Problem Set #6: Due: Wednesday, March 6, 11:59pm

1. *Hermitian operators:* Determine which of the following operators are Hermitian

a) the harmonic oscillator raising and lowering operator operators $\hat{a}_{\pm} = \frac{1}{\sqrt{2}}\left(\mp i\hat{p} + \hat{x}\right)$

b) the operator $\hat{a}_{+}\hat{a}_{-}$

c) the operators $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$

d) the operator combination $\hat{A}\hat{B} + \hat{B}\hat{A} \equiv \left\{\hat{A}, \hat{B}\right\}$, where $\hat{A}$ and $\hat{B}$ are Hermitian

e) the operator combination $i(\hat{A}\hat{B} + \hat{B}\hat{A})$ where $\hat{A}$ and $\hat{B}$ are Hermitian

f) the operator combination $(\hat{A}\hat{B} - \hat{B}\hat{A})$ where $\hat{A}$ and $\hat{B}$ are Hermitian

g) the operator combination $i(\hat{A}\hat{B} - \hat{B}\hat{A})$ where $\hat{A}$ and $\hat{B}$ are Hermitian

h) the operator combination $\hat{A}^{\dagger}\hat{A}$ where $\hat{A}$ is not Hermitian

i) If $\hat{A}$ and $\hat{B}$ are Hermitian, when is $\hat{A}\hat{B}$ Hermitian?

2. *Wave equation in momentum space:* Suppose you are solving for the stationary states of a harmonic oscillator Hamilton, starting from the time-independent Schrödinger equation. While we have usually employed the dimensionless momentum and position operators $\hat{p}_{\xi}$ and $\hat{\xi}$, here we write the Hamiltonian with the usual dimension-full operators $\hat{p}$ and $\hat{x}$,

$$\frac{\hbar\omega}{2}\left[\frac{b^2}{\hbar^2}\,\hat{p}^2 + \frac{1}{b^2}\hat{x}^2\right]|\alpha\rangle = E|\alpha\rangle$$

a) Repeat what we did in class, turning this into a wave equation for the position-space stationary state wave function $\langle x|\alpha\rangle \equiv \phi_{\alpha}(x)$.

b) Similarly, from the same starting point, derive a wave equation for the momentum-space stationary state wave function $\langle p|\alpha\rangle \equiv \phi_{\alpha}(p)$.

c) If someone gives you a position-space HO stationary state solution $\phi_{\alpha}(x)$, what substitutions do you make in that wave function to produce the corresponding momentum-space stationary state solution $\phi_{\alpha}(p)$?

3. *Alternative definition of a Hermitian operator:* Let $\hat{Q}$ be a Hermitian operator in a Hilbert space, and $|\alpha\rangle$ any state vector in that space. Then $\langle\alpha|\hat{Q}\alpha\rangle = \langle\hat{Q}\alpha|\alpha\rangle$. Show that this implies, for any state vectors $|\beta\rangle$ and $|\gamma\rangle$ in the Hilbert space, that $\langle\beta|\hat{Q}\gamma\rangle = \langle\hat{Q}\beta|\gamma\rangle$, providing an equivalent definition of a Hermitian operator. In your proof consider the cases $|\alpha\rangle = |\beta+\gamma\rangle$ and $|\alpha\rangle = |\beta+i\gamma\rangle$.

4. *Hermitian matrices:*

a) If you were to write down the most general $N \times N$ Hermitian matrix, how many independent real constants would you need?

b) Consider the case of $N = 2$. Show that the most general Hermitian matrix can be expressed as a linear combination of the four operators

$$a\hat{I} + b_x\hat{\sigma}_x + b_y\hat{\sigma}_y + b_z\hat{\sigma}_z = a\hat{I} + \vec{b} \cdot \vec{\hat{\sigma}}$$

where the coefficients $a,\ b_x,\ b_y,\ b_z$ are real and

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Are these four "basis operators" Hermitian?

c) In our discussions in class about representing vectors and operators, we insert the identity. Here we emphasize that by the identity in a finite vector space we mean the identity matrix $\hat{I}$, that is

$$\hat{I} = \sum_{i=1}^{N} |i\rangle\langle i|$$

Consider the Hermitian operator $\hat{b} \cdot \vec{\hat{\sigma}}$ where we have taken $\vec{b} \to \hat{b}$ to be a real unit vector. Find the eigenvalues λ_+ and λ_- and the orthonormalized eigenvectors $|v_+\rangle$ and $|v_-\rangle$ of of $\hat{b} \cdot \vec{\hat{\sigma}}$. Verify

$$\hat{I} = \sum_{i=\pm} |v_i\rangle\langle v_i|$$

This generates a family of possible representations of the operator, determined by choice of the unit vector $\hat{b}$. Can you see that the possible choices correspond to points within a unit circle?

d) One can think of these identity representations as a sum over projectors, where the projectors $\hat{P}_\pm$ project onto eigenspaces that you have defined through your selection of $\hat{b}$. That is

$$\hat{I} = \sum_{i=\pm} |v_i\rangle\langle v_i| \equiv \sum_{i=\pm} \hat{P}_i \quad \text{where} \quad \hat{P}_\pm \equiv |v_\pm\rangle\langle v_\pm|$$

Show that

$$\hat{P}_\pm = \frac{1}{2}\left(\hat{I} \pm \hat{b} \cdot \vec{\hat{\sigma}}\right)$$

These relations are often used in spin problems that arise in atomic physics, quantum information theory, etc.

5. *Eigenvalues of a finite box with a delta function potential;* This problem was employed in class (March 1 lecture) to illustrate how many of the concepts developed so far in 137A could be employed to extend our transmission and reflection treatment beyond simple plane-waves, to wave packets. The eigenvalue equation for n odd (the mirror symmetric case) was solved using Mathematica to find the roots.

a) Derive analytic solutions in the limits where the δ-function strength $\to 0$ (but not quite exactly zero) and $\to \infty$ (but not quite infinity).

b) Explain your answers physically.

c) Compare with the numerical results given in the March 1 example, for the δ-function strength α used there.

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Physics 137A Problem Set #6

1. Hermitian Operators

a) $a_\pm \sim \frac{1}{\sqrt{2}}(\mp i\hat{p} + \hat{x})$

$$\langle\chi|a_+|\chi\rangle = \int_{-\infty}^{\infty} \chi^*\left(\frac{1}{\sqrt{2}}(-i\hat{p}+\hat{x})\right)\chi\,dx = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\chi^*\left(-\hbar\frac{\partial}{\partial x}+x\right)\chi\,dx$$

$$= \frac{1}{\sqrt{2}}\left[\int_{-\infty}^{\infty} -\hbar\chi^*\frac{\partial\chi}{\partial x}\,dx + \int_{-\infty}^{\infty}\chi^* x\chi\,dx\right]$$

$$= \frac{-\hbar}{\sqrt{2}}\left(\cancelto{0}{\chi^*\chi\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty}\chi\frac{d\chi^*}{dx}\,dx\right) + \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\chi^* x\chi\,dx$$

$$= \frac{\hbar}{\sqrt{2}}\left(\int^{\pm\infty}(-i)\chi\left(i\frac{d}{dx}\right)\chi^*\,dx\right) + \frac{1}{\sqrt{2}}\int^{\pm\infty}\chi^* x\chi\,dx$$

$$= \frac{-i}{\sqrt{2}}\langle -\hat{p}\chi|\chi\rangle + \frac{1}{\sqrt{2}}\langle x\chi^*|\chi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\langle i\hat{p}\chi|\chi\rangle + \langle x\chi^*|\chi\rangle\right)$$

$$= \langle\chi|\frac{1}{\sqrt{2}}(-ip+x)^*|\chi\rangle = \langle\chi|\hat{a}_-|\chi\rangle$$

Therefore, $\hat{a}_-$ is adjoint of $\hat{a}_+$

b)
$$\hat{a}_+\hat{a}_- = \frac{1}{\sqrt{2}}\left(-i\hat{p}+\hat{x}\right)\frac{1}{\sqrt{2}}\left(i\hat{p}+\hat{x}\right)$$

$$= \frac{1}{2}\left(\hat{p}^2+\hat{x}^2 - i\hat{p}\hat{x} + i\hat{x}\hat{p}\right)$$

$$= \frac{1}{2}\left(\hat{p}^2 + \hat{x}^2 + i[\hat{x},\hat{p}]\right)$$

$$[\hat{x},\hat{p}] = i\hbar$$

$$= \frac{1}{2}\left(\hat{p}^2+\hat{x}^2-\hbar\right)$$

$$\langle\chi|\hat{a}_+\hat{a}_-|\chi\rangle = \int^{\pm\infty} \chi^*\left(\frac{1}{2}\left(\hat{p}^2+x^2-\hbar\right)\right)\chi\, dx$$

$$= \frac{1}{2}\int^{\pm\infty} \chi^*\left(-\hbar^2\frac{\partial^2}{\partial x^2} + x^2 - \hbar\right)\chi\, dx$$

Actually, knowing the Hamiltonian is hermitian proves $\hat{a}_+\hat{a}_-$ is also hermitian

$H = \hbar\omega\left(\hat{a}_+\hat{a}_- - \frac{1}{2}\right)$

$$\frac{-\hbar}{2}\int^{\pm\infty}\chi^*\frac{\partial^2\chi}{\partial x^2}dx + \underbrace{\frac{1}{2}\int^{\pm\infty}\chi^* x^2\chi\, dx}_{\text{hermitian}} \; \cancel{-\frac{\hbar}{2}\int^{\pm\infty}\chi^*\chi\, dx}$$

this will be proved to be hermitian in part (c)

Therefore,

$\hat{a}_+\hat{a}_-$ is hermitian ✓

$\frac{d}{dx}$

$$\langle\psi|\frac{d}{dx}|\psi\rangle = \int_{-\infty}^{\infty}\psi^*\frac{d\psi}{dx}\,dx = \cancel{\psi^*\psi}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\psi\frac{d\psi^*}{dx}\,dx$$

$$\langle\psi|\frac{d}{dx}|\psi\rangle = -\langle\frac{d}{dx}\psi|\psi\rangle \longrightarrow \frac{d}{dx} \text{ not hermitian } \times$$

$\frac{d^2}{dx^2}$

$$\langle\psi|\frac{d^2}{dx^2}|\psi\rangle = \int_{-\infty}^{\infty}\psi^*\frac{d^2\psi}{dx^2}\,dx$$

$$= \frac{d\psi}{dx}\psi^*\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d\psi}{dx}\frac{d\psi^*}{dx}\,dx$$

$$= \cancel{\frac{d\psi}{dx}\psi^*}\Big|_{-\infty}^{\infty} - \left(\cancel{\frac{d\psi^*}{dx}\psi}\Big|_{-\infty}^{\infty} - \int\psi\frac{d^2\psi^*}{dx^2}\,dx\right)$$

$$= \langle\frac{d^2}{dx^2}\psi|\psi\rangle \Longrightarrow \frac{d^2}{dx^2} \text{ is hermitian } \checkmark$$

b) $(\hat{A}\hat{B}+\hat{B}\hat{A})^\dagger = \hat{B}^\dagger\hat{A}^\dagger + \hat{A}^\dagger\hat{B}^\dagger = \hat{B}\hat{A}+\hat{A}\hat{B} = \hat{A}\hat{B}+\hat{B}\hat{A}$ ✓ hermitian

c) $[i(\hat{A}\hat{B}+\hat{B}\hat{A})]^\dagger = -i(\hat{B}^\dagger\hat{A}^\dagger+\hat{A}^\dagger\hat{B}^\dagger) = -i(\hat{A}\hat{B}+\hat{B}\hat{A})$ $\times$

d) $(\hat{A}\hat{B}-\hat{B}\hat{A})^\dagger = \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger = -\hat{A}\hat{B}+\hat{B}\hat{A}$ $\times$

e) $[i(\hat{A}\hat{B}-\hat{B}\hat{A})]^\dagger = -i(\hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger) = i(\hat{A}\hat{B}-\hat{B}\hat{A})$ ✓ hermitian h)

$$(A^\dagger A)^\dagger = \overline{(\overline{A^T} A)^T} = \left(\overline{A^T \overline{A}}\right) = (A^\dagger A) \quad \checkmark \text{ hermitian}$$

i)

$$\int \chi^*(\hat{A}\hat{B})\chi\, dx = \int ((AB)\chi)^* \chi\, dx$$

$$= \hat{A}\hat{B} = \hat{B}^\dagger A^\dagger \Longrightarrow \hat{A}\hat{B} = B\hat{A}$$

$[A,B] = 0$ for hermicity

2. Wave function in Momentum space

$$\frac{\hbar\omega}{2}\left[\frac{b^2}{\hbar^2}\hat{p}^2 + \frac{1}{b^2}\hat{x}^2\right]|\alpha\rangle = \mathcal{E}|\alpha\rangle$$

a) $\underline{\phi_\alpha(x)}$

$$\langle x|\frac{\hbar\omega}{2}\left[\frac{b^2}{\hbar^2}\hat{p}^2 + \frac{1}{b^2}\hat{x}^2\right]|\alpha\rangle = \mathcal{E}\langle x|\alpha\rangle = \mathcal{E}\phi_\alpha(x)$$

$$\langle x|\left(\frac{b^2}{\hbar^2}\hat{p}^2\right)|\alpha\rangle = \underbrace{\frac{b^2}{\hbar^2}\langle x|\hat{p}^2|\alpha\rangle = \frac{b^2}{\hbar^2}\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\phi_\alpha(x)}_{\text{from lec. 15}}$$

$$\langle x|\frac{1}{b^2}\hat{x}^2|\alpha\rangle = \frac{1}{b^2}\langle x|\hat{x}^2|\alpha\rangle = \frac{1}{b^2}\int_{-\infty}^{\infty} dx'\, \langle x|\hat{x}^2|x'\rangle\langle x'|\alpha\rangle$$

$$= \frac{1}{b^2}\int_{-\infty}^{\infty} dx'\, x'^2\langle x|x'\rangle\langle x'|\alpha\rangle = \frac{1}{b^2}\int_{-\infty}^{\infty} dx'\, x'^2\,\delta(x-x')\,\phi_\alpha(x')$$

$$= \frac{1}{b^2}x^2\phi_\alpha(x)$$

Therefore,
$$\frac{\hbar\omega}{2}\left(\frac{b^2}{\hbar^2}\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right) + \frac{x^2}{b^2}\right)\phi_\alpha(x) = \mathcal{E}\phi_\alpha(x)$$

$$= \frac{\hbar\omega}{2}\left(-b^2\frac{\partial^2}{\partial x^2} + \frac{x^2}{b^2}\right)\phi_\alpha(x) = \mathcal{E}\phi_\alpha(x)$$

$$\frac{\hbar\omega}{2}\left[\frac{b^2}{\hbar^2}\hat{p}^2 + \frac{1}{b^2}\hat{x}^2\right]\phi_\alpha(x) = \mathcal{E}\phi_\alpha(x)$$

$$\langle p|\frac{\hbar\omega}{2}\left[\frac{b^2}{\hbar^2}\hat{p}^2 + \frac{1}{b^2}\hat{x}^2\right]|\alpha\rangle = \mathcal{E}\langle p|\alpha\rangle = \mathcal{E}\phi_\alpha(p)$$

$$= \underbrace{\frac{\hbar\omega}{2}\frac{b^2}{\hbar^2}\langle p|\hat{p}^2|\alpha\rangle}_{①} + \underbrace{\frac{\hbar\omega}{2}\frac{1}{b^2}\langle p|\hat{x}^2|\alpha\rangle}_{②} = \mathcal{E}\phi_\alpha(p)$$

① $$\frac{\omega b^2}{2\hbar}\int_{-\infty}^{\infty} dp'\langle p|\hat{p}^2|p'\rangle\langle p'|\alpha\rangle = \int_{-\infty}^{\infty} dp'\ p^2\delta(p-p')\langle p'|\alpha\rangle = p^2\phi_\alpha(p)\cdot\frac{\omega b^2}{2\hbar}$$

② $$\frac{\hbar\omega}{2b^2}\int_{-\infty}^{\infty} dx\langle p|\hat{x}^2|x\rangle\langle x|\alpha\rangle = \frac{\hbar\omega}{2b^2}\int_{-\infty}^{\infty} x^2\langle p|x\rangle\langle x|\alpha\rangle\, dx$$

$$= \frac{\hbar\omega}{2b^2}\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\ x^2 e^{-ipx/\hbar}\langle x|\alpha\rangle = \frac{\hbar\omega}{2b^2}x^2\phi_\alpha(p)$$

Therefore,

$$\left[\frac{\omega b^2 p^2}{2\hbar} - \frac{\hbar\omega}{2b^2}\left(\hbar^2\frac{\partial^2}{\partial p^2}\right)\right]\phi_\alpha(p) = \mathcal{E}\phi_\alpha(p)$$

$$= \left[\frac{\omega b^2 p^2}{2\hbar} - \frac{\hbar^3\omega}{2b^2}\frac{\partial^2}{\partial p^2}\right]\phi_\alpha(p) = \mathcal{E}\phi_\alpha(p)$$

$$\hat{x} \to i\hbar\frac{\partial}{\partial p}$$
$$\hat{p} \to p$$

c) Given $\phi_\alpha(x)$, to get $\phi_\alpha(p)$, $\underbrace{\text{let } x \to p, \text{ let } m \to \frac{1}{m\omega^2}}$

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 x^2}{2} \quad - \frac{m\omega^2\hbar^2}{2}\frac{\partial^2}{\partial p^2} + \frac{1}{2m}\hat{p}^2$$

$$m \to \frac{1}{m\omega^2}$$

3 Alternative Definition of Hermitian Operator

$$\langle \alpha | \hat{Q}\alpha \rangle = \langle \hat{Q}\alpha | \alpha \rangle$$

~~Any $|\alpha\rangle$ can be defined in the Hilbert space~~

$$\cancel{|\alpha\rangle = |\beta + \gamma\rangle}$$

$$\cancel{\overset{?}{=} |\beta + i\gamma\rangle}$$

$$\cancel{\hat{Q}|\alpha\rangle = \hat{Q}|\beta + \gamma\rangle = \hat{Q}|\beta\rangle + \hat{Q}|\gamma\rangle \quad \hat{Q}|\beta\rangle}$$

$$\cancel{\langle \hat{Q}\alpha | = \hat{Q}\beta + \gamma |}$$

<u>Case I:</u> $|\alpha\rangle = |\beta + \gamma\rangle$

$$\langle \alpha | \hat{Q} | \alpha \rangle = \langle \beta + \gamma | \hat{Q} | \beta + \gamma \rangle$$

$$= \langle \beta | \hat{Q} | \beta \rangle + \langle \beta | \hat{Q} | \gamma \rangle + \langle \gamma | \hat{Q} | \beta \rangle + \langle \gamma | \hat{Q} | \gamma \rangle$$

$$\langle \hat{Q}\alpha | \alpha \rangle = \langle \hat{Q}(\beta + \gamma) | \beta + \gamma \rangle$$

$$= \langle \hat{Q}\beta | \beta \rangle + \langle \hat{Q}\beta | \gamma \rangle + \langle \hat{Q}\gamma | \beta \rangle + \langle \hat{Q}\gamma | \gamma \rangle$$

Therefore, since $\langle \beta | \hat{Q} | \gamma \rangle + \langle \gamma | \hat{Q} | \beta \rangle = \langle \hat{Q}\beta | \gamma \rangle + \langle \hat{Q}\gamma | \beta \rangle$

we must have

$$\langle \beta | \hat{Q} | \gamma \rangle = \langle \hat{Q}\beta | \gamma \rangle \quad \checkmark$$

ii 2: $|\alpha\rangle = |\beta + i\gamma\rangle$

$\langle\alpha|\hat{Q}|\alpha\rangle = \langle\beta + i\gamma|\hat{Q}|\beta + i\gamma\rangle$

$= \langle\beta|\hat{Q}|\beta\rangle + i\langle\beta|\hat{Q}|\gamma\rangle - i\langle\gamma|\hat{Q}|\beta\rangle + \langle\gamma|\hat{Q}|\gamma\rangle$

$\langle\hat{Q}\alpha|\alpha\rangle$

$= \langle\hat{Q}(\beta + i\gamma)|\beta + i\gamma\rangle$

$= \langle\hat{Q}\beta + i\hat{Q}\gamma|\beta + i\gamma\rangle$

$= \langle\hat{Q}\beta|\beta\rangle + i\langle\hat{Q}\beta|\gamma\rangle - i\langle\hat{Q}\gamma|\beta\rangle + \langle\hat{Q}\gamma|\gamma\rangle$

Therefore,

$i\langle\hat{Q}\beta|\gamma\rangle - i\langle\hat{Q}\gamma|\beta\rangle = i\langle\beta|\hat{Q}\gamma\rangle - i\langle\gamma|\hat{Q}|\beta\rangle$

$\Longrightarrow \langle\hat{Q}\beta|\gamma\rangle = \langle\beta|\hat{Q}\gamma\rangle$ ✓

4. Hermitian Matrices

a) $H^\dagger = H$ Elements below diagonal complex conjugates of elements above.

Along the diagonal $\longrightarrow$ N real constants

Above & below the diagonal $\longrightarrow$ $N^2 - N$ real #s

Since real #s equal above + below diagonal # = $N + N^2 - N$

$= N^2$ real #s

b) $N=2$

$$a\hat{I} + b_x\hat{\sigma}_x + b_y\hat{\sigma}_y + b_z\hat{\sigma}_z = aI + \vec{b}\cdot\vec{\sigma}$$ spin notation

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b_x \\ b_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ib_y \\ ib_y & 0 \end{pmatrix} + \begin{pmatrix} b_z & 0 \\ 0 & -b_z \end{pmatrix}$$

$$= \begin{pmatrix} a + b_z & b_x - ib_y \\ b_x + ib_y & a - b_z \end{pmatrix}$$

And since $\begin{pmatrix} a + b_z & b_x - ib_y \\ b_x + ib_y & a - b_z \end{pmatrix}^{\dagger} = \begin{pmatrix} a + b_z & b_x - ib_y \\ b_x + ib_y & a - b_z \end{pmatrix}$

<u>H is hermitian</u> ✓

c) $\hat{I} = \sum_{i=1}^{N} |i\rangle\langle i|$, $\vec{b}\cdot\vec{\sigma} = b_x\sigma_x + b_y\sigma_y + b_z\sigma_z$

$$\begin{pmatrix} a + b_z & b_x - ib_y \\ b_x + ib_y & a - b_z \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

pending

$(a+b_z)v_1 + (b_x - ib_y)v_2 = \lambda v_1$

$(b_x + ib_y)v_1 + (a - b_z)v_2 = \lambda v_2$

$$v_2 = \frac{v_1(\lambda - a - b_z)}{b_x - ib_y}$$

Therefore, we get an equation

$$(b_x + ib_y)v_1 + (a - b_z)\frac{\lambda - a - b_z}{b_x - ib_y}v_1 = \lambda\frac{\lambda - a - b_z}{b_x - ib_y}$$

using mathematica, I got the following

$$\lambda_- = a - \sqrt{b_x^2 + b_z^2 - ib_y^2}$$

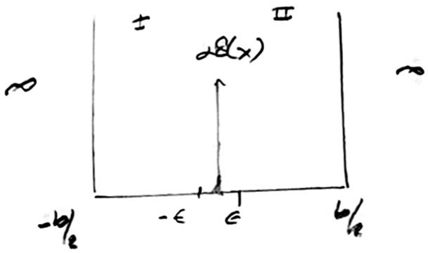
$$\lambda_+ = a + \sqrt{b_x^2 + b_z^2 - ib_y^2}$$

δ) I got ugly values for $|v_+\rangle$, $|v_-\rangle$, for which $I \neq |v_x\rangle\langle v_x| + |v_y\rangle\langle v_y|$

$|v_x\rangle\langle v_x| = |v_y\rangle\langle v_y| \sim \frac{-b_z}{b_x^2 b_y^2}$ proportional to $\hat{I}$?

$(b_x, b_y, b_z) = \vec{b}$

5. Eigenvalues of finite box w/ δ function potential



for I,

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$$

$$\frac{d^2}{dx^2}\psi = -\frac{2mE}{\hbar^2}\psi \longrightarrow \psi_I = Ae^{ikx} + Be^{-ikx}, \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi_{III} = Ce^{ikx} + De^{-ikx}$$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & -\frac{b}{2} < x < 0 \\ Ce^{ikx} + De^{-ikx} & 0 < x < \frac{b}{2} \end{cases}$$

Boundary Conditions

$$\psi_I\left(-\tfrac{b}{2}\right) = Ae^{-ikb/2} + Be^{ikb/2} = 0$$

$$\psi_{III}\left(\tfrac{b}{2}\right) = Ce^{ikb/2} + De^{-ikb/2} = 0$$

$$Ae^{-ikb/2} - De^{-ikb/2} = Ce^{ikb/2} - Be^{ikb/2}$$

$$e^{-ikb/2}(A - D) = e^{ikb/2}(C - B)$$

δ barrier

$$\frac{-\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}dx + \alpha\int_{-\epsilon}^{\epsilon}\delta(x)\psi\,dx = 0$$

too tired to solve but if you input ψ into this you will get the scattering states.

$$\frac{-\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2} + \alpha\psi(0) = 0 \Longrightarrow \left.\frac{d\psi}{dx}\right|_{\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon} = \frac{2m\alpha\psi(0)}{\hbar^2}$$

b) if you solve as $\alpha \to 0$, you will get infinite square well

as $\alpha \to \infty$, you will get two inf square wells (impenetrable region @ $x=0$)

I got the correct relation for the δ-potential limit:

$$\frac{d\chi}{dx}\bigg|_{\epsilon} - \frac{d\chi}{dx}\bigg|_{-\epsilon} = \frac{2m\alpha}{\hbar^2}\chi(0)$$

if I used the infinite square well solutions I would yield the same result as lecture.