Problem Set #12 (Last one: Congratulations!): Due Sunday, April 28, 11:59pm

1. Please fill out the course evaluation for 137A. (No points for this one - just gratitude.)

2. *Three-particle spin states, via an angular momentum coupling scheme:* One coupling scheme for defining 3-particle states of total S is given by

$$\begin{aligned}|\tfrac{1}{2},(\tfrac{1}{2}\tfrac{1}{2})S_{23},SM\rangle &\equiv \sum_{m_1 M_{23}} \langle \tfrac{1}{2}m_1\; S_{23}M_{23}|(\tfrac{1}{2}S_{23})SM\rangle\; |\tfrac{1}{2}m_1\rangle\; |(\tfrac{1}{2}\tfrac{1}{2})S_{23}M_{23}\rangle \\ &= \sum_{m_1 M_{23}} \langle \tfrac{1}{2}m_1\; S_{23}M_{23}|(\tfrac{1}{2}S_{23})SM\rangle \sum_{m_2 m_3} \langle \tfrac{1}{2}m_2\; \tfrac{1}{2}m_3|(\tfrac{1}{2}\tfrac{1}{2})S_{23}M_{23}\rangle |\tfrac{1}{2}m_1\rangle|\tfrac{1}{2}m_2\rangle|\tfrac{1}{2}m_3\rangle\end{aligned}$$

Don't be intimidated by this: it just says we couple states 2 and 3 of spin $\frac{1}{2}$ to form a state of total S_{23}, then we couple state 1 to the two-particle state with S_{23} to form a three-particle state with total S.

a)What are the three possible values for S_{23} and S? For each such of the three possibilities, evaluate the expression above by looking up the Clebsch-Gordan coefficients (or by asking Mathematica to do so). Compare to the results in Problem 3, Problem Set 11, parts b), c), d). What determines whether a state is symmetric or antisymmetric under the exchange $2 \leftrightarrow 3$?

b) What basis of coupled states analogous to $|\frac{1}{2},(\frac{1}{2}\frac{1}{2})S_{23},SM\rangle$ would you use to create three-particle states of good S and M that have definite symmetry under $m_1 \leftrightarrow m_2$? What would be the quantum number analogous to S_{23} and what choices would yield states even or odd under $m_1 \leftrightarrow m_2$?

3. *Enumerating antisymmetric states:* We have a system consisting of the single-particle states labeled by the quantum numbers $\ell = 1,\ m_\ell,\ s = \frac{1}{2},\ m_s,\ \tau = \frac{1}{2},\ m_\tau$. The particles we place in these states are indistinguishable fermions.

a) How many single-particle states are there? Show that these states can be represented as bits is a computer word, where the location of the bit corresponding to a given choice of $m_\ell,\ m_s,\ m_\tau$ is provided by the index

$$I = 4\,(m_\ell + 1) + 2\left(m_s + \frac{1}{2}\right) + \left(m_\tau + \frac{1}{2}\right) + 1$$

b) How many two-fermion, three-fermion, and four-fermion states can be formed in this basis? (Remember how we count many-fermion states in the m-scheme, as bits that are occupied or not, 1 or 0, in a computer word.)

c) Suppose we form two-particle states in the coupled scheme (which we like to do because it makes the exchange symmetry clear) ,

$$|(11)LM_L;(\tfrac{1}{2}\tfrac{1}{2})SM_S;(\tfrac{1}{2}\tfrac{1}{2})TM_T\rangle$$

Here L is obtain by coupling $\ell_1 = 1$ and $\ell_2 = 1$; and similarly for $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2}$ (S); and similarly for $\tau_1 = \frac{1}{2}$ and $\tau_2 = \frac{1}{2}$ (T). Make a table in which the first column gives the allowed $\{L, S, T\}$ values, the second column gives the exchange symmetry of each of the three components (e.g., $\{+, +, -\}$), the third column gives the total exchange symmetry, and the fourth column gives the number of states $(2L + 1)(2S + 1)(2T + 1)$. Use this table to identify the total number of states that are antisymmetric and thus can be two-particle fermionic states, and compared the total number of such states to the two-particle result of a).

d) What is the constraint on $L + S + T$ that determines whether a two-particle state describes fermions?

4. *Laughlin's fractional quantum Hall wave function (a variation of problem 5.11 of Griffiths):* Noninteracting electrons move on a 2D surface. We can pick an origin and measure electron positions by their x, y coordinates. Perpendicular to the surface is a strong magnetic field, $-\mathbf{B}\hat{z}$, that forces all of the electron spins to align. As the spins are all in the same state, we can ignore spin as it plays no role in the anti-symmertry. That is, anti-symmetry must come about through the spatial part of the wave function. Provided the density of electrons is not too large, non-interacting electron states labeled by $|\kappa\rangle$, $\kappa = 0, 1, 2, \cdots$, provide the single-particle basis from which we can construct many-body wave functions, where

$$\langle z|\kappa\rangle = N z^\kappa e^{-|z|^2/2} \qquad z \equiv \frac{x+iy}{a_0\sqrt{2}} \qquad a_0 = \sqrt{\frac{\hbar c}{e|B|}}$$

where z is a dimensionless complex coordinate that we can use to identify any point in the plane, and a_0 is the radius of the cyclotron orbits the electrons execute in the magnetic field. We agree to measure all distances in a_0 units, thereby making z dimensionless.

a) Calculate the normalization N by demanding

$$\int \frac{dx}{a_0}\frac{dy}{a_0}|\langle z|k\rangle|^2 = 1$$

It may be easiest to do this using circular coordinates $z = re^{i\theta}$, where $r = |z|$ is the dimensionless radial coordinate.

b) Show that the wave function

$$\Psi_1(z_1, z_2, \cdots, z_N) = N_N \left[\prod_{j<k}^{N}(z_j - z_k)\right] exp\left[-\frac{1}{2}\sum_{k=1}^{N}|z_k|^2\right]$$

has the correct antisymmetry to describe fermions. Use this result to show that all single-particle states are occupied and thus that this wave function is a single Slater determinant.

c) Consider the set of N-electron wave functions indexed by m

$$\Psi_m(z_1, z_2, \cdots, z_N) = N_N \left[\prod_{j<k}^{N}(z_j - z_k)^m\right] exp\left[-\frac{1}{2}\sum_{k=1}^{N}|z_k|^2\right]$$

where m is a positive integer. For what ms will this wave function have the proper antisymmetry for fermions? If you were to expand and re-express this wave function in terms of single-particle states (do not actually do this) what is the maximum value of κ in $\langle z|\kappa\rangle$ that could appear? If we define the "filling" of the state Ψ_m as the number of particles N divided by the number of single-particle states that are available for filling, what filling does the wave function Ψ_m represent? Evaluate the filling as $N \to \infty$, to simplify your result. Are states with $m > 1$ expressible as single Slater determinants (that is, a simple product states, anti-symmetrized)?

d) Laughlin argued that wave functions of the form of c) for small m might be very good approximate wave functions for electrons that *interact* with each other, repelling each other, through their mutual Coulomb interaction. Recalling work we did in class on the allowed short-distance behavior of the relative wave functions, explain why Laughlin's conjecture might be reasonable.

e) Compute the N=2 Laughlin wave function for the $m = 3$ case (the 1/3rd filling case), expressing it as sum of Slater determinants of the single particle wave functions $\langle z|\kappa\rangle$. (You need not worry about the overall many-particle wave function normalization.) Thus, unlike the $m = 1$ case, this wave functions cannot be expressed as a single anti-symmeterized product state, but instead requires a sum over such states. It can be shown that the m=3 wave functions are the exact solutions of the interacting problem when $N = 2, 3$.

5. *States of the He atom:* We can make a reasonable model of the two-electron He atom as two electrons occupying hydrogen-atom-like orbits around a helium nucleus, but with the charge of the He nucleus reduced from its true value $Z = 2$ to a somewhat value we will call Z_{eff}. This phenomenological charge takes into account the fact that each electron sometimes see the other electron interior to it, reducing the central charge from 2 to something ~ 1. That is, we can take into account some of the electron-electron interactions by using a screened nuclear charge. One might guess that $Z_{\text{eff}} \sim 1.5$.

a) Hydrogen-like single-electron orbits can be labeled by the quantum numbers $n\,\ell\,m_\ell\,m_s$. The available $n\ell$ single-particle states are $1s$, $2s$ and $2p$. How many single-electron states $n\,\ell\,m_\ell\,m_s$ are there?

b) We make a simple model of the lowest energy electronic states of He by requiring the two electrons to occupy $n = 1$ and $n = 2$ levels, with the additional constraint that there must be at least one electron in the $1s$ state. This defines our Hilbert space. Remembering that at most on fermion can occupy each quantum state, how many two-electron fermion states can be made by occupying the m-scheme states $n\,\ell\,m_\ell\,m_s$?

c) We can couple the spins, $\vec{S} = \vec{s}_1 + \vec{s}_2$ states. Under exchange of spin labels, what is the symmetry of the $S = 1$ (triplet) and $S = 0$ (singlet) states? Likewise what are the allowed symmetric and antisymmetric spatial wave functions that can be formed from two electrons occupying the $1s{-}1s$, $1s{-}2s$, and $1s{-}2p$ orbitals. Defining $\vec{L} = \vec{\ell}_1 + \vec{\ell}_2$, associate an L with each of these states.

d) Form all possible two-electron antisymmetric wave functions of the form $|\alpha(LS)JM_J\rangle$. Here α is an energy quantum number that distinguishes states where all of the other quantum numbers are equal: in the present case it can be the number of electrons in the 1s state. Make a table where the columns are labeled by α, ${}^{2S+1}L_J$, the radial wave function and its symmetry (e.g., $1s2s - 2s1s$: antisym), and the number of magnetic substates M_J. Note that L is typically denote as $\mathcal{S}$ for $L = 0$, $\mathcal{P}$ for $L = 1$, $\mathcal{D}$ for $L = 2$, etc How many total antisymmetric states are there? How do these predictions compare with the figure from Lecture 34 showing the low-lying levels of the He atom?

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Problem Set #12

Physics 137 A.

2. Three-particle spin-states.

$$|\tfrac{1}{2}, S_{23}, SM\rangle = \underbrace{\sum_{m_1 M_{23}} \langle \tfrac{1}{2} m_1 ; S_{23} M_{23} | SM \rangle}_{\text{CG coefficient for } m_1 \otimes M_{23}} \underbrace{\sum_{m_2 m_3} \langle \tfrac{1}{2} m_2 \tfrac{1}{2} m_3 | S_{23} M_{23} \rangle}_{\text{CG coefficient for } m_2 \otimes m_3} |\tfrac{1}{2} m_1\rangle |\tfrac{1}{2} m_2\rangle |\tfrac{1}{2} m_3\rangle$$

a) S_{23} results from coupling $|\tfrac{1}{2} m_2\rangle \otimes |\tfrac{1}{2} m_3\rangle$. Therefore, the total spin S_{23}

$$S_{23} \in \{ |\tfrac{1}{2} - \tfrac{1}{2}|, \tfrac{1}{2} + \tfrac{1}{2} \} = 0, 1 \quad \text{two possibilities}$$

S results from coupling $|\tfrac{1}{2} m_1\rangle \otimes |S_{23} M_{23}\rangle$. Therefore not 3 ???

$$S \in \{ |0 - \tfrac{1}{2}|, \dots, \tfrac{1}{2} + 1 \} = \tfrac{1}{2}, \tfrac{3}{2} \quad \text{two possibilities}$$

I only get two possible values for S_{23} & S. I am missing something but I will continue solving for all $|\tfrac{1}{2}, S_{23}, SM\rangle$ for the combinations I know thus far.

For the following calculations I use the notation $|\uparrow\rangle = |+\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle$
$|\downarrow\rangle = |-\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$

$|\tfrac{1}{2},1,\tfrac{3}{2}\tfrac{3}{2}\rangle$

$m_1, m_2, m_3 = \tfrac{1}{2}$

$M_{23} = 1$

$$|\tfrac{1}{2},1,\tfrac{3}{2}\tfrac{3}{2}\rangle = \langle\tfrac{1}{2}\tfrac{1}{2};11|\tfrac{3}{2}\tfrac{3}{2}\rangle\langle\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}|11\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= \cancelto{1}{\langle\uparrow\uparrow\uparrow|\uparrow\uparrow\uparrow\rangle}\cancelto{1}{\langle\uparrow\uparrow|\uparrow\uparrow\rangle}|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle = |+\rangle|+\rangle|+\rangle \checkmark$$

$|\tfrac{1}{2},1,\tfrac{3}{2}\tfrac{1}{2}\rangle$

$m_1, m_2, m_3 \in \{-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\}, \{\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\}, \{\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\}$

$$|\tfrac{1}{2},1,\tfrac{3}{2},\tfrac{1}{2}\rangle = \langle\tfrac{1}{2}\,-\tfrac{1}{2};11|\tfrac{3}{2}\,1\rangle\langle\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}|11\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$+\,2\langle\tfrac{1}{2}\tfrac{1}{2};\,10|\tfrac{3}{2}\tfrac{1}{2}\rangle\sum_{m_2,m_3\in\{-\frac{1}{2}\frac{1}{2}\}}\langle\tfrac{1}{2}m_2\tfrac{1}{2}m_3|10\rangle|\tfrac{1}{2}m_2\rangle|\tfrac{1}{2}m_3\rangle\left(|\tfrac{1}{2}m_1\rangle\right)$$

$$= |\downarrow;\uparrow\uparrow|\downarrow\uparrow\uparrow\rangle\langle\uparrow\uparrow|\uparrow\uparrow\rangle|\downarrow\rangle|\uparrow\rangle|\uparrow\rangle \qquad \tfrac{1}{\sqrt{2}} \qquad |\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$+\frac{2}{\sqrt{2}}\langle\uparrow;(\downarrow\uparrow+\uparrow\downarrow)|\uparrow\downarrow\uparrow+\uparrow\uparrow\downarrow\rangle\Big(CG\big[\cancel{\{\tfrac{1}{2},-\tfrac{1}{2}\},\{\tfrac{1}{2},\tfrac{1}{2}\}}\,\{1,0\}\big]|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$\tfrac{1}{\sqrt{2}}$$

$$+\,CG\big[\cancel{\{\tfrac{1}{2},\tfrac{1}{2}\},\{\tfrac{1}{2}\,-\tfrac{1}{2}\}},\{1,0\}\big]|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$= |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle + \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle + \frac{1}{\sqrt{2}}|\uparrow\rangle|\uparrow\rangle|\downarrow\rangle\right)$$

$$= |\downarrow\rangle|\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle \Longrightarrow$$

$$= \sqrt{\tfrac{1}{3}}\left(|-\rangle|+\rangle|+\rangle + |+\rangle|-\rangle|+\rangle + |+\rangle|+\rangle|-\rangle\right) \checkmark$$

$|\tfrac{1}{2}, 0, \tfrac{1}{2}\tfrac{1}{2}\rangle$

$\{m_1, m_2, m_3\} = \{\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\}\ \{\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\}$

$M_{23} = 0$

$$|\tfrac{1}{2}, 0, \tfrac{1}{2}\tfrac{1}{2}\rangle = \overbrace{\langle \tfrac{1}{2}\tfrac{1}{2}; 00 | \tfrac{1}{2}\tfrac{1}{2}\rangle}^{CG} \sum_{m_2, m_3 = \{-\frac{1}{2}, \frac{1}{2}\}} \overbrace{\langle \tfrac{1}{2}m_2 \tfrac{1}{2}m_3 | 00\rangle}^{CG} |\tfrac{1}{2}\tfrac{1}{2}\rangle |\tfrac{1}{2}m_2\rangle |\tfrac{1}{2}m_3\rangle$$

$$= \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) |\uparrow\rangle \Big(\langle \tfrac{1}{2} -\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2} | 00\rangle |\tfrac{1}{2}\tfrac{1}{2}\rangle |\tfrac{1}{2} -\tfrac{1}{2}\rangle |\tfrac{1}{2}\tfrac{1}{2}\rangle$$
$$+ \langle \tfrac{1}{2}\tfrac{1}{2}\ \tfrac{1}{2} -\tfrac{1}{2} | 00\rangle |\tfrac{1}{2}\tfrac{1}{2}\rangle |\tfrac{1}{2}\tfrac{1}{2}\rangle |\tfrac{1}{2} -\tfrac{1}{2}\rangle \Big)$$

$$= \frac{1}{\sqrt{2}} \left(-\frac{1}{\sqrt{2}} |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle + \frac{1}{\sqrt{2}} |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle \right)$$

$$= -\frac{1}{2} |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle + \frac{1}{2} |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle \Longrightarrow \sqrt{\tfrac{1}{2}}$$

$$= -\frac{1}{\sqrt{2}} |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle + \frac{1}{\sqrt{2}} |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle$$

$$= \frac{1}{\sqrt{2}} \left(|+\rangle|+\rangle|-\rangle - |+\rangle|-\rangle|+\rangle \right) \checkmark$$

All results match with PSET #11.

The Clebsch Gordan Coefficients determine whether the state is symmetric or antisymmetric for $2 \leftrightarrow 3$. More specifically, if

$$\langle \tfrac{1}{2}m_2 \tfrac{1}{2}m_3 | S_{23} M_{23}\rangle \Longleftrightarrow \langle \tfrac{1}{2}m_3 \tfrac{1}{2}m_2 | S_{23} M_{23}\rangle$$

If they are equivalent the symmetric. If they are negatives of each other the state is antisymmetric.

b) The basis $|\tfrac{1}{2}, S_{12}, SM\rangle$ where you first couple states 1 & 2 $\rightarrow S_{12}$. And afterwards couple the coupled state $|S_{12} M_{12}\rangle \otimes |\tfrac{1}{2} m_3\rangle$. The same rules as above apply except $2 \leftrightarrow 3 \Longrightarrow 1 \leftrightarrow 2$.

Enumerating Antisymmetric States.

a)

$l = 1, \; m_l, \; s = 1/2, \; m_s, \; \tau = 1/2, \; m_\tau$

$l = 1 \implies m_l = \{-1, 0, 1\}$

$s = 1/2 \implies m_s = \{-1/2, 1/2\} = m_\tau$

Therefore total # of states $= 3 \times 2 \times 2 = 12$ single-particle states.

As each quantum # l, s, τ represents a certain eigenvalue of a Quantum Operator, and no 2 fermions can occupy the same state, a computer word of 12 indices can be generated, where an index I is equal to 1 if the fermion is in that state, and 0 otherwise.

To generate 12 indices, we scale the possible locations of m_l to the position associated w/

$m_l = -1 \longrightarrow$ index 0

$m_l = 0 \longrightarrow$ index 4

$m_l = 1 \longrightarrow$ index 8

And likewise for m_s, m_τ. The 1 at the end ensures the index starts @ 1.

b)

for a ~~two~~ fermion state,

$$\# \text{ possible combinations}: \binom{12}{2} = \frac{12!}{2!(12-2)!} = 66$$

for a three fermion state: $\binom{12}{3} = 220$

four fermion state: $\binom{12}{4} = 495$

c)

Coupled: $|LM_L;\ SM_S;\ TM_T\rangle$

$L \in \{|l_1 - l_2|, \ldots, l_1 + l_2\} = \{0, 1, 2\}$

$S \in \{|s_1 - s_2|, \ldots, s_1 + s_2\} = \{0, 1\} = T$

$\{L, S, T\}$	Exchange Symmetry	Total Symmetry	# states
0 0 0	{+, + +}	+	1
1 0 0	− + +	−	3
1 1 0	+ − +	−	9
1 0 1	− + −	+	5
2 0 0	− + +	−	15
2 1 0	+ − +	+	15
2 0 1	+ + −	−	45
2 1 1	+ − −	+	

~~total~~ # of Antisymmetric = 32 + 12

d) $L+S+T$ must be odd for Pauli Exclusion Principle to hold

~~Laughlin's fractional quantum Hall wave function~~

$|n\rangle \quad n = 0, 1, 2, \ldots$

$$\langle z|n\rangle = N z^{n} e^{-|z|^2/2} \qquad z \equiv \frac{x+iy}{a_0\sqrt{2}} \qquad a_0 = \sqrt{\frac{\hbar c}{e|B|}}$$

a)

$$\int \frac{1}{a_0^2} |\langle z|n\rangle|^2 \, dx\,dy$$

$$\langle z|n\rangle = N \left(\frac{x+iy}{a_0\sqrt{2}}\right)^{n} e^{-\left(\frac{1}{4a_0^2}(x+iy)(x-iy)\right)}$$

$$\Longrightarrow 2^{-n} N^2 \iint e^{\frac{-x^2+y^2}{2a_0^2}} \left(\frac{x-iy}{a}\right)^{n} \left(\frac{x+iy}{a}\right)^{n} dx\,dy$$

$$= N^2 \cdot 2^{-n} \left(\frac{\sqrt{2}\,a_0}{3n} (x^2+y^2)^{\frac{\sqrt{3}\,n}{2}}\right)$$

Therefore $N = \sqrt{2^{n} \left(\frac{3n}{\sqrt{2}\,a_0 (x^2+y^2)^{\frac{\sqrt{3}\,n}{2}}}\right)}$

b)

$$\chi_1(z_1, z_2, \dots z_N) = N_N \underbrace{\left[\prod_{j<k}^{N} (z_j - z_k)\right]}_{} \exp\left[-\frac{1}{2}\sum_{k=1}^{N} |z_k|^2\right]$$

$$(z_1 - z_2)(z_1 - z_3)\dots(z_1 - z_N) \times (z_2 - z_3)\dots(z_2 - z_N)\dots \Big]$$

Under any exchange $1 \leftrightarrow 2$, $2 \leftrightarrow 3$, $N' \leftrightarrow N$ only one term becomes negative & the rest swap places resulting in the same product except for a minus. Therefore χ is antisymmetric under exchange.

The wave function can thus be expressed as

$$\chi(z_1, z_2 \dots z_N) = A \begin{vmatrix} z_1^{N-1} e^{-\frac{1}{2}|z_1|^2} & z_1^{N-2} e^{-\frac{1}{2}|z_1|^2} & \dots & e^{-\frac{1}{2}|z_1|^2} \\ z_2^{N-1} e^{-\frac{1}{2}|z_2|^2} & & \ddots & \vdots \\ \vdots & & & \\ z_N^{N-1} e^{-\frac{1}{2}|z_N|^2} & z_N^{N-2} e^{-\frac{1}{2}|z_N|^2} & \dots & e^{-\frac{1}{2}|z_N|^2} \end{vmatrix}$$

c) $$\chi = N_N\left[\prod_{j<k}^{N} (z_j - z_k)^m\right] \exp\left[-\frac{1}{2}\sum_{k=1}^{N} |z_k|^2\right]$$

for m odd, antisymmetry is preserved. Otherwise no negative would exist.

The maximum value of n in $\langle z | n \rangle$ would then be $m(N-1)$ which is the highest power of the product of N z's of $(z_j - z_k)^m$.

is the "filling" represents the ratio of N particles to the # of single particle states of filling, as $N \to \infty$, we get a fraction

$$\text{"filling"} = \frac{1}{m}$$ since there exist m single-state particles for every total #.

if $m > 1$, it's not possible to represent χ as a Slater Determinant since χ involves products of power differences which is not possible via matrix determinants.

d)
The relative χ of two fermions vanish at small distances as an odd power of their difference in position.

for small m, the $(z_j - z_k)^m$ implies the wave function is spread out which is consistent if we consider Coulomb Repulsion. The particles would spread out to minimize their mutual repulsion energy.

e) for $N = 2$, $m = 3$

$$\chi_3(z_1, z_2) = N(z_1 - z_2)^3 e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)}$$

$$a\begin{vmatrix} \langle z_1 | 0\rangle & \langle z_1 | 1\rangle \\ \langle z_2 | 0\rangle & \langle z_2 | 1\rangle \end{vmatrix} + b\begin{vmatrix} \langle z_1 | 0\rangle & \langle z_1, 1\rangle \\ \langle z_2 | 1\rangle & \langle z_2 | 0\rangle \end{vmatrix}$$

5. Helium Atom

a) for Hydrogen $1s \rightarrow 2p$

for $1s$: $l = 0 \Rightarrow m_l = 0$

$s = \frac{1}{2} \Rightarrow m_s = -\frac{1}{2}, \frac{1}{2}$ 2 possibilities

$n = 1$

for $2s$: $l = 0 \Rightarrow m_l = 0$

$s = \frac{1}{2} \Rightarrow m_s = -\frac{1}{2}, \frac{1}{2}$ 2 possibilities

$n = 2$

for $2p$: $l = 1 \Rightarrow m_l = -1, 0, 1$ 6 possibilities

$s = \frac{1}{2} \Rightarrow m_s = -\frac{1}{2}, \frac{1}{2}$

$n = 2$

Therefore, there are <u>10</u> possible $(n\, l\, m_l\, m_s)$ configurations.

b) If one e^- assumes the 1s state and there must be one e^- in the $n = 1$ (1s) & $n = 2$ (2s & 2p) state,

1s	2s	2p
1	1	1
1	0	

← only two possible $n l m_l m_s$ configuration under the exchange of spin labels, where

$$\vec{S} = s_1 + s_2$$

for triplet $S = 1$, the state must be symmetric

for singlet $S = 0$, the state must be antisymmetric

for 1s - 1s: Equivalent orbitals — symmetric spatial wave function

$L = 0$ antisymmetric spin wave function

for 1s - 2s: same as 1s-1s since $l = 0$ is still equivalent.

$L = 0$

for 1s - 2p: Now since $l = 0$; $l = 1$, the spatial wave function is antisymmetric

$L = 1$ so the spin wave function must be symmetric

b) α: # e^- in 1s

L: total $l_1 + l_2$

S: total $s_1 + s_2$

J: total $L + S$

Π_s

α	$^{2S+1}L_J$	χ symmetry	# m_J
2	$^1S_0$	1s - 1s (sym)	0
1	$^1S_0$	1s2s - 2s1s (antisym)	0
1	$^3S_0$	1s2s - 2s1s (antisym)	$-\frac{1}{2}, \frac{1}{2} \Rightarrow 2$
1	$^1P_1$	1s2p - 2p1s (antisym)	$-1, 0, 1 \Rightarrow 3$
1	$^3P_{210}$	1s2p - 2p1s	$-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \Rightarrow 4$.

This agrees w/ the figure in lecture 34.