Physics 89 (Mathematical Methods)
Problem Set #1
Due by 6pm, January 27, 2023

1 Approximating $\sqrt{2}$

(a) Using the rules for a Taylor series of a ratio of functions whose Taylor series are known, find the Taylor series around $x = 0$ for the three functions

$$f(x) = \sqrt{1+x}\,, \qquad g(x) = \frac{1}{\sqrt{1-x}}\,, \qquad h(x) = \sqrt{\frac{1+x}{1-x}}\,.$$

Express your results in terms of the "generalized binomial coefficient" which is defined as $\binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!}$. **Hint:** Write $h(x)$ as $h(x) = (1+x)(1-x^2)^{-1/2}$.

(b) Use the first three terms of the series (up to and including x^2) to calculate approximate expressions for $\sqrt{2} = f(1) = g(\frac{1}{2}) = h(\frac{1}{3})$. Which function gives the best approximation at this order?

2 Debye model

(a) Calculate the first 7 coefficients $(a_0, \ldots, a_6)$ of the Taylor series of the function

$$f(x) = \frac{x}{e^x - 1} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

around $x = 0$.

Hint: given

$$P(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots\,, \qquad Q(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Assume

$$\frac{P(x)}{Q(x)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

and solve for $a_0, a_1, \ldots$ by writing

$$b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots = (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

and expanding the right hand side.

Note: the numbers $B_n = n! a_n$ are called *Bernoulli numbers*, after the 17^{th} century Swiss mathematician Jacob Bernoulli.

(b) The following example comes from solid-state physics. Debye theory (named after the Dutch-American physicist Peter Debye) predicts a formula for the energy E stored as "heat" in a solid with N atoms, as a function of temperature T. (This formula is useful because the "heat capacity" can then be calculated by taking the derivative dE/dT, and it can be compared to experiment.) The formula is a bit complicated and is derived by analyzing the amount of heat that quantized vibrations of the solid can store, as a function of temperature. It assumes that each solid material is assigned a constant θ (known as the "Debye temperature" of the material). For example, measured from absolute zero, $\theta = 470K$ for iron and $\theta = 170K$ for gold. Debye's formula is then:

$$E(T) = 9NkT\left(\frac{T}{\theta}\right)^3 \int_0^{\theta/T} \frac{x^3}{e^x - 1}dx.$$

Here $k = 1.38\ldots \times 10^{-23} J \cdot K^{-1}$ is Boltzman's constant from thermodynamics. Use the result of part (a) to expand $E(T)$ for high temperature as:

$$E(T) = \alpha_0 + \frac{\alpha_1}{T} + \frac{\alpha_2}{T^2} + \frac{\alpha_3}{T^3} + \frac{\alpha_4}{T^4} + \frac{\alpha_5}{T^5} + \frac{\alpha_6}{T^6}$$

(You are asked to find the coefficients $\alpha_0, \ldots, \alpha_6$ in terms of N, k and θ. You may express them in terms of $a_0, a_1, \ldots$)

3 Legendre polynomials and Taylor series with an unknown parameter

In this problem u should be thought of as a given, unspecified, known parameter.
The function

$$f(x, u) = \frac{1}{\sqrt{1 - 2xu + x^2}}$$

can then be expanded in a Maclaurin series (Taylor series around $x = 0$) of the form

$$\frac{1}{\sqrt{1 - 2xu + x^2}} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

where the coefficients $a_0, a_1, \ldots$ are functions of u. Define

$$t \equiv 2xu - x^2$$

and use the first few terms in the Taylor series

$$\frac{1}{\sqrt{1-t}} = 1 + \frac{1}{2}t + \frac{3}{8}t^2 + \cdots \qquad \text{(continued to as many terms as you need)}$$

to calculate $a_0, \ldots, a_4$.
Hint: Substitute $t = 2xu - x^2$, open all the parentheses, and drop $O(x^5)$ terms.
Note that here the Taylor coefficient a_n is a function of u. It turns out to be a polynomial of degree n, and it is called the *Legendre polynomial* and is denoted by $P_n(u)$.

4 The error term

Recall the formula for the "error" in a truncated Taylor series:

$$f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n = \frac{f^{(N+1)}(\alpha)}{(N+1)!} \alpha^{N+1} \equiv R_{N+1}(\alpha)$$

for some α between 0 and x.

(a) Find the error $R_2(\alpha)$ for the Taylor series

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + R_2(\alpha)$$

for $0 < \alpha < x$.

(b) For $x = 0.01$, find the maximum value of $|R_2(\alpha)|$ (for $0 \leq \alpha \leq 0.01$).

Hint: verify that $R_2(\alpha)$ is monotonically increasing with α. A numerical answer with two significant digits will suffice.

(c) In special relativity the formula for kinetic energy is instead of $\frac{1}{2}mv^2$,

$$E_K = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2,$$

where m is the (rest) mass of the particle, $c \approx 3 \times 10^8 m/s$ is the speed of light and v is the velocity of the particle. Set $x = v/c$ and use part (a) to write

$$E_K = \frac{1}{2}mv^2 + \text{correction}$$

For $0 \leq v \leq 0.1c$, use part (b) to find an upper bound on the correction. Your answer should be of the form mc^2 times a numerical coefficient, and one significant digit in the constant suffices.