Single Systems – Lesson 01

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Notation

– X refers to the system being considered
– Σ refers to the set of classical states of X

Here are a few examples:

- if X is a bit, $\Sigma = 0, 1$ – the *binary alphabet*
- if X is a six-sided die, $\Sigma = 1, 2, 3, 4, 5, 6$
- if X is an electric fan switch, $\Sigma =$ high, medium, low, off

1 Classical Information

1.1 Classical States & Probability Vectors

In Quantum Computing (QC) our knowledge of X is uncertain. We thus represent our knowledge of the classical state of X by assigning *probabilities* to each classical state resulting in a *probabilistic state*.

For example, suppose X is a bit. In this case, based on our past experience or what we know about X, there is a 3/4 chance its classical state is 0 and a 1/4 chance it's 1. Therefore,

$$\Pr(X = 0) = \frac{3}{4} \qquad \Pr(X = 1) = \frac{1}{4}$$

We can represent this more succinctly with a column vector:

$$\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$$

The probability of the bit being 0 is placed at the top; the probability of the bit being 1 is placed at the bottom. We can represent any probabilistic state via a column vector satisfying two properties:

Probabilistic Vector Requirements

1. All entries of the vector are *nonnegative numbers*
2. The sum of the entries is equal to 1

1.2 Measuring Probabilistic States

Intuitively, we can never "see" a system in a probabilistic state; a measurement always yields exactly one of the allowed states.

Measuring changes our knowledge of the system, and therefore changes the probabilistic state we associate with the system. If we recognize that X is in the classical state $a \in \Sigma$, then the new probability vector representing our knowledge of X becomes a vector having 1 in the entry corresponding to a and 0 for all other entries.

$$\begin{pmatrix} 0.3 \\ 0.1 \\ 3.14 \\ 2.72 \\ \vdots \end{pmatrix} \quad \rightarrow \quad \boxed{\text{Measurement}} \quad \rightarrow \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

1.3 Standard Basis Vectors

We can define any probabilistic state vector as a *linear combination* of standard basis vectors. For example, assuming the system we have in mind is a bit, the standard basis vectors are given by

Computational Basis States

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For example, we have

$$\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} = \frac{3}{4}|0\rangle + \frac{1}{4}|1\rangle$$

1.4 Operations

Deterministic Operations

Deterministic Operations transform each classical state $a \in \Sigma$ into $f(a)$ for some function f of the form $f : \Sigma \rightarrow \Sigma$.

For example, if $\Sigma = 0, 1$, there are four functions of the form f_1, f_2, f_3, f_4 which can be represented as follows:

$$\begin{aligned} f_1(0) &= 0 \quad & f_1(1) &= 0 \\ f_2(0) &= 0 \quad & f_2(1) &= 1 \\ f_3(0) &= 1 \quad & f_3(1) &= 0 \\ f_4(0) &= 1 \quad & f_4(1) &= 1 \end{aligned}$$

The first and last of these functions are *constant*, where the output remains constant regardless of input. The middle two are *balanced* where the two possible output values occur the same # of times. The function f_2 is the **identity function** where $f(a) = a$ for $a \in \Sigma$. The function f_3 is the *NOT* function where each input is flipped for an output.

Every deterministic operation on probabilistic states can be represented as a matrix, where

$$M|a\rangle = |f(a)\rangle$$

for every $a \in \Sigma$. Such a matrix always exists and is unique. For the above constant and balanced functions:

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Deterministic matrix operations always have exactly one 1 in each column, and 0 for all other entries.

Let us denote $\langle a|$ the *row* vector having a 1 in the entry corresponding to a and 0 for all other entries, for each $a \in \Sigma$.

For example if $\Sigma = 0, 1$,

$$\langle 0| = (1 \quad 0) \quad \text{and} \quad \langle 1| = (0 \quad 1)$$

If we perform matrix multiplication on a column vector defined by $|b\rangle$ and a row vector defined by $\langle a|$, we obtain a square matrix having a 1 in the entry corresponding to the (b, a) location in the matrix and 0 everywhere else. For example,

$$|0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Using this notation, we can express M corresponding to the function f as

$$M = \sum_{a \in \Sigma} |f(a)\rangle\langle a|$$

If we switch the order of multiplication – $\langle a||b\rangle$, we obtain a 1 x 1 scalar. For the sake of tidiness we write the product as $\langle a|b\rangle$. We will later define $\langle a|b\rangle$ as the *inner product* of a and b.

$$\langle a|b\rangle = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$