Physics 137A

Review Lecture 3.

Classical Correspondences, Superposition, Particles as wave packets. —

Planck & Einstein found that light previously thought as a wave, can behave on the subatomic scale like a ~~collection~~ of particles, each with an energy associated with $\hbar\omega$.

de Broglie argued that a nonrelativistic electron a particle, acts like a wave with a characteristic wavelength $\lambda_e = \frac{h}{m_e v}$.

In the "old" quantum mechanics of Bohr — he raised several conceptual difficulties:

- The angular momentum of an orbiting electron is quantised in units of $\hbar$, for the orbits to be "stationary"

Bohr's condition for motion to be circular can be written

$$\oint p\,dq = nh \implies m_e v(2\pi r) = nh$$

$$m_e v r = \frac{h}{2\pi} n = n\hbar.$$

de Broglie gave us another way of thinking about Bohr's circular orbits

$$\oint dq = n\lambda_e \implies 2\pi r = n\left(\frac{h}{m_e v}\right)$$

$$m_e v r = n\hbar.$$

The stationary orbits correspond to an integer # of wavelengths.

The stationary orbit is therefore accounted for by de Broglie by stipulating that $e^-$ behave as waves with a given wavelength $\lambda_e$.

$$\lambda_e = \frac{h}{m_e v}$$

Correspondence Principle —

Classical Mechanics must emerge as a limit from quantum mechanics w/ large #s.

Classical Mechanics also emerges from quantum mechanics as $h \to 0$.

* QM becomes classical as $n \to \infty$, $h \to 0$

The principle of superposition & wave packets —

The accumulation of physics in the early 20th century indicating that light: particle $\leftrightarrow$ wave, led us to a path where particles w/ definite positions & momentum gave way to a description in terms of waves & wave packets. Let's review approaches of some of the properties achievable via waves.

Familiar waves, like those from sound or water, in 1D take the form

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

We look for solutions in the form of an oscillating wave.

$$\psi(x,t) = \phi(x)e^{i\omega t} \implies \frac{\partial^2 \phi(x)}{\partial x^2} = \frac{-\omega^2}{c^2}\phi(x)$$

to find

$\phi(x) = e^{ikx}$ where $c^2k^2 = \omega^2$ so that $\psi(x,t) = e^{i(kx-\omega t)}$ w/ $k(\omega) = \pm\frac{\omega}{c}$

A property of this equation is that it is linear in $\psi$, which yields to the principle of superposition:

if $\psi_1(\vec{x},t)$ & $\psi_2(\vec{x},t)$ satisfy the wave equation, so does

$$\psi_1(\vec{x},t) + \psi_2(\vec{x},t)$$

This property allows us to build wave packets, as one would get by throwing a rock into the middle of a quiet pond. It would give us difficulty trying to envision quantum mechanics w/o this property.

The correspondence principle requires us to be able to create localized particles in quantum mechanics, & we know how to build localized wave packets from waves through Fourier analysis:

$$\psi(x,t=0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ikx}\,dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x,t=0)e^{-ikx}\,dx$$

for example,

$$\psi(x,t=0) = e^{-x^2/a^2} \implies \psi(x,t=0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2k^2/4}e^{ikx}\,dk$$

However, this infinite sum of plane waves will not be a solution of our quantum mechanics wave equation unless our superposition principle holds.

## Wave Packets & Uncertainty Relationships

There is a size scale associated w/ our coordinate space wave packet, $\Delta x \sim a$. The smaller $a$ — the more localized in $x$ — the broader range of contributing momentum-space waves.

$$e^{-a^2k^2/4} = e^{-k^2/(2/a)^2} \implies \Delta k = \frac{2}{a} \quad \text{Consequently } \Delta x \Delta k \sim 1.$$

The more localized wave packets are in coordinate space, the more they delocalize in momentum space.

As the de Broglie relation gives us $p = \frac{h}{\lambda} = \frac{h}{2\pi}k$ so that $\Delta k = \frac{\Delta p}{\hbar}$, Quantum Mechanics has an uncertainty principle relating the product of coordinate & momentum uncertainties to $\hbar$.

### Heisenberg's Uncertainty Principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

## Schrödinger's Equation

These kinds of consideration helped Schrödinger create a wave equation that might account for the apparent dual nature of the $e^-$ & other elementary particles. In 1D

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

So what does this equation mean? Starting at the LHS, the potential is done while the derivative term can be rewritten in a way that clarifies units.

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} = \frac{-\hbar^2 c^2}{2mc^2}\cdot\frac{\partial^2}{\partial x^2}$$

As $(\hbar c)^2$ has units of $(\text{Energy}\cdot\text{Distance})^2$, $mc^2$ is an energy, and $\frac{\partial^2}{\partial x^2}$ has units of $(\text{Distance})^{-2}$, the first term on the LHS must be energy. More specifically, Kinetic Energy.

Classically KE is $\frac{P^2}{2m}$, and since $\frac{P^2}{2m} + V(x) = E_{total}$ the RHS must be conveying $E$. This requirement that Schrödinger's Equation w/ its differential operators correspond w/ classical mechanics allows us to define the Equations differential operators

Schrödinger's Equation Operators

$$\hat{P} \longleftrightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\hat{E} \longleftrightarrow i\hbar\frac{\partial}{\partial t}$$

Schrödinger's Equation can be rewritten

$$\left[\frac{\hat{P}^2}{2m} + V(x)\right]\psi(x,t) = \hat{E}\psi(x,t)$$

$\hat{P}$ & $\hat{E}$ are differential operators that act on the wave function

Consider a free particle, so $V \to 0$, w/ a candidate solution

$$\Psi(x,t) = e^{-i(Et - px)/\hbar}$$

where $E$ & $p$ are numbers. We note that

$$\hat{p}\Psi(x,t) = p\Psi(x,t) \qquad \hat{E}\Psi(x,t) = E\Psi(x,t)$$

while plugging into Schrödinger's Equation yields the constraint

$$\frac{p^2}{2m} = E$$

Therefore, there are an infinite set of plane-wave functions/solutions that are labeled by the ~~set~~ $p$, with $E(p) = \frac{p^2}{2m}$

The plane-wave solution of the Schrödinger Equation has a characteristic wavelength. The length of a wave corresponds to the distance required to change the phase by $2\pi$, at a fixed time $t$. That is,

$$2\pi = \frac{p\Delta x}{\hbar} \equiv \frac{p\lambda}{\hbar} \implies \lambda = \frac{2\pi\hbar}{p} = \frac{h}{p}.$$

de Broglie's wavelength.