

Physics 89

Problem Set #5

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1 Pauli Spin Matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) $AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $AC = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$BA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad CA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$BC = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$CB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

(b) $AB = iC \quad AC = -iB \quad BC = iA$

$BA = -iC \quad CA = iB \quad CB = -iA$

(c) $A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $B^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$C^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(6)

$$\sigma_1 = A \quad \sigma_2 = B \quad \sigma_3 = C \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_m \sigma_n = \sum_{k=1}^3 d_{mnk} \sigma_k + \beta_{mn} I \quad m=1,2,3 \quad n=1,2,3$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} +1 & \{ijk\} = 123, 312, 231 \\ -1 & \{ijk\} = 213, 321, 132 \\ 0 & \text{otherwise} \end{cases}$$

Example

$$AB = \sigma_1 \sigma_2 = \sum_{k=1}^3 d_{12k} \sigma_k + \beta_{12} I$$

$$= (d_{12} \sigma_1 + \beta_{12} I) + (d_{12} \sigma_2 + \beta_{12} I) + (d_{12} \sigma_3 + \beta_{12} I)$$

$$= \begin{pmatrix} 0 & d_{12} \\ d_{12} & 0 \end{pmatrix} + \begin{pmatrix} \beta_{12} & 0 \\ 0 & \beta_{12} \end{pmatrix} + \begin{pmatrix} 0 & -d_{12} i \\ d_{12} i & 0 \end{pmatrix} + \begin{pmatrix} \beta_{12} & 0 \\ 0 & \beta_{12} \end{pmatrix}$$

$$+ \begin{pmatrix} d_{12} & 0 \\ 0 & -d_{12} \end{pmatrix} + \begin{pmatrix} \beta_{12} & 0 \\ 0 & \beta_{12} \end{pmatrix}$$

$$= \begin{pmatrix} \beta_{12} & d_{12} \\ d_{12} & \beta_{12} \end{pmatrix} + \begin{pmatrix} \beta_{12} & -d_{12} i \\ d_{12} i & \beta_{12} \end{pmatrix} + \begin{pmatrix} d_{12} + \beta_{12} & 0 \\ 0 & \beta_{12} - d_{12} \end{pmatrix}$$

$$= \begin{pmatrix} d_{12} + 3\beta_{12} & d_{12} - d_{12} i \\ d_{12} + d_{12} i & 3\beta_{12} - d_{12} \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$$

$$\implies \epsilon_{mnk} = 0 \text{ if } m=n, \quad \delta_{mn} = 1 \text{ if } m=n$$

$$\begin{array}{ccc} C & B & A \\ BA, AC, CB & \longrightarrow & -i \quad 213, 132, 321 \quad -i\sigma_k \\ AB, CA, BC & \longrightarrow & +i \quad 123, 312, 231 \quad i\sigma_k \\ C & B & A \end{array}$$

$$\mathcal{L}_{mnk} = i\epsilon_{mnk}$$

$$\beta_{mn} = \delta_{mn}$$

$$\sigma_m \sigma_n = \sum_{k=1}^3 i\epsilon_{mnk} \sigma_k + \delta_{mn} I$$

$$(c) \text{tr} M = \sum_{i=1}^n M_{ii}$$

$$AB = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$ABC = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

$$\text{tr} M = i + i = \underline{\underline{2i}}$$

2 Upper Triangular Matrices

$$M = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

$$(a) M^T = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}$$

$$(b) \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = f(ad - 0) = \underline{f a d} \quad \left(\prod_{i=1}^n M_{ii} \right)$$

$$(c) MM^T = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + c^2 & bd + ce & cf \\ db + ec & d^2 + e^2 & ef \\ f & f & f^2 \end{pmatrix}$$

$$M^T M = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a^2 & ab & ac \\ ba & b^2 + d^2 & bc + de \\ ca & cb + ed & c^2 + e^2 + f^2 \end{pmatrix}$$

$$MM^T \neq M^T M$$

$$d) \left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ 0 & d & e & 0 & 1 & 0 \\ 0 & 0 & f & 0 & 0 & 1 \end{array} \right) \xrightarrow{\text{REF (Ti-Bu)}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & -\frac{b}{a^2} & \frac{eb - cd}{adf} \\ 0 & 1 & 0 & 0 & \frac{1}{d} & -\frac{e}{df} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{f} \end{array} \right)$$

$$M^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a^2} & \frac{eb - cd}{adf} \\ 0 & \frac{1}{d} & -\frac{e}{df} \\ 0 & 0 & \frac{1}{f} \end{pmatrix}$$

3. The Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$x(r, \theta) = r \cos \theta$$

$$y(r, \theta) = r \sin \theta$$

$$(a) \quad J = \begin{vmatrix} \frac{\partial(r \cos \theta)}{\partial r} & \frac{\partial(r \cos \theta)}{\partial \theta} \\ \frac{\partial(r \sin \theta)}{\partial r} & \frac{\partial(r \sin \theta)}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r \cos^2 \theta - (-r \sin^2 \theta) = r \cos^2 \theta + r \sin^2 \theta = \underline{r}$$

$$(b) \quad x(u, v) = a \cosh(u) \cos(v)$$

$$y(u, v) = a \sinh(u) \sin(v)$$

$$\frac{\partial(a \cosh(u) \cos(v))}{\partial u} = a \cos(v) \sinh(u) \quad \frac{\partial(a \sinh(u) \sin(v))}{\partial u} = a \sin(v) \cosh(u)$$

$$\frac{\partial(a \cosh(u) \cos(v))}{\partial v} = -a \sin(v) \cosh(u) \quad \frac{\partial(a \sinh(u) \sin(v))}{\partial v} = a \cos(v) \sinh(u)$$

$$J = \begin{vmatrix} a \cos(v) \sinh(u) & -a \sin(v) \cosh(u) \\ a \sin(v) \cosh(u) & a \cos(v) \sinh(u) \end{vmatrix}$$

$$= a^2 \cos^2(v) + a^2 \cos^2(v) \cosh^2(u) + a^2 \cosh^2(u) - a^2 \cosh^2(u) \cos^2(v)$$

$$= \underline{a^2 (\cos^2(v) + \cosh^2(u))}$$

$$= a^2 \cos^2(v) \sinh^2(u) + a^2 \sin^2(v) \cosh^2(u)$$

$$= a^2 \cos^2(v) (1 + \cosh^2(u)) + a^2 (1 - \cos^2(v)) \cosh^2(u)$$

(c) $x = r \sin \theta \cos \varphi$ $y = r \sin \theta \sin \varphi$ $z = r \cos \theta$

$$\frac{\partial x}{\partial r} = \sin \theta \cos \varphi \quad \frac{\partial x}{\partial \theta} = r \cos \theta \cos \varphi \quad \frac{\partial x}{\partial \varphi} = -r \sin \theta \sin \varphi$$

$$\frac{\partial y}{\partial r} = \sin \theta \sin \varphi \quad \frac{\partial y}{\partial \theta} = r \cos \theta \sin \varphi \quad \frac{\partial y}{\partial \varphi} = r \sin \theta \cos \varphi$$

$$\frac{\partial z}{\partial r} = \cos \theta \quad \frac{\partial z}{\partial \theta} = -r \sin \theta \quad \frac{\partial z}{\partial \varphi} = 0$$

$$J = \begin{vmatrix} \sin \theta \cos \varphi & r \cos \theta \cos \varphi & -r \sin \theta \sin \varphi \\ \sin \theta \sin \varphi & r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= -r \sin \theta \sin \varphi \left(-r \sin^2 \theta \sin \varphi + r \cos^2 \theta \sin \varphi \right) - r \sin \theta \cos \varphi \left(-r \sin^2 \theta \cos \varphi + r \cos^2 \theta \cos \varphi \right)$$

$$= -r \sin \theta \sin \varphi \left(-r \sin \varphi \right) - r \sin \theta \cos \varphi \left(-r \cos \varphi \right)$$

$$= r^2 \sin \theta \sin^2 \varphi + r^2 \sin \theta \cos^2 \varphi = \underline{\underline{r^2 \sin \theta}}$$

4. The Poffian 3 Relativistic Electrodynamics

$$M = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

(a)

$$\det M = \begin{vmatrix} -\frac{E_x}{c} & B_z & -B_y \\ -\frac{E_y}{c} & 0 & B_x \\ -\frac{E_z}{c} & -B_x & 0 \end{vmatrix} + \frac{E_y}{c} \begin{vmatrix} -\frac{E_x}{c} & 0 & -B_y \\ -\frac{E_z}{c} & -B_x & 0 \end{vmatrix} - \frac{E_z}{c} \begin{vmatrix} -\frac{E_x}{c} & B_z & 0 \\ -\frac{E_y}{c} & -B_z & B_x \end{vmatrix}$$

$$\frac{-E_x}{c} \left[\frac{-E_y}{c} (-B_y B_x) - B_x \left(\frac{E_x B_x}{c} + \frac{E_z B_z}{c} \right) \right]$$

$$+ \frac{E_y}{c} \left[\frac{-E_x}{c} (-B_x B_y) - B_y \left(\frac{-E_y B_y}{c} - \frac{E_z B_z}{c} \right) \right]$$

$$- \frac{E_z}{c} \left[\frac{-E_x}{c} (B_x B_z) + B_z \left(\frac{-E_y B_y}{c} - \frac{E_z B_z}{c} \right) \right]$$

$$- \frac{E_x}{c} \left[\frac{-E_y B_y B_x}{c} - B_x \left(\frac{E_x B_x + E_z B_z}{c} \right) \right] = - \frac{E_x}{c} \left[\frac{-E_y B_y B_x}{c} - \frac{E_x^2 B_x^2 - E_z B_x B_z}{c} \right]$$

$$+ \frac{E_y}{c} \left[\frac{E_x B_x B_y}{c} - B_y \left(\frac{-E_y B_y - E_z B_z}{c} \right) \right] = \frac{E_y}{c} \left[\frac{E_x B_x B_y}{c} + \frac{E_y^2 B_y^2 + E_z B_y B_z}{c} \right]$$

$$- \frac{E_z}{c} \left[\frac{-E_x B_x B_z}{c} + B_z \left(\frac{-E_y B_y - E_z B_z}{c} \right) \right] = - \frac{E_z}{c} \left[\frac{-E_x B_x B_z}{c} - \frac{E_y B_z B_y - E_z^2 B_z^2}{c} \right]$$

$$= \frac{E_x E_y B_y B_x}{c^2} + \frac{E_x^2 B_x^2 + E_x E_z B_x B_z}{c^2} + \frac{E_x E_y B_x B_y}{c^2} + \frac{E_y^2 B_y^2 + E_y E_z B_y B_z}{c^2}$$

$$+ \frac{E_z E_x B_x B_z}{c^2} + \frac{E_z E_y B_y B_z}{c^2} + \frac{E_z^2 B_z^2}{c^2}$$

$$= (2E_x E_y B_y B_x + 2E_z E_x B_x B_z + (E_x^2 B_x^2 + E_y^2 B_y^2 + E_z^2 B_z^2) + 2E_y E_z B_y B_z)$$

$$= \frac{1}{c^2} (2E_x E_y B_y B_x + 2E_z E_x B_x B_z + (E_x^2 B_x^2 + E_y^2 B_y^2 + E_z^2 B_z^2) + 2E_y E_z B_y B_z)$$

$$(b) (\vec{E} \cdot \vec{B})^2 = (\epsilon_x B_x + \epsilon_y B_y + \epsilon_z B_z)^2$$

$$= \epsilon_x^2 B_x^2 + \epsilon_y^2 B_y^2 + \epsilon_z^2 B_z^2 + 2\epsilon_x B_x \epsilon_y B_y + 2\epsilon_x B_x \epsilon_z B_z + 2\epsilon_y B_y \epsilon_z B_z$$

$$\text{if set } \mu = \alpha (\vec{E} \cdot \vec{B})^2$$

$$\boxed{\alpha = \frac{1}{c^2}}$$