Physics 81

Problem Set #1

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1 Approximating $\sqrt{2}$

(a) $f(x) = \sqrt{1+x} = (1+x)^{1/2}$

$$T(f(x))\Big|_{x=0} = 1 + \frac{1}{2}(1+x)^{-1/2}\Big|_{x=0} x - \frac{1}{8}(1+x)^{-3/2}\Big|_{x=0} x^2$$

$$= 1 + \tfrac{1}{2}x - \frac{1}{8}x^2 + \dots O(x^3)$$

$$(1+x)^n \approx 1 + px + \frac{p(p-1)}{2}x^2 = \binom{p}{k}x^k$$

$$(1+x)^{1/2} = 1 + \sum_{n=1}^{\infty}\binom{1/2}{n}x^n = T(f(x))$$

$$g(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$$

$$T(g(x))\Big|_{x=0} = 1 + \frac{1}{2}(1-x)^{-3/2}\Big|_{x=0} x + \frac{3}{8}(1-x)^{-5/2}\Big|_{x=0} x^2$$

$$= 1 + \tfrac{1}{2}x + \frac{3}{8}x^2 + \dots O(x^3)$$

$$(1-x)^{+1/2} \approx 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \dots$$

$$(1-x)^{+1/2} = 1 + \sum_{n=1}^{\infty}\binom{1/2}{n}(-x)^n$$

$$(1-x)^{-1/2} = 1 + \sum_{n=1}^{\infty}\binom{-1/2}{n}(-x)^n$$

$$T(g(x)) = 1 + \sum_{n=1}^{\infty}\binom{-1/2}{n}(-x)^n$$

$$h(x) = \frac{\sqrt{1+x}}{\sqrt{1-x}} = f(x)g(x)$$

$$T(f(x)g(x)) = T(f(x))T(g(x)) = \left(1 + \sum_{n=1}^{\infty}\binom{1/2}{n}x^n\right)\left(1 + \sum_{n=1}^{\infty}\binom{-1/2}{n}(-x)^n\right)$$

$$= 1 + \sum_{n=1}^{\infty}\binom{-1/2}{n}(-x)^n + \sum_{n=1}^{\infty}\binom{1/2}{n}x^n + \sum_{n=1}^{\infty}\binom{1/2}{n}x^n\sum_{n=1}^{\infty}\binom{-1/2}{n}(-x)^n$$

$$h(x) = (1+x)(1-x^2)^{-1/2}$$

$$T(1+x) = 1 + \sum_{n=1}^{\infty}\binom{1}{n}x^n$$

$$T(1-x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty}\binom{-1/2}{n}(-x^2)^n$$

$$\Longrightarrow h(x) = \left(1 + \sum_{n=1}^{\infty}\binom{1}{n}x^n\right)\left(1 + \sum_{n=1}^{\infty}\binom{-1/2}{n}(-x^2)^n\right)$$

(b) $\sqrt{2}$

$f(1) = 1 + 1/2 - 1/8 = 1.375$

$g(1/2) = 1 + 1/4 + 3/32 = 1.3438$

$h(1/3) = 1 + 1/3 + 1/18 = 1.388$

$\sqrt{2} \approx 1.414 \approx \underline{\underline{h(1/3)}}$

2. Debye model

(a) $f(x) = \dfrac{x}{e^x - 1} = a_0 + a_1 x + a_2 x^2 + \dots + a_6 x^6 \Big|_{x=0}$

$F(x) = x$
$\quad G(x) = e^x - 1$

$T(F(x)) = 0 + x$

$T(G(x)) = 0 + x + \dfrac{x^2}{2!} + \dots + \dfrac{x^6}{6!}$

$f(x) = \dfrac{F(x)}{G(x)} \Rightarrow (0 + x) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_6 x^6)(0 + x + \frac{1}{2}x^2 + \dots + \frac{1}{6!}x^6)$

$x = \left(\sum_{n=0}^{6} a_n x^n\right)\left(\sum_{n=0}^{6} \dfrac{x^n}{n!}\right)$?

$$\begin{array}{cccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \\ \end{array}$$

$$\left(\begin{array}{ccccccc|c} 1 & & & & & & & 1 \\ 1/2 & 1 & & & & & & 0 \\ 1/6 & 1/2 & 1 & & & & & 0 \\ 1/24 & 1/6 & 1/2 & 1 & & & & 0 \\ 1/120 & 1/24 & 1/6 & 1/2 & 1 & & & 0 \\ 1/720 & 1/120 & 1/24 & 1/6 & 1/2 & 1 & & 0 \end{array}\right)$$

$a_0 = 1$

$a_1 = 1/2$

$a_2 = 1/12$

$a_3 = 0$

$a_4 = -1/720$

$a_5 = 0$

$a_6 = 0$

(b) *

$$\mathcal{E}(T) = 9NkT\left(\frac{T}{\theta}\right)^3 \int_0^{\theta/T} \frac{x^3}{e^x - 1}\, dx$$

$$\mathcal{E}(T) = \alpha_0 + \frac{\alpha_1}{T} + \frac{\alpha_2}{T} + \dots + \frac{\alpha_6}{T}$$

$$\mathcal{E}(T) = 9NkT\left(\frac{T}{\theta}\right)^3 \int_0^{\theta/T} \frac{(x)^3}{e^x - 1}\, dx$$

$$\frac{x}{e^x - 1} \approx 1 + \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \frac{1}{720}x^4 \implies \frac{x^3}{e^x - 1} = 1 + \tfrac{1}{2}x^3 + \tfrac{1}{12}x^8 - \frac{1}{720}x^{64}$$

$$\mathcal{E}(T) \approx 9NkT\left(\frac{T}{\theta}\right)^3 \int_0^{\theta/T} \left(1 + \tfrac{1}{2}x^3 + \tfrac{1}{12}x^8 - \frac{1}{720}x^{64}\right) dx$$

$$= \left(x + \tfrac{1}{2}\frac{x^4}{4} + \frac{1}{12}\frac{x^9}{9} - \frac{1}{720}\frac{x^{65}}{65}\right)\Bigg|_0^{\theta/T}$$

$$= \frac{\theta}{T} + \frac{\left(\frac{\theta}{T}\right)^4}{8} + \frac{\left(\frac{\theta}{T}\right)^9}{108} - \frac{\left(\frac{\theta}{T}\right)^{65}}{46800}$$

$$\mathcal{E}(T) \approx 9NkT\left(\frac{T}{\theta}\right)^3 \left(\frac{\theta}{T} + \frac{\left(\frac{\theta}{T}\right)^4}{8} + \frac{\left(\frac{\theta}{T}\right)^9}{108} + \frac{\left(\frac{\theta}{T}\right)^{65}}{46800}\right)$$

$$\implies \frac{9Nk\cancel{T}}{\cancel{T}}\left(\frac{T}{\theta}\right)\left(\frac{T}{\theta}\right)\left(\frac{T}{\cancel{\theta}}\right)\left(\cancel{\theta}\right) = \frac{9NkT^3}{\theta^2} \longrightarrow \alpha_0 = 9Nk\left(\frac{\theta}{8} + \frac{T^3}{\theta^2}\right)$$

$$\implies \frac{9NkT}{8}\left(\cancel{\frac{T}{\theta}}\right)^{\cancel{3}}\left(\frac{\theta}{T}\right)^{\cancel{4}} = \frac{9Nk\theta}{8}$$

$$\implies \frac{\cancel{9}NkT}{\cancel{108}\ 12}\left(\cancel{\frac{T}{\theta}}\right)^3\left(\frac{\theta}{T}\right)^{\cancel{9}\,6} = \frac{1}{12}NkT\left(\frac{\theta}{T}\right)^6 = \frac{\frac{1}{12}Nk\theta^6}{T^5} \longrightarrow \alpha_5 = \frac{1}{12}Nk\theta$$

Something is wrong... :(

3. Legendre Polynomials

$$f(x,\mu) = \frac{1}{\sqrt{1 - 2x\mu + x^2}} \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots O(x^5)$$

let $t = 2x\mu - x^2$

$$\frac{1}{\sqrt{1-t}} = 1 + \frac{1}{2}t + \frac{3}{8}t^2 + \frac{5}{16}t^3 + \frac{35}{128}t^4 + \cancel{O(x^5)} \dots$$

$$\frac{1}{\sqrt{1 - 2x\mu + x^2}} \approx 1 + \frac{1}{2}(2x\mu - x^2) + \frac{3}{8}(2x\mu - x^2)^2 + \frac{5}{16}(2x\mu - x^2)^3 + \frac{35}{128}(2x\mu - x^2)^4$$

$$= 1 + x\mu - \frac{1}{2}x^2 + \frac{3x^2\mu^2}{2} - \frac{3x^3\mu}{2} + \frac{3x^4}{8} + \frac{5x^3\mu^3}{2} - \frac{15x^4\mu^2}{4} + \frac{35x^4\mu^4}{8}$$

$$= 1 + x\mu + x^2\left(-\frac{1}{2} + \frac{3\mu^2}{2}\right) + x^3\left(-\frac{3\mu}{2} + \frac{5\mu^3}{3}\right) + x^4\left(\frac{3}{8} - \frac{15\mu^2}{4} + \frac{35\mu^4}{8}\right)$$

$a_0 = 1 \quad a_1 = x \quad a_2 = \left(-\frac{1}{2} + \frac{3\mu^2}{2}\right) \quad a_3 = \left(-\frac{3\mu}{2} + \frac{5\mu^3}{3}\right) \quad a_4 = \left(\frac{3}{8} - \frac{15\mu^2}{4} + \frac{35\mu^4}{8}\right)$

4. The Error Term

$$f(x) - \sum_{n=0}^{N} \frac{f^{n}(0)}{n!} x^n = \frac{f^{(N+1)}(\alpha)}{(N+1)!} \alpha^{N+1} \equiv R_{N+1}(\alpha)$$

for $0 < \alpha < x$.

(a) $R_2 \alpha$ $\quad \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + R_2\alpha$

$$R_2\alpha = R_{1+1}\alpha = \frac{f''(\alpha)}{2}\alpha^2$$

$$f'(\alpha) = \frac{1}{2}(1-x)^{-3/2} \qquad f''(x) = \frac{3}{4}(1-x)^{-5/2} \Rightarrow f''(\alpha) = \frac{3}{4}(1-\alpha)^{-5/2}$$

$$R_2\alpha = \frac{3}{8}(1-\alpha)^{-5/2}\alpha^2 = \frac{3\alpha^2}{8\sqrt{(1-\alpha)^5}}$$

(b) $x = 0.01$ find $|R_2(\alpha)|$

$$f''(0) = \frac{3}{4}(1)^{-5/2} = \frac{3}{4}$$

$$f''(0.01) = \frac{3}{4}(0.99)^{-5/2} = 0.76$$

$M = f''(0.01)$

$$|R_2(\alpha)| = \frac{f''(0.01)}{2!}(0.01)^2 = \frac{0.769}{2}(0.01)^2$$

$$= 3.845 \times 10^{-5}$$

(c)

$$\mathcal{E}_k = m_0c^2 - mc^2 = \frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} - mc^2$$

let $x = \left(\frac{v}{c}\right)^2 \Longrightarrow \mathcal{E}_k = \frac{mc^2}{\sqrt{1-x}} - mc^2 = mc^2\left(\frac{1}{\sqrt{1-x}} - 1\right)$

$$T_1(\mathcal{E}_k) = mc^2\left(\left(\cancel{1} + \tfrac{1}{2}x + \tfrac{3}{8}x^2\right) - \cancel{1}\right) = mc^2\left(\tfrac{1}{2}x\right) = \frac{1}{2}mc^2\left(\frac{v^2}{\cancel{c^2}}\right) + \frac{3}{8}\left(\frac{v}{c}\right)^4 mc^2$$

$$= \tfrac{1}{2}mv^2 + \frac{3v^4m}{8c^2}.$$

let $0 < v < 0.1c$

$$\mathcal{E}_k'' = mc^2\left(\frac{3}{4}(1-x)^{-5/2}\right) \quad v \in [0, 0.1c]$$

$$\mathcal{E}_k(0) = mc^2\left[\frac{3}{4}\left(1-\cancel{\frac{0}{c^2}}\right)^{-5/2}\right] = \underline{\frac{3}{4}mc^2}$$

$$\mathcal{E}_k(0.1c) = mc^2\left[\frac{3}{4}\left(1-(0.1)^2\right)\right] = 0.7425\,mc^2$$

$$|R_2(x)|_{max} = \frac{\mathcal{E}_k''(0.1c)}{2!} = \frac{0.7425}{2}mc^2 = \underline{\underline{0.37\,mc^2}}$$