Physics 89 (Mathematical Methods)
Problem Set #5
Due by February 24, 2023

# Matrices that are important in Physics

## 1 Pauli matrices

The three matrices:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are known as *Pauli spin matrices.* They play an important role in the quantum mechanical theory of particles with spin.[1]

(a) Calculate the following matrices:

$$AB = ?, \qquad BA = ?, \qquad AC = ?, \qquad CA = ?, \qquad BC = ?, \qquad CB = ?$$

Is $AB$ equal to $BA$ or not?

(b) Express your answers to part (a) in terms of the matrices $\pm iA$, $\pm iB$, and $\pm iC$.

(c) Calculate

$$A^2 = ?, \qquad B^2 = ?, \qquad C^2 = ?,$$

(d) Denote

$$\sigma_1 = A, \qquad \sigma_2 = B, \qquad \sigma_3 = C, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The results of parts (b)-(c) can be summarized in the form

$$\sigma_m \sigma_n = \sum_{k=1}^{3} \alpha_{mnk} \sigma_k + \beta_{mn} I, \qquad m = 1, 2, 3, \qquad n = 1, 2, 3,$$

where $\alpha_{mnk}$ and $\beta_{mn}$ are (possibly complex) numbers that depend on the indices $m, n, k$. Express these numbers in terms of the Levi-Civita tensor $\epsilon_{mnk}$ and the Kronecker delta tensor $\delta_{mn}$.

(e) The **trace** of a square matrix is defined as the sum of the elements on the diagonal.

$$\operatorname{tr} M = \sum_{i=1}^{n} M_{ii} \qquad \text{(the trace of an } n \times n \text{ matrix } M\text{)}$$

Calculate $\operatorname{tr}(ABC)$.

**Note:** For any two $n \times n$ matrices $F$ and $G$ we have an identity that $\operatorname{tr}(FG) = \operatorname{tr}(GF)$.

[1] The standard notation of Quantum Mechanics is $\sigma_x = A$, $\sigma_y = B$, and $\sigma_z = C$. See part (d) below.

## 2 Upper Triangular matrices

A square matrix with all zero entries below the diagonal is called *upper triangular.* Here is an upper triangular $3 \times 3$ matrix:

$$M = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

(a) Calculate $M^T$ (the transpose of $M$);

(b) Calculate $\det M$.

(c) Calculate the matrix products $MM^T$ and $M^TM$. Are they equal or not?

(d) Calculate $M^{-1}$ (the inverse of $M$).

## 3 The Jacobian

A coordinate system on the plane is an assignment of two variables to each point. For example, the *polar coordinate system* assigns $(r, \theta)$ to the point with Cartesian coordinates

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

The *Jacobian* of the coordinate system is defined as the **determinant** of the matrix of first derivatives

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}.$$

(a) Calculate the Jacobian of polar coordinates. Simplify until you get an answer for $J$ that is independent of $\theta$.

(b) An elliptical coordinate system assigns the variables $(u, v)$ to a point with Cartesian coordinates

$$x = a\cosh u\cos v\,, \qquad y = a\sinh u\sin v,$$

where $a$ is a given constant.

Calculate the Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Express the result in terms of $\cosh u$ and $\cos v$ only (without using $\sinh u$ or $\sin v$).

(c) For spherical coordinates in 3D,

$$x = r\sin\theta\,\cos\phi\,, \qquad y = r\sin\theta\,\sin\phi\,, \qquad z = r\cos\theta\,,$$

calculate the Jacobian, which is now defined as

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}.$$

Simplify until you get a result that is independent of $\phi$.

# 4 The Pfaffian and relativistic electrodynamics

An **antisymmetric matrix** is a matrix that satisfies $M^T = -M$. When electrodynamics is combined with Special Relativity, it is convenient to combine the electric field components $(E_x, E_y, E_z)$ with the magnetic field components $(B_x, B_y, B_z)$ into a $4 \times 4$ antismmetric matrix:

$$M = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

where $c$ is the speed of light.[2]

(a) Calculate $\det M$ by Laplace development.
**Advice:** try to simplify the cofactors by collecting $E_x B_x + E_y B_y + E_z B_z = \vec{E} \cdot \vec{B}$ where it appears.

(b) Show that $\det M = \alpha(\vec{E} \cdot \vec{B})^2$, where $\alpha$ is a numerical coefficient that you need to determine. (It contains $c$.)

**Note:** in general the determinant of an antisymmetric $(2n) \times (2n)$ matrix can be expressed as a square of a polynomial in its coefficients (like $\vec{E} \cdot \vec{B}$ above). The square root of $\det M$ in this case is called the *Pfaffian* of $M$. The Pfaffian is only defined for an **antisymmetric matrix**. Also, it turns out that the determinant of an antisymmetric $m \times m$ matrix with odd $m$ is always zero.

[2] This matrix is called the *field strength tensor*.