Physics 89 (Mathematical Methods)
Problem Set #3
Due by 6pm, February 10, 2023

1 Tensors

Recall that 2D vectors (with components A_i) and 2D tensors of rank 2 (with components T_{ij}) transform under a change of coordinate system as

$$A'_i = \sum_{k=1}^{2} R_{ik} A_k, \qquad T'_{ij} = \sum_{k=1}^{2}\sum_{l=1}^{2} R_{ik} R_{jl} T_{kl}.$$

Here A'_i and T'_{ij} are the components in the "new" coordinate system which is rotated by an angle θ to the original one. R_{ij} are the elements of the rotation matrix given by

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

or, in other words,

$$R_{11} = R_{22} = \cos\theta, \qquad R_{12} = \sin\theta, \qquad R_{21} = -\sin\theta.$$

(a) Write out explicitly the expressions for T'_{11}, T'_{12}, T'_{21}, and T'_{22} in terms of $T_{11}, T_{12}, T_{21}, T_{22}$.

(b) A tensor is called **symmetric** if $T_{ij} = T_{ji}$ for every i, j. In our 2D case there is only one requirement, which is $T_{12} = T_{21}$. Check that this implies $T'_{12} = T'_{21}$.

(c) Suppose that $T_{12} = T_{21} = 0$. Express T'_{12} in the form $\alpha(T_{11} - T_{22})$ and find the factor α. This shows that a diagonal tensor (i.e., one with $T_{ij} = 0$ whenever $i \neq j$) doesn't remain in diagonal in other coordinate systems.

(d) Check that if $T_{ij} = \delta_{ij}$ (i.e., $T_{11} = T_{22} = 1$ and $T_{12} = T_{21} = 0$) then $T'_{ij} = \delta_{ij}$ as well.

2 Bases of vector spaces

Expand the vector

$$\vec{V} = a\hat{x} + b\hat{y} + c\hat{z}$$

in the basis

$$\vec{A} = \hat{x} + \hat{y}, \qquad \vec{B} = \hat{x} + \hat{z}, \qquad \vec{C} = \hat{y} + \hat{z}\,.$$

That is, find α, β, γ such that $\vec{V} = \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C}$.

3 Vector spaces of functions

(a) Expand the functions e^x and e^{-x} in the basis $\sinh x$, $\cosh x$.

(b) Expand the function e^x in the basis $\sinh x$, $\sinh(x + a)$, for a given number $a \neq 0$.

4 Orthogonal vectors

The three vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 7 \\ 7 \\ 1 \end{pmatrix},$$

are orthogonal to each other.

(a) Calculate the orthonormal vectors $e_1 = v_1/\|v_1\|$, $e_2 = v_2/\|v_2\|$, and $e_3 = v_3/\|v_3\|$.

(b) Find a fourth vector e_4 that is orthogonal to e_1, e_2 and e_3 by applying the Gram-Schmidt process to

$$f = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) Normalize the vector e_4.

References

[1] Mary L. Boas, "Mathematical Methods in the Physical Sciences," 3^{rd} Edition, John Wiley & Sons, 2006.