Physics 89

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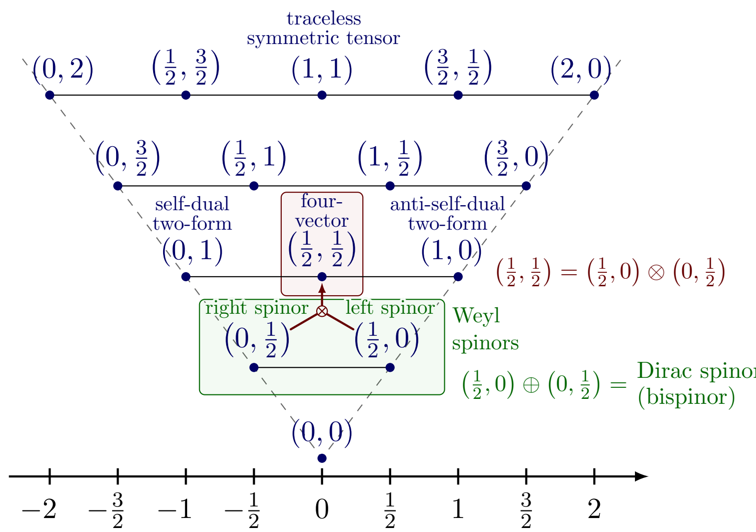
Adapted from Mathematics for Physicists: Introductory Concepts and Methods by Atland & Delft



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1 Chapter 1: Linear Algebra

1.1 L1: Mathematics before Numbers

1.1.1 L1.1: Sets and Maps

1.1.1.1 Sets

A set is a container holding certain elements. Formally, one writes $a \in A$ to indicate that **a** is an element of set A and $A = \{a, b, c, ...\}$. Building upon this definition of a set and its elements,

$\rightarrow$ We write $A = \{\}$ or $A = \emptyset$ to denote the **emptyset**

$\rightarrow$ A **subset** of A, denoted by $B \subset A$, contains some of the elements of A, for example, $B = \{a, b\} \subset A$. We may write $B \subseteq A$ if we want to indicate that the subset B may actually be equal to A, or $B \subsetneq A$ if this is not the case

$\rightarrow$ The **union** of two sets is denoted by $\cup$, for example, $\{a, b, c\} \cup \{c, d\} = \{a, b, c, d\}$. The **intersection** is denoted by $\cap$, for example $\{a, b, c\} \cap \{c, d\} = \{c\}$

$\rightarrow$ The removal of a subset $B \subset A$ from a set A that results in the **difference set** denoted by A/B. For example, $\{a, b, c, d\}/\{c\} = \{a, b, d\}$

$\rightarrow$ We will often define sets by **conditional rules** where set = {elements|rule}. For example, with $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ the set of all even integers up to 10 could be defined as $B = \{a \in A | a/2 \in A\} = \{2, 4, 6, 8, 10\}$.

$\rightarrow$ Given two sets A and B, we define their **Cartesian Product** as

$$A \times B = \{(a, b) | a \in A, b \in B\},$$

i.e. the container of all pairs formed by elements of A and B.

$\rightarrow$ The number of elements in a set is called its **cardinality**, which can be finite or infinite. Among those that are infinite we can distinguish between 'countable' and 'uncountable' sets. A set is **countable** if you can come up with a way to number its elements, for example, the set of even integers.

$\rightarrow$ Sometimes it is useful to organize sets in terms of **equivalence classes** expressing the equality $a \sim b$ of two elements relative to a certain criterion, R. For example let A be the set of all your relatives and let the criterion, R, be their sex. If a and b are both female or both male, we write $a \sim b$ An equivalence relation has the following properties:

→ **reflexivity**: $a \sim a$, every element is equivalent to itself.
→ **symmetry**: $a \sim b$ implies $b \sim a$
→ **transitivity**: $a \sim b$ and $b \sim c$ implies $a \sim c$.

The subset of all elements equivalent to a given reference element a is called an **equivalence class** and denoted $[a] \subset A$. In the example of relatives and their sex, there are two equivalence classes, for example $A = [\text{dad}] || [\text{mom}]$, one also might relabel $[\text{mom}] = [\text{aunt}]$. The set of all equivalence classes relative to a relation R is called its **quotient set** and is denoted by A/R. The quotient set $A/R = \{[\text{dad}], [\text{mom}]\}$ would have two elements, the class of males and females.

Example Consider the set of integers, and pick some integer q. Let any two integers be equivalent if they have the same remainder under division by q.

The equivalence class of all integers with the same remaineder r under division by q is

$$[r] = \{p \in \mathbb{Z} | p \bmod q = r\}$$

There are q such equivalence classes, and the set of these classes is denoted by

$$\mathbb{Z}_q \equiv \mathbb{Z}/q\mathbb{Z} = \{[0], [1], ..., [q-1]\}$$

1.1.1.2 Maps

Suppose you have two sets, A and B, together with a rule, F, such that each element of A has a corresponding element in B, where $F(a) \equiv b \in B$. Such a rule is called a **map**, which are speciified by the following notation:

$$F : A \to B$$

$$a \mapsto F(a)$$

The set A is called the **domain** of the map and B is its **codomain**. An element $a \in A$ fed into the map is called an **argument** and $F(a)$ is its **image**.

$$\text{domain} \to \text{codomain}$$

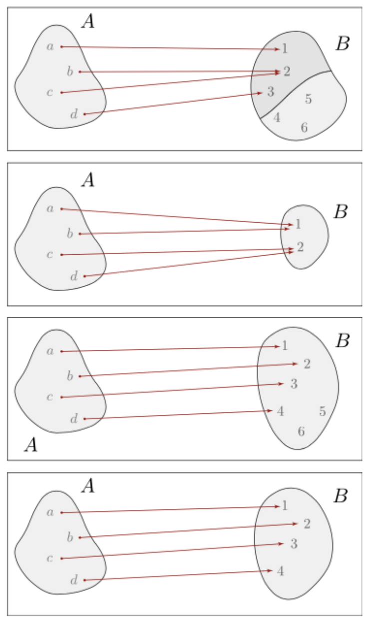
$$\text{argument} \mapsto \text{image}$$

The **image** of A under F, denoted by $F(A)$, is the set containing all image elements of $F : F(A) = \{F(a) | a \in A\} \subseteq B$.

$\rightarrow$ A map is called **surjective** if its image covers all of $B, F(A) = B$, where any element of the codomain is the image of *at least* one element of the domain.

$\rightarrow$ A map is called **injective** if every element of the codomain is the image of *at most* one element of the domain.

$\rightarrow$ A map is called **bijective** if its both surjective and injective, if every element of the codomain is the image of exactly one element of teh domain. This is called a *one-to-one* relationship. This allows the map to also establish an **inverse map**, $F^{-1} : B \to A$ such that $F^{-1}(F(a)) = a$ for every $a \in A$



Given two maps, $F : A \to B$ and $G : B \to C$, their **composition** is defined by using the image element of one as an argument for the other:

$$G \circ F : A \to C$$
$$a \mapsto G(F(a))$$

1.1.2 L1.2: Groups

The minimal structure which brings a set to life in terms of different operations between its elements is called a **group**. Let A be a set and consider an **operation** denoted by '$\cdot$', that assigns every pair of elements a and b in A another element, denoted by $a \cdot b$:

$$\cdot : A \times A \to A$$
$$(a, b) \mapsto a \cdot b$$

The set A and composition rule '$\cdot$' form a group, denoted by $G = (A, \cdot)$, provided that the following four **group axioms** are satisfied:
→ Closure: $a \cdot b$ is also in A
→ Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
→ Neutral Element: there exists an element e in A such that $e \cdot a = a \cdot e = a$
→ Inverse Element: for each a in A there exists b in A such that $a \cdot b = b \cdot a = e$

1.1.3 L1.3: Fields

A set for which both addition and multiplication is defined as separate operations is called a **field**. Formally, a field is a triple $F \equiv (A, +, \cdot)$, comprising a set A and two composition rules, addition and multiplication. The addition of the inverse element is known as **subtraction**. The multiplication by inverse is known as **division**.

Example
The integers ,$\mathbb{Z}$, do not form a field because multiplication always yields an integer, but division generally does not.

1.1.3.1 Complex Numbers

Addition and multiplication is defined by:

$$z + z' = (x + iy) + (x' + iy') \equiv (x + x') + i(y + y')$$
$$zz' = (x + iy)(x' + iy') \equiv (xx' - yy') + i(xy' + yx')$$

The **complex conjugate** is defined by

$$\bar{z} \equiv x = iy$$

Therefore,

$$z\bar{z} = x^2 + y^2$$

1.2 L2: Vector Spaces

1.2.1 L2.2: The standard vector space $\mathbb{R}^n$

$$\mathbb{R}^n \equiv \left\{ \vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \middle| x^1, x^2, \cdots, x^n \in \mathbb{R} \right\}.$$

Addition of vectors is defined by

$$\begin{pmatrix} 1.5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$$

Multiplication of a vector, by a scalar is defined by

$$2 \begin{pmatrix} 1.5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}.$$

Notice that the vectors can not be multiplied with each other.

1.2.2 L2.2 General Definition of Vector Spaces

In the theory of quantum mechanics, *functions* are considered as vectors. Vectors do not always need to be lines with arrows.

Although a function does not resemble an arrow, it meets criteria defining a vector and this analogy plays an important role in numerous applications.

However, let us first introduce the general vector space definition.

1.2.2.1 Vector Space Definition The defining property of vectors is that they can be added to each other and multiplied by elements of a number field $\mathbb{F}$. The formal definition

of a set of such objecs is
An $\mathbb{F}$-**vector space** is a triple, $(V, +, \cdot)$, consisting of a set, V, a **vector addition** rule,

$$\begin{aligned} + : & V \times V \mapsto V \\ & (\vec{v}, \vec{w}) \mapsto \vec{v} + \vec{w}. \end{aligned}$$

and a rule for **multiplication by scalars**

$$\begin{aligned} \cdot : & \mathbb{F} \times V \mapsto V \\ & (a, \vec{v}) \mapsto \vec{v} + \vec{w}. \end{aligned}$$

such that the following **vector space axioms** hold: (i) the addition of vectors, $(V, +)$, the neutral element of addition, $\vec{0}$, is called a null vector; the inverse element of a vector is called the negative vector, $-\vec{v}$. (ii) Scalar Multiplication satisfies the following rules, $\forall a, b \in \mathbb{F}, \mathbf{v}, \mathbf{w} \in V$:

$$\begin{aligned} &1. \quad (a+b)\vec{v} = a\vec{v} + b\vec{v} \\ &2. \quad a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w} \\ &3. \quad (ab)\vec{v} = a(b\vec{v}) \\ &4. \quad 1\vec{v} = \vec{v} \end{aligned}$$

1.2.2.2 Covariant Notation Below we will frequently consider sums $\vec{v_1}x^1 + \vec{v_2}x^2 + \dots$ of vectors $\vec{v_1}, \vec{v_2}, \dots$ with coefficients $x^1, x^2, \dots$. Superscript and subscript indices are called **contravariant indices** and **covarient indices**, respectively. Notation adopting this index positioning convention is called **covariant notation**