Math 126 Lecture 4 — Model PDEs I

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0.0.1 Derivation of the wave equation

Consider a rope with fixed ends, and call the height displacement from the flat $u(t, x)$.

Assume the rope has density ρ. An infinitesimal segment with length h therefore has mass ρh, at $x = x_j$.

From Newton's laws:

$$\begin{aligned} ma &= F(x, t) \\ \rho h \cdot \frac{d^2 u}{dt^2} &= \{\text{vertical restoring force}\} \end{aligned} \tag{1}$$

The vertical restoring force is given by

$$\begin{aligned} F &= T \sin \alpha_j + T \sin \beta_j \\ &\approx T \tan \alpha_j + T \tan \beta_j \\ &\approx T \left(\frac{u(x_{j-1}, t) - u(x_j, t)}{h} + \frac{u(x_{j+1}, t) - u(x_j, t)}{h} \right) \\ &\approx T \left[u'(x_j, t) - u'(x_{j-1}, t) \right] \end{aligned} \tag{2}$$

After substituting Eq. (2) into Eq. (1), we have

$$\begin{aligned} \rho h \cdot \frac{d^2 u}{dt^2} &= \{\text{vertical restoring force}\} \\ \frac{\partial^2 u}{\partial t^2} &= \frac{T}{\rho} \left[\frac{u'(x_j, t) - u'(x_{j-1}, t)}{h} \right] \\ &= \frac{T}{\rho} \left[u''(x_j, t) \right]. \end{aligned} \tag{3}$$

Defining $c^2 = \frac{T}{\rho}$, the wave equation in 1 dimension reads

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{4}$$

0.0.2 Boundary and Initial Value Problems

Recall the paradigm for solving ODEs.

1. Apply a method of solving to obtain a general solution (with arbitrary constants).
2. Plug in knowledge of a specific situation (e.g. initial conditions) to get specific solution.

Definition: The *initial value problem* for the wave equation is to solve

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} & = c^2 \frac{\partial^2 u}{\partial x^2} \\ u(0, x) & = g(x) \\ \frac{\partial u(0,x)}{\partial t} & = h(x) \end{cases} \tag{5}$$

for given functions $g(x)$, $h(x)$.

Often to adapt concepts from ODEs to PDEs we "promote" values to functions.

$$\begin{aligned} u(t) & \quad \text{is a number} \quad \text{(ODE)} \\ u(t, -) & \quad \text{is a function} \quad \text{(PDE)} \end{aligned} \tag{6}$$

The initial condition similarly promotes $u(o) \mapsto u(0, -)$.