Techniques for solving Taylor Series

A Multiplying Series by Polynomial or another Series

Given $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, find series for $(x+1)\sin(x)$

$$(x+1)\sin(x) = (x+1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$$= x + x^2 - \frac{x^3}{3!} - \frac{x^4}{3!} + \dots$$

Example

Calculate Series for $e^x\cos(x)$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos(x) = \sum \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

$$e^x\cos(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$a_4 = \frac{1}{2!}\left(-\frac{1}{2!}\right) + \frac{1}{4!} + \frac{1}{4!} = \frac{-1}{4} + \frac{1}{12} = \frac{-1}{6}$$

$a_0 = 1$ $\quad a_2 = 0$

$a_1 = 1$ $\quad a_3 = \frac{1}{3!} - \frac{1}{2!} = -\frac{1}{3}$

$$e^x\cos(x) = 1 + x + 0x^2 - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \dots$$

B. Division of Two Series or of a Series by a Polynomial

Calculate Series of $\frac{1}{x}\ln(1+x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\frac{1}{x}\ln(1+x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots \quad \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

Example

Calculate Series for $\tan(x)$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}} = \frac{\left(x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots\right)}{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)}$$

$$\begin{array}{r l}
 & x + \frac{x^3}{3} + \frac{2}{15}x^5 \dots \\
1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots & \overline{\big)\ \cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} \dots} \\
 & -\ \cancel{x} - \frac{x^3}{2!} + \frac{x^5}{4!} \\
 & \overline{\quad \cancel{\frac{x^3}{3}} - \frac{x^5}{30} \quad} \\
 & -\ \cancel{\frac{x^3}{3}} - \frac{x^5}{6} \\
 & \overline{\quad \frac{2x^5}{15} \quad \dots}
\end{array}$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$$

C Binomial Series

$$\binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

Calculate Series for $1/(1+x)$

$$\frac{1}{1+x} = (1+x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} x^n = 1 - x + \frac{(-1)(-2)}{2!} x^2 + \frac{(-1)(-2)(-3)}{3!} x^3 + \dots$$

$$= 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-x)^n$$

Example

$\sqrt{1+x} = ?$

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

D. Substitution

Calculate Series for e^{-x^2}

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots$$

$$e^{\tan(x)} = 1 + \left(x + \frac{x^3}{3!} + \dots\right) + \frac{1}{2!}\left(x + \frac{x^3}{3!} + \dots\right)^2$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{2} + \frac{3}{8}x^4 + \dots$$

E. Combination of Methods

Calculate Series for $\tan^{-1}(x)$ given that

$$\int_0^x \frac{1}{1+t^2}\,dt = \tan^{-1}(x)$$

let $x = t^2 \implies \frac{1}{1+t^2} = (1+x)^{-1} = \sum_{n=0}^{\infty}\binom{-1}{n}x^n = 1 - x + \frac{(-1)(-1-1)}{2!}x^2 + \frac{(-1)(-1-1)(-1-2)}{3!}x^3$

$$= 1 - x + x^2 - x^3 + \dots$$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \dots$$

$$\int_0^x \frac{1}{1+t^2}\,dt = \left. t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots \right|_0^x$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$