# 5B - Introductory Electromagnetism, Waves, and Optics

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## Contents

# 1 Maxwell's Equations

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} &= 0
\end{aligned}
$$

The goal of this course will be to understand Maxwell's Equations, and the unison between the electric and magnetic field.

**Lorent'z Force eq**

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

**Rules**

1. Charge in nature is quantized in units of $e$
2. Charge is conserved
3. Charge has 2 types: $\pm$

$\vec{E}$ and $\vec{B}$ are *vector fields*. This means $\vec{E}$ is a function of every point in space: $\vec{E}(x, y, z)$

$$\vec{E}(r) = E_x(x, y, z)\hat{i} + E_y(x, y, z)\hat{j} + E_z(x, y, z)\hat{k}$$

$\nabla$ is a vector operator:

$$\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial x}\hat{k}$$

**Vector Operations**

$$\vec{A}\alpha \to \vec{B}$$
$$\vec{A} \cdot \vec{B} \to \alpha$$
$$\vec{A} \times \vec{B} \to \vec{C}$$

Consider a scalar field: $\varphi(\vec{r})$

$$\begin{aligned}
d\varphi &= \varphi(r+dr) - \varphi(r) \\
d\varphi &= \frac{\partial\varphi(x,y,z)}{\partial x}dx + \frac{\partial\varphi(x,y,z)}{\partial y}dy + \frac{\partial\varphi(x,y,z)}{\partial z}dz \\
d\vec{r} &= \hat{i}dx + \hat{y}dy + \hat{z}dz
\end{aligned}$$

$$d\varphi = \nabla\varphi(\vec{r}) \cdot d\vec{r}$$

$\nabla\varphi$ = "gradient of $\varphi$. " Also $\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$. This is known as the **divergence** of $\varphi$.

The **curl** of $\varphi$ is equal to

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$\rho = \text{charge density} = \frac{\text{number of particles} q}{dV} = nq$$

where $n = \frac{\text{number of particles}}{dv}$

$$\vec{J} = \text{current density}$$

# 2 Statics

**Equations of Electrostatics**

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \times \vec{E} = 0$$

**Equations of Magnetostatics**

$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

## 2.1 Flux

"Flux" = Flow

Consider a fluid flow with a velocity vector field $\vec{v}(\vec{r})i$, flowing into a small aperture defined by $\hat{n}\, d\vec{a}$, where $\hat{n}$ is the unit normal vector to the aperture, and $d\vec{a}$ is the area.

$$d\Phi = \vec{v} \cdot d\vec{a}$$

Relating to the Electric field,

$$d\Phi_E = \vec{E}(\vec{r}) \cdot d\vec{a}$$
$$\int_S d\Phi_E = \Phi_E = \int_S \vec{E}(\vec{r}) \cdot d\vec{a}$$

Green's, Gauss's, Divergence Theorem

$$\int_S \vec{E} \cdot d\vec{a} = \iiint_S \nabla \cdot \vec{E}(\vec{r})\, d^3r$$

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$$\Phi = \oint_S \vec{E}(\vec{r}) \cdot d\vec{a} = \iiint_V d^3r \, \nabla \cdot \vec{E}(\vec{r}) \rightarrow \text{Divergence Theorem}$$

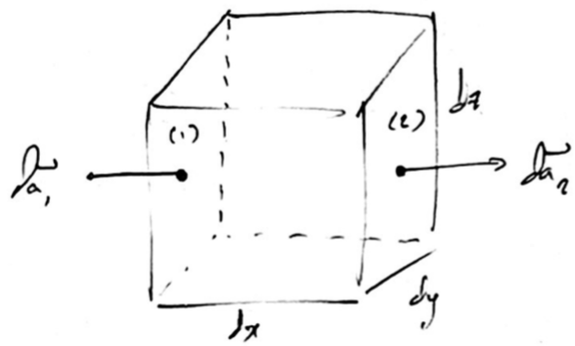


Figure 1: Flux from two sides of cube

$$d\Phi_{12} = \vec{E}(x,y,z) \cdot d\vec{a_1} + \vec{E}(x+dx,y,z) \cdot d\vec{a_2} \quad | \quad d\vec{a_1} = -\hat{i} dydz, d\vec{a_2} = \hat{i} dydz$$

$$d\Phi_{12} = \left[E_x(x+dx,y,z) - E_x(x,y,z)\right] dydz$$
$$d\Phi_{12} = \left[\frac{\partial E_x(x,y,z)}{\partial x} dx\right] dydz$$

Since $dv = dxdydz$,

$$\begin{aligned} d\Phi_{12} &= \frac{\partial E_x(x,y,z)}{\partial x} \cdot d^3v \\ d\Phi_{tot} &= d\Phi_{12} + d\Phi_{34} + \dots \\ &= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) d^3r \\ d\Phi_{tot} &= \nabla \cdot \vec{E}(\vec{r}) \, d^3r \end{aligned}$$

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{d\Phi}{d^3r} \\ \nabla \cdot E &= \frac{\text{flux}}{\text{volume}} \end{aligned}$$

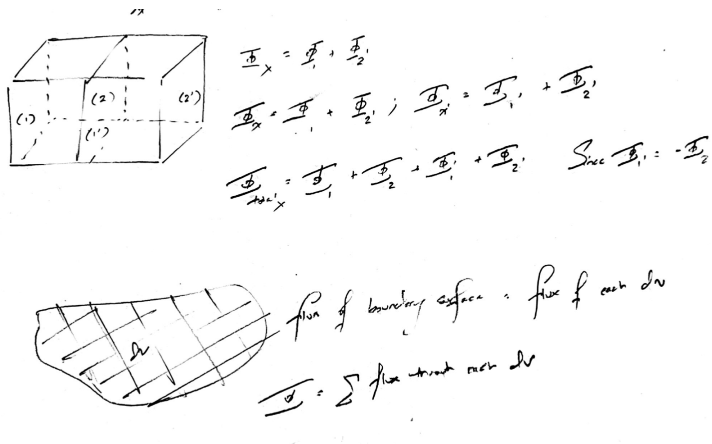

Figure 2: Flux from the boundary surface is equal to adding the flux of every $dV$ inside

## 2.2 Divergence Theorem

$$\iiint \nabla \cdot \vec{E}(\vec{r})\, d^3r = \oint_S \vec{E} \cdot d\vec{a}$$

$$\iiint \frac{\rho(\vec{r})}{\epsilon_0} d^3r = \oint \vec{E}(\vec{r}) \cdot d\vec{a}$$

**Gauss' Law**

$$\Phi = \frac{Q}{\varepsilon_0}$$

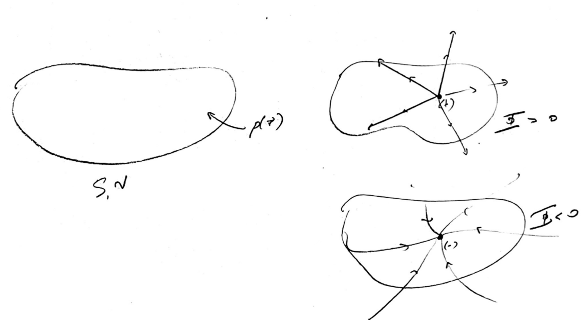

Figure 3: Flux from positive charge is positive (source), flux from negative charge is negative (sink)

## 2.3 Coulomb's Law

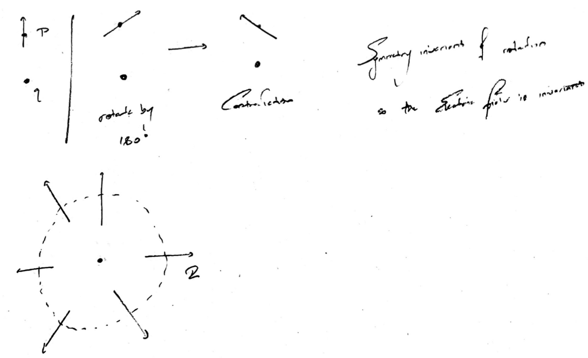


Figure 4: Electric field from a sphere is directly outward and has the same magnitude due to symmetry

## 2.4 Electric Field of Point Charge

$$\begin{aligned}
\vec{E}(\vec{r}) &\propto \hat{r} \\
\Phi &= \frac{Q}{\epsilon_0} \\
E(r) \cdot 4\pi r^2 &= \frac{Q}{\epsilon_0} \\
\vec{E}(\vec{r}) &= \hat{r}\frac{Q}{4\pi\epsilon_0 r^2}
\end{aligned}$$

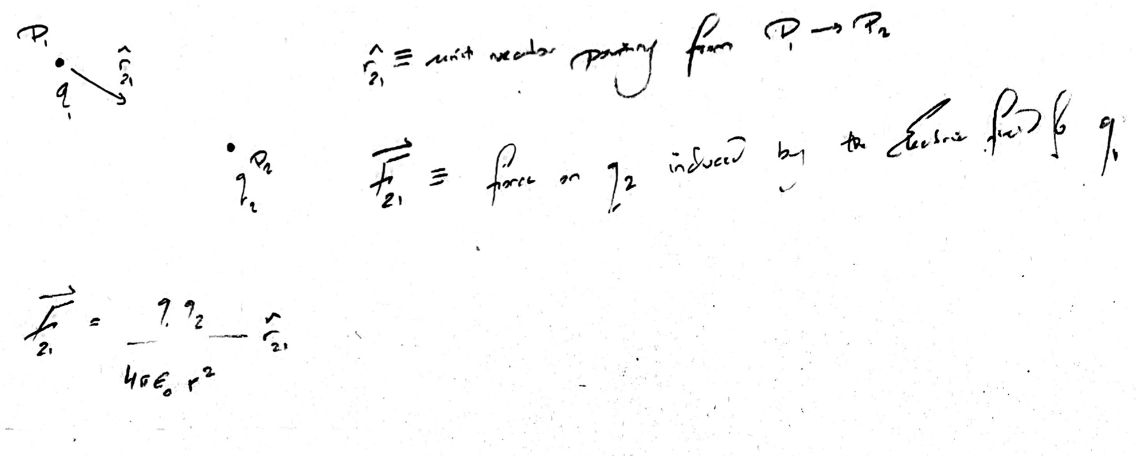


Figure 5: Diagram of Coulomb's Law

**Coulomb's Law**

$$\vec{F_{21}} = q_2\vec{E_1}(P_2) = \frac{q_2 q_1}{4\pi\epsilon_0 r^2}\hat{r_{21}}$$

The force is proportional to the product of the charges, and inversely proportional to the square of the distance, and directed along the line between the charge, and directed along the line between the charges. Attractive for opposite signs of charge, and vice versa.

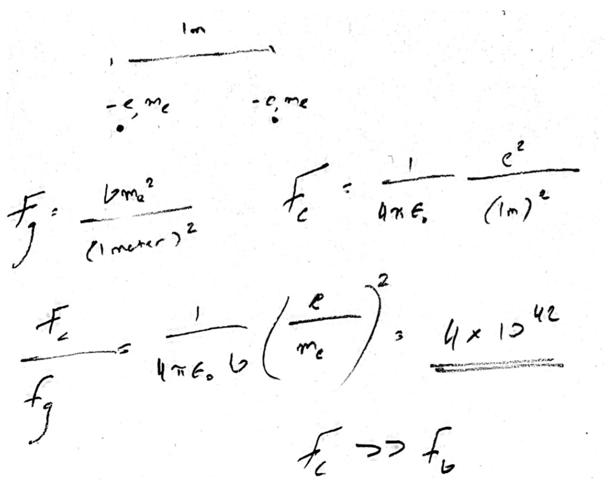


Figure 6: Comparison between $F_g$ and $F_c$

***Jan 19 Summary***

$$\iiint_V \nabla \cdot \vec{E}(\vec{r})\, dxdydz = \oint_S \vec{E} \cdot d\vec{a}$$

$$\Phi = \frac{Q}{\epsilon_0} \rightarrow \vec{E}_{\text{point charge}} = \frac{Q}{4\pi r^2 \epsilon_0}$$

**Calculating $\vec{E}$ for arbitrary $\rho(\vec{r})$**

Superposition: $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$, works because Maxwell's Equation are linear.

$$\nabla \cdot \vec{E_1} = \frac{\rho_1}{\varepsilon_0}$$
$$\nabla \cdot \vec{E_2} = \frac{\rho_2}{\varepsilon_0}$$

$$\nabla \cdot (\vec{E_1} + \vec{E_2}) = \frac{\rho_1 + \rho_2}{\varepsilon_0}$$

Proved $\nabla \cdot \vec{E_T} = \frac{\rho_T}{\varepsilon_0}$

**System of point charges**

Therefore the field at $P$ is

$$\vec{E_1}(\vec{r}) = \frac{q_1}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r_1}|^2} \cdot \frac{(\vec{r} - \vec{r_1})}{|\vec{r} - \vec{r_1}|} = \frac{q_1}{4\pi\varepsilon} \frac{(\vec{r} - \vec{r_1})}{|\vec{r} - \vec{r_1}|^3}$$

Superposition:

$$\vec{E}(\vec{r}) = \sum_i \frac{q_i}{4\pi\varepsilon_0} \frac{(\vec{r} - \vec{r_1})}{|\vec{r} - \vec{r_1}|^3}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\vec{r'})(\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3}$$
$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

## 2.5 Gauss's Law + Symmetry

#### 2.5.0.1 Symmetry of cylinder of charge

#### 2.5.0.2 Symmetry of plane of charge

#### 2.5.0.3 Using Gauss's Law symmetry Problem. Uniform Sphere of Charge

Gauss's Law

$$\oint_S = \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$$

Outside Surface

$$E(r) \cdot 4\pi r^2 = \frac{\rho_0 \frac{4\pi r^3}{3}}{\varepsilon_0}$$
$$E(r) = \frac{\rho_0 R^3}{3\varepsilon_0 r^2} = \frac{Q}{4\pi r^2 \varepsilon_0}$$

Inside Surface

$$E(r) \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \cdot \left(\frac{r}{R}\right)^3$$

Outside:

$$E(r) = \frac{Q}{4\pi r^2 \varepsilon}$$

Inside:

$$E(r) = \frac{Q}{4\pi\varepsilon} \frac{r}{R^3}$$

#### 2.5.0.4 Lower Dimensional Charge Distributions

#### 2.5.0.5 Applying Gauss's Law to a Plane of Charge

$$2E(r)A = \frac{\sigma A}{\varepsilon_0}$$
$$E = \frac{\sigma}{\varepsilon_0}$$