Physics 89 (Mathematical Methods)
Problem Set #4, Due by February 17, 2023, 6pm
**please submit via GradeScope**

# 1 Distances in the Hilbert space of functions

In the Hilbert space of functions on the segment $[-1, 1]$, the "distance" between $f(x)$ and $g(x)$ is defined as

$$\|f - g\| = \sqrt{\|f - g\|^2} = \sqrt{\int_{-1}^{1} (f(x) - g(x))^2 dx}.$$

For the functions

$$f(x) = 1, \qquad g(x) = x, \qquad h(x) = x^2,$$

calculate the distances

$$a = \|f - h\|, \qquad b = \|h - g\|, \qquad c = \|f - g\|,$$

and check that they satisfy the **triangle inequalities**:

$$a + b \geq c, \qquad a + c \geq b, \qquad b + c \geq a.$$

(The triangle inequality holds for any three vectors in a Hilbert space, and is a consequence of the Cauchy-Schwarz inequality.)

# 2 Expanding a function in Legendre polynomials

A continuous function $f(x)$ that is defined in the range $-1 \leq x \leq 1$ can be expanded in the basis of Legendre polynomials as

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x).$$

Here $C_n$ are constant coefficients [that are specific to the function $f(x)$] and $P_n$ is the Legendre polynomial of degree $n$. For example,

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

The coefficients $C_n$ can be calculated by

$$C_n = \frac{\langle P_n | f \rangle}{\langle P_n | P_n \rangle} = \frac{\int_{-1}^{1} P_n(x) f(x) dx}{\int_{-1}^{1} P_n(x) P_n(x) dx} = \frac{2n+1}{2} \int_{-1}^{1} P_n(x) f(x) dx.$$

Now set

$$f(x) = \cos\left(\frac{\pi x}{2}\right).$$

(a) Calculate $C_0, C_1, C_2$.

(b) Calculate the "norm" of $f(x)$, that is

$$\|f\| = \sqrt{\int_{-1}^{1} f(x)^2 dx}.$$

(c) Keeping only the first three terms in the sum, define the function

$$h(x) = C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x).$$

It turns out to be a good approximation for $f(x)$. Calculate numerically $h(x)$ and $f(x)$ at $x = 0, 0.5$, and 1.0, keeping three digits after the decimal points, and compare.

(d) Calculate how far $f$ is from $h$ in the Hilbert space, that is:

$$\|f - h\| = \sqrt{\int_{-1}^{1} (f(x) - h(x))^2 dx}.$$

Give a numerical answer with up to two digits after the decimal point.

**Hint:** There is a quick way to do this without any new integrals, just using parts (a) and (b), and the fact that $\|P_n\|^2 = \frac{2}{2n+1}$.

To solve this problem You'll need the integrals

$$\int \cos\left(\frac{\pi x}{2}\right) dx = \frac{2}{\pi} \sin\left(\frac{\pi x}{2}\right), \qquad \int x \cos\left(\frac{\pi x}{2}\right) dx = \frac{4}{\pi^2} \cos\left(\frac{\pi x}{2}\right) + \frac{2}{\pi} x \sin\left(\frac{\pi x}{2}\right),$$

and

$$\int x^2 \cos\left(\frac{\pi x}{2}\right) dx = \frac{8}{\pi^2} x \cos\left(\frac{\pi x}{2}\right) + \frac{2}{\pi^3}(\pi^2 x^2 - 8) \sin\left(\frac{\pi x}{2}\right), \qquad \int \cos^2\left(\frac{\pi x}{2}\right) dx = \frac{x}{2} + \frac{1}{2\pi} \sin(\pi x).$$

# 3 Generating function for Legendre polynomials

There is a way to encode all the Legendre polynomials $P_n(x)$ at once by introducing an auxiliary variable $t$ and defining the Taylor series

$$\sum_{n=0}^{\infty} P_n(x) t^n.$$

If $-1 < x < 1$ is fixed, this turns out to be the Taylor series of the function

$$\frac{1}{\sqrt{1 - 2xt + t^2}},$$

viewed as a function of $t$ (for fixed $x$).

(a) As an analytic function of a complex variable $1/\sqrt{1-2xz+z^2}$ has singularities when $1-2xz+z^2=0$. Solve it to determine the radius of convergence of the Taylor series. Note that the solutions are complex numbers.

**Additional note:** the function also has a cut, because of the square root, but we can ignore it for this computation.

(b) We introduce another variable $s$ and calculate the integral

$$\int_{-1}^{1}\left(\sum_{n=0}^{\infty}P_n(x)t^n\right)\left(\sum_{m=0}^{\infty}P_m(x)s^m\right)dx=\int_{-1}^{1}\frac{dx}{\sqrt{(1-2xt+t^2)(1-2xs+s^2)}}$$

as a function of $s,t$. Using the indefinite integral

$$\int\frac{dx}{\sqrt{(a-x)(b-x)}}=\log\left(\frac{a+b}{2}-x-\sqrt{(a-x)(b-x)}\right)$$

and a bit of algebra, we can show that

$$\int_{-1}^{1}\frac{dx}{\sqrt{(1-2xt+t^2)(1-2xs+s^2)}}=\frac{1}{\sqrt{st}}\log(1+\sqrt{st})-\frac{1}{\sqrt{st}}\log(1-\sqrt{st})$$

assuming $0<t<1$ and $0<s<1$.

On the other hand, we can expand the original integral as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\int_{-1}^{1}P_n(x)P_m(x)dx\right)t^n s^m.$$

Use the result of part (b) and the known Taylor series

$$\log(1-u)=-u-\frac{u^2}{2}-\frac{u^3}{3}-\frac{u^4}{4}+\cdots,\qquad \log(1+u)=u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\cdots,$$

to calculate

$$\int_{-1}^{1}P_n(x)P_m(x)dx.$$

Your result should prove that $P_n$ and $P_m$ that were defined at the top of the problem are orthogonal if $n\neq m$.

# 4 Laguerre polynomials

Let $f(x)$ and $g(x)$ be functions defined in the range $0\le x<\infty$, and define the inner-product

$$\langle f|g\rangle\equiv\int_0^{\infty}e^{-x}f(x)g(x)dx.$$

Now apply the Gram-Schmidt process to the system of functions

$$f_0=1\,,\qquad f_1=x\,,\qquad f_2=x^2\,,\qquad f_3=x^3\,,$$

to obtain an orthonormal (with respect to the inner-product) system

$$e_0 = ?\,, \qquad e_1 = ?\,, \qquad e_2 = ?\,, \qquad e_3 = ?\,,$$

**Hint:** It is useful to know that

$$\int_0^\infty x^n e^{-x} dx = n!$$

**Note:** Up to ($\pm$) signs, the polynomials that you found are called *Laguerre polynomials.* The infinite series of Laguerre polynomials plays an important role in the Quantum-Mechanical description of the Hydrogen atom.