Physics 89 - Introduction to Mathematical Physics

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Contents

1 Difference between Mathematics and Physics

Example 1 - Electrostatics

Math Question

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = ?$$

Math Solution

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\log(1-x), \quad for -1 \leq x \leq 1$$

So,

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\log(2)$$

Example 2 - Diffusion

$$f(x, y, z, t) = \text{density of diffusing material at time } t$$

Let there exist a cube containing moles

$$\frac{\partial f}{\partial t} = D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$$

where D is the *diffusion coefficient*, and the diffusion equation describes how f evolves with time

Math Question

Solve

$$\frac{\partial f}{\partial t} = D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$$

given initial condition

$$f(x, y, z, 0) = \text{concentrated lump at the origin}$$

Math Solution

$$f(x, y, z, t) = \frac{N}{(4\pi Dt)^(3/2)} e^{-\frac{x^2+y^2+z^2}{4Dt}}$$

where N is the number of moles released

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2 Taylor Series

- Techniques for obtaining series
- Estimate error, converge?

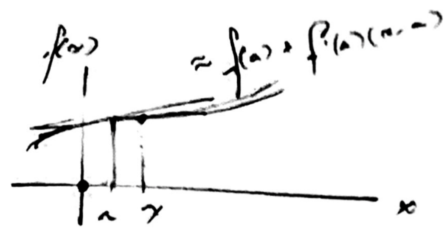

Figure 1: Taylor Series Visualization

$$f(x) \approx f(0) + f'(0)x + \cdots + \frac{1}{n!}f^n(0)x^n$$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + ... = \sum_{k=0}^{\infty} \frac{1}{k!}f^k(a)(x-a)^k$$

Question

How good is this approximation?

Big O notation

$$\sum_{k=9}^{n} \frac{1}{k!}f^k(0)x^k + O(x^{n+1})$$

Formally,

$$F(x) = o(x^{n+1}) \quad \text{as } x \to 0$$

$$|F| \leq C|x|^{n+1} \quad \text{for some unexpected constant c}$$

$$\lim_{x \to 0} \frac{F}{|x|^{n+1}} = 0$$

Example

$$e \approx 1.9GeV \approx 3700mc^2$$

Special Relativity

$$E_k = m_0c^2 - mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$$
$$\approx 0 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \frac{5}{16}m\frac{v^8}{c^4}$$
$$f(v) = \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \dots$$

$$\frac{1}{\sqrt{1-x}} \to \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$(1+x)^P, \quad \text{then set } p = \frac{1}{2}$$

$$\begin{aligned} f(x) &= (1+x)^n \\ f'(x) &= p(1+x)^{p-1} \\ f^k(x) &= p(p-1)\dots(p-k+1)(1+x)^{p-k} \to f^k(0) \\ &= p\dots(p-k+1) \end{aligned}$$

$$(1+x)^n \approx 1 + px + \frac{p(p-1)}{2!}x^2 + ... + \frac{p!}{k!(p-k)!}x^k = \binom{p}{k}x^k$$

$$\sum_{k=0}^{n}\binom{p}{k}x^k \quad \text{generalized binomial coefficient}$$

$$(1+x)^P = \sum_{k=0}^{n}\binom{p}{k}x^k + O(x^{n+1}$$

Question

Given $\frac{1}{\sqrt{1+x}}$ taylor series, how good is this approximation if $x = 0.1$?

Solution

$$\text{Actual Answer} \rightarrow \frac{1}{\sqrt{1.1}} = 0.9534626$$

Taylor Polynomials $x, x^2 \rightarrow 1 - \dfrac{0.1}{2} = 0.95 \quad / \quad 1 - \dfrac{0.5}{2} + \dfrac{3(0.5)^2}{8} = 0.95375$ good approx

More Taylor Series

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\end{aligned}$$

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\sinh x &= \frac{e^x - e^x}{2} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots
\end{aligned}$$

2.1 Testing for Convergence

If $\sum_0^\infty a_n x^n \leq \infty$ converges,

$$\sum_0^\infty a_n (\lambda X)^n \leq \infty \qquad |\lambda| \leq 1$$

Taylor Series have interval of convergence of the form

$$[-L, L] \quad (-L, L) \quad [-L, L) \quad (-L, L]$$

Truncated Taylor Series Approximation

$$R_0(x) = f(x) - f(0) = f'(c)x$$

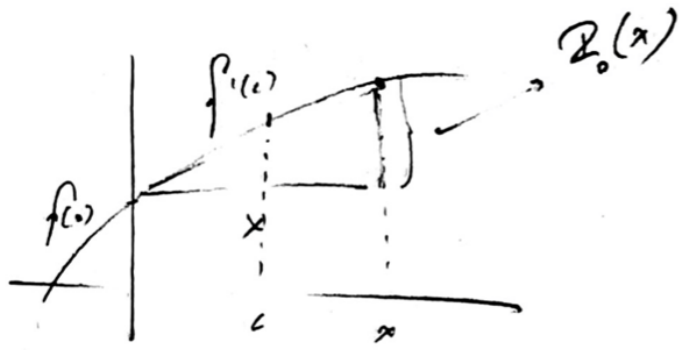

Figure 2: Remainder Visualized

Remainder Theorem

$$R_n(x) = f^{n+1}(c)\frac{x^{n+1}}{(n+1)!} \quad \text{for some } 0 \le c \le x$$

$$\begin{aligned} x &= \frac{\pi}{2} \\ R &= \sin\frac{\pi}{2} - (x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880..} + 0) \\ &= f^{10}(c)\frac{x^10}{10!} \quad 0 \le c \le \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} |f^{11}(c)| &= |-\cos c| < 1 \\ |R_{10}| &\le \frac{1}{11!}\left(\frac{\pi}{2}\right)^{11} \approx 3.6 \times 10^{-6} \end{aligned}$$

Technique for Solving Taylor Series by dividing two polynomials

$$
\begin{aligned}
f(x) &= a_0 + a_1 x + \dots \\
g(x) &= b_0 + b_1 x + \dots \\
\frac{f(x)}{g(x)} &= (c_0 + c_1 x + c_2 x^2 + \dots) \\
a_0 + a_1 x + \dots &= (b_0 + b_1 x + \dots)(c_0 + c_1 x + \dots)
\end{aligned}
$$

$$a_0 = b_0 c_0$$

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3 Complex Numbers

- Definition
- Functions: $\log z, \sqrt{z}, \sin z,$, etc.
- Applications: AC Circuits, Hydrodynamics
- Math Applications: $\int_{\infty}^{\infty}$

3.1 Taylor Series

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + - \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

The interval of convergence for the taylor series of $\frac{1}{1+x^2}$ is from $(-1, 1)$, which is not readily apparent since

$$@x \pm 1, f(x) = \frac{1}{2}$$

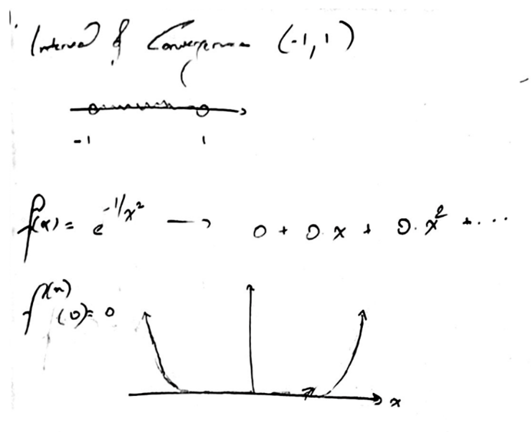

Figure 3: taylor series of e^{1/x^2}

3.2 Complex Numbers

Introduced by *Cardano* in the 1500s with the intent of solving cubic equations.

Quadratic Equations

$$0 = x^2 + bx + c \quad x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Cubic Equations

$$0 = x^3 + ax + b \qquad \left(\frac{-b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}\right)^{\frac{1}{3}}$$

$$x^3 - x = 0 \to x = \frac{1}{\sqrt{3}}\left[\sqrt{-1}^{1/3} + (-\sqrt{-1})^{1/3}\right]$$

- consistency
- final answer is **real**
- simplifies computations

3.2.0.1 Rules of Complex Numbers

$$z = a + bi$$

$$i^2 = -1$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Example

$$\begin{aligned}&(1+i)^2 = 2i\\&i^4 = 1\end{aligned}$$

$$\begin{aligned}\frac{1}{a+bi} &= \frac{(a+bi)}{(a-bi)(a+bi)} = \frac{(a-bi)}{a^2+b^2}\\&= \left(\frac{a}{a^2+b^2}\right) - \left(\frac{b}{a^2+b^3}\right)i\end{aligned}$$

3.3 Applications

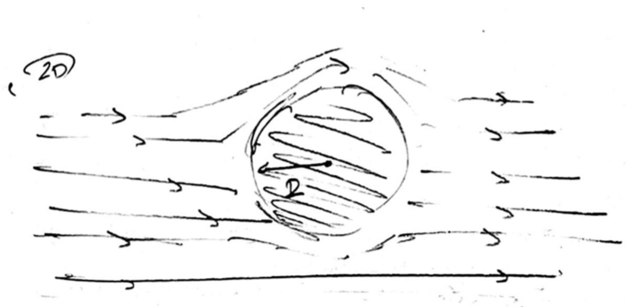

Figure 4: 2D diagram of Sphere from above

3.3.0.1 Hydrodynamics

$$\vec{v}(x,y) = v_x\hat{i} + v_y\hat{j}$$

Problem

$$V_x, V_y = ?$$

Model

1. Incompressible

$$(a). \quad 0 = \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

2. Irrotational

$$(b.) \quad 0 = (\nabla \times \vec{v})_z = \frac{\partial v_x}{\partial x} - \frac{\partial v_x}{\partial y}$$

Solving (a) and (b)

Set of **coupled** partial differential equations (PDEs)

- What are the Boundary Conditions?
  - an additional set of equations at the edges

$$
\begin{aligned}
&(1.) \quad r = \sqrt{x^2 + y^2} \to \infty \quad \vec{v} \to v_0 \hat{i} \\
&(2.) \quad \vec{v} \cdot \hat{r} = 0
\end{aligned}
$$

Fact: Complex Numbers
Define $z = x + iy$, z is **not** the third coordinate
Define $U = v_x \hat{i} - i v_y$ and $U = f(z) \to$ Equations (a.) and (b.) are automatically satisfied.

Solution

$$U = v_0 \left(1 - \frac{R^2}{z^2}\right)$$

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

$$\frac{1}{z^2} = \frac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2}$$

$$v_x = v_0 - \frac{v_0 R^2 (x^2 - y^2)}{(x^2 + y^2)^2}$$

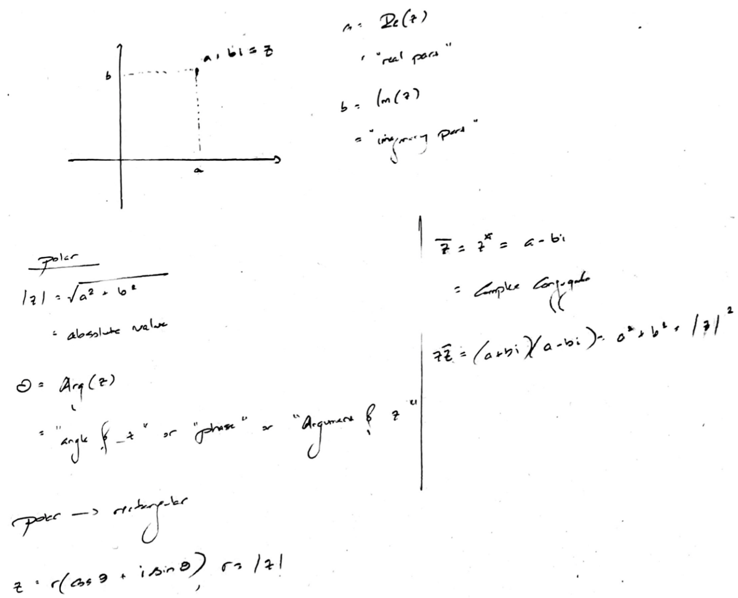

Figure 5: complex plane

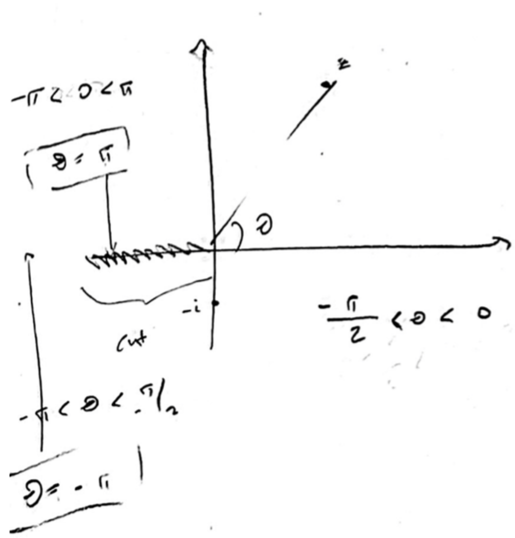

Figure 6: quadrant's of complex plane in polar coordinates

3.3.0.2 The Complex Plane

3.3.0.3 Euler's Identity

$$\cos\theta + i\sin\theta = e^{i\theta}$$

$$\begin{aligned} e^x &= 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3!} + ... \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{iy} &= 1 + \frac{iy}{1} - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \dots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \dots\right) + \left(\frac{y}{1} - \frac{y^3}{3!} + \dots\right) i = \cos y + i \sin y \end{aligned}$$

Euler's Identities

$$\begin{aligned} e^{i\pi} &= -1 \\ 1 = e^{2\pi i} &= e^{2\pi n i} \quad n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

$$\log z = ?$$

$$\begin{aligned} z &= re^{i\theta} \\ \log z &= \log r + i(\theta + 2\pi n) \end{aligned}$$

$$\sqrt{z}$$

$$\begin{aligned} \sqrt{re^{iz}} &= \sqrt{r}e^{i\theta/2} \\ &= \sqrt{r}e^{\frac{i(\theta+2\pi)}{2}} \\ &= -\sqrt{r}e^{i\theta/2} \end{aligned}$$

3.3.0.4 Trigonometric Functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh y$$

$$sin(iy) = i\frac{e^{y} - e^{-y}}{2} = i \sinh y$$

3.4 Hyperbolic Functions

$$\tanh = \frac{\sinh y}{\cosh y}$$

Everything is **Real** from now on.

3.4.0.1 Identities

$$\sinh(\alpha + \beta) = \sinh \alpha \cosh \beta + \cosh \alpha + \sinh \beta$$
$$\cosh(\alpha + \beta) = \cosh \alpha \cosh \beta + \sinh \alpha + \sinh \beta$$
$$\tanh(\alpha + \beta) = \frac{\tanh \alpha + \tanh \beta}{1 + \tanh \alpha \tanh \beta}$$

3.4.0.2 Applications to Special Relativity Relativistic Addition to Velocities

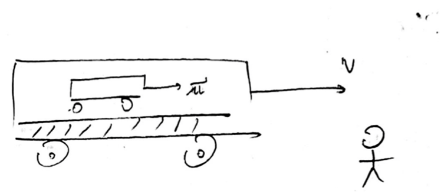

Figure 7: a train moving with a car moving inside of it, what would an observer calculate for the speed of the interior car?

$$iW = \frac{u+v}{1+\frac{uv}{c^2}} = c\frac{\tanh \alpha - \tanh \beta}{1 + \tanh \alpha + \tanh \beta} = c \tanh(\alpha + \beta)$$

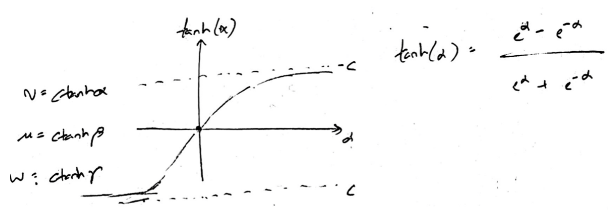

Figure 8: rapidity - using hyperbolic tangent establishes the bounds of velocity as c and $-c$

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Functions of Complex Variables

- Cauchy - Riemann Eqns
- Taylor Series
- $\int_{-\infty}^{\infty}$
- Singularities, Poles, Residue

The Complex Conjugate

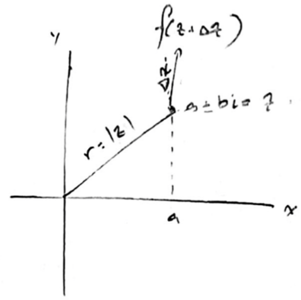

Figure 9: graphing complex numbers

$$\bar{z} = a - bi \quad z\bar{z} = a^2 + b^2 = |z|^2$$

Functions of $z = x + iy$

$$\begin{aligned} ReZ &= x \\ ImZ &= y \\ |z| &= \sqrt{x^2 + y^2} \\ \bar{=} z &= x - iy \end{aligned}$$

Analytic Functions

$$\frac{1}{z}$$
$$z^2$$
$$e^z$$
$$\sin z$$
$$\cos z$$

$$f(z) = u + iv$$

$$u = u(x, y)$$
$$v = v(x, y)$$

Analytic Functions are the functions where you can write

$$f(z + \Delta z) - f(z) = f'(z)\Delta z + O(\Delta z^2) \quad \Delta z \to 0$$

Note

$$f(x + \Delta x, y + \Delta y) - f(x, y) = \left(\frac{\partial f}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial y}\right)\Delta y + \dots$$

Check

$$e^{z+\Delta z} = e^z e^{\Delta z} = e^z(1 + \Delta z + \dots)$$
$$e^{z+\Delta z} - e^z = e^z \delta z + \dots = f''(z)\delta z + \dots$$

Therefore, $e^{z+\Delta z}$ is *analytic*. However,

$$\bar{z} + \Delta\bar{z} - \bar{z} = \bar{\Delta} z$$

$$\Delta z = \Delta x \to \bar{\Delta} z = (1)\Delta z \quad \text{Horizontal}$$
$$\Delta z = i\Delta y \to \bar{\Delta} z = -i\Delta y = (-1)\Delta z \quad \text{Vertical}$$

$$f(z) = u + iv$$

$$\frac{f(x+\Delta x, y) - f(x)}{\Delta x} = f'(z)$$

$$\frac{f(x, y+\Delta y) - f(x,y)}{i\Delta y} = f'(z) = \frac{\partial f}{\partial y}\frac{1}{i}$$

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$$
$$\left(\frac{\partial u}{\partial x}\right) + i\left(\frac{\partial v}{\partial x}\right) = \frac{1}{i}\left[\left(\frac{\partial u}{\partial y}\right) + i\left(\frac{\partial v}{\partial y}\right)\right]$$
$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Cauchy Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Example

$$f(z) = x^2 - y^2 + 2ixy$$

$$u = x^2 - y^2$$
$$v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

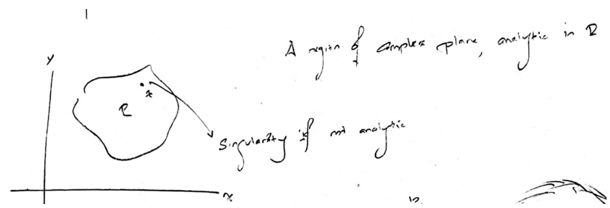

Figure 10: a region in the complex plane

3.4.0.3 Taylor Series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{converges if analytic}$$

Given a constant λ

$$0 < \lambda < 1$$

$$f(\lambda z) = \sum_{n=0}^{\infty} a_n (\lambda z)^n \qquad \text{definitely converges}$$

$$|a_n z^n \lambda^n| = |a_n z^n||\lambda|^n$$

Let $\lambda = re^{i\theta}$, $0 < r < 1$, $|\lambda| < 1$

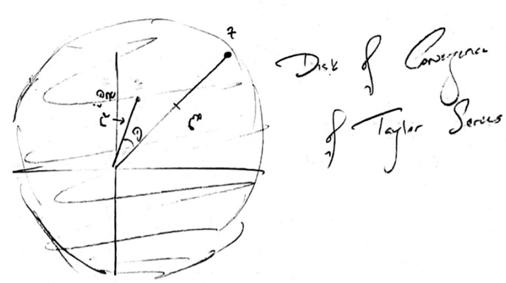

Figure 11: disk of convergence of power series

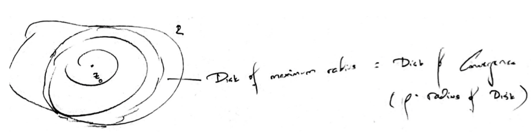

Figure 12: disk of convergence is disk with maximum radius inside R

Example

$$f(x) = \frac{1}{x^2+1}$$
$$f(z) = \frac{1}{z^2+1}$$

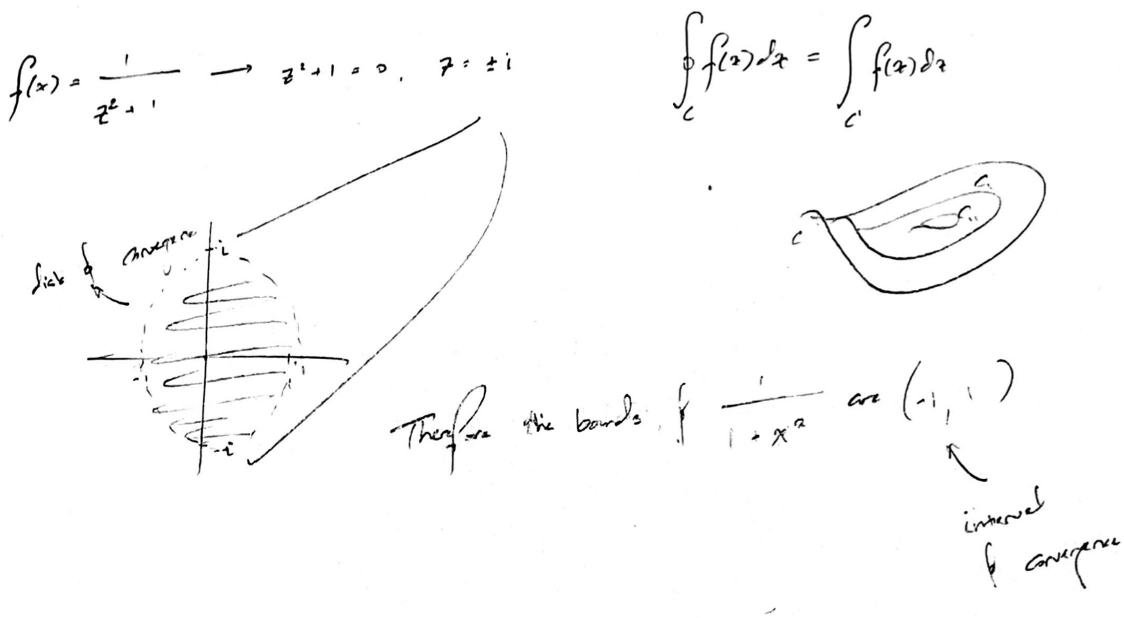

Figure 13: using complex number instead of real numbers to calculate interval of convergence

Example

$$if(z) = \log(z)$$

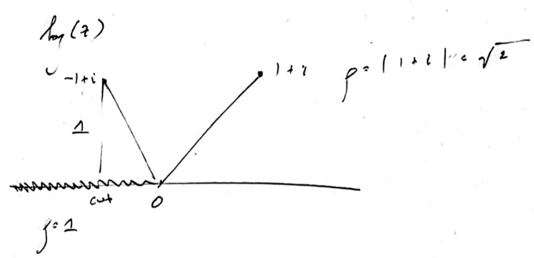

Figure 14: graph of complex log to determine convergence

3.4.0.4 Path Integrals

$$\int_{\infty}^{\infty} \left(\frac{\sin x}{x}\right) dx = \pi$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \arctan x\big|_{-\infty}^{\infty} = \pi$$

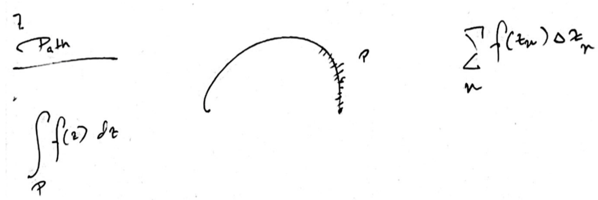

Figure 15: path P

Technique
Parametrize P from $0 < t < \pi$ as $z(t)$

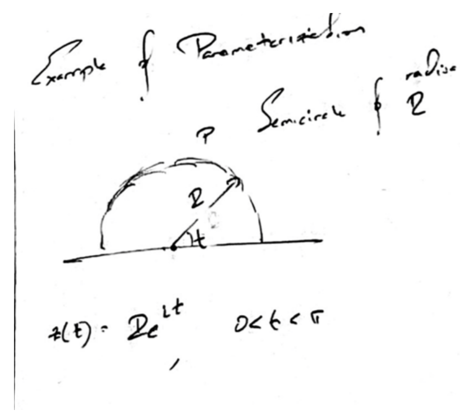

Figure 16: example of parameterizing $z(t)$

$$\int_P f(z)dz = \int_a^b f(z(t)) \left(\frac{dz}{dt}\right) dt$$

$$f(z) = z^3$$
$$\frac{dz}{dt} = iRe^{it}$$
$$f(z(t)) = \left(Re^{it}\right)^3 = R^3 e^{3it}$$

Collect

$$
\begin{aligned}
\int_0^{\pi} f(z(t)) \left(\frac{dz}{dt}\right) dt &= \int_0^{\pi} R^3 e^{3it} (iRe^{it}) dt \\
&= iR^4 \int_0^{\pi} e^{4it} = \frac{iR^4}{4i} e^{4it}\Big|_0^{\pi} = 0 \\
e^{4\pi i} &= 1
\end{aligned}
$$

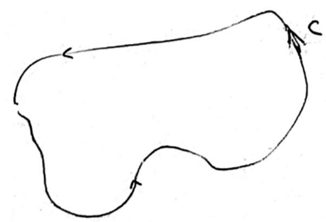

Figure 17: contour integral path

3.4.0.5 Contour Integrals Contour Integrals = 0 for any analytic function that is analytic for the entire region C

$$
\begin{aligned}
&\oint_C f(z) dz = 0 \\
&= \int_0^{2\pi} R^n e^{nit} \frac{dz}{dt} dt \\
&= \int_0^{2\pi} R^n e^{nit} iRe^{it} dt \\
&= iR^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \\
&= \frac{iR^{n+1}}{i(n+1)} e^{i(n+1)t}\Big|_0^{2\pi} = 0
\end{aligned}
$$

Effect of Singularities
Let $n = -1$

$$
\begin{aligned}
f(z) &= \frac{1}{z} \\
\oint_C \frac{dz}{z} = iR^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \\
= i \int_0^{2\pi} dt = 2\pi i \neq 0
\end{aligned}
$$

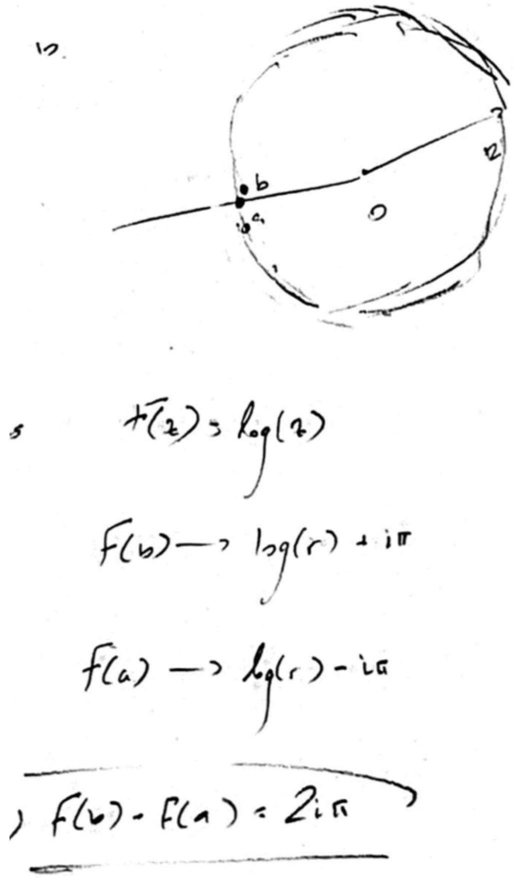

Figure 18: Fundamental Theorem of Contour Integrals

$$\int_a^b f(z)dz = F(b) - F(a)$$

Example

$$f(z) = \frac{1}{z^2+1} \qquad \alpha = i$$

$$= \frac{1}{(z-i)(z+i)} = \frac{1}{z-i}\left(\frac{1}{z+i}\right)$$

$$f(z) = \frac{g(z)}{z-\alpha}$$

$$\begin{aligned} f(z) = \frac{1}{z-\alpha}[g(\alpha) + (z-\alpha)g'(\alpha) + \dots] \\ = \frac{g(\alpha}{z-\alpha} + g'(\alpha) + \dots \end{aligned}$$

$$\boxed{\oint_C f(z)dz = \oint_C' f(z)dz = \oint_D f(z)dz = \oint \frac{g(\alpha)}{z-\alpha}dz}$$