Multiple Systems – Lesson 02

Deval Deliwala

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Contents

We can always choose to view multiple systems *together* as if they form a single, compound system – to which the discussion in the previous section applies.

1 Classical Information

Recognizing how mathematics works in the familiar setting of classical information is helpful in understanding why quantum information is described the way it is.

1.1 Classical States via the Cartesian Product

We will start with classical states of multiple systems. We will begin by discussing just two systems and then generalize further.

To be precise, let us suppose that X is a system whose classical state set is Σ and Y is a second system having classical state set Γ. Because these are *classical state sets*, our assumption is that Σ and Γ are both finite and nonempty. It could be that $\Sigma = \Gamma$, but this is not necessarily so.

Now imagine the two systems X and Y are placed side-by-side, with X on the left and Y on the right. We can views these two systems as if they form a single system (X, Y) or XY.

The set of classical states of XY is the *Cartesian Product* of Σ and Γ, which is the set defined as

Cartesian Product

$$\Sigma \times \Gamma = (a, b) : a \in \Sigma \text{ and } b \in \Gamma$$

The Cartesian Product is the mathematical notion of viewing an element of one set and an element of a second set together, as if they form a single set.

For more than two systems, the situation generalizes in a natural way. If we suppose that $X_1, \cdots, X_n$ are systems having classical state sets $\Sigma_1, \cdots, \Sigma_n$, respectively for any positive integer n, the classical state set of the n-tuple $X_1 X_2 \cdots X_n$, viewed as a single joint system, is the Cartesian Product

$$\Sigma_1 \times \cdots \times \Sigma_n = (a_1, \cdots, a_n) : a_1 \in \Sigma_1 \cdots a_n \in \Sigma_n$$

1.2 Representing states as strings

It is convenient to write a classical state $(a_1, \cdots, a_n)$ as a string '$a_1 \cdots a_n$'. The notion of a string, which is fundamentally important concept in CS, is formalized in mathematical terms through the cartesian product.

For example suppose that $X_1, \cdots, X_{10}$ are bits, so the classical state sets of these systems are all the same

$$\Sigma_1 = \Sigma_2 = \cdots = \Sigma_{10} = 0, 1$$

The set $0, 1$ is referred to as the *binary alphabet*. There are $2^{10} = 1024$ classical states of the joint system $X_1 X_2 \cdots X_{10}$ which are elements of the set

$$\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_{10} = 0, 1^1 0.$$

Written as strings, these classical states look like this:

$$\begin{gathered} 0000000000 \\ 0000000001 \\ 0000000010 \\ 0000000011 \\ 0000000100 \\ \vdots \\ 1111111111 \end{gathered}$$

For the classical state 0001010000 we see that X_4 and X_6 are in the state 1, while all other systems are in the state 0.

1.3 Probabilistic States

Probabilistic states associate a probability with each classical state of a system. Thus, a probabilistic state of multiple systems – viewed collectively as if they form a single system – associates a probability with each element of the Cartesian product of the classical state sets of the individual systems.

Suppose X and Y are both bits. So their corresponding classical state sets are $\Sigma = 0, 1$ and $\Gamma = 0, 1$. Here is a probabilistic state of the pair XY:

$$\begin{aligned} \Pr((X, Y) = (0, 0)) &= \frac{1}{2} \\ \Pr((X, Y) = (0, 1) &= 0 \\ \Pr((X, Y) = (1, 0) &= 0 \\ \Pr((X, Y) = (1, 1) &= \frac{1}{2} \end{aligned}$$

This probabilistic state is one in which both X and Y are random bits – each is 0 with probability 1/2 and 1 with probability 1/2 – but the classical states of the two bits always agree. This is an example of a *correlation* between these systems.

1.4 Ordering Cartesian Product State Sets

To represent a probabilistic state of multiple systems as a Cartesian product, one must decide on an ordering of the product's elements. The entries in each $n-$ tuple are viewed as being ordered by significance that *decreases from left to right.*

The Cartesian Product $1, 2, 3 \times 0, 1$ is ordered like this:

$$(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)$$

The probabilistic state represented by the probability vector above

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Where from top to bottom we get the probability associated with 00, then 01, then 10, then 11.

1.5 Independence of Two Systems

Intuitively speaking, two systems are independent if learning the classical state of either system has no effect on the probabilities associated with the other.

Suppose X and Y are systems having classical state sets Σ and Γ respectively. They are said to be *independent* if it is the case that

$$\Pr((X, Y) = (a, b)) = \Pr(X = a)\Pr(Y = b)$$

for every $a \in \Sigma$ and $b \in \Gamma$.

To express this in terms of probability vectors, assume the probabilistic state of (X, Y) is described as a probability vector written in Dirac notation as

$$\sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|ab\rangle$$

The above condition is equivalent to the existence of two probability vectors

$$|\phi\rangle = \sum_{a\in\Sigma} q_a|a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} r_b|b\rangle$$

representing the probabilities associated with the classical states of X and Y such that

$$p_{ab} = q_a r_b$$

For example, the probabilistic state of a pair of bits (X, Y) represented by the vector

$$\frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

is one in which X and Y are independent. Specifically the condition required for independence is true for the probability vectors

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

For example, to match the 00 entry we need $\frac{1}{6} = \frac{1}{4} \times \frac{2}{3}$. The other entries can be verified in a similar manner.

On the other hand, the probabilistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

does not represent independence between X and Y.

Suppose that there did exist probability vectors $|\phi\rangle$ and ψ for which $p_a b = q_a r_b$ is satisfied for every choice a and b. It would then necessarily be that

$$q_0 r_1 = \Pr((X, Y) = (0, 1)) = 0$$

This implies that either $q_0 = 0$ or $r_1 = 0$. Which means that $q_0 r_0 = 0$ or $q_1 r_1 = 0$ which isn't the case. Hence, there do not exist vectors ϕ and $|\psi\rangle$ satisfying the property required for independence.

We can now define correlation precisely as a *lack of independence*.

1.6 Tensor Products of Vectors

The condition for independence just described can be expressed more succinctly through the notion of a *tensor product*. In concrete terms, given two vectors

$$|\phi\rangle = \sum_{a \in \Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b \in \Gamma} \beta_b |b\rangle$$

the tensor product $|\phi\rangle \otimes |\psi\rangle$ is a new vector over the joint set $\Sigma \times \Gamma$ defined as

Tensor Product

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_a \beta_b |ab\rangle$$

Equivalently the vector $|\pi\rangle = |\phi\rangle \otimes |\psi\rangle$ is defined by the equation

$$\langle ab|\pi\rangle = \langle a|\phi\rangle\langle b|\psi\rangle$$

being true for every $a \in \Sigma$ and $b \in \Gamma$
In this situation it is said that $|\pi\rangle$ is a *product state* or *product vector*. We often just write $|\phi\rangle|\psi\rangle$ instead of $|\phi\rangle \otimes |\psi\rangle$. The tensor product for two column vectors is defined as follows:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_1\beta_k \\ \alpha_2\beta_1 \\ \vdots \\ \alpha_2\beta_k \\ \vdots \\ \alpha_m\beta_1 \\ \vdots \\ \alpha_m\beta_k \end{pmatrix}$$

For standard basis vectors:

$$|a\rangle \otimes |b\rangle = |ab\rangle = |a, b\rangle$$

The tensor product of two vectors has the important property that it is *bilinear*, which means that it is linear in each of the two arguments separately, assuming that the other argument is fixed.

Bilinearity

1. Linearity in the first argument:

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes |\psi\rangle = |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle$$
$$(\alpha|\phi\rangle) \otimes |\psi\rangle = \alpha(|\phi\rangle \otimes |\psi\rangle)$$

2. Linearity in the second argument:

$$|\phi\rangle \otimes (|\psi_1\rangle + |\psi_2\rangle) = |\phi\rangle \otimes |\psi_1\rangle + |\phi\rangle \otimes |\psi_2\rangle$$
$$|\phi\rangle \otimes (\alpha|\psi\rangle) = \alpha(|\phi\rangle \otimes |\psi\rangle)$$

Scalars “float freely” within tensor products.

1.7 Independence and Tensor Products for 3 or More Systems

If $X_1, \cdots, X_n$ are systems with classical state sets $\Sigma_1, \cdots, \Sigma_n$, then a probabilistic state of the combined system $X_1 X_2 \cdots X_n$ is a *product state*

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$$

for probability vectors $|\phi_1\rangle \cdots |\phi_n\rangle$ describing probabilistic states $X_1, \cdots, X_n$

The definition of the tensor product generalizes as the following:

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$$

is defined by the equation

Tensor Product Definition

$$\langle a_1 \cdots a_n|\psi\rangle = \langle a_1|\phi_1\rangle \cdots \langle a_n|\phi_n\rangle$$

for every $a_1 \in \Sigma_1, \cdots a_n \in \Sigma_n$.

Similar to the tensor product for two vectors, the tensor product for 3 or more vectors is linear in each of the arguments individually. We say the tensor product of 3 or more vectors is *multilinear*.

For multiple systems is $X_1, ... X_n$ are *independenet*, when they are in a product state, they are actually *mutually independent*. There is also other notions of independence for 3 or more systems such as *pairwise independence*.

For any positive integer n and any classical states $a_1, ..., a_n$ we have

$$|a_1\rangle \otimes \cdot \otimes |a_n\rangle = |a_1 \cdot a_n\rangle$$

1.8 Measurements of Probabilistic States

If two bits (X, Y) are described by the following probability vector

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle,$$

then the outcome measurement of 0 for X and 0 for Y is obtained with probability $1/2$ and the outcome 11 is also obtained with probability $1/2$.

Suppose however, that we choose not to measure *every* system, just some *proper subset* of the systems. This will result in a measurement outcome for each system that gets measured, and will also affect our knowledge of the remaining systems.

Suppose that X is a system having classical state set Σ and Y is a system having classical state set Γ, and the two systems together are in some probabilistic state. We will consider what happens when we just measure X and do nothing with Y. The situation vice versa is handled symmetrically.

We know that the probability to observe a particular classical state $a \in \Sigma$ when just X is measured must be consistent with the probabilities we would obtain under the assumption that Y was also measured. That is, we must have

$$\Pr(X = a) = \sum_{b \in \Gamma} \Pr((X, Y) = (a, b))$$

Intuitively, if this formula is wrong, it would mean that the probabilities for X measurements are influenced simply by whether or not Y is also measured, irrespective of the outcome of Y. If Y happened to be in a distant location, say, another galaxy, this would allow for faster-than-light signaling, which we obviously reject.

Given the assumption that only X is measured, there still exists uncertainty over the classical state of Y. We must update our description so this uncertainty of Y is properly reflected.

The following *conditional probability* formula reflects this uncertainty.

$$\Pr(Y = b \,|\, X = a) = \frac{\Pr((X, Y) = (a, b))}{\Pr(X = a)}$$

Here, the expression $\Pr(Y = b \,|\, X = a)$ denotes the probability that $Y = b$ *conditioned* on $X = a$.

To express these formulas in terms of probability vector, consider a probability vector $|\psi\rangle$ describing the joint state of (X, Y).

$$|\psi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} p_{ab} |ab\rangle$$

Measuring X alone yields each possible outcome with probabilities

$$\Pr(X = a) = \sum_{b \in \Gamma} p_{ab}$$

Thus the vector representing the probabilistic state of X alone is given by

$$\sum_{a\in\Sigma}\left(\sum_{c\in\Gamma}p_{ac}\right)|a\rangle$$

The updated probabilistic state of Y having obtained a particular outcome $a \in \Sigma$ of the measurement of X, is represented by the probability vector:

$$|\pi_a\rangle = \frac{\sum_{b\in\Gamma} p_{ab}|b\rangle}{\sum_{c\in\Gamma} p_{ac}}$$

If X results in a classical state a, we update our description of the probabilistic state of the joint system (X, Y) to $|a\rangle \otimes |\pi_a\rangle$.

The definition of $|\pi_a\rangle$ is the *normalization* of the vector $\sum_{b\in\Gamma} p_{ab}|b\rangle$ where we divide the sum of the entries in this vector to obtain the probability vector.

1.9 Example of Classical State Measurements

For example, suppose the classical state of X is $\Sigma = 0, 1$, the classical state of Y is $\Gamma = 1, 2, 3$, and the probabilistic state of (X, Y) is

$$|\psi\rangle = \frac{1}{2}|0,1\rangle + \frac{1}{12}|0,3\rangle + \frac{1}{12}|1,1\rangle + \frac{1}{6}|1,2\rangle + \frac{1}{6}|1,3\rangle$$

Our goal is to determine probabilities of two possible outcomes – 0 and 1, and to calculate what the resulting probabilistic state of Y is for the two outcomes, assuming X is measured.

Using the bilinearity of the tensor product, we may rewrite $|\psi\rangle$ as

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{2}|1\rangle + \frac{1}{12}|3\rangle\right) + |1\rangle \otimes \left(\frac{1}{12}|1\rangle + \frac{1}{6}|2\rangle + \frac{1}{6}|3\rangle\right)$$

We have isolated the distinct standard basis vectors for the system being measured. This is always possible, regardless of what vector we start with.

Thus the probabilities of the two outcomes are given by

$$\Pr(X = 0) = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$$
$$\Pr(X = 1) = \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{5}{12}$$

The probabilistic state of Y, conditioned on each possible outcome can also be quickly inferred by normalizing the vectors in parenthesis so that these vectors become probability vectors. That is, conditioned on X being 0, the probabilistic state of Y becomes

$$\frac{\frac{1}{2}|1\rangle + \frac{1}{12}|3\rangle}{\frac{7}{12}} = \frac{6}{7}|1\rangle + \frac{1}{7}|3\rangle$$

and conditioned on the measurement of X being 1, the probabilistic state of Y becomes

$$\frac{\frac{1}{12}|1\rangle + \frac{1}{6}|2\rangle + \frac{1}{6}|3\rangle}{\frac{5}{12}} = \frac{1}{5}|1\rangle + \frac{2}{5}|2\rangle + \frac{2}{5}|3\rangle$$

1.10 Operations on Probabilistic States

We will consider *operations* on multiple systems in probabilistic states. Returning to the typical setup where we have two systems X and Y, let us consider classical operations on the compound system (X, Y). Any operation is represented by a stochastic matrix whose rows and columns are indexed by the Cartesian Product $\Sigma \times \Gamma$.

For example, suppose X and Y are bits, and consider the operation:

> If $X = 1$, then perform a NOT operation on Y.
> Otherwise do nothing.

This is a deterministic operation known as a *controlled-NOT* or CNOT operation, where X is the *control* bit that determines whether or not a NOT operation gets applied to the *target* bit Y. Here is the matrix representation of this operation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Its action on the standard basis states:

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |01\rangle \\ |10\rangle &\mapsto |11\rangle \\ |11\rangle &\mapsto |10\rangle \end{aligned}$$

Another example is the operation having this description

Perform one of the following two operations, each with probability $\frac{1}{2}$:

- Set Y to be equal to X
- Set X to be equal to Y

The matrix representation is as follows:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

The action of this operation on the standard basis vectors:

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto \frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle \\ |10\rangle &\mapsto \frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle \\ |11\rangle &\mapsto |11\rangle \end{aligned}$$

In these examples, we view two systems together as a single system. We can do this with any number of systems.

1.11 Independent Operations

Suppose we want to *independently* perform *separate operations* on the systems. For example, taking our usual setup of X and Y having classical state sets Σ and Γ, let's perform one operation on X, and, completely independently, another operation on Y. Suppose the matrix M acts on X and N acts on Y. The rows and columns of M have indices that are placed in correspondence with elements of Σ and likewise for N.

A natural question to ask is this: if we view X and Y as a single compound system (X, Y) what is the matrix that represents the *combined action* of two operations on the compound system? We must first introduce the notion of the *tensor product of matrices*

1.12 Tensor Product of Matrices

The tensor product of $M \otimes N$ of the matrices

$$M = \sum_{a,b\in\Sigma} \alpha_{ab}|a\rangle\langle b| \qquad N = \sum_{c,d\in\Gamma} \beta_{cd}|c\rangle\langle d|$$

is the matrix

Tensor Product of M and N

$$M \otimes N = \sum_{a,b\in\Sigma} \sum_{c,d\in\Gamma} \alpha_{ab}\beta_{cd}|ac\rangle\langle bd|$$

Equivalently, M and N is defined by the equation

$$\langle ac|M \otimes N|bd\rangle = \langle a|M|b\rangle\langle c|N|d\rangle$$

Another equivalent way to describe $M \otimes N$ is that it is the unique matrix that satisfies the equation

$$(M \otimes N)(|\phi\rangle \otimes |\psi\rangle) = (M|\phi\rangle) \otimes (N|\psi\rangle)$$

for every possible choice of vectors ϕ and ψ. Where ϕ corresponds to the indices Σ and the indices of ψ correspond to Γ.

The tensor product of two matrices can be written explicitly as follows:

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mm} \end{pmatrix} \otimes \begin{pmatrix} \beta_{11} & \cdots & \beta_{1k} \\ \vdots & \ddots & \vdots \\ \beta k1 & \cdots & \beta kk \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{11}\beta_{11} & \cdots & \alpha_{11}\beta_{1k} & & \alpha_{1m}\beta_{11} & \cdots & \alpha_{1m}\beta_{1k} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \alpha_{11}\beta_{k1} & \cdots & \alpha_{11}\beta_{k1} & & \alpha 1m\beta_{k1} & \cdots & \alpha_{1m}\beta_{kk} \\ & \vdots & & \ddots & & \vdots & \\ \alpha_{m1}\beta_{11} & \cdots & \alpha_{m1}\beta_{1k} & & \alpha_{mm}\beta_{11} & \cdots & \alpha_{mm}\beta_{1k} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \alpha_{m1}\beta_{k1} & \cdots & \alpha_{m1}\beta_{kk} & & \alpha_{mm}\beta_{k1} & \cdots & \alpha_{mm}\beta_{kk} \end{pmatrix}$$

For three or more systems it is the same – the tensor product $M_1 \otimes \cdots \otimes M_n$ is defined by the condition that

$$\langle a_1 \cdots a_n|M_1 \otimes \cdots \otimes M_n|b_1 \cdots b_n\rangle = \langle a_1|M_1|b_1\rangle \cdots \langle a_n|M_n|b_n\rangle$$

1.13 Summary

We found that if M is a probabilistic operation on X, N is a probabilistic operation on Y, and the two operations are performed independently, the resulting operation on the compound system (X, Y) is the tensor product $M \otimes N$ – **tensor products represent independence**.

If we have two systems X and Y that are independently in the probabilistic states $|\phi\rangle$ and $|\pi\rangle$ then the compound system (X, Y) is in the probabilistic state $|\phi\rangle \otimes |\pi\rangle$.

Let us take a look at an example, which recalls probabilistic operation on a single bit. If the classical state of the bit is 0, it is left alone; and if the classical state of the bit is 1, it is flipped to 0 with probability $\frac{1}{2}$. This operation is represented by the matrix:

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

If we perform this operation on bit X, and a NOT operation is independently performed on the second bit Y, then the joint operation on the compound system (X, Y) has the matrix representation

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

which is also a stochastic matrix. If we do an operation on one bit and do nothing to the other, this is the same as taking the tensor product against the identity matrix. For example, resetting the bit X to the 0 state and doing nothing to Y yields the probabilistic state on (X, Y) represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 Quantum Information

The mathematical description of quantum information for multiple systems is similar to the probabilistic case and makes use of similar concepts and techniques.

2.1 Quantum States

Multiple states can be viewed collectively as single, compound systems. That is, quantum states of multiple systems are represented by column vectors having complex number entries and Euclidean norm equal to 1 – just like single systems. In the multiple system case, the indices of these vectors are placed in correspondence with the *Cartesian Product* of the classical state sets associated with the individual systems.

If X and Y are qubits, then the classical state set of the pair of qubits (X, Y) viewed collectively as a single system is the Cartesian Product $\{0,1\} \times \{0,1\}$. We associate this Cartesian Product set with the set $\{00, 01, 10, 11\}$.

For the quantum state vector

$$\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{i}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$$

We can write it explicitly as a column vector

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

2.2 Tensor Products of Quantum State Vectors

Suppose that $|\phi\rangle$ is a quantum state vector of system X and $|\psi\rangle$ is a quantum state vector of Y. The tensor product $|\phi\rangle \otimes |\psi\rangle$ which is equivalent to $|\psi\rangle|phi\rangle$ or $|\phi \otimes \psi\rangle$, is then a quantum state vector of the joint system (X, Y). This is known as the *product state*.

When a pair of systems (X, Y) is in a product state $|\phi\rangle \otimes |\psi\rangle$, this means X is in the quantum state $|\phi\rangle$, Y is in the quantum state ψ, and the states of the two systems have nothing to do with one another.

The fact that the tensor product vector $|\phi\rangle \otimes |\psi\rangle$ is also a quantum state is consistent with the euclidean norm being *multiplicative* with respect to tensor products:

$$
\begin{aligned}
|||\phi\rangle \otimes |\psi\rangle|| &= \sqrt{\sum_{(a,b)\in\Sigma\times\Gamma} |\langle ab|\phi \otimes \psi\rangle|^2} \\
&= \sqrt{\sum_{a\in\Sigma}\sum_{b\in\Gamma} |\langle a|\phi\rangle\langle b|\psi\rangle|^2} \\
&= \sqrt{\left(\sum_{a\in\Sigma} |\langle a|\phi\rangle|^2\right)\left(\sum_{b\in\Gamma} |\langle b|\psi\rangle|^2\right)} \\
&= |||\phi\rangle|||||\psi\rangle||
\end{aligned}
$$

Product states of any sizes joint system $(X_1, \ldots, X_n)$ are all quantum state vectors as well.

2.3 Quantum Entanglement

Not all quantum state vectors of multiple systems are product states. For example, the quantum state vector

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

of two qubits is not a product state – both states are related to one another.

If the above quantum state vector was a product state – there would exist quantum state vectors $|\phi\rangle$ and $|\psi\rangle$ for which

$$|\phi\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

But then it would be the case that

$$\langle 0|\phi\rangle\langle 1|\psi\rangle = \langle 01|\phi \otimes \psi\rangle = 0$$

since the coefficient in front of $|01\rangle$ is 0. This implies that $\langle 0|\phi\rangle = 0$ or $\langle 1|\psi\rangle = 0$, or both. This contradicts the fact that

$$\langle 0|\phi\rangle\langle 0|\psi\rangle = \langle 00|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

and

$$\langle 1|\phi\rangle\langle 1|\psi\rangle = \langle 11|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

since both are nonzero. It follows that the above quantum state represents a *correlation* between two systems – we say the systems are ***entangled***

Entanglement can be complicated, particularly for the sorts of noisy quantum states that can be described in the general, density matrix formulation of quantum information that was mentioned in Lesson 01 — but for quantum state vectors in the simplified formulation that we are focusing on here, entanglement is equivalent to correlation.
Here, any quantum state vector that is not a product vector represents an entangled state.

2.4 Bell States

Bell states are the following four two-qubit states:

$$\begin{aligned}
|\phi^+\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
|\phi^-\rangle &= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\
|\psi^+\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\
|\psi^-\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle
\end{aligned}$$

The collection of all four Bell states

$$\{|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle\}$$

is known as the *Bell Basis* – any 2bit quantum state vector can be expressed as a linear combination of the four Bell states.

$$|00\rangle = \frac{1}{\sqrt{2}}|\phi^+\rangle + \frac{1}{\sqrt{2}}|\phi^-\rangle$$

2.5 GHZ & W States

Next we will consider two interesting examples of states of *three qubits*.

The first example that represents a quantum state of 3 qubits (X, Y, Z) is the *GHZ state*:

GHZ State

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

The second example is the *W State*:

W State

$$\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$$

Neither of these is a product state – they can not be written as a tensor product of 3 qubit quantum state vectors. Systems having the classical state set $\{0, 1, 2\}$ are called *trits*, or, assuming we consider the possibility that they are in quantum states, *qutrits*. The term *qudit* refers to a system having classical state set $\{0, \ldots, d-1\}$ for an arbitary choice of d.

2.6 Measurements of Quantum States

If a system having a classical state set Σ is in a quantum state represented by the vector $|\psi\rangle$, and that system is measured (with respect to a standard basis measurement), then each classical state $a \in \Sigma$ appears with probability $|\langle a|\psi\rangle|^2$.

Let us suppose that $X_1, \ldots, X_n$ are systems having classical state sets $\Sigma_1, \ldots, \Sigma_n$, respectively. We can then view $(X_1, \ldots, X_n)$ as a single system whose classical state set is the Cartesian Product $\Sigma_1 \times \cdots \times \Sigma_n$. If a quantum state of this system is represented by $|\psi\rangle$, and all of the systems are measured, then each possible outcome $(a_1, \ldots, a_n) \in \Sigma_1 \times \cdots \times \Sigma_n$ appears with probability $|\langle a_1 \cdots a_n|\psi\rangle|^2$.

For example, if systems X and Y are jointly in the quantum state

$$\frac{3}{5}|0\rangle|\heartsuit\rangle - \frac{4i}{5}|1\rangle|\spadesuit\rangle$$

then measuring both systems with respect to a standard basis measurement yields the outcome $(0, \heartsuit)$ with probability $\frac{9}{25}$ and the outcome $(1, \spadesuit)$ with probability $\frac{16}{25}$

2.7 Partial Measurements for Two Systems

Now let us consider the situation in which we have multiple systems in some quantum state, and we measure a proper subset of the systems. Let's begin with two systems X and Y having classical state sets Σ and Γ.

In general a quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_{ab}|ab\rangle$$

where $\{\alpha_{ab} : (a, b) \in \Sigma \times \Gamma$ is a collection of complex numbers satisfying

$$\sum_{(a,b)\in\Sigma\times\Gamma} |\alpha_{ab}|^2 = 1$$

($|\psi\rangle$ is a unit vector).

If both X and Y were measured, then each possible outcome $(a, b) \in \Sigma \times \Gamma$ would appear with probability

$$|\langle ab|\psi\rangle|^2 = |\alpha_{ab}|^2$$

Supposing that just the first system X is measured, the probability for each outcome $a \in \Sigma$ to appear therefore must be equal to

$$\sum_{b\in\Gamma} |\langle ab|\psi\rangle|^2 = \sum_{b\in\Gamma} |\alpha_{ab}|^2$$

The probability for each particular outcome to appear when X is measured cannot possibly depend on whether or not Y was also measured as that would allow faster-than-light communication.

Having obtained a particular outcome $a \in \Sigma$ of this measurement of X we expect that the quantum state of X changes so that it is equal to $|a\rangle$, like we had for single systems. What happens to Y?

Let us describe the joint quantum state (X, Y) under the assumption that X was measured and the result was the classical state a.

First we express the vector $|\psi\rangle$ as

$$|psi\rangle = \sum_{a\in\Sigma} |a\rangle \otimes |\phi_a\rangle$$

where

$$|\phi_a\rangle = \sum_{b\in\Gamma} \alpha_{ab}|b\rangle$$

for each $a \in \Sigma$. The probability that the standard basis measurement of X results in each outcome of a may be written as follows:

$$\sum_{b\in\Gamma} |\alpha_{ab}|^2 = |||\phi_a\rangle||^2$$

Now, as a result of the standard basis measurement of X resulting in outcome a, we have that the quantum state of the pair (X, Y) together becomes

$$|a\rangle \otimes \frac{|\phi_a\rangle}{|||\phi_a\rangle||}$$

The state "collapses" like in the single-system case. Informally speaking, $|a\rangle \otimes |\phi_a\rangle$ represents the component of $|\psi\rangle$ that is consistent with measurement of X resulting in outcome a. We normalize this vector to yield a valid quantum state vector.

As an example, let us consider the state of two qubits (X, Y) from the beginning of the section:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{i}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$$

To understand what happens when the first system X is measured, we begin by writing

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle\right) + |1\rangle \otimes \left(\frac{i}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle\right)$$

The probability of the measurement to result in outcome 0 is

$$\left|\left|\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle\right|\right|^2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

in which case the state of (X, Y) becomes

$$|0\rangle \otimes \frac{\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle}{\sqrt{\frac{2}{3}}} = |0\rangle \otimes \left(\sqrt{\frac{3}{4}}|0\rangle - \frac{1}{2}|1\rangle\right)$$

The same process goes for if the measurement of X yields 1

2.8 Remark on Reduced Quantum States

This example shows a limitation of the simplified description of Quantum Information: it offers no way to describe the reduced (or marginal) quantum state of just one of two systems like we did in the probabilistic case.

For a probabilistic state of two systems (X, Y) described by a probability vector

$$|\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|ab\rangle$$

the *reduced* state of X alone is described by the probability vector

$$\sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|a\rangle$$

For quantum state vectors – there is no analog.

2.9 The Swap Operation

Let us take a look at two classes of examples of unitary operations on multiple systems, beginning with the swap operation.

Suppose that X and Y are systems that share the same classical state set Σ. The *swap* operation on the pair (X, Y) is the operation that exchanges the contents of the two systems but otherwise leaves the systems alone.

We will denote this operation as SWAP such that for every choice of classical states $a, b \in \Sigma$

$$\mathrm{SWAP}|a\rangle|b\rangle = |b\rangle|a\rangle$$

such that

$$\mathrm{SWAP} = \sum_{c,d\in\Sigma} |c\rangle\langle d| \otimes |d\rangle\langle c| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.10 Controlled-Unitary Operations

Now let us suppose that Q is a qubit and R is an arbitrary system, with whatever classical state we wish.

For every unitary operation U acting on R, a *controlled* U operation is a unitary operation on the pair (Q, R) defined as follows:

$$CU = |0\rangle\langle 0| \otimes I_R + |1\rangle\langle 1| \otimes U$$

For example, if R is also a qubit, and we think about the Pauli X operation on R, then a controlled-X operation is given by

$$CX = |0\rangle\langle 0| \otimes I_R + |1\rangle\langle 1| \otimes X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

If we instead take R to be two qubits and U to be the SWAP operation between these two qubits we obtain:

$$\text{CSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This operation can be described as follows:

$$\begin{aligned} \text{CSWAP}|0bc\rangle &= |0bc\rangle \\ \text{CSWAP}|1bc\rangle &= |1cb\rangle \end{aligned}$$

A *controlled-controlled-NOT* operation, CCX:

$$CCX = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where

$$CCX = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I + |11\rangle\langle 11| \otimes X$$