Physics 89 - Introduction to Mathematical Physics

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Contents

1 Difference between Mathematics and Physics

Example 1 - Electrostatics

Math Question

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = ?$$

Math Solution

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\log(1-x), \quad for -1 \leq x \leq 1$$

So,

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\log(2)$$

Example 2 - Diffusion

$$f(x, y, z, t) = \text{density of diffusing material at time } t$$

Let there exist a cube containing moles

$$\frac{\partial f}{\partial t} = D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

where D is the *diffusion coefficient*, and the diffusion equation describes how f evolves with time

Math Question

Solve

$$\frac{\partial f}{\partial t} = D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

given initial condition

$$f(x, y, z, 0) = \text{concentrated lump at the origin}$$

Math Solution

$$f(x, y, z, t) = \frac{N}{(4\pi D t)^(3/2)} e^{-\frac{x^2+y^2+z^2}{4Dt}}$$

where N is the number of moles released

Jan 19

2 Taylor Series

- Techniques for obtaining series
- Estimate error, converge?

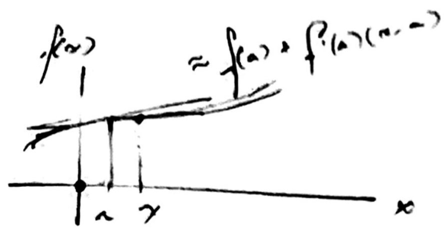

Figure 1: Taylor Series Visualization

$$f(x) \approx f(0) + f'(0)x + \cdots + \frac{1}{n!}f^n(0)x^n$$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + ... = \sum_{k=0}^{\infty} \frac{1}{k!}f^k(a)(x-a)^k$$

Question

How good is this approximation?

Big O notation

$$\sum_{k=9}^{n} \frac{1}{k!}f^k(0)x^k + O(x^{n+1})$$

Formally,

$$F(x) = o(x^{n+1}) \quad \text{as } x \to 0$$

$$|F| \leq C|x|^{n+1} \quad \text{for some unexpected constant c}$$

$$\lim_{x \to 0} \frac{F}{|x|^{n+1}} = 0$$

Example

$$e \approx 1.9GeV \approx 3700mc^2$$

Special Relativity

$$E_k = m_0c^2 - mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$$
$$\approx 0 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \frac{5}{16}m\frac{v^8}{c^4}$$
$$f(v) = \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \dots$$

$$\frac{1}{\sqrt{1-x}} \to \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$(1+x)^P, \quad \text{then set } p = \frac{1}{2}$$

$$\begin{aligned} f(x) &= (1+x)^n \\ f'(x) &= p(1+x)^{p-1} \\ f^k(x) &= p(p-1)\dots(p-k+1)(1+x)^{p-k} \to f^k(0) \\ &= p\dots(p-k+1) \end{aligned}$$

$$(1+x)^n \approx 1 + px + \frac{p(p-1)}{2!}x^2 + ... + \frac{p!}{k!(p-k)!}x^k = \binom{p}{k}x^k$$

$$\sum_{k=0}^{n}\binom{p}{k}x^k \quad \text{generalized binomial coefficient}$$

$$(1+x)^P = \sum_{k=0}^{n}\binom{p}{k}x^k + O(x^{n+1}$$

Question

Given $\frac{1}{\sqrt{1+x}}$ taylor series, how good is this approximation if $x = 0.1$?

Solution

$$\text{Actual Answer} \rightarrow \frac{1}{\sqrt{1.1}} = 0.9534626$$

Taylor Polynomials $x, x^2 \rightarrow 1 - \dfrac{0.1}{2} = 0.95 \quad / \quad 1 - \dfrac{0.5}{2} + \dfrac{3(0.5)^2}{8} = 0.95375$ good approx

More Taylor Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$
$$\sinh x = \frac{e^x - e^x}{2} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

2.1 Testing for Convergence

If $\sum_0^\infty a_n x^n \leq \infty$ converges,

$$\sum_0^\infty a_n(\lambda X)^n \leq \infty \qquad |\lambda| \leq 1$$

Taylor Series have interval of convergence of the form

$$[-L, L] \quad (-L, L) \quad [-L, L) \quad (-L, L]$$

Truncated Taylor Series Approximation

$$R_0(x) = f(x) - f(0) = f'(c)x$$

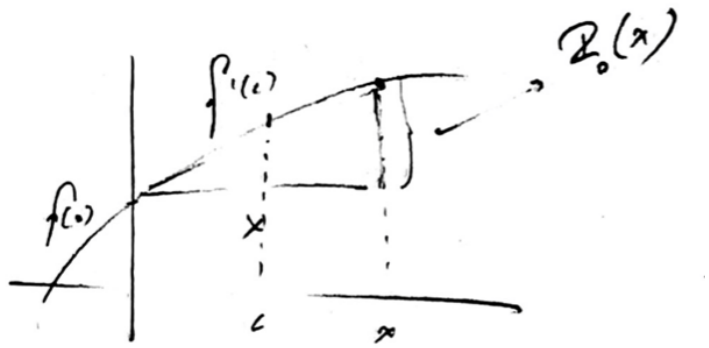

Figure 2: Remainder Visualized

Remainder Theorem

$$R_n(x) = f^{n+1}(c)\frac{x^{n+1}}{(n+1)!} \quad \text{for some } 0 \le c \le x$$

$$\begin{aligned} x &= \frac{\pi}{2} \\ R &= \sin\frac{\pi}{2} - (x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880..} + 0) \\ &= f^{10}(c)\frac{x^10}{10!} \quad 0 \le c \le \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} |f^{11}(c)| &= |-\cos c| < 1 \\ |R_{10}| &\le \frac{1}{11!}\left(\frac{\pi}{2}\right)^{11} \approx 3.6 \times 10^{-6} \end{aligned}$$

Technique for Solving Taylor Series by dividing two polynomials

$$\begin{aligned} f(x) &= a_0 + a_1 x + \dots \\ g(x) &= b_0 + b_1 x + \dots \\ \frac{f(x)}{g(x)} &= (c_0 + c_1 x + c_2 x^2 + \dots) \\ a_0 + a_1 x + \dots &= (b_0 + b_1 x + \dots)(c_0 + c_1 x + \dots) \end{aligned}$$

$$a_0 = b_0 c_0$$