5B - Introductory Electromagnetism, Waves, and Optics

deval deliwala

January 26, 2023

Contents

1 Maxwell's Equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$

The goal of this course will be to understand Maxwell's Equations, and the unison between the electric and magnetic field.

Lorent'z Force eq

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Rules

1. Charge in nature is quantized in units of e
2. Charge is conserved
3. Charge has 2 types: $\pm$

$\vec{E}$ and $\vec{B}$ are *vector fields*. This means $\vec{E}$ is a function of every point in space: $\vec{E}(x, y, z)$

$$\vec{E}(r) = E_x(x, y, z)\hat{i} + E_y(x, y, z)\hat{j} + E_z(x, y, z)\hat{k}$$

∇ is a vector operator:

$$\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial x}\hat{k}$$

Vector Operations

$$\vec{A}\alpha \to \vec{B}$$
$$\vec{A} \cdot \vec{B} \to \alpha$$
$$\vec{A} \times \vec{B} \to \vec{C}$$

Consider a scalar field: $\varphi(\vec{r})$

$$\begin{aligned}
&d\varphi = \varphi(r + dr) - \varphi(r) \\
&d\varphi = \frac{\partial\varphi(x,y,z)}{\partial x}dx + \frac{\partial\varphi(x,y,z)}{\partial y}dy + \frac{\partial\varphi(x,y,z)}{\partial z}dz \\
&d\vec{r} = \hat{i}dx + \hat{y}dy + \hat{z}dz
\end{aligned}$$

$$d\varphi = \nabla\varphi(\vec{r}) \cdot d\vec{r}$$

$\nabla\varphi$ = "gradient of φ. " Also $\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$. This is known as the **divergence** of φ.

The **curl** of φ is equal to

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$\rho = \text{charge density} = \frac{\text{number of particles} q}{dV} = nq$$

where $n = \frac{\text{number of particles}}{dv}$

$$\vec{J} = \text{current density}$$

2 Statics

Equations of Electrostatics

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \times \vec{E} = 0$$

Equations of Magnetostatics

$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

2.1 Flux

"Flux" = Flow

Consider a fluid flow with a velocity vector field $\vec{v}(\vec{r})i$, flowing into a small aperture defined by $\hat{n}\, d\vec{a}$, where $\hat{n}$ is the unit normal vector to the aperture, and $d\vec{a}$ is the area.

$$d\Phi = \vec{v} \cdot d\vec{a}$$

Relating to the Electric field,

$$d\Phi_E = \vec{E}(\vec{r}) \cdot d\vec{a}$$
$$\int_S d\Phi_E = \Phi_E = \int_S \vec{E}(\vec{r}) \cdot d\vec{a}$$

Green's, Gauss's, Divergence Theorem

$$\int_S \vec{E} \cdot d\vec{a} = \iiint_S \nabla \cdot \vec{E}(\vec{r})\, d^3r$$

$$\Phi = \oint_S \vec{E}(\vec{r}) \cdot d\vec{a} = \iiint_V d^3r\, \nabla \cdot \vec{E}(\vec{r}) \rightarrow \text{Divergence Theorem}$$

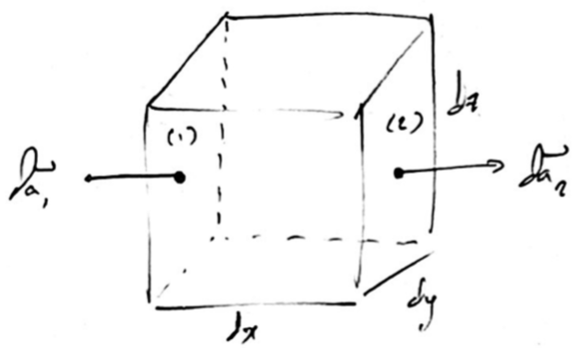

Figure 1: Flux from two sides of cube

$$d\Phi_{12} = \vec{E}(x,y,z) \cdot d\vec{a_1} + \vec{E}(x+dx,y,z) \cdot d\vec{a_2} \quad | \quad d\vec{a_1} = -\hat{\imath} dydz, d\vec{a_2} = \hat{\imath} dydz$$

$$d\Phi_{12} = [E_x(x+dx,y,z) - E_x(x,y,z)]\, dydz$$
$$d\Phi_{12} = \left[\frac{\partial E_x(x,y,z)}{\partial x}\, dx\right] dydz$$

Since $dv = dxdydz$,

$$d\Phi_{12} = \frac{\partial E_x(x,y,z)}{\partial x} \cdot d^3v$$
$$d\Phi_{tot} = d\Phi_{12} + d\Phi_{34} + \dots$$
$$= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) d^3r$$
$$d\Phi_{tot} = \nabla \cdot \vec{E}(\vec{r})\, d^3r$$

$$\nabla \cdot \vec{E} = \frac{d\Phi}{d^3r}$$
$$\nabla \cdot E = \frac{\text{flux}}{\text{volume}}$$

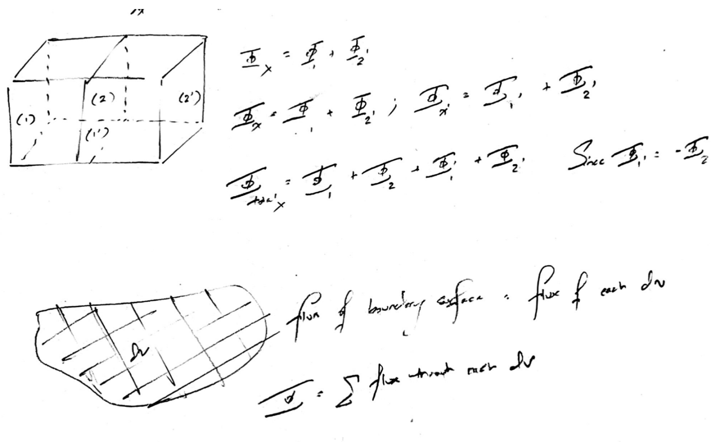

Figure 2: Flux from the boundary surface is equal to adding the flux of every dV inside

2.2 Divergence Theorem

$$\iiint \nabla \cdot \vec{E}(\vec{r})\, d^3r = \oint_S \vec{E} \cdot d\vec{a}$$

$$\iiint \frac{\rho(\vec{r})}{\epsilon_0} d^3r = \oint \vec{E}(\vec{r}) \cdot d\vec{a}$$

Gauss' Law

$$\Phi = \frac{Q}{\varepsilon_0}$$

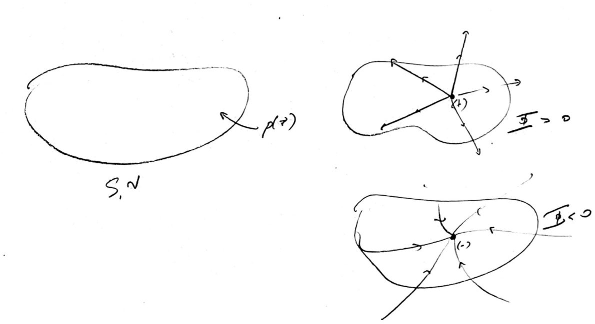

Figure 3: Flux from positive charge is positive (source), flux from negative charge is negative (sink)

2.3 Coulomb's Law

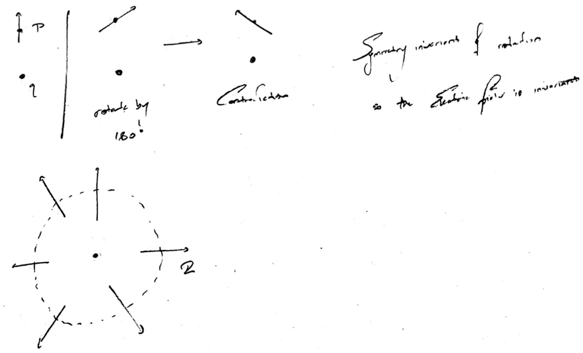

Figure 4: Electric field from a sphere is directly outward and has the same magnitude due to symmetry

2.4 Electric Field of Point Charge

$$\vec{E}(\vec{r}) \propto \hat{r}$$
$$\Phi = \frac{Q}{\epsilon_0}$$
$$E(r) \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$
$$\vec{E}(\vec{r}) = \hat{r}\frac{Q}{4\pi\epsilon_0 r^2}$$

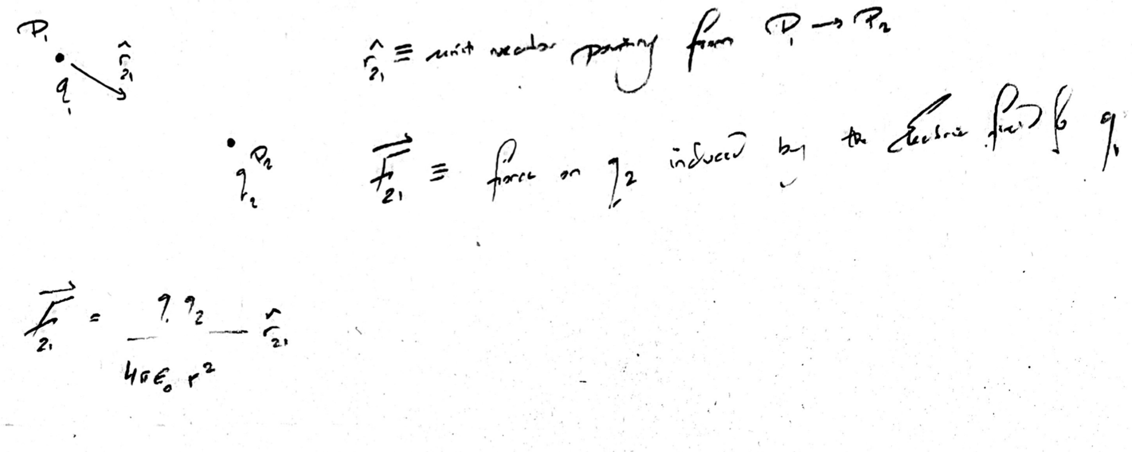

Figure 5: Diagram of Coulomb's Law

Coulomb's Law

$$\vec{F_{21}} = q_2\vec{E_1}(P_2) = \frac{q_2 q_1}{4\pi\epsilon_0 r^2}\hat{r_{21}}$$

The force is proportional to the product of the charges, and inversely proportional to the square of the distance, and directed along the line between the charge, and directed along the line between the charges. Attractive for opposite signs of charge, and vice versa.

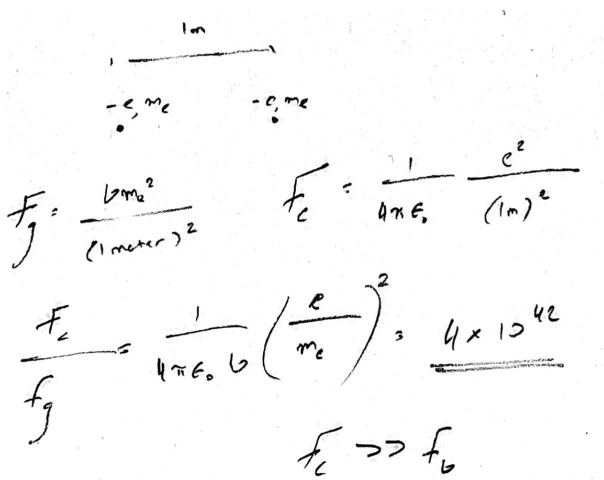

Figure 6: Comparison between F_g and F_c

Jan 19 Summary

$$\iiint_V \nabla \cdot \vec{E}(\vec{r})\, dxdydz = \oint_S \vec{E} \cdot d\vec{a}$$

$$\Phi = \frac{Q}{\epsilon_0} \to \vec{E}_{\text{point charge}} = \frac{Q}{4\pi r^2 \epsilon_0}$$

Calculating $\vec{E}$ for arbitrary $\rho(\vec{r})$

Superposition: $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$, works because Maxwell's Equation are linear.

$$\nabla \cdot \vec{E_1} = \frac{\rho_1}{\varepsilon_0}$$
$$\nabla \cdot \vec{E_2} = \frac{\rho_2}{\varepsilon_0}$$

$$\nabla \cdot (\vec{E_1} + \vec{E_2}) = \frac{\rho_1 + \rho_2}{\varepsilon_0}$$

Proved $\nabla \cdot \vec{E_T} = \frac{\rho_T}{\varepsilon_0}$

Figure 7: volumes of charge

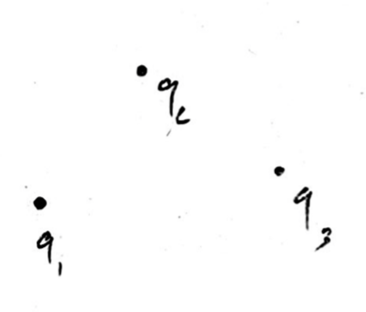

Figure 8: system of charges

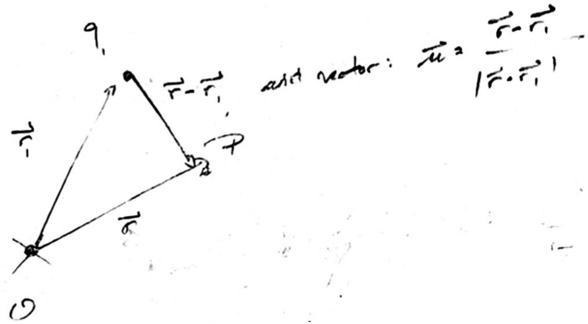

Figure 9: notations of vectors, $\vec{r}, \vec{r_1}, \vec{r} - \vec{r_1}$

System of point charges

Therefore the field at P is

$$\vec{E_1}(\vec{r}) = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r_1}|^2}\cdot\frac{(\vec{r}-\vec{r_1})}{|\vec{r}-\vec{r_1}|} = \frac{q_1}{4\pi\varepsilon}\frac{(\vec{r}-\vec{r_1})}{|\vec{r}-\vec{r_1}|^3}$$

Superposition:

$$\vec{E}(\vec{r}) = \sum_i \frac{q_i}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r_1})}{|\vec{r}-\vec{r_1}|^3}$$

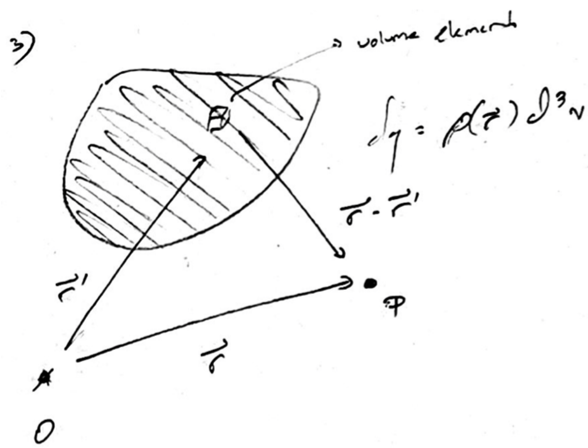

Figure 10: volume charge distribution, $dq = \rho(r)d^3v$

Figure 11: using symmetry to determine component of $\vec{E}$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

2.5 Gauss's Law + Symmetry

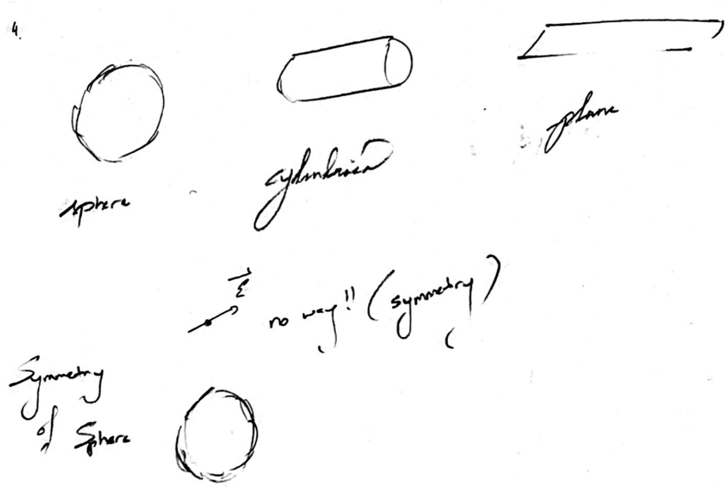

Figure 12: different Gauss's Law symmetries, symmetry of a sphere of charge

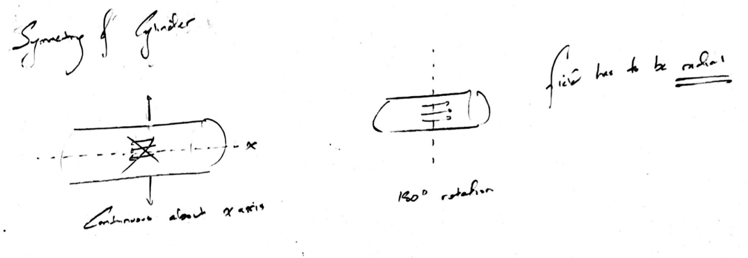

Figure 13: symmetry of a cylinder of charge

2.5.0.1 Symmetry of cylinder of charge

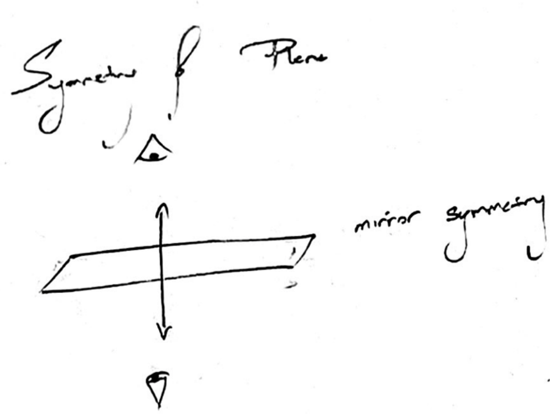

Figure 14: symmetry of a plane – mirror symmetry

2.5.0.2 Symmetry of plane of charge

2.5.0.3 Using Gauss's Law symmetry Problem. Uniform Sphere of Charge

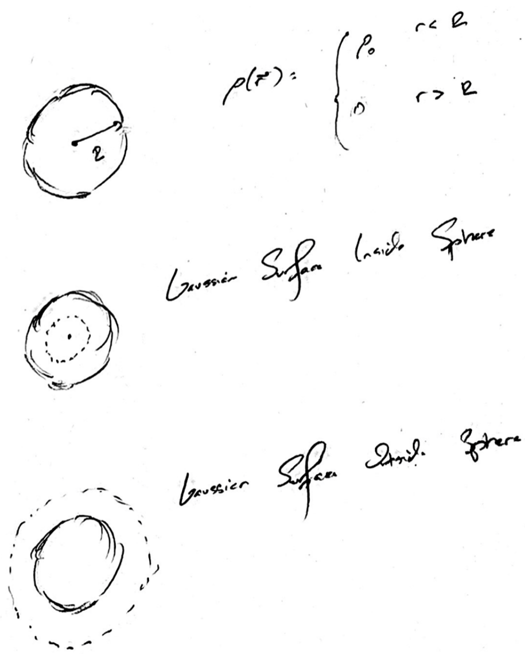

Figure 15: uniform sphere of charge, inside and outside the radius

Gauss's Law

$$\oint_S = \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$$

Outside Surface

$$E(r) \cdot 4\pi r^2 = \frac{\rho_0 \frac{4\pi r^3}{3}}{\varepsilon_0}$$

$$E(r) = \frac{\rho_0 R^3}{3\varepsilon_0 r^2} = \frac{Q}{4\pi r^2 \varepsilon_0}$$

Inside Surface

$$E(r) \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \cdot \left(\frac{r}{R}\right)^3$$

Outside:

$$E(r) = \frac{Q}{4\pi r^2 \varepsilon}$$

Inside:

$$E(r) = \frac{Q}{4\pi\varepsilon}\frac{r}{R^3}$$

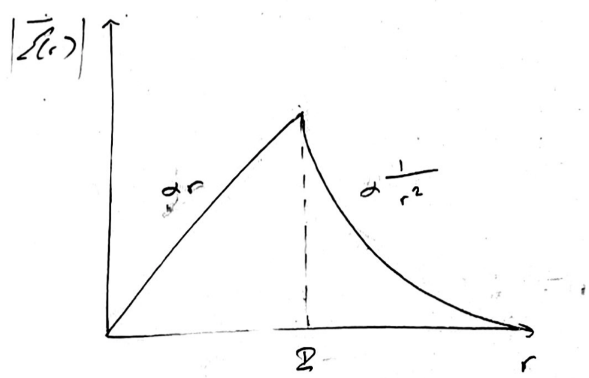

Figure 16: graph of $E(r)$ v. r for a spherical charge distribution

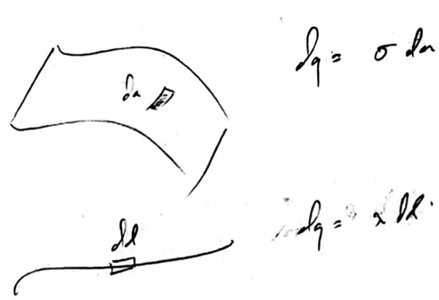

Figure 17: 1 and 2 dimensional charge distributions - σ and λ

2.5.0.4 Lower Dimensional Charge Distributions

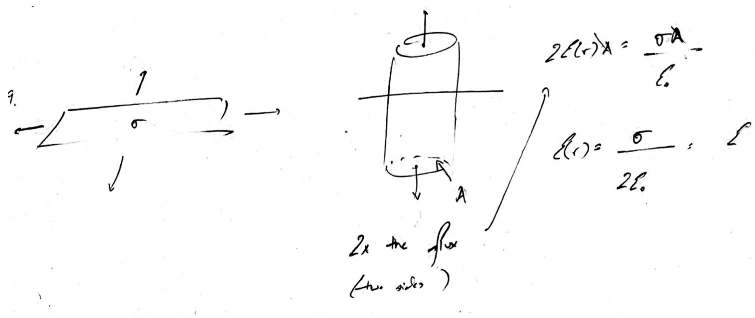

Figure 18: gaussian surface of a plane of charge

2.5.0.5 Applying Gauss's Law to a Plane of Charge

$$2E(r)A = \frac{\sigma A}{\varepsilon_0}$$
$$E = \frac{\sigma}{\varepsilon_0}$$

2.6 Visualizing the Electric Field

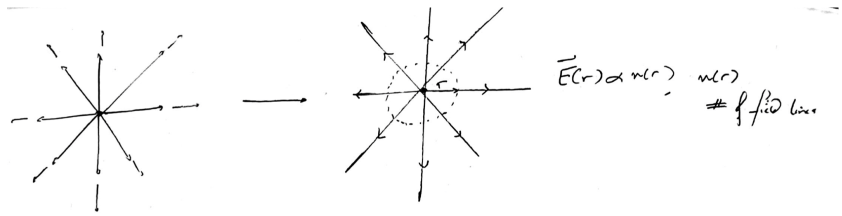

Figure 19: visualizing electric field lines

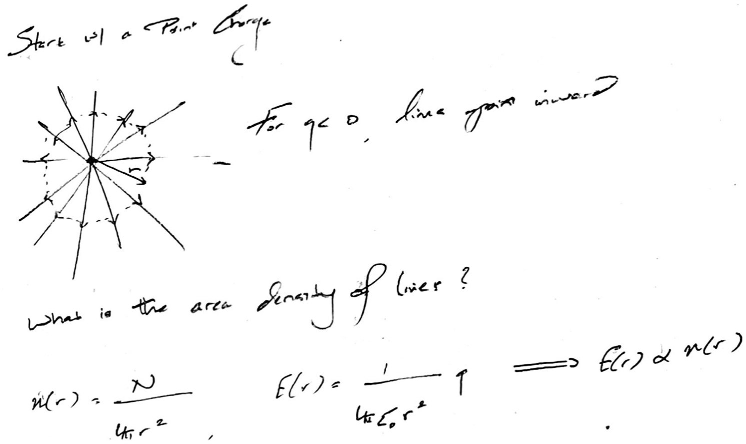

Figure 20: area density of field lines

2.7 Energy, Work, and Electrostatic Potential

2.7.0.1 Work-Energy Theorem Consider a charged particle, moving in a force field $\vec{F}(r)$

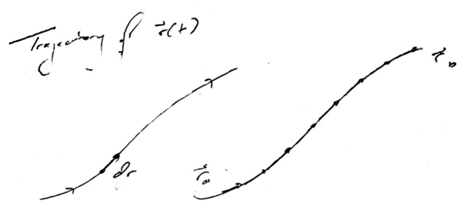

Figure 21: trajectory of $\vec{r}(t)$

$$\vec{F}(\vec{r}) = m\frac{d\vec{v}}{dt}$$

$$\vec{F}(\vec{r}) \cdot d\vec{r} = m\frac{d\vec{v}}{dt} \cdot d\vec{r} \qquad \frac{d\vec{r}}{dt} = \vec{v}$$

$$= m\frac{d\vec{v}}{dt} \cdot \vec{v}dt = md\vec{v} \cdot \vec{v}$$

$$d(\vec{v} \cdot \vec{v}) \to \text{Chain rule} \to \vec{v} \cdot d\vec{v} + d\vec{v} \cdot \vec{v} = 2d\vec{v}\vec{v}$$

$$d\vec{v} \cdot \vec{v} = \frac{1}{2}dv^2$$

$$\vec{F} \cdot d\vec{r} = \frac{m}{2}dv^2$$

$$\int_{r_a}^{r_b} \vec{F} \cdot d\vec{r} = \frac{m}{2}\int_{r_a}^{r_b} d(v) = \frac{m}{2}\left[v^2(r_b) - v^2(r_a)\right]$$

Therefore, $\int_{\vec{r_a}}^{\vec{r_b}} \vec{F}(\vec{r}) \cdot d\vec{r} =$ Work done by the force.

$$\boxed{\int_{\vec{r_a}}^{\vec{r_b}} \vec{F}(\vec{r}) \cdot d\vec{r} = \Delta KE}$$

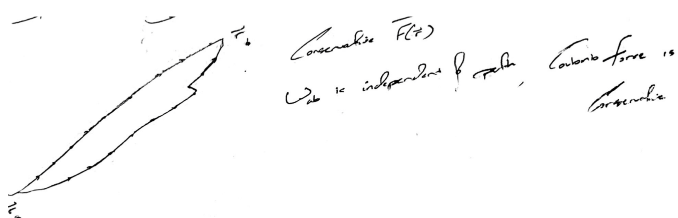

Figure 22: conservative force fields

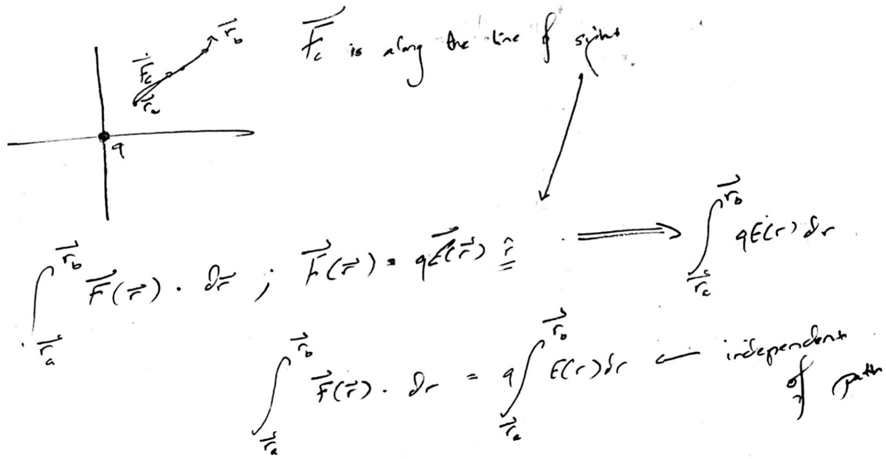

Figure 23: deriving conservative force field from Lorentz's Law

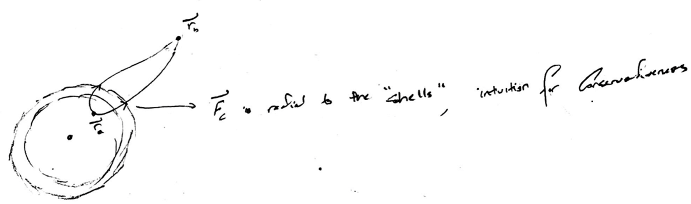

Figure 24: intuition for conservative news

Revision

$$W_{ab} = q \int_{\vec{r_a}}^{\vec{r_b}} \vec{E}(\vec{r}) \cdot d\vec{r} = q \int_{P_1}^{P_2} \vec{E} \cdot d\vec{r}$$

W_{ab}^{ext} = work done to move a particle on a trajectory in the presence of a force field

$$W_{ab}^{ext} = -q \int_{P_1}^{P_2} \vec{E}(\vec{r}) \cdot d\vec{r}$$

2.8 Electric Potential

Define $\varphi(r_a, r_b)$

$$\varphi(\vec{r_a}, \vec{r_b}) = -\int_{r_a}^{r_b} \vec{E}(\vec{r}) \cdot d\vec{r}$$

$$\varphi = \int_{r_a}^{r_b} d\varphi = \varphi(r_b) - \varphi(r_a)$$

$$W_{ab}^{ext} = q[\varphi(r_b) - \varphi(r_b)] \qquad \text{relation between potential and work}$$

2.8.0.1 Conservation of Energy

$$W_{ab} = K_b - K_a = -W_{ab}^{ext} = q[\varphi(r_a) - \varphi(r_b)]$$

$$K_b + q\varphi(r_b) = K_a + q\varphi(r_a) \qquad \text{Conservation of Energy}$$

Conservatino of Energy

$$K_b + q\varphi(r_b) = K_a + q\varphi(r_a)$$

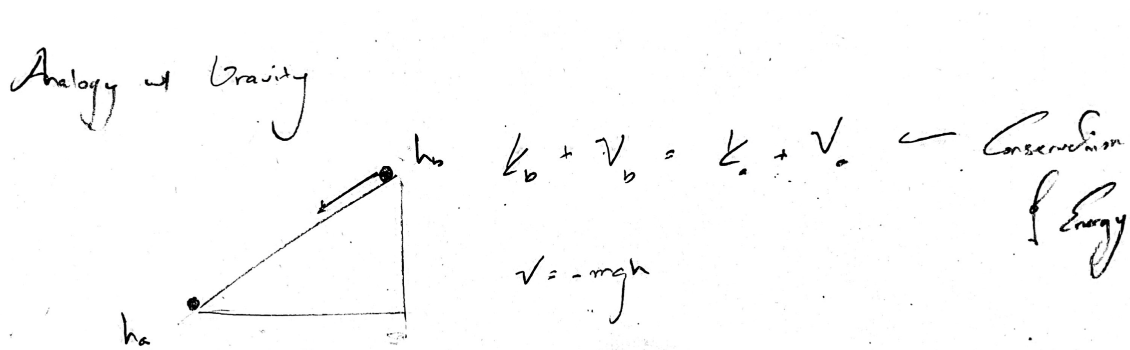

Figure 25: Analogy with Gravity