Physics 89

Problem Set #5

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1 Pauli Spin Matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a)

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \underline{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}} \qquad AC = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

$$BA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underline{\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}} \qquad CA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$BC = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underline{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}$$

$$CB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \underline{\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}}$$

(b) $AB = iC \qquad AC = -iB \qquad BC = iA$

$BA \;\; -iC \qquad CA = iB \qquad CB = -iA$

(c) $A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \qquad B^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

$$C^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

(b)

$$\sigma_1 = A \qquad \sigma_2 = B \qquad \sigma_3 = C \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_m \sigma_n = \sum_{k=1}^{3} \alpha_{mnk} \sigma_k + \beta_{mn} I \qquad m = 1, 2, 3 \qquad n = 1, 2, 3$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \qquad \epsilon_{ijk} = \begin{cases} +1 & \{ijk\} = 123, 312, 231 \\ -1 & \{ijk\} = 213, 321, 132 \\ 0 & \text{otherwise} \end{cases}$$

Example

$$AB = \sigma_1 \sigma_2 = \sum_{k=1}^{3} \alpha_{12} \sigma_k + \beta_{12} I$$

$$= (\alpha_{12} \sigma_1 + \beta_{12} I) + (\alpha_{12} \sigma_2 + \beta_{12} I) + (\alpha_{12} \sigma_3 + \beta_{12} I)$$

$$= \begin{pmatrix} 0 & \alpha_{12} \\ \alpha_{12} & 0 \end{pmatrix} + \begin{pmatrix} \beta_{12} & 0 \\ 0 & \beta_{12} \end{pmatrix} + \begin{pmatrix} 0 & -\alpha_{12} i \\ \alpha_{12} i & 0 \end{pmatrix} + \begin{pmatrix} \beta_{12} & 0 \\ 0 & \beta_{12} \end{pmatrix}$$

$$+ \begin{pmatrix} \alpha_{12} & 0 \\ 0 & -\alpha_{12} \end{pmatrix} + \begin{pmatrix} \beta_{12} & 0 \\ 0 & \beta_{12} \end{pmatrix}$$

$$= \begin{pmatrix} \beta_{12} & \alpha_{12} \\ \alpha_{12} & \beta_{12} \end{pmatrix} + \begin{pmatrix} \beta_{12} & -\alpha_{12} i \\ \alpha_{12} i & \beta_{12} \end{pmatrix} + \begin{pmatrix} \alpha_{12} + \beta_{12} & 0 \\ 0 & \beta_{12} - \alpha_{12} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{12} + 3\beta_{12} & \alpha_{12} - \alpha_{12} i \\ \alpha_{12} + \alpha_{12} i & 3\beta_{12} - \alpha_{12} \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$$

$$\Longrightarrow \epsilon_{mnk} = 0 \text{ if } m = n, \quad \delta_{mn} = 1 \text{ if } m = n$$

$\overset{C}{BA},\ \overset{B}{AC},\ \overset{A}{CB} \longrightarrow -i \quad 213,\ 132,\ 321 \quad -i\sigma_k$

$\underset{C}{AB},\ \underset{B}{CA},\ \underset{A}{BC} \longrightarrow +i \quad 123,\ 312,\ 231 \quad i\sigma_k$

$$\alpha_{mnk} = i\epsilon_{mnk}$$

$$\beta_{mn} = \delta_{mn}$$

$$\sigma_m \sigma_n = \sum_{k=1}^{3} i\epsilon_{mnk}\sigma_k + \delta_{mn} I$$

(e) $\operatorname{tr} M = \sum_{i=1}^{n} M_{ii}$

$$AB = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$ABC = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

$$\operatorname{tr} M = i + i = \underline{\underline{2i}}$$

2. Upper-Triangular Matrices

$$M = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

(a) $M^T = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}$

(b) $\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = f(ad - 0) = \underline{fad} \quad \left(\prod_{i=1}^{n} M_{ii} \right)$

(c) $MM^T = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + c^2 & bd + ce & cf \\ db + ec & d^2 + e^2 & ef \\ cf & ef & f^2 \end{pmatrix}$

$M^T M = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a^2 & ab & ac \\ ba & b^2 + d^2 & bc + de \\ ca & cb + ed & c^2 + e^2 + f^2 \end{pmatrix}$

$\underline{MM^T \neq M^T M}$

d) $\left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ 0 & d & e & 0 & 1 & 0 \\ 0 & 0 & f & 0 & 0 & 1 \end{array} \right) \xrightarrow{\text{RREF (Ti-84)}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & -\frac{b}{ad} & \frac{eb - cd}{adf} \\ 0 & 1 & 0 & 0 & \frac{1}{d} & \frac{-e}{df} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{f} \end{array} \right)$

$$M^{-1} = \begin{pmatrix} 1/a & -b/ad & eb - cd/adf \\ 0 & 1/d & -e/df \\ 0 & 0 & 1/f \end{pmatrix}$$

3. The Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x(r, \theta) = r\cos\theta$

$y(r, \theta) = r\sin\theta$

(a)

$$J = \begin{vmatrix} \frac{\partial (r\cos\theta)}{\partial r} & \frac{\partial (r\cos\theta)}{\partial \theta} \\ \frac{\partial (r\sin\theta)}{\partial r} & \frac{\partial (r\sin\theta)}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta) = r\cos^2\theta + r\sin^2\theta = \underline{\underline{r}}$$

(b) $x(u, v) = a\cosh(u)\cos(v)$

$y(u, v) = a\sinh(u)\sin(v)$

$$\frac{\partial(a\cosh(u)\cos(v))}{\partial u} = a\cos(v)\sinh(u) \qquad \frac{\partial(a\sinh(u)\sin(v))}{\partial u} = a\sin(v)\cosh(u)$$

$$\frac{\partial(a\cosh(u)\cos(v))}{\partial v} = -a\sin(v)\cosh(u) \qquad \frac{\partial(a\sinh(u)\sin(v))}{\partial v} = a\cos(v)\sinh(u)$$

$$\begin{vmatrix} a\cos(v)\sinh(u) & -a\sin(v)\cosh(u) \\ a\sin(v)\cosh(u) & a\cos(v)\sinh(u) \end{vmatrix}$$

$$= a^2\cos^2(v)\sinh^2(u) + a^2\sin^2(v)\cosh^2(u)$$

$$= a^2\cos^2(v)\left(1 + \cosh^2(u)\right) + a^2\left(1 - \cos^2(v)\right)\cosh^2(u)$$

$$= a^2\cos^2(v) + \cancel{a^2\cos^2(v)\cosh^2(u)}$$

$$+ a^2\cosh^2(u) - \cancel{a^2\cosh^2(u)\cos^2(v)}$$

$$= \underline{\underline{a^2\left(\cos^2(v) + \cosh^2(u)\right)}}$$

(c)

$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\varphi \qquad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\varphi \qquad \frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\varphi \qquad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\varphi \qquad \frac{\partial y}{\partial \varphi} = r\sin\theta\cos\varphi$$

$$\frac{\partial z}{\partial r} = \cos\theta \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta \qquad \frac{\partial z}{\partial \varphi} = 0$$

$$J = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= -r\sin\theta\sin\varphi\left(-(r\sin^2\theta\sin\varphi + r\cos^2\theta\sin\varphi)\right) - r\sin\theta\cos\varphi\left(-(r\sin^2\theta\cos\varphi + r\cos^2\theta\cos\varphi)\right)$$

$$= -r\sin\theta\sin\varphi\left(-(r\sin\varphi)\right) - r\sin\theta\cos\varphi\left(-(r\cos\varphi)\right)$$

$$= r^2\sin\theta\sin^2\varphi + r^2\sin\theta\cos^2\varphi = \underline{\underline{r^2\sin\theta}}$$

4. The Pfaffian & Relativistic Electrodynamics

$$
M = \begin{pmatrix} 0 & \mathcal{E}_x/c & \mathcal{E}_y/c & \mathcal{E}_z/c \\ -\mathcal{E}_x/c & 0 & B_z & -B_y \\ -\mathcal{E}_y/c & -B_z & 0 & B_x \\ -\mathcal{E}_z/c & B_y & -B_x & 0 \end{pmatrix}
$$

$$
= -\frac{\mathcal{E}_x}{c} \begin{vmatrix} \frac{-\mathcal{E}_x}{c} & B_z & -B_y \\ \frac{-\mathcal{E}_y}{c} & 0 & B_x \\ \frac{-\mathcal{E}_z}{c} & -B_x & 0 \end{vmatrix} + \frac{\mathcal{E}_y}{c} \begin{vmatrix} \frac{-\mathcal{E}_x}{c} & 0 & -B_y \\ \frac{-\mathcal{E}_y}{c} & -B_z & B_x \\ \frac{-\mathcal{E}_z}{c} & B_y & 0 \end{vmatrix}
$$

$$
- \frac{\mathcal{E}_z}{c} \begin{vmatrix} \frac{-\mathcal{E}_x}{c} & 0 & B_z \\ \frac{-\mathcal{E}_y}{c} & -B_z & 0 \\ \frac{-\mathcal{E}_z}{c} & B_y & -B_x \end{vmatrix}
$$

$$\frac{-\xi_x}{c}\left[\frac{\xi_y}{c}\left(-B_y B_x\right) - B_x\left(\frac{\xi_x B_x}{c} + \frac{\xi_z B_z}{c}\right)\right]$$

$$+\frac{\xi_y}{c}\left[\frac{-\xi_x}{c}\left(-B_x B_y\right) - B_y\left(\frac{-\xi_y B_y}{c} - \frac{\xi_z B_z}{c}\right)\right]$$

$$\frac{-\xi_z}{c}\left[\frac{-\xi_x}{c}\left(B_x B_z\right) + B_z\left(\frac{-\xi_y B_y}{c} - \frac{\xi_z B_z}{c}\right)\right]$$

$$-\frac{\xi_x}{c}\left[\frac{-\xi_y B_y B_x}{c} - B_x\left(\frac{\xi_x B_x + \xi_z B_z}{c}\right)\right] = \frac{-\xi_x}{c}\left[\frac{-\xi_y B_y B_x}{c} - \frac{\xi_x B_x^2 - \xi_z B_x B_z}{c}\right]$$

$$+\frac{\xi_y}{c}\left[\frac{\xi_x B_x B_y}{c} - B_y\left(\frac{-\xi_y B_y - \xi_z B_z}{c}\right)\right] = \frac{\xi_y}{c}\left[\frac{\xi_x B_x B_y}{c} + \frac{\xi_y B_y^2 + \xi_z B_y B_z}{c}\right]$$

$$-\frac{\xi_z}{c}\left[\frac{-\xi_x B_x B_z}{c} + B_z\left(\frac{-\xi_y B_y - \xi_z B_z}{c}\right)\right] = \frac{-\xi_z}{c}\left[\frac{-\xi_x B_x B_z}{c} - \frac{\xi_y B_z B_y - \xi_z B_z^2}{c}\right]$$

$$= \frac{\xi_x \xi_y B_y B_x}{c^2} + \frac{\xi_x^2 B_x^2 + \xi_x \xi_z B_x B_z}{c^2} + \frac{\xi_x \xi_y B_x B_y}{c^2} + \frac{\xi_y^2 B_y^2 + \xi_y \xi_z B_y B_z}{c^2}$$

$$+\frac{\xi_z \xi_x B_x B_z}{c^2} + \frac{\xi_z \xi_y B_z B_y + \xi_z^2 B_z^2}{c^2}$$

$$= \frac{\left(2\xi_x \xi_y B_y B_x + 2\xi_z \xi_x B_x B_z + \left(\xi_x^2 B_x^2 + \xi_y^2 B_y^2 + \xi_z^2 B_z^2\right) + 2\xi_y \xi_z B_y B_z\right)}{c^2}$$

$$(\vec{\xi}\cdot\vec{B})^2 = (\xi_x B_x + \xi_y B_y + \xi_z B_z)^2$$

$$= \xi_x^2 B_x^2 + \xi_y^2 B_y^2 + \xi_z^2 B_z^2 + 2\xi_y B_y \xi_x B_x + 2\xi_z B_z \xi_x B_x$$

$$+ 2\xi_z B_z \xi_y B_y$$

if $\det M = \alpha(\vec{\xi}\cdot\vec{B})^2$, $\boxed{\alpha = \dfrac{1}{c^2}}$