Problem Set #10: Due: April 12, 11:59pm

Late Deadline with 10% penalty: April 15, 11:59pm

1. *More on the hydrogen atom:* As we went to the work of calculating the kinetic energy of hydrogenic states last time, let's use that work to check the uncertainty principle in 3D for the ground state of the hydrogen atom.

a) Consider any s-state hydrogenic wave function. By parity $\langle \mathbf{p} \rangle = 0$ and $\langle \mathbf{r} \rangle = 0$. Consequently $\sigma_{\mathbf{p}}^2 = \langle \mathbf{p}^2 \rangle$ and $\sigma_{\mathbf{r}}^2 = \langle \mathbf{r}^2 \rangle$. Relate $\sigma_{\mathbf{p}}^2$ to $\sigma_{p_x}^2$ and $\sigma_{\mathbf{r}}^2$ to σ_x^2. Consequently, from the 1D uncertainty principle, what is the uncertainty relationship for $\sigma_{\mathbf{p}}\sigma_{\mathbf{r}}$?

b) Take the ground state, $n = 1, \ell = 0$. What is $\langle \mathbf{p}^2 \rangle$? (Feel free to use your result from the last problem set.) What is $\langle \mathbf{r}^2 \rangle$? Compute $\sigma_{\mathbf{p}}\sigma_{\mathbf{r}}$ for this state and compare the answer to the result you got in part a). (Note: You will want to have exact values, so the transformation to the dimensionless distance y is helpful for both operators, as described in the last problem set.)

2. *Using the Wigner Eckart theorem:*

a) The spin operator for spin-$\frac{1}{2}$ can be written in terms of its Cartesian components as

$$\hat{\boldsymbol{S}} = \frac{\hbar}{2}\boldsymbol{\sigma} = \frac{\hbar}{2}\left(\sigma_x \hat{\boldsymbol{e}}_x + \sigma_y \hat{\boldsymbol{e}}_y + \sigma_z \hat{\boldsymbol{e}}_z\right)$$

where the σ_i are the Pauli matrices (see problem 4). There are twelve matrix elements of the form

$$\langle \tfrac{1}{2} m_f | \sigma_i | \tfrac{1}{2} m_i \rangle, \quad i \in \{x, y, z\}$$

Evaluate these. How many of them are nonzero?

b) The rank-one spin operator can also be written in terms of its spherical components as

$$\hat{\boldsymbol{S}} = \frac{\hbar}{2}\boldsymbol{\sigma} = \frac{\hbar}{2}\sum_{m=-1} (-1)^m \sigma_m \hat{\boldsymbol{e}}_{-m} = \frac{\hbar}{2}\left(-\sigma_{-1}\hat{\boldsymbol{e}}_1 + \sigma_0 \hat{\boldsymbol{e}}_0 - \sigma_1 \hat{\boldsymbol{e}}_{-1}\right)$$

where

$$\sigma_m \equiv \hat{\boldsymbol{e}}_m \cdot \boldsymbol{\sigma} \qquad \hat{\boldsymbol{e}}_1 = -\frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_x + i\hat{\boldsymbol{e}}_y) \qquad \hat{\boldsymbol{e}}_0 = \hat{\boldsymbol{e}}_z \qquad \hat{\boldsymbol{e}}_{-1} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_x - i\hat{\boldsymbol{e}}_y)$$

Verify that this expression for $\hat{\boldsymbol{S}}$ is equivalent to the Cartesian expression in part a). Then evaluate the 12 matrix elements

$$\langle \tfrac{1}{2} m_f | \sigma_m | \tfrac{1}{2} m_i \rangle, \quad m \in \{-1, 0, 1\}$$

How many of them are nonzero?

c) Suppose you had evaluated in b) only the one matrix element (the easiest one) $\langle \frac{1}{2}\frac{1}{2}|\sigma_0|\frac{1}{2}\frac{1}{2}\rangle$. Use this matrix element and the Wigner Eckart theorem

$$\langle j_f m_f | T^k_m | j_i m_i \rangle = \frac{(-1)^{j_i - m_i}}{\sqrt{2k+1}} \langle j_f m_f j_i - m_i | (j_f j_i) k m \rangle \langle j_f || T^k || j_i \rangle$$

to determined the reduced matrix element $\langle \frac{1}{2}||\sigma||\frac{1}{2}\rangle$. Then use this result and the Wigner Eckart theorem to determine all matrix elements. Compare to b). (You can calculate the Clebsch Gordan coefficients using Mathematica, or use the tables in Griffiths.)

3. *The spin-$\frac{1}{2}$ rotation matrix:* We discussed in class the rotation matrix, which describes how an angular momentum eigenstate transforms under a general rotation of the axes, described here by the three Euler angles. As such states stay in their subspaces, the "complete set" of states we need to insert is just $\{|jm'\rangle\langle jm'|, -j \leq m' \leq j\}$.

$$\hat{U}(\alpha,\beta,\gamma)|jm\rangle = \sum_{m'=-j}^{j} \; |jm'\rangle \; \langle jm'|\hat{U}(\alpha,\beta,\gamma)|jm\rangle = \sum_{m'=-j}^{j} \; |jm'\rangle \; D^j_{m'\,m}(\alpha,\beta,\gamma)$$

$$\hat{U}(\alpha,\beta,\gamma) \equiv \; \exp[-\frac{i}{\hbar}\alpha \hat{J}_z] \;\; \exp[-\frac{i}{\hbar}\beta \hat{J}_y] \; \exp[-\frac{i}{\hbar}\gamma \hat{J}_z]$$

Because the first and third rotations described by the Euler angles are about the z-axis, one finds a relatively simple expression for the needed matrix elements

$$D^j_{m'\,m}(\alpha,\beta,\gamma) = \langle jm'|\hat{U}(\alpha,\beta,\gamma)|jm\rangle = e^{-i\alpha m'} \; d^j_{m'\,m}(\beta) \; e^{-i\gamma m}$$

Thus we can derive the rotation matrix if we can evaluate

$$d^j_{m'm}(\beta) = \langle jm'| \; \exp[-\frac{i}{\hbar}\beta \hat{J}_y] \; |jm\rangle$$

a) Rewrite $d^j_{m'm}(\beta)$ for $j = \frac{1}{2}$ so that $\hat{J}_y$ can be replaced by the corresponding Pauli matrix σ_y.

b) Exponentiated operators are defined by the corresponding power series. Write out the first six terms of such an expansion for the matrix element of part a).

c) What is the matrix product $\sigma_y \sigma_y$? Use the result to simplify the result in part b), so that only two matrices appear. Sum the coefficients of those matrices to obtain a simple, closed-form expression for $d^{\frac{1}{2}}_{m'm}(\beta)$ and thus for the entire rotation matrix.

4. *Larmor precession:*

Consider a spin-1/2 system quantized using $\hat{\mathbf{S}}^2$ and $\hat{S}_z$. The initial wave function at $t = 0$ is

$$|\psi(0)\rangle = |s = \tfrac{1}{2} m_s = \tfrac{1}{2}\rangle \equiv \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so pointed along the z axis. Recall from Lecture 29 that for spin-1/2, the spin operators are given by the Pauli matrices

$$\hat{\boldsymbol{S}} = \frac{\hbar}{2}\boldsymbol{\sigma} \quad \text{where} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) What is the matrix representing $\hat{\boldsymbol{S}}^2$? What property of the Pauli matrices ensures that the operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ correspond to observables?

b) If one measures the observable $\hat{S}_x$ at $t = 0$ what are the possible outcomes and what are the probabilities of those outcomes?

c) Suppose we again start with $|\psi(0)\rangle = \chi_+$ but let the system evolve for a time t under influence of a magnetic field of strength B pointed along the positive y axis. Calculate the probability of finding the system in the states χ_+ and χ_- as a function of time.

d) Calculate the probabilities of finding the system in the states χ^x_+, χ^x_-, χ^y_+, and χ^y_- as a function of time. (Recall that χ^x_+ represents the state with spin aligned in the + direction along x, etc.)

e) Plot the six probabilities found in parts c) and d) as a function of $t\gamma B$, letting this variable run from 0 to 2π. Including the times at $t\gamma B = 0$ and $t\gamma B = 2\pi$, you will see five times at which a measurement of $\hat{S}_x$ or $\hat{S}_z$ is guaranteed to yield a definite value. Explain the pattern of these values – why they progress as they do.

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Problem Set #10

Physics 137A

Hydrogen Atom

a) s-wave ($l=0$). $\langle \hat{p} \rangle = 0$, $\langle \hat{r} \rangle = 0$.

$$\sigma_{\hat{p}}^2 = \langle \hat{p}^2 \rangle \quad \sigma_{\hat{r}}^2 = \langle \hat{r}^2 \rangle$$

Therefore,

$$\sigma_{\hat{p}}^2 = \langle (-i\hbar\nabla)^2 \rangle = \langle -\hbar^2 \nabla^2 \rangle = \langle -\hbar^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \rangle$$

$$= \langle -\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \frac{\partial^2}{\partial y^2} - \hbar^2 \frac{\partial^2}{\partial z^2} \rangle$$

$$\sigma_{\hat{p}}^2 = \langle \left(-i\hbar \frac{\partial}{\partial x}\right)^2 \rangle + \langle \left(-i\hbar \frac{\partial}{\partial y}\right)^2 \rangle + \ldots$$

$$= \sigma_{p_x}^2 + \sigma_{p_y}^2 + \sigma_{p_z}^2 = \underline{3\sigma_{p_x}^2} \Longrightarrow \sigma_{\hat{p}} = \sqrt{3}\,\sigma_{p_x}$$

$$\sigma_{\hat{r}}^2 = \langle \hat{x}^2 + \hat{y}^2 + \hat{z}^2 \rangle = \langle \hat{x}^2 \rangle + \langle \hat{y}^2 \rangle + \langle \hat{z}^2 \rangle$$

$$= \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \quad \text{since } \langle x_i \rangle = 0$$

$$= 3\sigma_x^2 \quad \text{rotational symmetry} \Longrightarrow \sigma_{\hat{r}} = \sqrt{3}\,\sigma_x$$

Therefore,

$$\sigma_{\hat{p}}\sigma_{\hat{r}} = 3\sigma_{p_x}\sigma_x = \frac{3\hbar}{2} \quad \text{from 1D Uncertainty Principle}$$

b. Ground ~~State~~ $n=1,\ \ell=0$

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

Solving for $\langle \hat{r}^2 \rangle$,

$$\langle \hat{r}^2 \rangle = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{1}{\pi a^3} e^{-2r/a} \cdot r^2 \cdot r^2 \sin\theta \, dr\, d\phi\, d\theta = \underline{\underline{3a_0^2}}$$

Solving for $\langle \hat{p}^2 \rangle$,

$$\langle \hat{p}^2 \rangle = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \left(-\hbar^2 \nabla^2\right) \cdot \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \cdot r^2 \sin\theta \, dr\, d\phi\, d\theta$$

$$= \underline{\frac{\hbar^2}{a_0^2}}$$

Solving for $\sigma_{\hat{r}}$

$$\sigma_{\hat{r}} = \sqrt{\langle \hat{r}^2 \rangle - \langle \hat{r} \rangle^2} = \sqrt{3a_0^2 - \frac{9}{4}a_0^2} = \sqrt{\frac{3}{4}}\, a_0$$

$$\sigma_{\hat{p}} = \sqrt{\langle \hat{p}^2 \rangle - 0} = \frac{\hbar}{a_0}$$

Hence,

$$\sigma_{\hat{p}}\sigma_{\hat{r}} = \sqrt{\frac{3}{4}}\,\hbar \neq \frac{3}{2}\hbar$$

The values are not equivalent. I'm confused as to why...

Wigner Eckart Theorem

a) $s = 1/2$

$$\hat{S} = \frac{\hbar}{2}\hat{\sigma} = \frac{\hbar}{2}\left(\sigma_x \hat{e}_x + \sigma_y \hat{e}_y + \sigma_z \hat{e}_z\right)$$

$\langle \tfrac{1}{2} m_f | \sigma_i | \tfrac{1}{2} m_i \rangle$, $i \in \{x, y, z\}$: $\langle \tfrac{1}{2}\tfrac{1}{2} | \sigma_i | \tfrac{1}{2}\tfrac{1}{2} \rangle$

$\langle \tfrac{1}{2} {-\tfrac{1}{2}} | \sigma_i | \tfrac{1}{2}\tfrac{1}{2} \rangle$

$\langle \tfrac{1}{2}\tfrac{1}{2} | \sigma_i | \tfrac{1}{2} {-\tfrac{1}{2}} \rangle$ per σ_i

$\langle \tfrac{1}{2} {-\tfrac{1}{2}} | \sigma_i | \tfrac{1}{2} {-\tfrac{1}{2}} \rangle$

soln shown in <u>mathematica</u>.

<u>b</u> mathematica soln

b) spherical basis

$$\hat{S} = \frac{\hbar}{2}\hat{\sigma} = \frac{\hbar}{2}\left(-\sigma_{-1}\hat{e}_1 + \sigma_0 \hat{e}_0 - \sigma_1 \hat{e}_{-1}\right)$$

$\sigma_m \equiv \hat{e}_m \cdot \hat{\sigma}$, $\hat{e}_1 = -\frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)$ $\hat{e}_0 = \hat{e}_z$ $\hat{e}_{-1} = \frac{1}{\sqrt{2}}(\hat{e}_x - i\hat{e}_y)$

Therefore,

$$\hat{S} = \frac{\hbar}{2}\left(-(\hat{e}_{-1} \cdot \vec{\sigma})\hat{e}_1 + (\hat{e}_0 \cdot \vec{\sigma})\hat{e}_0 - (\hat{e}_1 \cdot \vec{\sigma})\hat{e}_{-1}\right)$$

$$= \frac{\hbar}{2}\left(-\tfrac{1}{\sqrt{2}}\hat{\sigma}_x \hat{e}_1 + \frac{i}{\sqrt{2}}\hat{\sigma}_y \hat{e}_1 + \hat{\sigma}_z \hat{e}_z + \frac{1}{\sqrt{2}}\hat{\sigma}_x \hat{e}_{-1} + \frac{i}{\sqrt{2}}\hat{\sigma}_y \hat{e}_{-1}\right)$$

And since

$\frac{1}{\sqrt{2}}\sigma_x(-\hat{e}_1 + \hat{e}_{-1}) = \hat{\sigma}_x \hat{e}_x$

$\frac{1}{\sqrt{2}}\sigma_y(\hat{e}_1 + \hat{e}_{-1}) = \hat{\sigma}_y \hat{e}_y$

$$\frac{\hbar}{2}\left(-\sigma_{-1}\hat{e}_1 + \hat{\sigma}_0 \hat{e}_0 - \hat{\sigma}_1 \hat{e}_{-1}\right) = \frac{\hbar}{2}\left(\hat{\sigma}_x \hat{e}_x + \hat{\sigma}_y \hat{e}_y + \hat{\sigma}_z \hat{e}_z\right)$$

✓

Evaluating $\langle \tfrac{1}{2} m_f | \sigma_m | \tfrac{1}{2} m_i \rangle \quad m \in \{-1, 0, 1\}$

$$\sigma_{-1} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix}$$

using mathematica
[see attached]

All solutions of $\langle \tfrac{1}{2} m_f | \sigma_m | \tfrac{1}{2} m_i \rangle$ are on <u>mathematica</u>.

<u>4</u> nonzero sol'ns

$\Rightarrow \langle \tfrac{1}{2} \tfrac{1}{2} | \sigma_0 | \tfrac{1}{2} \tfrac{1}{2} \rangle = 1$

Wigner Eckart Theorem

$$\langle j_f m_f | T^k_m | j_i m_i \rangle = \frac{(-1)^{j_i - m_i}}{\sqrt{2k+1}} \langle j_f m_f j_i - m_i | (j_f j_i) k m \rangle \langle \cancel{j_f} \| T^k \| j_i \rangle$$

Solving for reduced matrix element "proportionality factor"

$$\langle \tfrac{1}{2} \tfrac{1}{2} | \sigma_0 | \tfrac{1}{2} \tfrac{1}{2} \rangle = 1$$

$$\langle \tfrac{1}{2} \tfrac{1}{2} | \sigma_0 | \tfrac{1}{2} \tfrac{1}{2} \rangle = \frac{1}{\sqrt{2}} \langle 1 0 \tfrac{1}{2} \tfrac{1}{2} | \tfrac{1}{2} \tfrac{1}{2} \rangle \langle \tfrac{1}{2} \| T^1 \| \tfrac{1}{2} \rangle$$

$$= \tfrac{1}{\sqrt{2}} \left(-\sqrt{\tfrac{1}{3}}\right) \langle \tfrac{1}{2} \| T^1 \| \tfrac{1}{2} \rangle$$

Therefore, $\langle \tfrac{1}{2} \| \sigma_2 \| \tfrac{1}{2} \rangle = -\sqrt{6}$

Thus, since

$$\langle \tfrac{1}{2} \| \sigma_z \| \tfrac{1}{2} \rangle = -\sqrt{6}$$

use table for CG coefficients

$$\langle \tfrac{1}{2}\tfrac{1}{2} | \sigma_z | \tfrac{1}{2} -\tfrac{1}{2} \rangle = \frac{1}{\sqrt{2}} \langle 1 0 \tfrac{1}{2} -\tfrac{1}{2} | \tfrac{1}{2} \tfrac{1}{2} \rangle (-\sqrt{6}) = -1 \quad \times$$

$$\langle \tfrac{1}{2} -\tfrac{1}{2} | \sigma_z | \tfrac{1}{2} \tfrac{1}{2} \rangle = \frac{1}{\sqrt{2}} \langle 1 0 \tfrac{1}{2} \tfrac{1}{2} | \tfrac{1}{2} -\tfrac{1}{2} \rangle (-\sqrt{6}) = -1 \quad \times$$

$$\langle \tfrac{1}{2} -\tfrac{1}{2} | \sigma_z | \tfrac{1}{2} -\tfrac{1}{2} \rangle = \frac{1}{\sqrt{2}} \langle 1 0 \tfrac{1}{2} -\tfrac{1}{2} | \tfrac{1}{2} -\tfrac{1}{2} \rangle (-\sqrt{6}) = +1 \quad \checkmark$$

I do not fully understand how to use wigner eckart theorem. My $\langle \tfrac{1}{2} -\tfrac{1}{2} | \sigma_0 | \tfrac{1}{2} -\tfrac{1}{2} \rangle$ was correct but the others were not. Specifically, I'm unsure on how to know what k, q is for T^k_q. I'm unable to go to office hours this week but will next week.

3. Spin $\tfrac{1}{2}$ ~~rotation~~ matrix

$$\hat{\mathcal{U}}(\alpha, \beta, \gamma) | jm \rangle = \sum_{m'=-j}^{j} | j m' \rangle D^j_{m'm}(\alpha, \beta, \gamma)$$

$$D^j_{m'm}(\alpha, \beta, \gamma) = e^{-i\alpha m'} d^j_{m'm}(\beta) e^{-i\gamma m}$$

$$d^j_{m'm}(\beta) = \langle j m' | e^{-i/\hbar \, \beta \hat{J}_y} | j m \rangle$$

a) $d^{j}_{m'm}(\beta) = \langle j m' | e^{-i/\hbar\, \beta \hat{J}_y} | j m \rangle$ $\Big|$ $= \langle \tfrac{1}{2} m' | e^{-i/\hbar\, \beta \frac{\hbar}{2} \sigma_y} | j m \rangle$

$j = 1/2$

$= \langle \tfrac{1}{2} m' | e^{-i \frac{\beta}{2} \sigma_y} | j m \rangle$

b) $e^{-i\beta/2\, \sigma_y} = \sum_n \frac{1}{n!} \left(-i \frac{\beta}{2} \sigma_y\right)^n$

$\langle \tfrac{1}{2} m' | \sum_n \frac{1}{n!} \left(-i \frac{\beta}{2} \sigma_y\right)^n | \tfrac{1}{2} m \rangle$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_y^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_y^5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \sigma_y^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Therefore, $e^{-i\beta/2\, \sigma_y} \approx I - i\beta/2\, \sigma_y - \frac{\beta^2}{8} I + i\frac{\beta^3}{48} \sigma_y + \frac{\beta^4}{384} I$

$- i\frac{\beta^5}{3840} \sigma_y.$

Hence,

$\langle \tfrac{1}{2} m' | e^{-i\frac{\beta}{2}\sigma_y} | \tfrac{1}{2} m \rangle \approx \langle \tfrac{1}{2} m' | I | \tfrac{1}{2} m \rangle - i\frac{\beta}{2} \langle \tfrac{1}{2} m' | \sigma_y | \tfrac{1}{2} m \rangle$

$- \frac{\beta^2}{8} \langle \tfrac{1}{2} m' | I | \tfrac{1}{2} m \rangle + \frac{i\beta^3}{48} \langle \tfrac{1}{2} m' | \sigma_y | \tfrac{1}{2} m \rangle$

$+ \frac{\beta^4}{384} \langle \tfrac{1}{2} m' | I | \tfrac{1}{2} m \rangle - \frac{i\beta^5}{3840} \langle \tfrac{1}{2} m' | \sigma_y | \tfrac{1}{2} m \rangle$

c)

$$\langle \tfrac{1}{2} m' | e^{-i\frac{\beta}{2}\sigma_y} | \tfrac{1}{2} m \rangle \approx \langle \tfrac{1}{2} m' | \tfrac{1}{2} m \rangle - i\frac{\beta}{2} \langle \tfrac{1}{2} m' | \sigma_y | \tfrac{1}{2} m \rangle$$

$$- \frac{\beta^2}{8} \langle \tfrac{1}{2} m' | \tfrac{1}{2} m \rangle + \frac{i\beta^3}{48} \langle \tfrac{1}{2} m' | \sigma_y | \tfrac{1}{2} m \rangle$$

$$+ \frac{\beta^4}{384} \langle \tfrac{1}{2} m' | \tfrac{1}{2} m \rangle - \frac{i\beta^5}{3840} \langle \tfrac{1}{2} m' | \sigma_y | \tfrac{1}{2} m \rangle$$

Collecting terms,

$$\approx \left(1 - \frac{\beta^2}{8} + \frac{\beta^4}{384}\right) \langle \tfrac{1}{2} m' | \tfrac{1}{2} m \rangle$$

$$+ i\left(-\frac{\beta}{2} + \frac{\beta^3}{48} - \frac{\beta^5}{3840}\right) \langle \tfrac{1}{2} m' | \sigma_y | \tfrac{1}{2} m \rangle \approx \underline{\underline{d^{1/2}_{m'm}(\beta)}}$$

4. Larmor Precession

$$s = \tfrac{1}{2}, \quad |\psi(0)\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle \equiv \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{S} = \frac{\hbar}{2}\hat{\sigma} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) $$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{\hbar^2}{4}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right)$$

$$= \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3\hbar^2}{4} I.$$

Pauli Matrices are <u>Hermitian</u> and so is $\hat{S}^2$ obviously.

b)

$$S_x |\chi(0)\rangle = \frac{1}{2}\hbar |\chi(0)\rangle$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\chi_+ + 0\chi_- , \quad \alpha = 1, \beta = 0$$

Therefore, measuring $\hat{S}_x$ yields $\frac{\hbar}{2}$ w/ probability 1.

c)

$$|\chi(0)\rangle = \chi_+ , \quad \vec{B} = B\hat{j}$$

$$\hat{H} = -\gamma\vec{B}\cdot\hat{S} = -\gamma B\hat{S}_y = \frac{-\gamma\hbar B}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The Eigenstates,

$$\begin{pmatrix} i \\ 1 \end{pmatrix}, \; E_1 = \frac{\gamma\hbar B}{2} \qquad \begin{pmatrix} -i \\ 1 \end{pmatrix}, \; E_2 = -\frac{\gamma\hbar B}{2}$$

Therefore,

$$|\chi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = z_1\begin{pmatrix} i \\ 1 \end{pmatrix} + z_2\begin{pmatrix} -i \\ 1 \end{pmatrix} \Longrightarrow z_1 = \frac{-i}{2}, \quad z_2 = \frac{i}{2}$$

$$\begin{pmatrix} i \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Longrightarrow \alpha = i, \; \beta = 1$$

$$\begin{pmatrix} -i \\ 1 \end{pmatrix} = \alpha'\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta'\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Longrightarrow \alpha' = -i, \; \beta' = 1$$

Putting everything together,

$$|\chi(0)\rangle = \frac{-i}{2}\left(i\chi_+ + \chi_-\right) + \frac{i}{2}\left(-i\chi_+ + \chi_-\right)$$

$$= \chi_+$$ just realized I went in a circle. In any case,

$\alpha = 1, \quad \beta = 0$

Determining energy of χ_+,

$\begin{pmatrix} i \\ 1 \end{pmatrix} \quad E_1 = \frac{\gamma \hbar B}{2} \qquad \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad E_2 = \frac{-\gamma \hbar B}{2}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} i \\ 1 \end{pmatrix} + \beta\begin{pmatrix} -i \\ 1 \end{pmatrix} \Rightarrow \alpha = \frac{-i}{2}, \quad \beta = \frac{i}{2}$$

Therefore, $E_+ =$

Therefore,

$$|\chi(0)\rangle = \chi_+ = z_1\begin{pmatrix} i \\ 1 \end{pmatrix} + z_2\begin{pmatrix} -i \\ 1 \end{pmatrix} \implies z_1 = \frac{-i}{2} \quad z_2 = \frac{i}{2}$$

$$= \frac{-i}{2}\begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Thus

$$|\chi(t)\rangle = \frac{-i}{2}\begin{pmatrix} i \\ 1 \end{pmatrix} e^{-i\gamma\frac{B}{2}t} + \frac{i}{2}\begin{pmatrix} -i \\ 1 \end{pmatrix} e^{i\gamma\frac{B}{2}t}$$

$$= \begin{pmatrix} 1/2 \\ -i/2 \end{pmatrix} e^{-i\gamma\frac{B}{2}t} + \begin{pmatrix} 1/2 \\ i/2 \end{pmatrix} e^{i\gamma\frac{B}{2}t}$$

$$= \begin{pmatrix} \frac{e^{-i\gamma\frac{B}{2}t}}{2} \\ \frac{-i}{2} e^{-i\gamma\frac{B}{2}t} \end{pmatrix} + \begin{pmatrix} \frac{e^{i\gamma\frac{B}{2}t}}{2} \\ \frac{i}{2} e^{i\gamma\frac{B}{2}t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{-i\gamma\frac{B}{2}t} + e^{i\gamma\frac{B}{2}t}}{2} \\ \frac{i}{2}\left(-e^{-i\gamma\frac{B}{2}t} + e^{i\gamma\frac{B}{2}t}\right) \end{pmatrix}$$

$$= \frac{1}{2}\left(e^{-i\gamma\frac{B}{2}t} + e^{i\gamma\frac{B}{2}t}\right)\chi_+ - \frac{1}{2i}\left(-e^{-i\gamma\frac{B}{2}t} + e^{i\gamma\frac{B}{2}t}\right)\chi_-$$

$$= \cos\left(\frac{\gamma B t}{2}\right)\chi_+ - \sin\left(\frac{\gamma B t}{2}\right)\chi_-$$

$$P(\chi_+) = \cos^2\left(\frac{\gamma B t}{2}\right) \qquad P(\chi_-) = \sin^2\left(\frac{\gamma B t}{2}\right)$$

$$P(\chi_+) + P(\chi_-) = 1 \checkmark$$

$$\langle \hat{S}_x \rangle = \frac{\hbar}{2}\left(\cos\left(\frac{\gamma B t}{2}\right) \;\; -\sin\left(\frac{\gamma B t}{2}\right)\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\left(\frac{\gamma B t}{2}\right) \\ -\sin\left(\frac{\gamma B t}{2}\right) \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos\frac{\gamma B t}{2} \;\; -\sin\frac{\gamma B t}{2}\right)\begin{pmatrix} \sin\frac{\gamma B t}{2} \\ -\cos\frac{\gamma B t}{2} \end{pmatrix}$$

$$= -\frac{\hbar}{2}\left(2\cos\left(\frac{\gamma B t}{2}\right)\sin\left(\frac{\gamma B t}{2}\right)\right)$$

$$= -\hbar\left(2\cos\frac{\gamma B t}{2}\sin\frac{\gamma B t}{2}\right)$$

$$|\chi(t)\rangle = \cos\frac{\gamma B t}{2}\chi_+ - \sin\frac{\gamma B t}{2}\chi_-$$

$$= \left(\frac{\cos\frac{\gamma B t}{2} - \sin\frac{\gamma B t}{2}}{\sqrt{2}}\right)\chi_+^x + \left(\frac{\cos\frac{\gamma B t}{2} + \sin\frac{\gamma B t}{2}}{\sqrt{2}}\right)\chi_-^x$$

let $\cos\frac{\gamma B t}{2} = c$

$-\sin\frac{\gamma B t}{2} = s$

$$\Longrightarrow \begin{pmatrix} c \\ s \end{pmatrix} = \frac{c+s}{\sqrt{2}}\chi_+^x + \frac{c-s}{\sqrt{2}}\chi_-^x$$

$$= \frac{c+s}{\sqrt{2}}\begin{pmatrix} c_+^1 \\ c_+^2 \end{pmatrix} + \frac{c-s}{\sqrt{2}}\begin{pmatrix} c_-^1 \\ c_-^2 \end{pmatrix}$$

$$c_+ = c_- = 1$$

$$\frac{c+s}{\sqrt{2}}c_+^2 + \frac{c-s}{\sqrt{2}}c_-^2 = s$$

$$c_+^2 = \sqrt{2} \qquad c_-^2 = -\sqrt{2}$$

$$\Longrightarrow \chi_+^x = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$$\chi_-^x = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

YOU CAN IGNORE THIS ENTIRE PAGE.

d)

$$P(x_+^x) = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)^2 = \left(\frac{\cos\frac{\gamma B t}{2} - \sin\frac{\gamma B t}{2}}{\sqrt{2}}\right)^2$$

$$P(x_-^x) = \left(\frac{\alpha-\beta}{\sqrt{2}}\right)^2 = \left(\frac{\cos\frac{\gamma B t}{2} + \sin\frac{\gamma B t}{2}}{\sqrt{2}}\right)^2$$

$$P(x_+) = \cos^2\frac{\gamma B t}{2}$$

$$P(x_-) = \sin^2\frac{\gamma B t}{2}$$

$$P(x_+^y) = \left(\frac{\alpha - i\beta}{\sqrt{2}}\right)^2 = \left(\frac{\cos\frac{\gamma B t}{2} + i\sin\frac{\gamma B t}{2}}{\sqrt{2}}\right)^2$$

$$P(x_-^y) = \left(\frac{\alpha + i\beta}{\sqrt{2}}\right)^2 = \left(\frac{\cos\frac{\gamma B t}{2} - i\sin\frac{\gamma B t}{2}}{\sqrt{2}}\right)^2$$

e)

Plotting all 6 Probabilities as function of $\gamma B t$, I use mathematica