Math 126 Lecture 1 **Review**

01/23/2026

1 Ordinary Differential Equations

Definition: A *first-order* ordinary differential equation (ODE) is an equation

$$\frac{du}{dt} = F(u,t),$$

for an unknown $u : (a,b) \to \mathbb{R}$.

1.1 Solving

1.1.1 Direct Integration

If $F(u,t) = f(t)$ is u-independent,

$$u'(t) = \frac{du}{dt} = f(t).$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_0^t u'(s)\,ds &= \int_0^t f(s)\,ds \\ u(t) &= \int_0^t f(s)\,ds + C. \end{aligned}$$

Definition: An ODE $u'(t) = F(u,t)$ is *separable* if

$$F(u,t) = f(u)g(t).$$

1.1.2 Separation of Variables for ODEs

A separable equation can be rearranged so the right-hand side (RHS) becomes u-independent.

$$\begin{aligned} \frac{du}{dt} &= f(u)g(t) \\ \frac{du}{dt}\frac{1}{f(u)} &= g(t). \end{aligned}$$

Then we solve via direct integration:

$$\begin{aligned} \frac{du}{dt} &= f(u)g(t) \\ \int \frac{du}{dt}\frac{1}{f(u)}\,dt &= \int g(t)\,dt + C \\ \int \frac{1}{f(u)}\,du &= \int f(t)\,dt + C. \end{aligned}$$

Exercise I: Solve

$$\frac{dy}{dt} = 3t^2(y^2 - 1).$$

Solution: Recognize the RHS is separable and perform separation of variables:

$$\begin{aligned} \int \frac{dy}{y^2-1} &= \int 3t^2\,dt \\ \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| &= t^3 + C \\ \frac{y-1}{y+1} &= De^{2t^3} \\ y(t) &= \frac{1+De^{2t^3}}{1-De^{2t^3}}, \end{aligned}$$

where $D = e^C$.

Remark: It is very easy to write down an ODE resulting in an integral without an anti-derivative in terms of standard functions. Hence, not all ODEs have closed form solutions in terms of functions we have names for. Solutions still exist, but must be understood qualitatively or numerically.

Definition: A second-order *linear* ODE with constant coefficients takes the form:

$$au''(t) + bu'(t) + cu(t) = 0$$

for $a, b \in \mathbb{R}$.

1.1.3 The Characteristic Polynomial

Take the guess or *ansatz* $u(t) = Ce^{\lambda t}$ and plug in:

$$\begin{aligned} u'(t) &= C\lambda e^{\lambda t} \\ u''(t) &= C\lambda^2 e^{\lambda t} \end{aligned}$$

into the differential equation. This gives

$$\begin{aligned} 0 &= au'' + bu' + cu \\ &= a(C\lambda^2 e^{\lambda t}) + b(C\lambda e^{\lambda t}) + cCe^{\lambda t} \\ &= Ce^{\lambda t}(a\lambda^2 + b\lambda + c). \end{aligned}$$

Hence, either $C = 0$, which means $u(t) = 0$ trivially constant, or $a\lambda^2 + b\lambda + c = 0$, for which we can use the quadratic formula that yields two roots λ_1, λ_2. Hence, our solution is

$$u(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

More generally, this method works for any constant coefficient ODE of any order:

$$a_k u^{(k)} + a_{k-1}u^{(k-1)} + \dots + a_0 u = 0.$$

There are also other possible ansatzes (ansatzi?) that are possible.

Exercise II: Solve the non-constant coefficient second-order linear ODE

$$r^2 u''(r) + ru'(r) + \Lambda u(r) = 0.$$

using the ansatz $u(r) = r^\alpha$.

1.2 Initial Value Problems

Definition: An ODE is called *homogeneous* if its equal to 0. In other words, all terms in the equation depend on $u(t)$. Otherwise, if there exist terms independent of $u(t)$, we can lump them together into a unified $f(t)$ only dependent on t, and place it on the RHS.

$$\begin{aligned} a_{k(t)}u^{(k)} + a_{k-1}(t)u^{(k-1)} + \dots + a_0(t)u &= 0 && \text{homogeneous} \\ &= f(t) && \text{inhomogeneous.} \end{aligned}$$

Inhomogeneous terms appear as external sources or driving forces in applications. Typically the method of solving an inhomogeneous ODE involves solving the homogeneous variant first.

Example: Inhomogeneous ODEs from external/driving forces.

$$\underbrace{\frac{dP}{dt}}_{\text{population}} = \underbrace{kP}_{\text{births}} + \underbrace{I(t)}_{\text{im/emmigration}}$$

$$\underbrace{\frac{dT}{dt}}_{\text{temperature}} = k(T - T_0) + \underbrace{s(t)}_{\text{external heat source}}$$

Theorem 1.2.1: An nth order ODE has a solution with n arbitrary constants if it is linear. The solution then has the form:

$$u(t) = C_1 f_1(t) + C_2 f_2(t) + \dots + C_n f_{n(t)}.$$

Definition: An *initial value problem* (IVP) or *boundary value* problem imposes n real-valued constraints at ends of the problem interval.

Exercise III: Solve the IVP on $[0,1]$.

$$\begin{cases} u''(t) - w^2 u(t) = 0 \\ u(0) = 0 \\ u'(0) = 4 \end{cases}.$$

Solution: By characteristic polynomials, $u(t) = C_1 e^{wt} + C_2 e^{-wt}$. Plugging in,

$$\begin{aligned} 3 = u(0) &= C_1 + C_2 \\ 4 = u'(0) &= C_1 \omega e^0 + C_2 \omega e^0 = C_1 w - C_2 \omega, \end{aligned}$$

which gives two equations and two unknowns and is therefore solvable.

Remark: An interpretation of the system at $t = \text{initial time}$ is required to know its complete evolution. The initial position and velocity of a ball thrown dictates the ball's future position. These two unknowns correspond to Newton's second law being a second-order differential equation. Similarly, the initial population of a civilization influences how fast the population grows afterwards and corresponds to a first-order differential equation.

2 Structure on $\mathbb{R}^n$

Definition: $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$.

$\mathbb{R}^n$ is a vector space of dimension n. This defines algebraic operations

$$\begin{aligned} &/\ v + w \\ &/\ cv, \end{aligned}$$

which under *linear maps*, satisfy

$$\begin{aligned} T(v+w) &= T(v) + T(w) \\ T(cv) &= cT(v). \end{aligned}$$

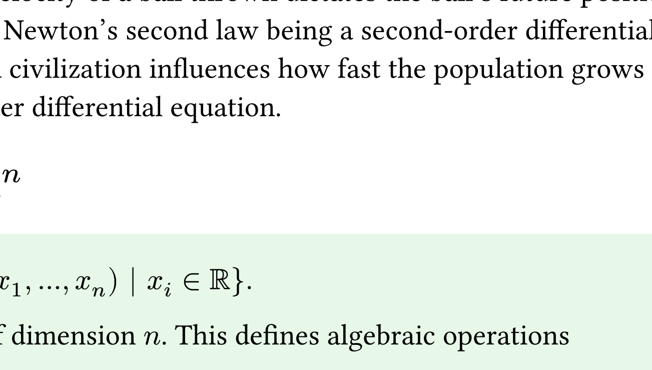
This gives $\mathbb{R}^n$ *algebraic structure.*

Definition: The Euclidean norm of a vector,

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2},$$

gives $\mathbb{R}^n$ a *topology*, a notion of open and closed sets, and of proximity.

Definition: A subset $\Omega \subseteq \mathbb{R}^n$ is *open* if for any $x \in \Omega$, there is a radius $r > 0$ so the ball $\{y \mid \|x - y\| < r\}$ remains $\subseteq \Omega$.



Intuitively open sets contain none of the boundary. Otherwise any infinitesimal $r > 0$ escapes Ω.

Definition: A subset $\Omega \subseteq \mathbb{R}^n$ is *closed* if for all converging sequences $\{x_k\} \subseteq \Omega$, the limit x is *also* $\in \Omega$. In other words, there exist no holes or punctures in Ω.

An equivalent definition is

$$\Omega \text{ closed} \iff \mathbb{R}^n \setminus \Omega$$

is open. Intuitively, closed means the boundary is included.

Remark: In $\mathbb{R}$, open/closed agrees with (a,b) or $[a,b]$ for intervals.

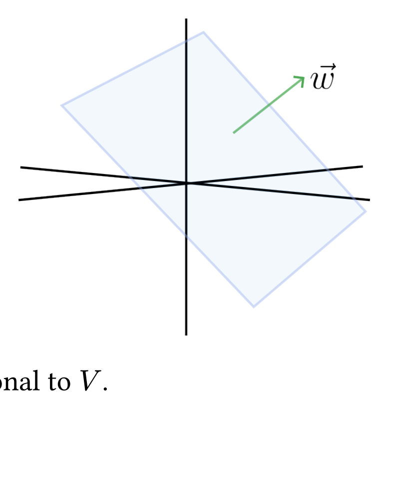
Definition: The Euclidean innter product on $\mathbb{R}^n$ is the *bilinear* pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle &: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ \langle x, y \rangle &= x_1 y_1 + x_2 y_2 + \dots + x_n y_n \end{aligned}$$

Two vectors are orthogonal if $\langle v, w \rangle = 0$.

Example: The orthogonal complement of a vector or subspace is

$$\operatorname{span}\{v\}^\perp \quad \text{or} \quad V^\perp = \{w \in \mathbb{R}^n \mid \langle v, w, \rangle = 0 \forall v \in V\}$$



It is the set of all vectors orthogonal to V.