Physics 89

deval deliwala

January 17, 2023

Adapted from *Mathematical Methods in the Physical Sciences* by Mary L. Boas 3rd ed.

Contents

1 Infinite Series, Power Series

1.1 The Geometric Series

In a geometric progression we multiply each term by some fixed number to get the next term.

$$2, 4, 8, 16, 32, \ldots,$$
$$1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \cdots,$$
$$a, ar, ar^2, ar^3, \cdots$$

Let us consider the expression

$$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$$

This expression is an example of an *infinite series*, and we are asked to find its sum. Let us first find the sum of n terms, the formula being

$$\boxed{S_n = \frac{a(1 - r^n)}{1 - r}}$$

Using this equation, where a is the first term and r is the multiplier, we find

$$S_n = \frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}[1 - (\frac{2}{3})^n]}{1 - \frac{2}{3}} = 2\left[1 - \left(\frac{2}{3}\right)^n\right]$$

As n increases, $\frac{2}{3}^n$ decreases and approaches zero, thus the sum of the infinite series is 2. Infinite Geometric Sequences are known as *geometric series* and can be written in the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \ldots$$

The sum of the geometric series is thus

$$S = \lim_{n \to \infty} S_n$$

A geometric series only has a sum if $|r| < 1$, and in this case, the sum is also equal to

$$S = \frac{a}{1-r}$$

The series is then called *convergent*.

1.2 Definitions and Notations

We can write series in shorthand form using summation:

$$\sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2 + \dots$$

1.3 Convergent and Divergent Series

Convergent series have a finite sum, divergent series do not. You can not apply ordinary algebra to divergent series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad \text{is divergent}$$
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad \text{is convergent}$$

Given that $\lim_{n\to\infty} S_n = S$, we make the following definitions

a. If the partial sums S_n of an infinite series tend to a limit S, the series is called *convergent*, otherwise it is *divergent*
b. The limiting value S is called the *sum*
c. The difference $R_n = S - S_n$ is called the *remainder*

1.4 The Preliminary Test for Convergence

If the terms of an infinite series do *not* tend to zero, the series diverges. If $\lim_{n\to\infty} a_n = 0$, we must test further.

1.5 Convergence Tests for Series of Positive Terms; Absolute Convergence

Four useful tests exist for series whose terms are all positive. We could also use these tests on the *absolute value* of negative series to determine whether a series is *absolutely convergent*, which also means it converges as well, but with a different sum.

The Comparison Test

This test has two parts (a) and (b).
(a) Let

$$m_1 + m_2 + m_3 + m_4 + \dots$$

be a series of positive terms which you know converges. Then the series you are testing, namely

$$a_1 + a_2 + a_3 + a_4 + \dots$$

is absolutely convergent if $|a_n| \leq m_n$ (that is, if the absolute value of each term of the a series is no larger than the corresponding term of the m series). (b) Let

$$d_1 + d_2 + d_3 + d_4 + \dots$$

be a series of positive terms which you know diverges. Then the series

$$|a_1| + |a_2| + |a_3| + |a_4| + \dots$$

diverges if $|a_n| \geq d_n$ for all n from some point on.

Example

Test $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6}i + \frac{1}{24} + \dots$ for convergence.

As a comparison series, choose the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

We know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges because the ratio is $\frac{1}{2} \leq 1$, and since every corresponding sequence in $\sum_{n=1}^{\infty} \frac{1}{n!}$ is less than $\frac{1}{2^n}$, we know that the series $\frac{1}{n!}$ converges as well.

The Integral Test

We can use this test when the terms of the series are positive *and* not increasing, when $a_{n+1} \leq a_n$. The test states that

If $0 \leq a_{n+1} \leq a_n$ for $n > N$, then $\sum^{\infty} a_n$ converges if $\int^{\infty} a_n \, dn$ is finite and diverges if the integral is infinite. (integral is evaluated only at the upper limit).

Example

Test for convergence the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Using the integral test, we evaluate

$$\int^{\infty} \frac{1}{n} \, dn = \ln n \big|^{\infty} = \infty$$

Since the integral is infinite, the harmonic series diverges