Physics 89

Problem Set #2

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Various functions

Express the following in $(x+iy)$ form

(a) $\cosh\left(\frac{\pi i}{4}\right)$

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} \Longrightarrow \cosh\left(\frac{\pi i}{4}\right) = \frac{e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}}}{2} = \left.\frac{e^{i\theta} + e^{-i\theta}}{2}\right|_{i\theta = i\frac{\pi}{4}}$$

$$e^{i\theta} = (\cos\theta + i\sin\theta) \Longrightarrow e^{i\pi/4} = \cos \pi/4 + i\sin \pi/4$$

$$= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

$$e^{-i\pi/4} = \cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}$$

$$= \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = \frac{1-i}{\sqrt{2}}$$

$$\cosh\left(\frac{i\pi}{4}\right) = \frac{\frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}}{2} = \frac{\cancel{2}}{\cancel{2}\sqrt{2}} = \frac{1}{\sqrt{2}} \Longrightarrow \boxed{\cosh\left(i\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + 0i}$$

$$\sinh\left(\frac{i\pi}{2} + \ln(2)\right)$$

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2} \implies \frac{e^{i\pi/2 + \ln(2)} - e^{-i\pi/2 - \ln(2)}}{2}$$

$$\implies \frac{e^{i\pi/2} e^{\ln(2)} - \frac{e^{-i\pi/2}}{e^{\ln(2)}}}{2} = \frac{2e^{i\pi/2} - \frac{e^{-i\pi/2}}{2}}{2}$$

$$= \frac{2\left(\cancel{\cos(\pi/2)} + i\sin(\pi/2)\right) - \frac{\cancel{\cos(\pi/2)} - i\sin(\pi/2)}{2}}{2}$$

$$= \frac{2i + i/2}{2} = \underline{\underline{\frac{5i}{4}}}$$

c) $\cos(\pi + i\ln(2))$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \Longrightarrow \frac{e^{i(\pi + i\ln(2))} + e^{-i(\pi + i\ln(2))}}{2}$$

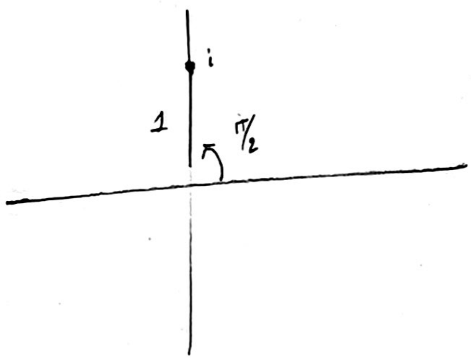
$$= \frac{e^{i\pi - \ln(2)} + e^{-i\pi + \ln(2)}}{2} = \frac{\frac{e^{i\pi}}{e^{\ln(2)}} + e^{-i\pi}e^{\ln(2)}}{2}$$

$e^{i\pi} = -1$

$e^{-i\pi} = \cos(\pi) - i\sin(\pi) = -1$

$$\frac{\frac{-1}{2} - 2}{2} = -5/4 \qquad \boxed{\cos(\pi + i\ln(2)) = \frac{-5}{4} + 0i}$$

d) i^i



$\ln(i) = \ln(1) + i(\pi/2 + 2n\pi)$

$i^i = e^{i\ln(i)} = e^{i(i(\pi/2 + 2n\pi))}$ | * $a^b = e^{b\ln(a)}$

$= e^{i^2(\pi/2 + 2n\pi)} = e^{-\pi/2 - 2n\pi}$

$$\boxed{= e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \dots}$$

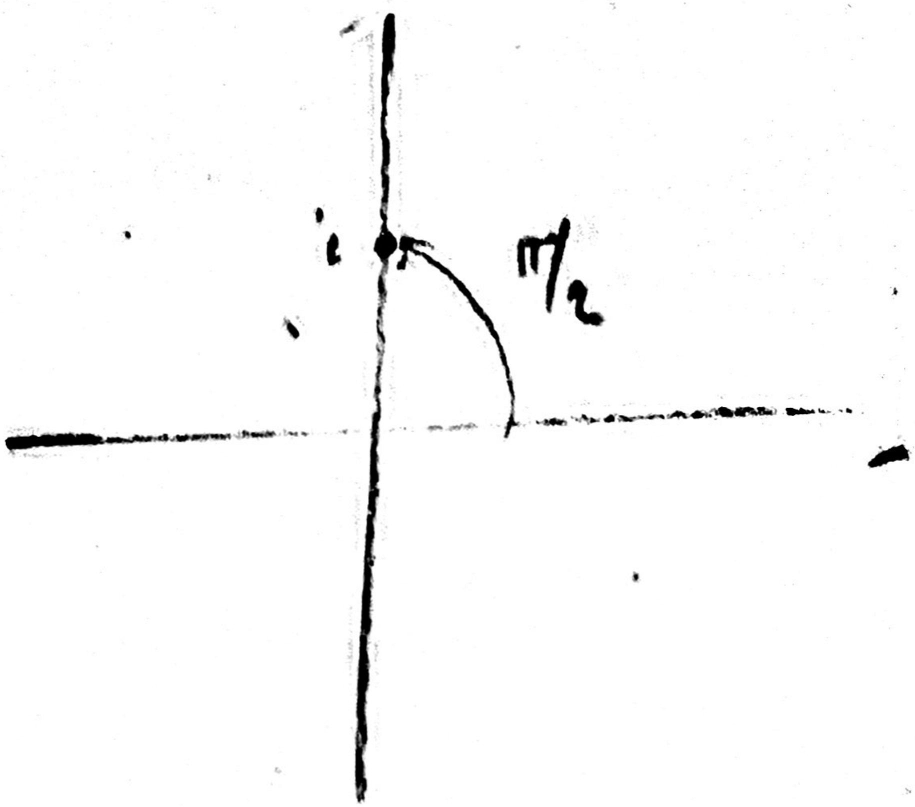
f) $i^{2/3} = e^{2/3\ln(i)} = e^{2/3(i(\pi/2 + 2n\pi))} = e^{2/3\, i(\pi/2 + 2n\pi)}$

$= \cos\left(\frac{2}{3}\left(\frac{\pi}{2} + 2n\pi\right)\right) + i\sin\left(\frac{2}{3}\left(\frac{\pi}{2} + 2n\pi\right)\right)$

$= \cos\left(\frac{\pi}{3} + \frac{4}{3}n\pi\right) + i\sin\left(\frac{\pi}{3} + \frac{4}{3}n\pi\right)$

$= \cos(60° + 240° n) + i\sin(60° + 240° n)$

e)

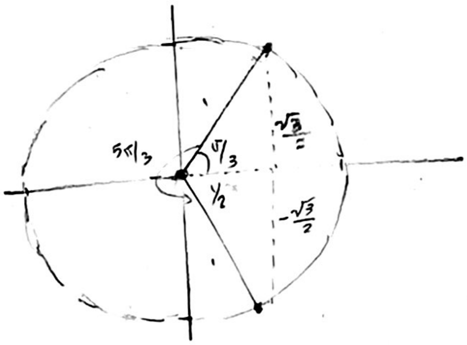


$$i = e^{i\pi/2}$$

$$\ln(e^{i\pi/2}) = \underline{i\pi/2}$$

$$\cos(60° + 240° n) + i\sin(60° + 240° n) = e^{\frac{2}{3} i \left(\frac{\pi}{2} + 2n\pi\right)} = re^{i\theta}$$

$r = 1$

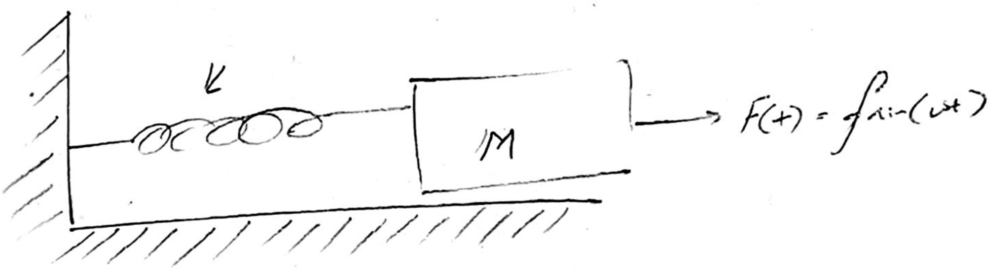


$$\cos(\pi/3) = x = \frac{1}{2}$$

$$\sin(\pi/3) = y = \frac{\sqrt{3}}{2}$$

$$\boxed{i^{2/3} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i}$$

Driven Damped Harmonic Oscillator



$$M\ddot{x} = f\sin(\omega t) - kx - \gamma\dot{x}$$

a) $F(t) = \tilde{F}(t) = fe^{i\omega t}$

$$M\ddot{x} = fe^{i\omega t} - kx - \gamma\dot{x}$$

b) $x(t) = ze^{i\omega t}$

$\dot{x}(t) = i\omega z e^{i\omega t}$

$\ddot{x}(t) = i^2\omega^2 z e^{i\omega t} = -\omega^2 z e^{i\omega t}$

b)

$$M(-\omega^2 z \cancel{e^{i\omega t}}) = f\cancel{e^{i\omega t}} - kz\cancel{e^{i\omega t}} - \gamma i\omega z\cancel{e^{i\omega t}}$$

$$-M\omega^2 z = f - kz - \gamma i\omega z$$

$$-M\omega^2 z + kz + \gamma i\omega z = f$$

$$z(-M\omega^2 + k + \gamma i\omega) = f$$

c)

$$\Longrightarrow z = \frac{f}{-M\omega^2 + k + \gamma i\omega}$$

d)

$$x(t) = ze^{i\omega t} = \frac{f}{-M\omega^2 + k + \gamma i\omega} e^{i\omega t} = \frac{f}{-M\omega^2 + k + \gamma i\omega}(\cos(\omega t) + i\sin(\omega t))$$

$$x(t) = \frac{f(\cos(\omega t) + i\sin(\omega t))}{k + \gamma i\omega - M\omega^2} \cdot \frac{k - \gamma i\omega - M\omega^2}{k - \gamma i\omega - M\omega^2}$$

$$= \frac{f(\cos(\omega t) + i\sin(\omega t))(k - \gamma i\omega - M\omega^2)}{k^2 - 2M\omega^2 k + \gamma^2\omega^2 + M^2\omega^4}$$

$$= f\left[k\cos(\omega t) - \underline{\gamma i\omega\cos(\omega t)} - M\omega^2\cos(\omega t) + \underline{ki\sin(\omega t)} + \gamma\omega\sin(\omega t) - M\omega^2\cos(\omega t) - \underline{M\omega^2 i\sin(\omega t)}\right]$$

$$i(-\gamma\omega\cos(\omega t) + k\sin(\omega t) - M\omega^2\sin(\omega t))$$

$$\mathrm{Im}(x(t)) = \frac{f\left[\gamma i\omega\cos(\omega t) + ki\sin(\omega t) - M\omega^2 i\sin(\omega t)\right]}{k^2 - 2M\omega^2 k + \gamma^2\omega^2 + M^2\omega^4}$$

$$= \frac{f(-\gamma\omega\cos(\omega t) + k\sin(\omega t) - M\omega^2\sin(\omega t))}{k^2 - 2M\omega^2 + \gamma^2\omega^2 + M^2\omega^4}$$

Integrals Using Complex #s

Evaluate $\int e^{(a+ib)x}\,dx$

$$\int e^{(a+ib)x}\,dx = \int e^{ax}\left(\cos(bx) + i\sin(bx)\right)dx$$

$$= \int e^{ax}\cos(bx)\,dx + i\int e^{ax}\sin(bx)\,dx$$

$$\frac{1}{a+bi}e^{(a+bi)x} = \int e^{x}\cos(bx)\,dx + i\int e^{ax}\sin(bx)\,dx$$

$$\frac{a-bi}{a-bi}\cdot\frac{1}{a+bi}e^{(a+bi)x} = \frac{(a-bi)e^{(a+bi)x}}{a^2+b^2} = \frac{(a-bi)}{a^2+b^2}e^{ax}\left(\cos(bx) + i\sin(bx)\right)$$

$$\underbrace{\frac{e^{ax}}{a^2+b^2}\left(a\cos(bx) + b\sin(bx)\right)}_{\int e^{ax}\cos(bx)\,dx} + \underbrace{\frac{e^{ax}}{a^2+b^2}\,i\left(a\sin(bx) - b\cos(bx)\right)}_{i\int e^{ax}\sin(bx)\,dx}$$

✓

Taylor Series for Analytic functions

$$f(x) = \frac{1}{x^2 + 6x + 5} \qquad g(x) = \frac{1}{x^2 + 4x + 5}$$

$$f(x) = \frac{1}{x^2 + 6x + 5} = \frac{1}{5} - \frac{6}{25}x + \frac{31}{125}x^3 + \dots$$

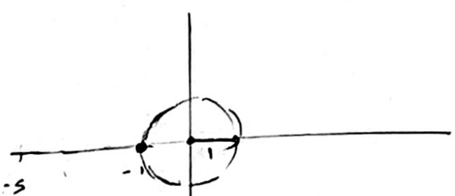
$$g(x) = \frac{1}{x^2 + 4x + 5} = \frac{1}{5} - \frac{4}{25}x + \frac{11}{125}x^3 + \dots$$

$$f(z) = \frac{1}{z^2 + 6z + 5} = \frac{1}{(z+5)(z+1)}$$

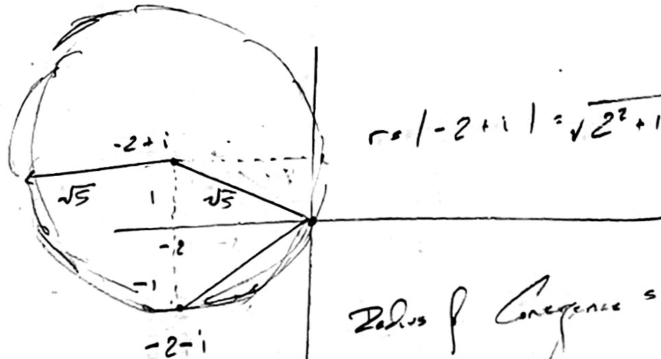
$(-5, -1)$

$$g(z) = \frac{1}{z^2 + 4z + 5} \qquad \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2}$$

$$= (-2 \pm i)$$



Radius of Convergence = 1



$r = |-2+i| = \sqrt{2^2 + 1^2} = \sqrt{5}$

Radius of Convergence = $\sqrt{5}$

Interval

$(-1, 1)$

Radius of Convergence : Radius of disk to nearest singularity