Physics 5B

Homework #1

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1. find the gradients

(a) $f(x,y,z) = x^2 + y^3 + z^4$

$\nabla f(x,y,z) = 2x\hat{i} + 3y^2\hat{j} + 4z^3\hat{k}$

(b) $f(x,y,z) = x^2y^3z^4$

$\nabla f(x,y,z) = 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$

2. $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$

→ [y] miles north of Campanile

→ [x] miles east of Campanile

(a) Top of the hill?

$\nabla h(x,y) = 10(\langle 2y - 6x - 18,\ 2x - 8y + 28\rangle)$

Critical points: $2y - 6x - 18 = 0$
$2x - 8y + 28 = 0$

$\Rightarrow \left(\begin{array}{cc|c} 2 & -6 & 18 \\ -8 & 2 & -28 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right) \begin{cases} y = 3 \\ x = -2 \end{cases}$

Top of the hill located 3 miles north, 2 miles west of the Campanile

(b) $h(-2,3) = 10(-12 - 12 - 36 + 36 + 84 + 12) = \underline{\underline{720 \text{ feet}}}$

(c) $\vec{u} = \langle 1, 1\rangle \rightarrow \hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

$D_u h(x,y) = \nabla h \cdot \hat{u} = \left[(20y - 60x - 180)\left(\frac{1}{\sqrt{2}}\right) + (20x - 80y + 280)\left(\frac{1}{\sqrt{2}}\right)\right]_{(1,1)}$

$= (20 - 60 - 180)\left(\frac{1}{\sqrt{2}}\right) + (20 - 80 + 280)\left(\frac{1}{\sqrt{2}}\right) = 0$ (flat)

Slope is steepest in direction of $\nabla h(x,y)$: $\nabla h(1,1) = \langle 20 - 60 - 180,\ 20 - 80 + 280\rangle$

$= \langle -220, 220\rangle$

$= \underline{\underline{\langle -1, 1\rangle}}$

3 Calculate Divergence

(a) $\vec{v} = x^2\hat{i} + 3x^2 y\hat{j} - 2xz\hat{k}$

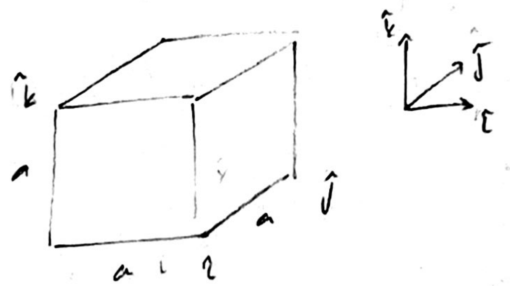
$$\nabla\cdot\vec{v} = \frac{\partial(x^2)}{\partial x} + \frac{\partial(3x^2y)}{\partial y} + \frac{\partial(-2xz)}{\partial z} = \cancel{2x} + 3x^2 - \cancel{2x} = 3x^2$$

(b) $\vec{v} = xy\hat{i} + 2y\hat{j} - 3zx\hat{k}$

$$\nabla\cdot\vec{v} = \frac{\partial(xy)}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial(-3zx)}{\partial z} = y + 2 - 3x$$

4 Divergence Theorem

$\vec{v} = \langle xy, 2yz, -3xz \rangle$ $\quad \oint_S \vec{v}\cdot\hat{n}\, dS = \iiint_V (\nabla\cdot\vec{v})\, dV$



Bottom $(-\hat{k})$: $\int_0^a\int_0^a +3xz\, dx\, dy = 0$

Top $(\hat{k})$: $\int_0^a\int_0^a -3xa\, dx\, dy = \int\left[\frac{-3x^2a}{2}\right]_0^a = \int_0^a \frac{-3a^2a}{2}\, dy = \left(\frac{-3a^2a}{2}\, y\right)_0^a = \frac{-3a^3}{2}$

Front $(-\hat{j})$: $\int_0^a\int_0^a -2yz\, dx\, dz = 0$

Back $(\hat{j})$: $\int_0^a\int_0^a 2yz\, dx\, dz = \int\left[2zx\right]_0^a = \int_0^a 2za\, dz = \left[z^2a\right]_0^a = a^3$

Left $(-\hat{i})$: $\int_0^a\int_0^a -xy\, dy\, dz = 0$

Right $(\hat{i})$: $\int_0^a\int_0^a y\, dy\, dz = \int_0^a\left[\frac{y^2}{2}\right]_0^a dz = \frac{a^2}{2}a = \frac{a^3}{2}$

$$F_{total} = -\frac{3a^3}{2} + a^3 + \frac{a^3}{2} = a^3\left(-\frac{3}{2} + 1 + \frac{1}{2}\right) = 0$$

$$\iiint_V (\nabla\cdot\vec{v})\,dV = \int_0^a\int_0^a\int_0^a (y+2z-3x)\,dx\,dy\,dz$$

$$\int_0^a\int_0^a\int_0^a (y+2z-3x)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a \left[yx+2zx-\frac{3x^2}{2}\right]_0^a dy\,dz = \int_0^a\int_0^a ay+2az-\frac{3a^2}{2}\,dy\,dz$$

$$= \int_0^a \left[\frac{ay^2}{2}+2azy-\frac{3a^2}{2}y\right]_0^a dz = \int_0^a \frac{a^3}{2}+2a^2z-\frac{3a^3}{2}\,dz$$

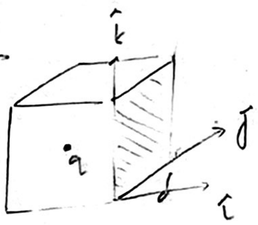
$$= \left[\frac{a^3}{2}z + a^2z^2 - \frac{3a^3}{2}z\right]_0^a = \frac{a^4}{2}+a^4-\frac{3a^4}{2}$$

$$= a^4\left(\frac{1}{2}+1-\frac{3}{2}\right) = \underline{\underline{0}}$$

$$\Longrightarrow \oint_S \vec{v}\cdot\hat{n}\,dS = \iiint_V (\nabla\cdot\vec{v})\,dV \quad ✓$$

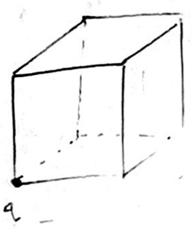
5. flux

(a)



$\int \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$ for one side, $\int \vec{E}\cdot d\vec{a} = \underline{\underline{\frac{q}{6\varepsilon_0}}}$

(b)



Intuition Answer

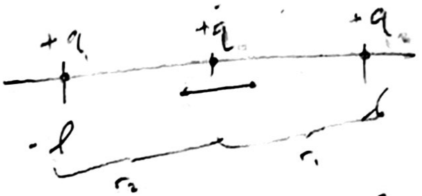
$\Phi_{bottom} = \Phi_{left} = \Phi_{front} = 0$ since $\vec{E}\cdot d\vec{a} = 0$

Only 1/8 of total flux is entering cube, $\Phi_{cube} = \frac{q}{8\varepsilon_0}$

Therefore, $\Phi_{top} + \Phi_{back} + \Phi_{right} = \frac{q}{8\varepsilon_0}$

Therefore, $\Phi_{top} = \Phi_{back} = \Phi_{right} = \frac{1}{3}\frac{q}{8\varepsilon_0} = \underline{\underline{\frac{q}{24\varepsilon_0}}}$

6 Coulomb's Law



$$F_c = \sum_{n=1}^{k} \frac{1}{4\pi\epsilon_0}\frac{qq}{r^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{r_1^2} + \frac{1}{4\pi\epsilon_0}\frac{q^2}{r_2^2} = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) = ma$$

let $r_2 = r + \partial_r$, $r_1 = r - \partial_r$

$$\Rightarrow F_c = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{(r+\partial_r)^2} - \frac{1}{(r-\partial_r)^2}\right) = \frac{q^2}{4\pi\epsilon_0}\left(\left(1+\frac{\partial_r}{2}\right)^{-2} - \left(1-\frac{\partial_r}{2}\right)^{-2}\right)$$

$$= \frac{q^2}{4\pi\epsilon_0}\left(\left(1+\frac{2\partial_r}{2}+\cancel{\frac{\partial_r^2}{2^2}}\right) - \left(1-\frac{2\partial_r}{2}+\cancel{\frac{\partial_r^2}{2^2}}\right)\right)$$

$$= \frac{q^2}{4\pi\epsilon_0}\left(\left(1-\frac{2\partial_r}{2}\right) - \left(1+\frac{2\partial_r}{2}\right)\right) = \frac{q^2}{\cancel{4}\pi\epsilon_0}\left(-\frac{\cancel{4}\partial_r}{2}\right) = -\frac{q^2}{2\pi\epsilon_0}\partial_r$$

$$F_c = \frac{-q^2}{2\pi\epsilon_0}\partial_r \quad / \quad F_s = -k\partial x$$

Therefore, $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{q^2}{2m\pi\epsilon_0}}$

$$f = \frac{1}{2\pi}\sqrt{\frac{q^2}{2m\pi\epsilon_0}}$$