

Multiple Systems – Lesson 02

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December 21, 2023

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We can always choose to view multiple systems *together* as if they form a single, compound system – to which the discussion in the previous section applies.

1 Classical Information

Recognizing how mathematics works in the familiar setting of classical information is helpful in understanding why quantum information is described the way it is.

1.1 Classical States via the Cartesian Product

We will start with classical states of multiple systems. We will begin by discussing just two systems and then generalize further.

To be precise, let us suppose that X is a system whose classical state set is Σ and Y is a second system having classical state set Γ . Because these are *classical state sets*, our assumption is that Σ and Γ are both finite and nonempty. It could be that $\Sigma = \Gamma$, but this is not necessarily so.

Now imagine the two systems X and Y are placed side-by-side, with X on the left and Y on the right. We can view these two systems as if they form a single system (X, Y) or XY .

The set of classical states of XY is the *Cartesian Product* of Σ and Γ , which is the set defined as

Cartesian Product

$$\Sigma \times \Gamma = \{ (a, b) : a \in \Sigma \text{ and } b \in \Gamma \}$$

The Cartesian Product is the mathematical notion of viewing an element of one set and an element of a second set together, as if they form a single set.

For more than two systems, the situation generalizes in a natural way. If we suppose that X_1, \dots, X_n are systems having classical state sets $\Sigma_1, \dots, \Sigma_n$, respectively for any positive integer n , the classical state set of the n -tuple $X_1 X_2 \dots X_n$, viewed as a single joint system, is the Cartesian Product

$$\Sigma_1 \times \dots \times \Sigma_n = \{ (a_1, \dots, a_n) : a_1 \in \Sigma_1 \dots a_n \in \Sigma_n \}$$

1.2 Representing states as strings

It is convenient to write a classical state (a_1, \dots, a_n) as a string ' $a_1 \dots a_n$ '. The notion of a string, which is a fundamentally important concept in CS, is formalized in mathematical terms through the cartesian product.

For example suppose that X_1, \dots, X_{10} are bits, so the classical state sets of these systems are all the same

$$\Sigma_1 = \Sigma_2 = \dots = \Sigma_{10} = 0, 1$$

The set $0, 1$ is referred to as the *binary alphabet*. There are $2^{10} = 1024$ classical states of the joint system $X_1 X_2 \dots X_{10}$ which are elements of the set

$$\Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_{10} = 0, 1^{10}.$$

Written as strings, these classical states look like this:

```

0000000000
0000000001
0000000010
0000000011
0000000100
      ⋮
1111111111

```

For the classical state 0001010000 we see that X_4 and X_6 are in the state 1, while all other systems are in the state 0.

1.3 Probabilistic States

Probabilistic states associate a probability with each classical state of a system. Thus, a probabilistic state of multiple systems – viewed collectively as if they form a single system – associates a probability with each element of the Cartesian product of the classical state sets of the individual systems.

Suppose X and Y are both bits. So their corresponding classical state sets are $\Sigma = 0, 1$ and $\Gamma = 0, 1$. Here is a probabilistic state of the pair XY :

$$\begin{aligned} \Pr((X, Y) = (0, 0)) &= \frac{1}{2} \\ \Pr((X, Y) = (0, 1)) &= 0 \\ \Pr((X, Y) = (1, 0)) &= 0 \\ \Pr((X, Y) = (1, 1)) &= \frac{1}{2} \end{aligned}$$

This probabilistic state is one in which both X and Y are random bits – each is 0 with probability $1/2$ and 1 with probability $1/2$ – but the classical states of the two bits always agree. This is an example of a *correlation* between these systems.

1.4 Ordering Cartesian Product State Sets

To represent a probabilistic state of multiple systems as a Cartesian product, one must decide on an ordering of the product's elements. The entries in each n -tuple are viewed as being ordered by significance that *decreases from left to right*.

The Cartesian Product $1, 2, 3 \times 0, 1$ is ordered like this:

$$(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)$$

The probabilistic state represented by the probability vector above

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Where from top to bottom we get the probability associated with 00, then 01, then 10, then 11.

1.5 Independence of Two Systems

Intuitively speaking, two systems are independent if learning the classical state of either system has no effect on the probabilities associated with the other.

Suppose X and Y are systems having classical state sets Σ and Γ respectively. They are said to be *independent* if it is the case that

$$\Pr((X, Y) = (a, b)) = \Pr(X = a)\Pr(Y = b)$$

for every $a \in \Sigma$ and $b \in \Gamma$.

To express this in terms of probability vectors, assume the probabilistic state of (X, Y) is described as a probability vector written in Dirac notation as

$$\sum_{(a,b) \in \Sigma \times \Gamma} p_{ab} |ab\rangle$$

The above condition is equivalent to the existence of two probability vectors

$$|\phi\rangle = \sum_{a \in \Sigma} q_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b \in \Gamma} r_b |b\rangle$$

representing the probabilities associated with the classical states of X and Y such that

$$p_{ab} = q_a r_b$$

Tensor Product

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} \alpha_a \beta_b |ab\rangle$$

Equivalently the vector $|\pi\rangle = |\phi\rangle \otimes |\psi\rangle$ is defined by the equation

$$\langle ab|\pi\rangle = \langle a|\phi\rangle \langle b|\psi\rangle$$

being true for every $a \in \Sigma$ and $b \in \Gamma$

In this situation it is said that $|\pi\rangle$ is a *product state* or *product vector*. We often just write $|\phi\rangle|\psi\rangle$ instead of $|\phi\rangle \otimes |\psi\rangle$. The tensor product for two column vectors is defined as follows:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} \alpha_1 \beta_1 \\ \vdots \\ \alpha_1 \beta_k \\ \alpha_2 \beta_1 \\ \vdots \\ \alpha_2 \beta_k \\ \vdots \\ \alpha_m \beta_1 \\ \vdots \\ \alpha_m \beta_k \end{pmatrix}$$

For standard basis vectors:

$$|a\rangle \otimes |b\rangle = |ab\rangle = |a, b\rangle$$

The tensor product of two vectors has the important property that it is *bilinear*, which means that it is linear in each of the two arguments separately, assuming that the other argument is fixed.

Bilinearity

1. Linearity in the first argument:

$$\begin{aligned} (|\phi_1\rangle + |\phi_2\rangle) \otimes |\psi\rangle &= |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle \\ (\alpha|\phi\rangle) \otimes |\psi\rangle &= \alpha(|\phi\rangle \otimes |\psi\rangle) \end{aligned}$$

2. Linearity in the second argument:

$$\begin{aligned} |\phi\rangle \otimes (|\psi_1\rangle + |\psi_2\rangle) &= |\phi\rangle \otimes |\psi_1\rangle + |\phi\rangle \otimes |\psi_2\rangle \\ |\phi\rangle \otimes (\alpha|\psi\rangle) &= \alpha(|\phi\rangle \otimes |\psi\rangle) \end{aligned}$$

Scalars “float freely” within tensor products.

1.7 Independence and Tensor Products for 3 or More Systems

If X_1, \dots, X_n are systems with classical state sets $\Sigma_1, \dots, \Sigma_n$, then a probabilistic state of the combined system $X_1 X_2 \dots X_n$ is a *product state*

$$|\psi\rangle = |\phi_1\rangle \otimes \dots \otimes |\phi_n\rangle$$

for probability vectors $|\phi_1\rangle \dots |\phi_n\rangle$ describing probabilistic states X_1, \dots, X_n

The definition of the tensor product generalizes as the following:

$$|\psi\rangle = |\phi_1\rangle \otimes \dots \otimes |\phi_n\rangle$$

is defined by the equation

Tensor Product Definition

$$\langle a_1 \dots a_n | \psi \rangle = \langle a_1 | \phi_1 \rangle \dots \langle a_n | \phi_n \rangle$$

for every $a_1 \in \Sigma_1, \dots, a_n \in \Sigma_n$.

Similar to the tensor product for two vectors, the tensor product for 3 or more vectors is linear in each of the arguments individually. We say the tensor product of 3 or more vectors is *multilinear*.

For multiple systems is X_1, \dots, X_n are *independenet*, when they are in a product state, they are actually *mutually independent*. There is also other notions of independence for 3 or more systems such as *pairwise independence*.

For any positive integer n and any classical states a_1, \dots, a_n we have

$$|a_1\rangle \otimes \dots \otimes |a_n\rangle = |a_1 \cdot a_n\rangle$$

1.8 Measurements of Probabilistic States

If two bits (X, Y) are described by the following probability vector

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle,$$

$$\frac{\frac{1}{2}|1\rangle + \frac{1}{12}|3\rangle}{\frac{7}{12}} = \frac{6}{7}|1\rangle + \frac{1}{7}|3\rangle$$

and conditioned on the measurement of X being 1, the probabilistic state of Y becomes

$$\frac{\frac{1}{12}|1\rangle + \frac{1}{6}|2\rangle + \frac{1}{6}|3\rangle}{\frac{5}{12}} = \frac{1}{5}|1\rangle + \frac{2}{5}|2\rangle + \frac{2}{5}|3\rangle$$

1.10 Operations on Probabilistic States

We will consider *operations* on multiple systems in probabilistic states. Returning to the typical setup where we have two systems X and Y , let us consider classical operations on the compound system (X, Y) . Any operation is represented by a stochastic matrix whose rows and columns are indexed by the Cartesian Product $\Sigma \times \Gamma$.

For example, suppose X and Y are bits, and consider the operation:

If $X = 1$, then perform a NOT operation on Y .
Otherwise do nothing.

This is a deterministic operation known as a *controlled-NOT* or CNOT operation, where X is the *control* bit that determines whether or not a NOT operation gets applied to the *target* bit Y . Here is the matrix representation of this operation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Its action on the standard basis states:

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |01\rangle \\ |10\rangle &\mapsto |11\rangle \\ |11\rangle &\mapsto |10\rangle \end{aligned}$$

Another example is the operation having this description

Perform one of the following two operations, each with probability $\frac{1}{2}$:

- Set Y to be equal to X
- Set X to be equal to Y

The matrix representation is as follows:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

The action of this operation on the standard basis vectors:

$$\begin{aligned} |00\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto \frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle \\ |10\rangle &\mapsto \frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle \\ |11\rangle &\mapsto |11\rangle \end{aligned}$$

In these examples, we view two systems together as a single system. We can do this with any number of systems.

1.11 Independent Operations

Suppose we want to *independently* perform *separate operations* on the systems. For example, taking our usual setup of X and Y having classical state sets Σ and Γ , let's perform one operation on X , and, completely independently, another operation on Y . Suppose the matrix M acts on X and N acts on Y . The rows and columns of M have indices that are placed in correspondence with elements of Σ and likewise for N .

A natural question to ask is this: if we view X and Y as a single compound system (X, Y) what is the matrix that represents the *combined action* of two operations on the compound system? We must first introduce the notion of the *tensor product of matrices*

1.12 Tensor Product of Matrices

The tensor product of $M \otimes N$ of the matrices

$$M = \sum_{a,b \in \Sigma} \alpha_{ab} |a\rangle \langle b| \quad N = \sum_{c,d \in \Gamma} \beta_{cd} |c\rangle \langle d|$$

is the matrix

Tensor Product of M and N

$$M \otimes N = \sum_{a,b \in \Sigma} \sum_{c,d \in \Gamma} \alpha_{ab} \beta_{cd} |ac\rangle \langle bd|$$

Equivalently, M and N is defined by the equation

$$\langle ac | M \otimes N | bd \rangle = \langle a | M | b \rangle \langle c | N | d \rangle$$

Another equivalent way to describe $M \otimes N$ is that it is the unique matrix that satisfies the equation

$$(M \otimes N)(|\phi\rangle \otimes |\psi\rangle) = (M|\phi\rangle) \otimes (N|\psi\rangle)$$

for every possible choice of vectors ϕ and ψ . Where ϕ corresponds to the indices Σ and the indices of ψ correspond to Γ .

The tensor product of two matrices can be written explicitly as follows:

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mm} \end{pmatrix} \otimes \begin{pmatrix} \beta_{11} & \cdots & \beta_{1k} \\ \vdots & \ddots & \vdots \\ \beta_{k1} & \cdots & \beta_{kk} \end{pmatrix} = \begin{pmatrix} \alpha_{11}\beta_{11} & \cdots & \alpha_{11}\beta_{1k} & \cdots & \alpha_{1m}\beta_{11} & \cdots & \alpha_{1m}\beta_{1k} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \alpha_{11}\beta_{k1} & \cdots & \alpha_{11}\beta_{kk} & \cdots & \alpha_{1m}\beta_{k1} & \cdots & \alpha_{1m}\beta_{kk} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha_{m1}\beta_{11} & \cdots & \alpha_{m1}\beta_{1k} & \cdots & \alpha_{mm}\beta_{11} & \cdots & \alpha_{mm}\beta_{1k} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \alpha_{m1}\beta_{k1} & \cdots & \alpha_{m1}\beta_{kk} & \cdots & \alpha_{mm}\beta_{k1} & \cdots & \alpha_{mm}\beta_{kk} \end{pmatrix}$$

For three or more systems it is the same – the tensor product $M_1 \otimes \cdots \otimes M_n$ is defined by the condition that

$$\langle a_1 \cdots a_n | M_1 \otimes \cdots \otimes M_n | b_1 \cdots b_n \rangle = \langle a_1 | M_1 | b_1 \rangle \cdots \langle a_n | M_n | b_n \rangle$$

1.13 Summary

We found that if M is a probabilistic operation on X , N is a probabilistic operation on Y , and the two operations are performed independently, the resulting operation on the compound system (X, Y) is the tensor product $M \otimes N$ – **tensor products represent independence**.

2 Quantum Information

The mathematical description of quantum information for multiple systems is similar to the probabilistic case and makes use of similar concepts and techniques.

2.1 Quantum States

Multiple states can be viewed collectively as single, compound systems. That is, quantum states of multiple systems are represented by column vectors having complex number entries and Euclidean norm equal to 1 – just like single systems. In the multiple system case, the indices of these vectors are placed in correspondence with the *Cartesian Product* of the classical state sets associated with the individual systems.

If X and Y are qubits, then the classical state set of the pair of qubits (X, Y) viewed collectively as a single system is the Cartesian Product $\{0, 1\} \times \{0, 1\}$. We associate this Cartesian Product set with the set $\{00, 01, 10, 11\}$.

For the quantum state vector

$$\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{i}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$$

We can write it explicitly as a column vector

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

2.2 Tensor Products of Quantum State Vectors

Suppose that $|\phi\rangle$ is a quantum state vector of system X and $|\psi\rangle$ is a quantum state vector of Y . The tensor product $|\phi\rangle \otimes |\psi\rangle$ which is equivalent to $|\psi\rangle|\phi\rangle$ or $|\phi \otimes \psi\rangle$, is then a quantum state vector of the joint system (X, Y) . This is known as the *product state*.

When a pair of systems (X, Y) is in a product state $|\phi\rangle \otimes |\psi\rangle$, this means X is in the quantum state $|\phi\rangle$, Y is in the quantum state $|\psi\rangle$, and the states of the two systems have nothing to do with one another.

The fact that the tensor product vector $|\phi\rangle \otimes |\psi\rangle$ is also a quantum state is consistent with the euclidean norm being *multiplicative* with respect to tensor products:

$$\begin{aligned}
|||\phi\rangle \otimes |\psi\rangle|| &= \sqrt{\sum_{(a,b) \in \Sigma \times \Gamma} |\langle ab|\phi \otimes \psi\rangle|^2} \\
&= \sqrt{\sum_{a \in \Sigma} \sum_{b \in \Gamma} |\langle a|\phi\rangle \langle b|\psi\rangle|^2} \\
&= \sqrt{\left(\sum_{a \in \Sigma} |\langle a|\phi\rangle|^2\right) \left(\sum_{b \in \Gamma} |\langle b|\psi\rangle|^2\right)} \\
&= |||\phi\rangle|| |||\psi\rangle||
\end{aligned}$$

Product states of any sizes joint system (X_1, \dots, X_n) are all quantum state vectors as well.

2.3 Quantum Entanglement

Not all quantum state vectors of multiple systems are product states. For example, the quantum state vector

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

of two qubits is not a product state – both states are related to one another.

If the above quantum state vector was a product state – there would exist quantum state vectors $|\phi\rangle$ and $|\psi\rangle$ for which

$$|\phi\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

But then it would be the case that

$$\langle 0|\phi\rangle \langle 1|\psi\rangle = \langle 01|\phi \otimes \psi\rangle = 0$$

since the coefficient in front of $|01\rangle$ is 0. This implies that $\langle 0|\phi\rangle = 0$ or $\langle 1|\psi\rangle = 0$, or both. This contradicts the fact that

$$\langle 0|\phi\rangle \langle 0|\psi\rangle = \langle 00|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

and

$$\langle 1|\phi\rangle \langle 1|\psi\rangle = \langle 11|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

