Physics 89

Problem Set #2

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Various functions

Express the following in $(x+iy)$ form

(a) $\cosh\left(\frac{\pi i}{4}\right)$

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} \Longrightarrow \cosh\left(\frac{\pi i}{4}\right) = \frac{e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}}}{2} = \frac{e^{i\theta} + e^{-i\theta}}{2} \Bigg|_{i\theta = i\frac{\pi}{4}}$$

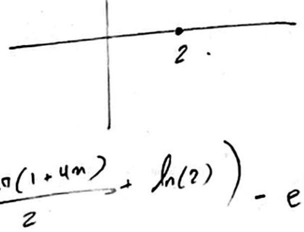
$$e^{i\theta} = (\cos\theta + i\sin\theta) \Longrightarrow e^{i\pi/4} = \cos\pi/4 + i\sin\pi/4 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

$$e^{-i\pi/4} = \cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = \frac{1-i}{\sqrt{2}}$$

$$\cosh\left(\frac{i\pi}{4}\right) = \frac{\frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}}{2} = \frac{\cancel{2}}{\cancel{2}\sqrt{2}} = \frac{1}{\sqrt{2}} \Longrightarrow \boxed{\cosh\left(\frac{i\pi}{4}\right) = \frac{1}{\sqrt{2}} + 0i}$$

b) $\sinh\left(\frac{i\pi}{2} + \ln(2)\right)$

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2} \Longrightarrow \frac{e^{\frac{i\pi}{2} + \ln(2)} - e^{\frac{i\pi}{2} + \ln(2)}}{2}$$



$$\ln(2) = \ln(2) + 2n\pi i \Longrightarrow \frac{i\pi}{2} + \ln(2) = \frac{i\pi(1+4n)}{2} + \ln(2)$$

$$\frac{e^{\left(\frac{i\pi(1+4n)}{2} + \ln(2)\right)} - e^{\left(\frac{i\pi(1+4n)}{2} + \ln(2)\right)}}{2} = \frac{e^{\frac{i\pi(1+4n)}{2}}\cancel{e^{\ln(2)}} - e^{\frac{i\pi(1+4n)}{2}}\cancel{e^{\ln(2)}}}{\cancel{2}}$$

$$= e^{\frac{i\pi(1+4n)}{2}} - e^{\frac{i\pi(1+4n)}{2}} = \boxed{0}$$

c) $\cos(\pi + i\ln(2))$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \Longrightarrow \frac{e^{i(\pi + i\ln(2))} + e^{-i(\pi + i\ln(2))}}{2}$$

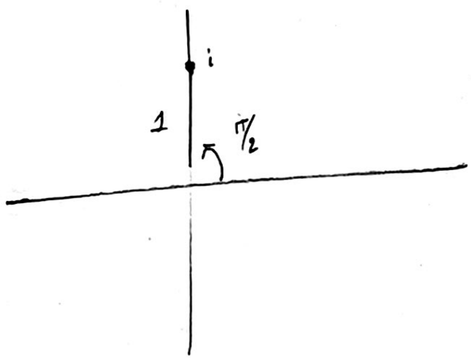
$$= \frac{e^{i\pi - \ln(2)} + e^{-i\pi + \ln(2)}}{2} = \frac{\frac{e^{i\pi}}{e^{\ln(2)}} + e^{-i\pi}e^{\ln(2)}}{2}$$

$e^{i\pi} = -1$

$e^{-i\pi} = \cos(\pi) - i\sin(\pi) = -1$

$$\frac{\frac{-1}{2} - 2}{2} = -5/4 \quad \boxed{\cos(\pi + i\ln(2)) = \frac{-5}{4} + 0i}$$

d) i^i



$\ln(i) = \ln(1) + i(\pi/2 + 2n\pi)$

$$i^i = e^{i\ln(i)} = e^{i(i(\pi/2 + 2n\pi))} \quad \Big|\; {}^{*}\, a^b = e^{b\ln(a)}$$

$$= e^{i^2(\pi/2 + 2n\pi)} = e^{-\pi/2 - 2n\pi}$$

$$\boxed{= e^{-\pi/2},\ e^{-5\pi/2},\ e^{-9\pi/2},\ \dots}$$

e) $\log(i)$, $i = e^{i\pi/2}$

$\log_{10}(e^{i\pi/2})$

f) $$i^{2/3} = e^{2/3\ln(i)} = e^{2/3(i(\pi/2 + 2n\pi))} = e^{2/3\, i(\pi/2 + 2n\pi)}$$

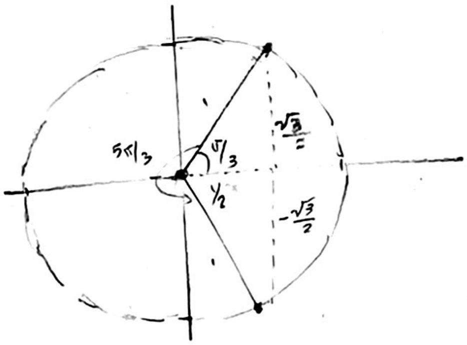
$$= \cos\left(\frac{2}{3}\left(\frac{\pi}{2} + 2n\pi\right)\right) + i\sin\left(\frac{2}{3}\left(\frac{\pi}{2} + 2n\pi\right)\right)$$

$$= \cos\left(\frac{\pi}{3} + \frac{4}{3}n\pi\right) + i\sin\left(\frac{\pi}{3} + \frac{4}{3}n\pi\right)$$

$$= \cos(60^\circ + 240^\circ n) + i\sin(60^\circ + 240^\circ n)$$

$\cos(60° + 240° n) + i\sin(60° + 240° n) = e^{2/3\, i(\pi/2 + 2n\pi)} = re^{i\theta}$

$r = 1$

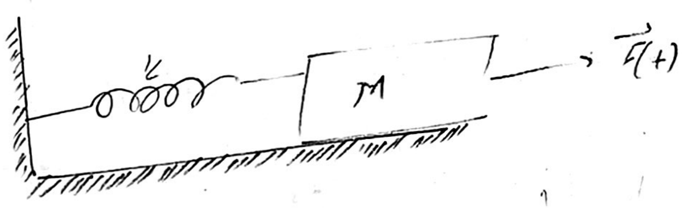


$\cos(\pi/3) = x = \frac{1}{2}$

$\sin(\pi/3) = y = \frac{\sqrt{3}}{2}$

$\boxed{i^{2/3} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i}$

Driven Damped Harmonic Oscillator



$|\vec{F}(t)| = f\sin(\omega t)$

$f(v) = -\gamma v$

$M\ddot{x} = f\sin(\omega t) - kx - \gamma\dot{x}$

a) $F(t) = f e^{i\omega t}$

b) $x(t) = ze^{i\omega t}$ $\quad \dot{x} = i\omega z e^{i\omega t}$ $\quad \ddot{x} = i^2\omega^2 z e^{i\omega t} = -\omega^2 z e^{i\omega t}$

$m(-\omega^2 z e^{i\omega t}) = f\sin(\omega t) - kze^{i\omega t} - \gamma(i\omega z e^{i\omega t})$

$-\omega^2 m z e^{i\omega t} = f\sin(\omega t) - kze^{i\omega t} - \gamma i\omega z e^{i\omega t}$

$f\sin(\omega t) = kze^{i\omega t} + \gamma i\omega z e^{i\omega t} - \omega^2 m z e^{i\omega t} = z(ke^{i\omega t} + \gamma i\omega e^{i\omega t} - \omega^2 m e^{i\omega t})$

c) $z = \frac{f\sin(\omega t)}{e^{i\omega t}(k + \gamma i\omega - \omega^2 m)} \implies x(t) = \frac{f\sin(\omega t)}{\cancel{e^{i\omega t}}(k + \gamma i\omega - \omega^2 m)} \cancel{e^{i\omega t}} = \frac{f\sin(\omega t)}{k + \gamma i\omega - \omega^2 m}$

$\frac{f\sin(\omega t)}{k + \gamma i\omega - \omega^2 m} \frac{(k - \gamma i\omega - \omega^2 m)}{(k - \gamma i\omega - \omega^2 m)} = \frac{f\sin(\omega t)(k - \gamma i\omega - \omega^2 m)}{k^2 - \cancel{k\gamma i\omega} - \omega^2 mk + \cancel{k\gamma i\omega} + \gamma^2\omega^2 - \cancel{\gamma i\omega^3 m} - \omega^2 mk + \cancel{\omega^3\gamma i} + \omega^4 m^2} = \frac{f\sin(\omega t)(k - \gamma i\omega - \omega^2 m)}{k^2 - 2\omega^2 mk + \gamma^2\omega^2 + \omega^4 m^2}$

$$x(t) = \frac{f\sin(\omega t)(k - \gamma i\omega - \omega^2 m)}{k^2 - 2\omega^2 mk + \gamma^2\omega^2 + \omega^4 m^2} = \frac{f\sin(\omega t)(k - \gamma i\omega - \omega^2 m)}{k(k - 2\omega^2 m) + \omega^2(\gamma^2 + \omega^2 m^2)}$$

$$= \frac{f\sin(\omega t)k - f\sin(\omega t)\omega^2 m - f\sin(\omega t)\gamma i\omega}{k(k-2\omega^2 m) + \omega^2(\gamma^2 + \omega^2 m^2)} \Longrightarrow Im(x(t)) = \frac{-f\sin(\omega t)\gamma\omega}{k(k-2\omega^2 m) + \omega^2(\gamma^2 + \omega^2 m^2)}$$

$$\boxed{Re(x(t)) = \frac{f\sin(\omega t)(k - \omega^2 m)}{k(k-2\omega^2 m) + \omega^2(\gamma^2 + \omega^2 m^2)}}$$

Integrals Using Complex #s

Evaluate $\int e^{(a+ib)x}\,dx$

$$\int e^{(a+ib)x}dx = \int e^{ax}(\cos(bx) + i\sin(bx))dx$$

$$= \int e^{ax}\cos(bx)\,dx + i\int e^{ax}\sin(bx)\,dx$$

$$\frac{1}{a+bi}e^{(a+bi)x} = \int e^{x}\cos(bx)\,dx + i\int e^{ax}\sin(bx)\,dx$$

$$\frac{a-bi}{a-bi}\cdot\frac{1}{a+bi}e^{(a+bi)x} = \frac{(a-bi)e^{(a+bi)x}}{a^2+b^2} = \frac{(a-bi)}{a^2+b^2}e^{ax}(\cos(bx) + i\sin(bx))$$

$$\underbrace{\frac{e^{ax}}{a^2+b^2}(a\cos(bx) + b\sin(bx))}_{\int e^{ax}\cos(bx)dx} + \underbrace{\frac{e^{ax}}{a^2+b^2}i(a\sin(bx) - b\cos(bx))}_{+\, i\int e^{ax}\sin(bx)\,dx}$$

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Taylor Series for Analytic functions

$f(x) = \dfrac{1}{x^2 + 6x + 5}$ $\quad$ $g(x) = \dfrac{1}{x^2 + 4x + 5}$

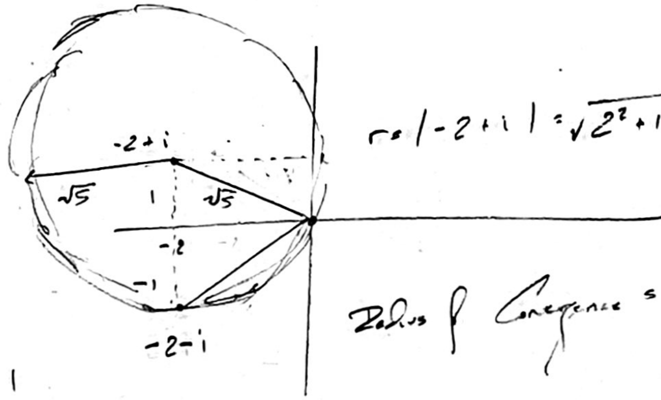
$f(x) = \dfrac{1}{x^2 + 6x + 5} = \dfrac{1}{5} - \dfrac{6}{25}x + \dfrac{31}{125}x^3 + \ldots$

$g(x) = \dfrac{1}{x^2 + 4x + 5} = \dfrac{1}{5} - \dfrac{4}{25}x + \dfrac{11}{125}x^3 + \ldots$

$f(z) = \dfrac{1}{z^2 + 6z + 5} = \dfrac{1}{(z+5)(z+1)}$ $\quad$ $g(z) = \dfrac{1}{z^2 + 4z + 5}$ $\quad$ $\dfrac{-4 \pm \sqrt{16 - 20}}{2} = \dfrac{-4 \pm 2i}{2}$

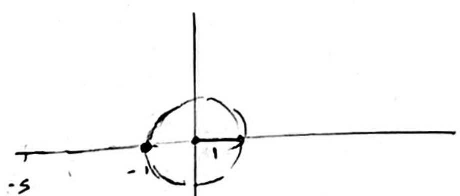
$(-5, -1)$

$= (-2 \pm i)$



$r = |-2+i| = \sqrt{2^2 + 1^2} = \sqrt{5}$

Radius of Convergence $= \sqrt{5}$



Radius of Convergence $= 1$

Interval

$(-1, 1)$

Radius of Convergence : Radius of disk to nearest singularity