Problem Set #1: Due: Friday midnight, Jan 26

You are welcome to use Mathematica or similar tools in doing these problems

1. *Classical hydrogen and its radiation:*

In this problem we will consider the emission of radiation in a classical hydrogen atom: radiation by the electron held in circular orbit by the Coulomb force will lead to an in-spiral of the electron, toward the proton at its center. We will explore how quickly this will happen. Although the calculation is classical, we will use the same constants we will later employ in treating the hydrogen atom quantum mechanically, namely the fine structure constant α and the Bohr radius a_0:

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \sim \frac{1}{137} \qquad a_0 = \frac{\hbar}{\alpha m_e c} \sim 5.29 \times 10^{-11} \text{ m}$$

a) The classical electron is held in circular orbit at radius r by the Coulomb force, which counterbalances the centripetal force, producing an acceleration a of the electron:

$$m_e\, a = m_e \frac{v^2}{r} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} \equiv \frac{\alpha}{r^2} \hbar c$$

Solve for the electron velocity v and acceleration a as functions of r, writing the radius r in your expressions in terms of the dimensionless ratio $\frac{a_0}{r}$

b) The hydrogen atom Hamiltonian is

$$E = \frac{1}{2} m_e v^2 - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r} = \frac{1}{2} m_e v^2 - \frac{\alpha}{r} \hbar c$$

Write the energy as a function of $\frac{a_0}{r}$.

c) The rate of energy loss by an accelerating charge is given by the Larmor formula

$$\frac{dE}{dt} = -\frac{2}{3}\, a^2\, \frac{\alpha \hbar}{c^2}$$

where a is the acceleration. Express the classical hydrogen atom energy loss rate in terms of $\frac{r}{a_0}$.

d) Calculate the time Δt for the electron to complete one orbit, the energy loss ΔE over the period Δt, and the fractional energy loss $\frac{\Delta E}{E}$ over one period, all expressed in terms of r/a_0. Even though the electron, as it loses energy, must spiral inward, is the approximation of a circular orbit (used to calculate energy loss) a reasonable one, for an electron in an orbit near $r \sim a_0$? At what r would this approximation be questionable, e.g., where the fractional energy loss per orbit grows, say, to ≥ 0.1? Presumably our formulas will begin to break down around this radius.

e) Suppose the atom starts out in a circular orbit at $r = a_0$. Roughly estimate the time required for the electron to radiate an amount of energy equal to its original binding energy. It would be sufficient to calculate the energy loss per orbit at $r = a_0$, the number of orbits at that radius that would be required to radiate the specified energy, and the time required to execute those orbits – neglecting the fact that the radius is changing. This exercise will underscore why the stability of the hydrogen atom is a puzzle in classical physics.

f) Repeat part e), again assuming that orbits are always effectively circular, but accounting for the changing radius. This will require you to relate the Larmor energy loss dE/dt to the change in the classical hydrogen atom energy with radius dE/dr, yielding a differential equation one can solve. Compare the answers from parts e) and f).

2. *Bose Einstein Condensation:* Many-particle systems that are quantum mechanically entangled exhibit some fascinating behaviors. Bose-Einstein condensation (BEC) occurs when the typical nearest-neighbor separation of atoms in a cold gas is comparable to or smaller than the de Broglie wavelength of an atom. What atomic density (in atoms/cm^3) is required to achieve BEC, if the atoms have mass number A=100, the temperature T is 10^{-7}K, and the atom kinetic energy is 3/2 kT?

3. *Black-body radiation:*

a Convert the Planck energy density differential $\rho_E(\nu)d\nu$ into a corresponding differential $\rho_E(\lambda)d\lambda$, where λ is the wavelength. Plot $\rho_E(\lambda)$ and the corresponding Raleigh-Jeans formula for a temperature of 6000K, which is about the temperature of the Sun's photosphere. Indicate on the spectrum the band of wavelengths corresponding to visible light.

b Wein's law for wavelength states that the maximum of $\rho_E(\lambda)$ occurs at $\lambda_{\max} = b/T$. Determine b in terms of relevant parameters such as h or k_B and a number you evaluate. Wein's law for $\rho_E(\nu)$ is $\nu_{\max} = \beta T$, so similarly determine β. Compare b and β at T=6000 K.

c The Raleigh-Jeans law failed spectacularly to explain the Stephan-Boltzmann relation $P = \sigma T^4$, the power emitted per unit surface area from a black body. Using Planck's formula for the energy density inside a cavity, derive the Stefan-Boltzmann relation, obtaining a formula for the Stefan-Boltzmann constant σ. Accomplish this by punching a hole in the cavity of area A, determining the energy that flows through the hole in a time δt. The relevant volume inside the cavity is a thin slab of cross section A and thickness $c\delta t$. To complete the calculation you must figure out what fraction of the energy in that slab flows through the opening in time δt. Assume the radiation flow in the cavity is isotropic. You should be able to see that at the far surface of the slab (so $c\delta t$ away from the hole) only radiation flowing directly toward A will be able to escape in time δt. In contrast, at the near surface of the slab, right at the hole, radiation flowing through a solid angle of 2π will escape – so half the total energy density in that part of the slab. The figure below shows a case intermediate between these two limits.

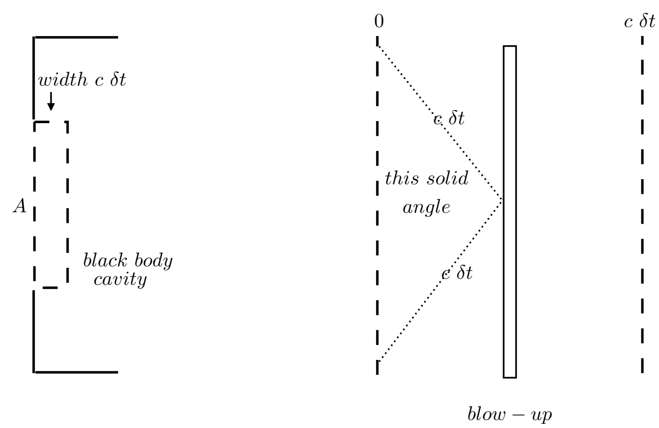

Figure 1: Radiation passing through the indicated dV subsection of the slab.

Integrate between these two surfaces, at each step figuring out the contributing solid angle, to determine the fraction of the energy that escapes, thereby completing the calculation. You may need to remember (or look up) the relationship between a cone and the solid angle it subtends.

d The solar constant, the total radiated solar power per unit (perpendicular) area measured at earth, is 1.361 kilowatts/m^2. The earth-Sun separation is $\sim 1.47 \times 10^{11}$ m. The solar radius is $\sim 6.96 \times 10^8$ m. Assuming the Sun radiates as a black body, estimate its surface temperature.

4. *de Broglie wavelength:* Compare the de Broglie wavelength for an electron in orbit at $r = a_0$

(you calculated the velocity in problem 1) to that for Christian McCaffrey, who has a mass of 95 kg and peak speed of 9.81 meters/sec.