Physics 89 (Mathematical Methods)
Problem Set #2
Due by 6pm, February 3, 2023

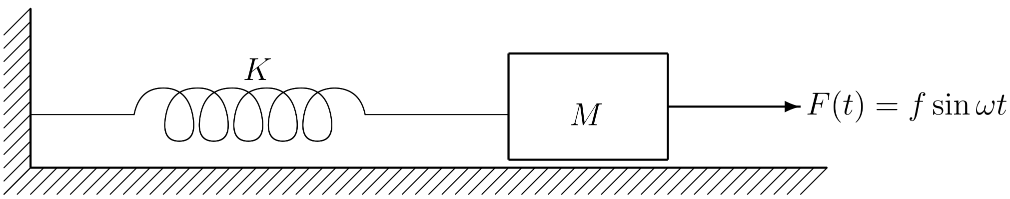
1 Various functions

Express each of the following in rectangular $(x + iy)$ form:

(a) $\cosh(\pi i/4)$, (b) $\sinh(\frac{i\pi}{2} + \ln 2)$, (c) $\cos(\pi + i\ln 2)$, (d) i^i, (e) $\log i$, (f) $i^{2/3}$

For parts (d), (e) and (f), there are more than one solution if you treat the functions $\log z$ and $z^{2/3}$ as multivalued. Any one will suffice.

2 Driven Damped Harmonic Oscillator



A mass M is connected to a spring K and is driven by an oscillating force $F(t) = f \sin \omega t$. There is a friction force proportional to the velocity v of the mass and given by $-\gamma v$ (where γ is a constant). Find a solution $x(t)$ to Newton's equation

$$M\ddot{x} = f \sin \omega t - Kx - \gamma \dot{x}$$

by following these steps:

(a) Replace $F(t)$ with the complex $\tilde{F} = f e^{i\omega t}$.

(b) Look for a complex solution of the form $x(t) = z e^{i\omega t}$, where z is a complex constant. Substitute this into Newton's equation to get an equation for z.

(c) Express the solution for z in terms of the constants M, K, f, γ, and ω.

(d) Calculate the imaginary part of $x(t)$ to get the physical solution.

To be sure, in your solution you don't need to write anything for part (a).

3 Integrals using complex numbers

Evaluate $\int e^{(a+ib)x} dx$ and take real and imaginary parts to show that:

$$(a) \qquad \int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}, \qquad \int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}.$$

(Problems 2.11.17 and 2.11.18 of [Boas].)

4 Taylor series for analytic functions

Define the two functions $f(x)$ and $g(x)$ by

$$f(x) = \frac{1}{x^2+6x+5}, \qquad g(x) = \frac{1}{x^2+4x+5}.$$

The two Taylor series

$$\begin{aligned} f(x) &= \frac{1}{x^2+6x+5} = \tfrac{1}{5} - \tfrac{6}{25}x + \tfrac{31}{125}x^3 + \cdots \\ g(x) &= \frac{1}{x^2+4x+5} = \tfrac{1}{5} - \tfrac{4}{25}x + \tfrac{11}{125}x^3 + \cdots \end{aligned}$$

turn out to have different segments of convergence. The series for $g(x)$ converges for $|x| < \sqrt{5}$ and doesn't converge for $|x| > \sqrt{5}$, while the series for $f(x)$ converges for $|x| < 1$ and doesn't converge for $|x| > 1$.

Can you explain this fact using complex numbers?

[You don't have to explain what happens at $x = \pm 1$ for $f(x)$ and $x = \pm\sqrt{5}$ for $g(x)$.]

References

[1] Mary L. Boas, "Mathematical Methods in the Physical Sciences," 3^{rd} Edition, John Wiley & Sons, 2006.