Physics 8A

Vectors & Tensors

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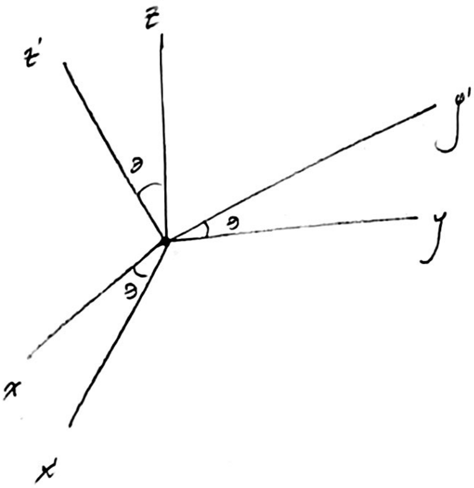
1 Tensors



$$A_i' = \sum_{k=1}^{2} R_{ik} A_k$$

$$T_{ij}' = \sum_{k=1}^{2}\sum_{l=1}^{2} R_{ik} R_{jl} T_{kl}$$

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

a) $$T_{11}' = \sum_{k=1}^{2}\sum_{l=1}^{2} R_{1k} R_{1l} T_{kl}$$

$$= T_{11} R_{11} R_{11} + T_{21} R_{12} R_{11} + T_{12} R_{11} R_{12} + T_{22} R_{12} R_{12}$$

$$= T_{11}\cos^2\theta + T_{21}\sin\theta\cos\theta + T_{12}\cos\theta\sin\theta + T_{22}\sin^2\theta$$

$$T_{12}' = \sum_{k=1}^{2}\sum_{l=1}^{2} R_{1k} R_{2l} T_{kl}$$

$$= T_{11} R_{11} R_{21} + T_{21} R_{12} R_{21} + T_{12} R_{11} R_{22} + T_{22} R_{12} R_{22}$$

$$= -T_{11}\cos\theta\sin\theta - T_{21}\sin^2\theta + T_{12}\cos^2\theta + T_{22}\sin\theta\cos\theta$$

$$T_{21}' = \sum_{k=1}^{2}\sum_{l=1}^{2} R_{2k} R_{1l} T_{kl} = T_{11} R_{21} R_{11} + T_{21} R_{22} R_{12} + T_{12} R_{21} R_{12} + T_{22} R_{22} R_{12}$$

$$= -T_{11}\sin\theta\cos\theta + T_{21}\cos\theta\sin\theta - T_{12}\sin^2\theta + T_{22}\cos\theta\sin\theta$$

$$T'_{22} = \sum_{k=1}^{2}\sum_{l=1}^{2} \ell_{2k}\ell_{2l}T_{kl}$$

$$= T_{11}\ell_{21}\ell_{21} + T_{21}\ell_{22}\ell_{21} + T_{12}\ell_{21}\ell_{22} + T_{22}\ell_{22}\ell_{22}$$

$$= T_{11}\sin^2\theta - T_{21}\cos\theta\sin\theta - T_{12}\cos\theta\sin\theta + T_{22}\cos^2\theta$$

b) Symmetric tensors $\longrightarrow$ $T_{12} = T_{21}$

$$T'_{12} = \sum_{k=1}^{2}\sum_{l=1}^{2} \ell_{1k}\ell_{2l}T_{kl} = -T_{11}\cos\theta\sin\theta - T_{21}\sin^2\theta + T_{12}\cos^2\theta + T_{22}\sin\theta\cos\theta$$

$$T'_{21} = \sum_{k=1}^{2}\sum_{l=1}^{2} \ell_{2k}\ell_{1l}T_{kl} = -T_{11}\sin\theta\cos\theta + T_{21}\cos^2\theta - T_{12}\sin^2\theta + T_{22}\cos\theta\sin\theta$$

if $T_{21} = T_{12}$

$T'_{21} = T'_{12}$

c) $T_{12} = T_{21} = 0$

$$T'_{12} = -T_{11}\cos\theta\sin\theta + T_{22}\cos\theta\sin\theta = -\cos\theta\sin\theta(T_{11} - T_{22}) \Longrightarrow \underline{\alpha = -\cos\theta\sin\theta}$$

d) $T_{ij} = \delta_{ij}$ $(T_{11} = T_{22} = 1,\ T_{12} = T_{21} = 0)$ $T'_{ij} = \delta_{ij}$

$T'_{11} = \cos^2\theta + \sin^2\theta = \underline{\underline{1}}$

$T'_{21} = \sin\theta\cos\theta - \cos\theta\sin\theta = \underline{\underline{0}}$

$T'_{12} = T'_{21} = \underline{\underline{0}}$

$T'_{22} = \sin^2\theta + \cos^2\theta = \underline{\underline{1}}$

$T'_{11} = T'_{22} = 1$

$T'_{12} = T'_{21} = 0$

$\longrightarrow \underline{\underline{T'_{ij} = \delta_{ij}}}$

2. Bases of Vector Spaces

$$\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{A} = \hat{i} + \hat{j} \qquad \vec{B} = \hat{i} + \hat{k} \qquad \vec{C} = \hat{j} + \hat{k}$$

find α, β, γ such that $\vec{V} = \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C}$

$$= \alpha(\hat{i} + \hat{j}) + \beta(\hat{i} + \hat{k}) + \gamma(\hat{j} + \hat{k})$$

$$= \alpha\hat{i} + \beta\hat{i} \quad = \hat{i}(\alpha + \beta) = a\hat{i}$$
$$+ \alpha\hat{j} + \gamma\hat{j} \quad \hat{j}(\alpha + \gamma) = b\hat{i}$$
$$+ \beta\hat{k} + \gamma\hat{k} \quad \hat{k}(\beta + \gamma) = c\hat{i}$$

$a = \beta + \alpha$

$b = \alpha + \gamma$

$c = \beta + \gamma$

$$\Longrightarrow \begin{pmatrix} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 0 & a \\ 0 & 1 & 1 & c \\ 1 & 0 & 1 & b \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & a \\ 0 & 1 & 1 & c \\ 0 & -1 & 1 & b-a \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & a-c \\ 0 & 1 & 1 & c \\ 0 & 0 & 1 & \frac{b-a+c}{2} \end{pmatrix} \Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & a-c \\ 0 & 1 & 0 & \frac{-b+a+c}{2} \\ 0 & 0 & 1 & \frac{b-a+c}{2} \end{array}\right)$$

$$\boxed{\begin{aligned} \alpha &= a - c \\ \beta &= \frac{-b+a+c}{2} \\ \gamma &= \frac{b-a+c}{2} \end{aligned}}$$

3. Vector Spaces of Functions

a) Expand functions e^x, e^{-x} in the basis $[\sinh(x),\ \cosh(x)]$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$e^x = \cosh(x) + \sinh(x)$$

$$e^{-x} = \cosh(x) - \sinh(x)$$

b) $[\sinh(x),\ \sinh(x+a)]$ $a \neq 0$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \sinh(x+a) = \frac{e^{x+a} - e^{-x-a}}{2}$$

$$e^{-x} = -\alpha\sinh(x) - \beta\sinh(x+a)$$

$$e^x = \alpha\sinh(x) + \beta\sinh(x+a)$$

$$= \alpha\left(\frac{e^x - e^{-x}}{2}\right) + \beta\left(\frac{e^{x+a} - e^{-x-a}}{2}\right)$$

$$= \frac{1}{2}\alpha e^x - \frac{1}{2}\alpha e^{-x} + \frac{1}{2}\beta e^{x+a} - \frac{1}{2}\beta e^{-x-a} = e^x$$

$$e^{-x} = \alpha\sinh(-x) + \beta\sinh(-x+a)$$

$$= \frac{1}{2}\alpha e^{-x} - \frac{1}{2}\alpha e^x + \frac{1}{2}\beta e^{-x+a} - \frac{1}{2}\beta e^{x-a} = e^{-x}$$

$$\begin{pmatrix} \sinh(x) & \sinh(x+a) & e^x \\ -\sinh(x) & -\sinh(x+a) & e^{-x} \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & \frac{\sinh(x+a)}{\sinh(x)} & \frac{e^x}{\sinh(x)} \\ \frac{\sinh(x)}{\sinh(x+a)} & 1 & \frac{e^{-x}}{-\sinh(x+a)} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{\sinh(x+a)}{\sinh(x)} & \frac{e^x}{\sinh(x)} \\ -\sinh(x) & -\sinh(x+a) & e^{-x} \end{pmatrix} \xrightarrow{R_1(\sinh(x)) + R_2} \begin{pmatrix} 1 & \frac{\sinh(x+a)}{\sinh(x)} & \frac{e^x}{\sinh(x)} \\ 0 & 0 & 2\cosh(x) \end{pmatrix}$$

$$\xrightarrow{\frac{R_2}{2\cosh(x)}} \begin{pmatrix} 1 & \frac{\sinh(x+a)}{\sinh(x)} & \frac{e^x}{\sinh(x)} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - \frac{e^x}{\sinh(x)} R_2} \begin{pmatrix} 1 & \frac{\sinh(x+a)}{\sinh(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha + \beta \frac{\sinh(x+a)}{\sinh(x)} = 0$$

$$\alpha \sinh(x) + \beta \sinh(x+a) = e^x$$

Not possible w/o parameterization?

$$\cancel{e^x = \alpha\left(\frac{e^x - e^{-x}}{2}\right) + \beta\left(\frac{e^{x+a} - e^{-x-a}}{2}\right)}$$

$$\cancel{= \frac{1}{2}\alpha e^x - \frac{1}{2}\alpha e^{-x} + \frac{1}{2}\beta e^{x+a} - \frac{1}{2}\beta e^{-x-a}}$$

4. Orthogonal Vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 7 \\ 7 \\ 1 \end{pmatrix} \quad f = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

a) $|\vec{v}_1| = 2$ $\qquad |\vec{v}_2| = 2$ $\qquad |\vec{v}_3| = 10$

$$\hat{e}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix} \quad \hat{e}_2 = \frac{\vec{v}_2}{|\vec{v}_2|} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \quad \hat{e}_3 = \frac{\vec{v}_3}{|\vec{v}_3|} = \begin{pmatrix} 1/10 \\ 7/10 \\ 7/10 \\ 1/10 \end{pmatrix}$$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 7 \\ 7 \\ 1 \end{pmatrix}$$

$$-\ \vec{u}_4 = f - \text{proj}_{u_1} f - \text{proj}_{u_2} f - \text{proj}_{u_3} f$$

$$= f - \frac{\langle f, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle f, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle f, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3$$

$$= f - \frac{u_1}{|u_1|^2} - \frac{u_2}{|u_2|^2} - \frac{u_3}{|u_3|^2}$$

$$= f - \frac{u_1}{4} - \frac{u_2}{4} - \frac{u_3}{100} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/4 \\ 1/4 \\ -1/4 \\ -1/4 \end{pmatrix} - \begin{pmatrix} 1/4 \\ -1/4 \\ 1/4 \\ -1/4 \end{pmatrix} - \begin{pmatrix} 1/100 \\ 7/100 \\ 7/100 \\ 1/100 \end{pmatrix}$$

$$\vec{u}_4 = \begin{pmatrix} 49/100 \\ -7/100 \\ -7/100 \\ 1/100 \end{pmatrix}$$

$$|\vec{u}_4| = 1/2$$

$$\tilde{e}_4 = \begin{pmatrix} 49/50 \\ -7/50 \\ -7/50 \\ 1/50 \end{pmatrix}$$