbar^2}\alpha A \Longrightarrow k = \frac{m\alpha}{\hbar^2} \Longrightarrow E = \frac{-m\alpha^2}{2\hbar^2}$$

Energy of Bound state

$$\psi = Ae^{-k|x|} = Ae^{-\frac{m\alpha}{\hbar^2}|x|}$$

$$\int_{-\infty}^{\infty} |A|^2 e^{-2k|x|}dx \longrightarrow 2\int_0^{\infty} |A|^2 e^{-2kx} \Longrightarrow A = \frac{\sqrt{m\alpha}}{\hbar}$$

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-\frac{m\alpha}{\hbar^2}|x|} \qquad E = \frac{-m\alpha^2}{2\hbar^2}$$

$\underline{\underline{E > 0}}$

$$\psi_I = \psi_{II} \qquad A + B = C + D$$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Ce^{ikx} + \cancel{De^{-ikx}} & x > 0 \end{cases}$$

$$\frac{-\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2}{dx^2}\psi\, dx = -\alpha\psi(0) \longrightarrow \left.\frac{d\psi}{dx}\right|_{\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon} = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

$$ikCe^{ika} - \left(ikAe^{-ika} - ikBe^{ika}\right) = -\frac{2m\alpha}{\hbar^2}\psi(a)$$

$$ikC - (ikA - ikB) = ik(C - A + B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

$$\Rightarrow \quad i(C - A + B) = \frac{-2m\alpha}{\hbar^2 k}(A+B) = -2\beta(A+B)$$

$$iC = -2\beta(A+B) + iA - iB$$

$$iC = A(i - 2\beta) - B(-i + 2\beta)$$

$$\begin{cases} C = A(1 + 2\beta i) - B(1 - 2\beta i) \\ A + B = C \end{cases}$$

$$A + B = A(1 + 2\beta i) - B(1 - 2\beta i) \qquad C = A \cdot \frac{1}{1 - \beta i}$$

$$B = A \cdot \frac{\beta i}{1 - \beta i}$$

Transmisión: $T = \dfrac{|C|^2}{|A|^2} = \dfrac{1}{1+\beta^2}$ $\qquad \beta = \dfrac{m\alpha}{\hbar^2 k}$

Reflexión: $R = \dfrac{|B|^2}{|A|^2} = \dfrac{\beta^2}{1+\beta^2}$