Taylor Series

BA Disc.

1/23/23

1. $\sin(x)$

Method 1

$$f(x) = \sin(x) \quad = 0$$
$$f'(x) = \cos(x) \quad = 1$$
$$f''(x) = -\sin(x) \quad = 0$$
$$f'''(x) = -\cos(x) \quad = -1$$
$$x=0$$

$$T(\sin(x)) = 0 + 1x + 0 - \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Method 2

$$\sin(x) = a_0 + a_1 x + a_2 x^2 + \dots$$
$$\cos(x) = \sin'(x) = a_1 + 2a_2 x + 3a_3 x^2 \dots$$

$$\sin(0) = a_0 = 0$$
$$\cos(0) = a_1 = 1$$
$$\vdots$$
$$x=0$$

$$a_0 = 0 \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = \frac{-1}{2!} \dots$$

$$T(\sin(x)) = 0 + x + 0 - \frac{x^3}{2!} + \dots \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

2. $e^x \cos(x)$

$$T(e^x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad T(\cos(x)) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$a_0 = 1$ $\qquad a_3 = \frac{1}{3!} - \frac{1}{2!} = -1/3$

$a_1 = 1$ $\qquad a_4 = \frac{1}{4!} - \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{24} - \frac{1}{4} = \frac{-5}{24}$

a_2 0

3. $(x+1)\sin(x)$

$$= (x+1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$$

$$= a_0 + a_1 x + a_2 x^2 + \dots = x + x^2 - \frac{x^3}{3!} - \frac{x^4}{3!} + \dots$$

$a_0 = 0$ $\quad a_3 = \frac{-1}{3!}$

$a_1 = 1$ $\quad a_4 = \frac{-1}{3!}$

$a_2 = 1$

4. $\tan(x)$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)}{\left(1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right)} = \frac{\frac{(-1)^n x^{(2n+1)}}{(2n+1)!}}{\frac{\cancel{(-1)^n x}(2n)}{(2n)!}} = \frac{x^{(2n+1)}(2n)!}{x^{2n}(2n+1)!} = \frac{x^{(2n+1)}x^{-2n}(2n)!}{(2n+1)!}$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$a_0 = 0$ $\quad a_2 = 0$ $\quad a_4 = 0$

$a_1 = 1$ $\quad a_3 = 1/3$ $\quad a_5 = 2/15$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

5. $\frac{1}{x}\ln(1+x)$

$$= \frac{1}{x}\left(x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots\right) = 1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \dots$$

6. e^{-x^2}

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \longrightarrow e^{u} = \sum_{n=0}^{\infty} \frac{u^n}{n!} \Longrightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

7. $e^{\tan(x)}$

$$e^{\tan(x)} = \sum_{n=0}^{\infty} \frac{(\tan(x))^n}{n!} = \sum_{n=0}^{\infty} \frac{(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots)^n}{n!}$$

$$= 1 + (x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \dots) + \frac{(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots)^2}{2!} + \dots$$

$a_0 = 1$ $\quad a_2 = 1/2!$

$a_1 = 1$

$$e^{\tan(x)} = 1 + x + \frac{1}{2}x^2 + \dots$$

8. $\tan^{-1}(x)$

① $\displaystyle\int_0^x \frac{1}{1+t^2}\,dt = \tan^{-1}(x)\Big|_0^x$

② $\displaystyle\frac{1}{1+t^2} = \frac{1}{1+u} = (1+u)^{-1}$

③ $\displaystyle(1+u)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$

④ $u = t^2$, $\displaystyle(1+u)^{-1} = 1 - x + \frac{\cancel{2}x^2}{\cancel{2!}} - \frac{\cancel{6}x^3}{\cancel{3!}} = 1 - x + x^2 - x^3 + \dots$

$$= 1 - t^2 + t^4 - t^6 + \dots$$

$$\int_0^x (1 - t^2 + t^4 - t^6 + \dots)\,dt = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots\Big|_0^x$$

$$= \tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$