Physics 89

Problem Set #6

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1. Eigenvalues & Eigenvectors

$A\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, $\quad \lambda I - A = \begin{pmatrix} \lambda & -1 \\ -2 & \lambda+1 \end{pmatrix}$

$$\begin{vmatrix} \lambda & -1 \\ -2 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1) - 2 = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda+2)(\lambda-1) = 0$$

$$\underline{\lambda = -2, 1} \quad \text{Eigenvalues}$$

$\underline{\lambda = -2}$

$$\left(\begin{array}{cc|c} -2 & -1 & 0 \\ -2 & -1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

let $x_2 = t$

$$x_1 + \frac{1}{2}t = 0 \Longrightarrow x_1 = -\frac{1}{2}t$$

$$\Longrightarrow \vec{x}_{(\lambda=-2)} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 1}$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

let $x_2 = t$

$$x_1 - t = 0 \Longrightarrow \begin{matrix} x_1 = t \\ x_2 = t \end{matrix} \quad \vec{x}_{(\lambda=1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$B\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \qquad \lambda I - B \begin{pmatrix} \lambda & -1 & -1 \\ 1 & \lambda & 1 \\ 1 & -1 & \lambda \end{pmatrix}$$

$$\begin{vmatrix} \lambda & -1 & -1 \\ 1 & \lambda & 1 \\ 1 & -1 & \lambda \end{vmatrix} = \lambda(\lambda^2+1) + (\lambda-1) - (-1-\lambda) = 0$$

$$= \lambda^3 + \lambda + \lambda \cancel{-1} \cancel{+1} + \lambda = 0$$

$$= \lambda^3 + 3\lambda = 0$$

$$= \lambda^2 + 3 = 0$$

$$\lambda = \frac{0 \pm \sqrt{0-12}}{2} = \frac{\sqrt{-12}}{2} = \pm\sqrt{3}\,i, \quad 0$$

$\underline{\lambda = 0}$

$$\left(\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

let $x_3 = t$ $\quad x_1 = -t$ $\quad \vec{\lambda}_{(\lambda=0)} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$x_2 = -t$

$x_3 = t$

$\lambda = \sqrt{3}\,i$

$$\left(\begin{array}{ccc|c} \sqrt{3}i & -1 & -1 & 0 \\ 1 & \sqrt{3}i & 1 & 0 \\ 1 & -1 & \sqrt{3}i & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{-\sqrt{3}}{3i} & \frac{-\sqrt{3}}{3i} & 0 \\ 1 & \sqrt{3}i & 1 & 0 \\ 1 & -1 & \sqrt{3}i & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{-\sqrt{3}}{3i} & \frac{-\sqrt{3}}{3i} & 0 \\ 0 & \sqrt{3}i + \frac{\sqrt{3}}{3i} & 1 + \frac{\sqrt{3}}{3i} & 0 \\ 0 & -1 + \frac{\sqrt{3}}{3i} & \sqrt{3}i + \frac{\sqrt{3}}{3i} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & \frac{-\sqrt{3}}{3i} & \frac{-\sqrt{3}}{3i} & 0 \\ 0 & \frac{-2\sqrt{3}}{3i} & \frac{3i+\sqrt{3}}{3i} & 0 \\ 0 & \frac{-3i+\sqrt{3}}{3i} & \frac{-2\sqrt{3}}{3i} & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{-i}{2}(\sqrt{3}-i) & 0 \\ 0 & 1 & \frac{i}{2}(\sqrt{3}-i) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3 = t$

$x_1 = \frac{i}{2}(\sqrt{3}-i)t$ $\quad \vec{\lambda}_{(\sqrt{3}i)} = \begin{pmatrix} \frac{i}{2}(\sqrt{3}-i) \\ \frac{-i}{2}(\sqrt{3}-i) \\ 1 \end{pmatrix}$

$x_2 = \frac{-i}{2}(\sqrt{3}-i)t$

$$-\sqrt{3}i$$

$$\left(\begin{array}{ccc|c} -\sqrt{3}i & -1 & -1 & 0 \\ 1 & -\sqrt{3}i & 1 & 0 \\ 1 & -1 & -\sqrt{3}i & 0 \end{array}\right) \Longrightarrow \vec{\lambda}_{(-\sqrt{3}i)} = \begin{pmatrix} -\frac{i}{2}(\sqrt{3}-i) \\ \frac{i}{2}(\sqrt{3}-i) \\ 1 \end{pmatrix}$$

2 Rotation Matrix

$$\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \vec{v}_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(a) $\lambda I - M \begin{pmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{pmatrix}$

$$\begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} = (\lambda - \cos\theta)^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 0$$

$$\lambda^2 - 2\cos\theta\,\lambda + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm \frac{\sqrt{4(\cos^2\theta - 1)}}{2}$$

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$\lambda_1 = \cos\theta + \sqrt{\cos^2\theta - 1} \qquad \lambda_2 = \cos\theta - \sqrt{\cos^2\theta - 1}$$

(b) $C = (v_1 | v_2) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$

$$C^{-1} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ i & -i & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2i & -i & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{-1}{2i} \end{array}\right) \Longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2i} \end{array}\right)$$

$$C^{-1}MC = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}\cos\theta + \frac{1}{2i}\sin\theta & -\frac{1}{2}\sin\theta + \frac{1}{2i}\cos\theta \\ \frac{1}{2}\cos\theta - \frac{1}{2i}\sin\theta & -\frac{1}{2}\sin\theta - \frac{1}{2i}\cos\theta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta - i\sin\theta & 0 \\ 0 & \cos\theta + i\sin\theta \end{pmatrix} = D$$

(c) $e^{A} = M$

$f(M) = C f(D) C^{-1}$

$f(M) = \ln(M) = C \ln(D) C^{-1}$

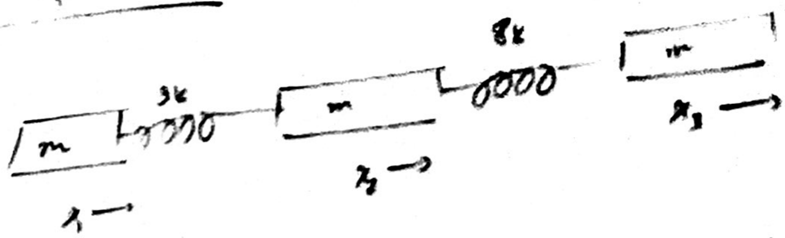
$$A = C \begin{pmatrix} \ln(\cos\theta - i\sin\theta) & 0 \\ 0 & \ln(\cos\theta + i\sin\theta) \end{pmatrix} C^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} -i\theta & 0 \\ 0 & i\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix}$$

$$= \begin{pmatrix} -i\theta & i\theta \\ \theta & \theta \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} = \begin{pmatrix} \cancel{\frac{-i\theta}{2} + \frac{i\theta}{2}} \; 0 & \frac{-i\theta}{2i} - \frac{i\theta}{2i} \\ \theta & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

3 Resonance Frequencies



(a)

$x_1 = A_1\cos\omega t \quad x_2 = A_2\cos\omega t \quad x_3 = A_3\cos\omega t$

$\ddot{x}_1 = -\omega^2 A_1\cos\omega t \quad \ddot{x}_2 = -\omega^2 A_2\cos\omega t \quad \ddot{x}_3 = -\omega^2 A_3\cos\omega t$

$$\begin{cases} -m\omega^2 A_1 \cancel{\cos\omega t} = 3k(\cancel{\cos\omega t}(A_2 - A_1)) \\ -m\omega^2 A_2 \cancel{\cos\omega t} = 8k(\cancel{\cos\omega t}(A_3 - A_2)) - 3k(\cancel{\cos\omega t}(A_2 - A_1)) \\ -m\omega^2 A_3 \cancel{\cos\omega t} = -8k(\cancel{\cos\omega t}(A_3 - A_2)) \end{cases}$$

$$\begin{cases} -m\omega^2 A_1 = 3k(A_2 - A_1) \\ -m\omega^2 A_2 = 8k(A_3 - A_2) - 3k(A_2 - A_1) \\ -m\omega^2 A_3 = -8k(A_3 - A_2) \end{cases}$$

$$A_1(-m\omega^2 + 3k) = 3kA_2$$

$$-m\omega^2 A_1 + 3kA_1 = 3kA_2$$

$$-m\omega^2 A_2 + 8kA_2 + 3kA_2 = 8kA_3 + 3kA_1 \Longrightarrow A_2(-m\omega^2 + 11k) = 8kA_3 + 3kA_1$$

$$-m\omega^2 A_3 + 8kA_3 = 8kA_2 \qquad A_3(-m\omega^2 + 8k) = 8kA_2$$

$$\frac{-m\omega^2 A_1}{k} = -3kA_1 + 3kA_2 + 0A_3$$

$$\frac{-m\omega^2 A_2}{k} = 3kA_1 - 11kA_2 + 8kA_3$$

$$\frac{-m\omega^2 A_3}{k} = 0A_1 + 8kA_2 - 8kA_3$$

$$\lambda A_1 = -3A_1 + 3A_2 + 0A_3$$

$$\lambda A_2 = 3A_1 - 11A_2 + 8A_3 \Longrightarrow \vec{\lambda}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} -3 & 3 & 0 \\ 3 & -11 & 8 \\ 0 & 8 & -8 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$\lambda A_3 = 0A_1 + 8A_2 - 8A_3$$

(b)
$$\begin{vmatrix} \lambda+3 & -3 & 0 \\ -3 & \lambda+11 & -8 \\ 0 & -8 & \lambda+8 \end{vmatrix} = 0 \Longrightarrow \lambda_1 = -4 \quad \lambda_2 = -18 \quad \lambda_3 = 0$$

$$\vec{\lambda_1}\begin{pmatrix} -3/2 \\ 1/2 \\ 1 \end{pmatrix} \quad \vec{\lambda_2}\begin{pmatrix} 1/4 \\ -5/4 \\ 1 \end{pmatrix} \quad \vec{\lambda_3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\frac{m\omega_1^2}{k} = -3/2 \qquad \frac{m\omega_1^2}{k} = 1/4 \qquad \frac{m\omega_1^2}{k} = 1$$

$$\frac{m\omega_2^2}{k} = 1/2 \qquad \frac{m\omega_2^2}{k} = -5/4 \qquad \frac{m\omega_2^2}{k} = 1$$

$$\frac{m\omega_3}{k} = 1 \qquad \frac{m\omega_3^2}{k} = 1 \qquad \frac{m\omega_3^2}{k} = 1$$

$$\omega_1 = \sqrt{\frac{-3k}{2m}} \qquad \omega_1 = \sqrt{\frac{k}{4m}} \qquad \omega_1 = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{k}{2m}} \qquad \omega_2 = \sqrt{\frac{-5k}{4m}} \qquad \omega_2 = \sqrt{\frac{k}{m}}$$

$$\omega_3 = \sqrt{\frac{k}{m}} \qquad \omega_3 = \sqrt{\frac{k}{m}} \qquad \omega_3 = \sqrt{\frac{k}{m}}$$