Physics 89 (Mathematical Methods)
Problem Set #6
Due by March 3, 2023, 6pm

1 Eigenvalues and eigenvectors

Calculate the eigenvalues and their corresponding eigenvectors for the following matrices.

$$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

The eigenvectors are only unique up to multiplication by a scalar, so you can pick any eigenvector you like (for each eigenvalue). Please show your work.

2 Rotation matrix

The vectors

$$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

are eigenvectors of the rotation matrix

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) What are the corresponding eigenvalues λ_1 and λ_2?

(b) Define the matrix

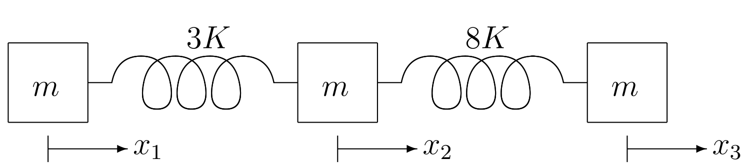
$$C = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

and calculate $D = C^{-1}MC$. Check that it is diagonal.

(c) Find a matrix A such that $e^A = M$.

Hint: Apply the function $f(x) = \ln x$ to M using the identity $M = CDC^{-1}$ and the rule $f(M) = Cf(D)C^{-1}$.

3 Resonance frequencies



Three equal masses m are connected in a row with two unequal springs between them. The spring constant of the left spring is $3K$ and the spring constant of the right one is $8K$ (where K is a given constant). Let x_1, x_2, x_3 be the displacements (from rest position) of the masses. The equations of motion that follow from Newton's second law are:

$$\begin{aligned} m\ddot{x}_1 &= 3K(x_2 - x_1)\,, \\ m\ddot{x}_2 &= 8K(x_3 - x_2) - 3K(x_2 - x_1)\,, \\ m\ddot{x}_3 &= -8K(x_3 - x_2)\,. \end{aligned}$$

Assuming each mass oscillates with the same frequency ω, we can guess a solution of the form

$$x_1 = A_1 \cos \omega t\,, \quad x_2 = A_2 \cos \omega t\,, \quad x_3 = A_3 \cos \omega t\,,$$

where A_1, A_2, A_3 are unknown constants, which we assume to be **not all zero**.

(a) Show that after substituting the expressions for x_1, x_2, x_3 into Newton's equations of motion and canceling common terms, we get an eigenvalue/eigenvectors problem for the matrix

$$M = \begin{pmatrix} -3 & 3 & 0 \\ 3 & -11 & 8 \\ 0 & 8 & -8 \end{pmatrix}.$$

The eigenvalue we are looking for is $\lambda \equiv -m\omega^2/K$.

(b) Use this insight to find all possible frequencies ω in terms of K and m. (These are the *resonance frequencies.*)