Physics 89

Problem Set #4

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1 Distances in Hilbert Space of functions

$$\|f-g\| = \sqrt{\|f-g\|^2} = \sqrt{\int_{-1}^{1} (f(x)-g(x))^2\, dx}$$

$f(x) = 1$ / $g(x) = x$ / $h(x) = x^2$

(c) $\|f-g\| = \sqrt{\int_{-1}^{1} (1-x)^2\, dx} = \sqrt{\int_{-1}^{1} (1-2x+x^2)\, dx} = \underline{\underline{\sqrt{8/3}}}$

(b) $\|h-g\| = \sqrt{\int_{-1}^{1} (x^2-x)^2\, dx} = \sqrt{\int_{-1}^{1} (x^4-2x^3+x^2)\, dx} = \underline{\underline{\sqrt{16/15}}}$

(a) $\|f-h\| = \sqrt{\int_{-1}^{1} (1-x^2)^2\, dx} = \sqrt{\int_{-1}^{1} (1-2x^2+x^4)\, dx} = \underline{\underline{\sqrt{16/15}}}$

Triangle Equalities

c

$a+b \geq c \Longrightarrow \quad 8/3 + 16/15 \geq 16/15$ ✓

$a+c \geq b \Longrightarrow \quad 8/3 + 16/15 \geq 16/15$ ✓

$b+c \geq a \Longrightarrow \quad 2\sqrt{16/15} \geq \sqrt{8/3}$ ✓

Expanding a function in Legendre Polynomials

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x)$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$C_n = \frac{\langle P_n | f \rangle}{\langle P_n | P_n \rangle} = \frac{2n+1}{2}\int_{-1}^{1} P_n(x) f(x)\, dx$$

let $f(x) = \cos\left(\frac{\pi x}{2}\right)$

a) $C_0 = \frac{1}{2}\int_{-1}^{1} \cos\left(\frac{\pi x}{2}\right) dx = \frac{1}{2}$ $\frac{1}{2}\int_{-1}^{1} \cos(u) \cdot \frac{2}{\pi}\, du = \frac{1}{\pi}\sin\left(\frac{\pi x}{2}\right)\Big|_{-1}^{1} = \underline{\underline{\frac{2}{\pi}}}$

$u = \frac{\pi x}{2} \Rightarrow du = \frac{\pi}{2}dx \quad dx = \frac{2}{\pi}du \Rightarrow$

$$C_1 = \frac{3}{2}\int_{-1}^{1} x\cos\left(\frac{\pi x}{2}\right) dx = \frac{3}{\pi^2}\left(\pi x \sin\left(\frac{\pi x}{2}\right) + 2\cos\left(\frac{\pi x}{2}\right)\right)\Bigg|_{-1}^{1} = \underline{\underline{0}}$$

$$C_2 = \frac{5}{2}\int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\cos\left(\frac{\pi x}{2}\right) dx = \underline{\underline{\frac{10\pi^2 - 120}{\pi^3}}}$$

$$\|f\| = \sqrt{\int_{-1}^{1} \cos^2\left(\frac{\pi x}{2}\right) dx} = \sqrt{\frac{\sin(\pi x) + \pi x}{2\pi}\Bigg|_{-1}^{1}} = \underline{\underline{1}}$$

$$h(x) = \frac{2}{\pi} + \frac{10\pi^2 - 120}{\pi^3}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$h(x) = \frac{2}{\pi} + \frac{10\pi^2 - 120}{\pi^3}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$f(0) = 1$ $\qquad h(0) = 0.980$

$f(0.5) = \frac{1}{\sqrt{2}} \approx 0.707$ $\qquad h(0.5) = 0.723$ $\qquad$ Pretty Similar!

$f(1) = 0$ $\qquad h(1) = -0.05$

d) $\|f-h\| = \sqrt{\int_{-1}^{1}(f(x) - h(x))^2\,dx}$

$$= \sqrt{\int_{-1}^{1}\left(\frac{\cos(\pi x)}{2} - \left(\frac{2}{\pi} + \frac{10\pi^2 - 120}{\pi^3}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\right)\right)^2 dx}$$

$$\int_{-1}^{1}\left(\cos\left(\frac{\pi x}{2}\right) - C_0 P_0(x) - C_2 P_2(x)\right)^2 dx$$

$$= \int_{-1}^{1}\left(-2C_0P_0\cos\left(\frac{\pi x}{2}\right) - 2C_2P_2\cos\left(\frac{\pi x}{2}\right) + \cos^2\left(\frac{\pi x}{2}\right) + C_0^2\cancel{P_0(x)^2} + 2C_0\cancel{P_0}C_2P_2 + C_2^2P_2^2\right)dx$$

$$= \int_{-1}^{1} -2C_0\cancel{P_0}\cos\left(\frac{\pi x}{2}\right) - 2C_2P_2\cos\left(\frac{\pi x}{2}\right) dx + \overbrace{\left(\frac{x}{2} + \cancel{\frac{1}{2\pi}\sin(\pi x)}^{\,0}\right)\Big|_{-1}^{1}}^{1}$$

$$+ \int_{-1}^{1} C_0^2 + 2C_0C_2P_2(x) + C_2^2P_2^2\,dx$$

$$-\int_{-1}^{1}\cos\left(\frac{\pi x}{2}\right)dx - 2C_2\int_{-1}^{1}P_2(x)\cos\left(\frac{\pi x}{2}\right)dx + 1 + \int_{-1}^{1}\frac{1}{4} + C_2P_2(x) + C_2^2P_2^2(x)\,dx$$

$$-\left(\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right)\Big|_{-1}^{1} - \frac{20\pi^2 - 240}{\pi^3}\int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\cos\left(\frac{\pi x}{2}\right)dx + 1$$

$$+ \frac{1}{4}x\Big|_{-1}^{1} + \frac{10\pi^2 - 120}{\pi^3}\int_{-1}^{1}P_2(x)\,dx + \left(\frac{10\pi^2 - 120}{\pi^3}\right)^2\int_{-1}^{1}P_2(x)^2\,dx$$

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$$-\left(\frac{=}{\pi}\sin\left(\frac{\pi x}{2}\right)\right)\Bigg|_{-1}^{1} - \frac{20\pi - 240}{\pi^3}\int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\cos\left(\frac{\pi x}{2}\right)dx + \frac{3}{2}$$

$$+ \frac{10\pi^2 - 120}{\pi^3}\int_{-1}^{1} P_2(x)\,dx + \left(\frac{10\pi^2 - 120}{\pi^3}\right)^2\int_{-1}^{1} P_2(x)^2\,dx$$

$$= -\frac{4}{\pi} - \frac{20\pi - 240}{\pi^3}\int_{-1}^{1}\frac{3}{2}x^2\cos\left(\frac{\pi x}{2}\right)dx - \frac{1}{2}\int_{-1}^{1}\cos\left(\frac{\pi x}{2}\right)dx + \frac{3}{2}$$

$$+ \frac{10\pi^2 - 120}{\pi^3}\left(\int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)dx + \int_{-1}^{1}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)^2 dx\right)$$

$$= -\frac{4}{\pi} + \frac{3}{2} - \frac{60\pi - 720}{2\pi^3}\left(\overbrace{\frac{8}{\pi^2}x\cos\left(\frac{\pi x}{2}\right)}^{0} + \frac{2}{\pi^3}(\pi^2x^2 - 8)\sin\left(\frac{\pi x}{2}\right)\right)\Bigg|_{-1}^{1}$$

$$-\frac{1}{2}\left(\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right)\Bigg|_{-1}^{1} + \overbrace{\frac{10\pi^2 - 120}{\pi^3}\left(\frac{x^3}{2} - \frac{1}{2}x\right)\Bigg|_{-1}^{1}}^{0}$$

$$+\left(\frac{9x^5}{20} + \frac{1}{4}x + \frac{x^5}{2}\right)\Bigg|_{-1}^{1} = \boxed{2.666}$$

3. Generating function for Legendre Polynomials

$$P_n(x) = \sum_{n=0}^{\infty} P_n(x)t^n, \quad -1 < x < 1 \implies P_n(x) = \frac{1}{\sqrt{1 - 2xt + t^2}}$$

(a) $1 - 2xz + z^2 = 0 \implies \dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$

Radius of Convergence $\dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$ for fixed $-1 < x < 1$

(b) $$\int_{-1}^{1}\left(\sum_{n=0}^{\infty} P_n(x)\,t^n\right)\left(\sum_{m=0}^{\infty} P_m(x)\,s^m\right) = \int_{-1}^{1}\frac{1}{\sqrt{(1-2xt+t^2)(1-2xs+s^2)}}\,dx$$

$$\int_{-1}^{1}\frac{dx}{\sqrt{(1-2xt+t^2)(1-2xs+s^2)}} = \frac{1}{\sqrt{st}}\left(\log(1+\sqrt{st}) - \log(1-\sqrt{st}\right)$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\int_{-1}^{1} P_n(x)P_m(x)\,dx\right)t^n s^m.$$

$$\log(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \cdots \quad / \quad \log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots$$

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = ?$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\int_{-1}^{1} P_n(x)P_m(x)\,dx\right)t^n s^m = \frac{1}{\sqrt{st}}\log(1+\sqrt{st}) - \frac{1}{\sqrt{st}}\log(1-\sqrt{st})$$

$$= \frac{1}{\sqrt{st}}\left(u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} \cdots\right) - \frac{1}{\sqrt{st}}\left(-u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} \cdots\right)$$

$$= \frac{1}{\sqrt{st}}\left(\left(u - \cancel{\frac{u^2}{2}} + \frac{u^3}{3} - \cancel{\frac{u^4}{4}} \cdots\right) + \left(u + \cancel{\frac{u^2}{2}} + \frac{u^3}{3} + \cancel{\frac{u^4}{4}} \cdots\right)\right)$$

$$= \frac{2}{\sqrt{st}}\left(u + \frac{u^3}{3} + \frac{u^5}{5} + \cdots\right)$$

$$= \frac{2}{\sqrt{st}}\sum_{n=0}^{\infty}\frac{u^{2n+1}}{2n+1}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{-1}^{1} P_n(x)P_m(x)\,dx\; t^n s^m = \frac{2}{t^{\frac{2n+1}{2}}s^{\frac{2n+1}{2}}}\sum_{n=0}^{\infty}\frac{\sqrt{st}^{\,2n+1}}{2n+1} = \frac{2t^{-\frac{2n+1}{2}}s^{-\frac{2n+1}{2}}}{t^n s^m}\sum_{n=0}^{\infty}\frac{\sqrt{st}^{\,2n+1}}{2n+1}$$

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 2t^{-\frac{4n+1}{2}}s^{-\frac{4n+1}{2}}\sum_{n=0}^{\infty}\frac{\sqrt{st}^{\,2n+1}}{2n+1}$$

4 Laguerre Polynomials

$$\langle f | g \rangle = \int_0^\infty e^{-x} f(x) g(x)\, dx$$

$$f_0 = 1 \quad f_1 = x \quad f_2 = x^2 \quad f_3 = x^3$$

$$u_1 = f_0 = 1$$

$$u_2 = f_1 - \frac{\langle f_1, u_1 \rangle}{\langle u_1 | u_1 \rangle} u_1$$

$$u_3 = f_2 - \frac{\langle f_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle f_2, u_2 \rangle}{\langle u_2 | u_2 \rangle} u_2$$

$$u_4 = f_3 - \frac{\langle f_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle f_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle f_3, u_3 \rangle}{\langle u_3 | u_3 \rangle} u_3$$

$$u_2 = x - \frac{\int_0^\infty e^{-x}(x)(1)\, dx}{\int_0^\infty e^{-x}\, dx} = \underline{\underline{x - 1}}$$

$$u_3 = x^2 - \frac{\int_0^\infty e^{-x} x^2\, dx}{\int_0^\infty e^{-x}\, dx} - \frac{\int_0^\infty e^{-x}(x-1)x^2\, dx}{\int_0^\infty e^{-x}(x-1)^2\, dx} = x^2 - 2 - 4 = \underline{\underline{x^2 - 6}}$$

$$u_4 = x^3 - \frac{\int_0^\infty e^{-x} x^3\, dx \to 6}{\int_0^\infty e^{-x}\, dx \to 1} - \frac{\int_0^\infty x^3(x-1)e^{-x}\, dx \to 18}{\int_0^\infty e^{-x}(x-1)^2\, dx \to 1} - \frac{\int_0^\infty e^{-x} x^3(x^2-6)\, dx \to \cancel{84}\ 7}{\int_0^\infty e^{-x}(x^2-6)^2\, dx \to \cancel{36}\ 3} = \underline{\underline{x^3 - 65/3}}$$

$u_1 = 1$

$u_2 = x - 1$

$u_3 = x^2 - 6$

$u_4 = x^3 - \frac{65}{3}$

$$\|u_1\| = \sqrt{\langle u_1, u_1 \rangle} = \sqrt{\int_0^\infty e^{-x}\,dx} = 1 \Longrightarrow \underline{\hat{e}_1 = 1}$$

$$\|u_2\| = \sqrt{\langle u_2, u_2 \rangle} = \sqrt{\int_0^\infty e^{-x}(x-1)^2\,dx} = 1 \Longrightarrow \underline{\hat{e}_2 = x - 1}$$

$$\|u_3\| = \sqrt{\langle u_3, u_3 \rangle} = \sqrt{\int_0^\infty e^{-x}(x^2-6)^2\,dx} = 6 \Longrightarrow \underline{\hat{e}_3 = \frac{x^2 - 6}{3}}$$

$$\|u_4\| = \sqrt{\langle u_4, u_4 \rangle} = \sqrt{\int_0^\infty e^{-x}\left(x^3 - \frac{65}{3}\right)^2} = \frac{8365}{9} \Longrightarrow \hat{e}_4 = \left(x^3 - \frac{65}{3}\right)\left(\frac{9}{8365}\right)$$

$$\underline{= \frac{9x^3}{8365} - \frac{585}{25095}}$$