Math 126 Lecture 2 — The three classes of linear PDEs

Deval Deliwala

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$$\begin{cases}\text{single variable calculus} \\ f(x) \quad \frac{d}{dx}\end{cases} \longmapsto \text{ODEs}$$

$$\begin{cases}\text{multivariable calculus} \\ f(x,y,z) \quad \frac{\partial}{\partial x} \quad \nabla\end{cases} \longmapsto \text{PDEs}$$

1 Definitions

Definition: An *ordinary differential equation* (ODE) is an equation relating an unknown function $f(x)$ of a single variable to its own derivatives.

Example: Some examples of ODEs:

$$\frac{dP}{dt} = kP \qquad \frac{dT}{dt} = -k(T - T_0) \qquad \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$

Most quantities in nature depend on multiple variables though. Newton's cooling ODE does not acknowledge when dipping a hot ball in water, the outsides cool down first. E.g. heat clearly varies in both time and space:

$$f(x) \longmapsto u(t,x,y,z)$$

Definition: A *partial differential equation* (PDE) is an equation relating an unknown function $u(t,x,y,z,...)$ to its own partial derivatives.

The *order* of the equation is the highest derivative magnitude that appears.

Example: Some examples of PDEs:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad \text{first order}$$

$$3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 7u = 0 \qquad \text{second order with constant coeffs.}$$

$$a(t,x)\frac{\partial^2 u}{\partial x^2} + b(t,x)\frac{\partial^2 u}{\partial y^2} + \cdots + f(t,x,y,z)u = 0 \qquad \text{second order with non-constant coeffs}$$

2 Three Classes of Linear PDEs

Definition: A PDE is *linear* if each term is a fixed, known functino times a partial derivative. It depends linearly on the partial derivatives:

$$\cos(t+x)\frac{\partial^2 u}{\partial y^2} \checkmark \qquad \left(\frac{\partial u}{\partial x}\right)^2 \times$$

$$3x\frac{\partial u}{\partial z} \checkmark \qquad \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \times$$

$$e^{y+z}\frac{\partial^{15} u}{\partial x^8 \partial y^7} \checkmark \qquad e^{u}\frac{\partial u}{\partial z} \times$$

Remark: Bizzare fact of nature: Effectively every equation coming from physics, chemistry, engineering, biology, economics, etc. is a first or second order equation.

Notation: derivatives of $u(t,x,y,z)$ are denoted as

$$\frac{\partial u}{\partial x} = \partial_x u = \partial_1 u = u_x$$

$$\frac{\partial^2 u}{\partial t \partial y} = \partial_t \partial_y u = \partial_0 \partial_2 u = u_{ty}$$

A second-order linear PDE can be written in the form:

$$\sum_{i,j=0}^{3} A_{ij}(t,x,...)\partial_i\partial_j u + \sum_{k=0}^{3} B(t,x,...)\partial_k u + C(t,x,...)u = f(t,x,y,z),$$

where $f(t,x,y,z) = 0$ if the PDE is homogeneous and nonzero if inhomogeneous. Therefore using four variables t, x, y, z, A_{ij} can be expressed as a 4×4 matrix:

$$A \equiv \begin{pmatrix} A_{00} & A_{10} & \cdots \\ A_{01} & A_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

The three types of linear PDEs depend on the structure of A.

2.1 Elliptical

A prototypical example is *The Laplace Equation*:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\nabla \cdot \nabla u = 0$$

$$\triangle u = 0$$

which corresponds to A being the 4×4 identity $\mathbb{1}_4$.

Definition: A second order linear PDE that's independent of time:

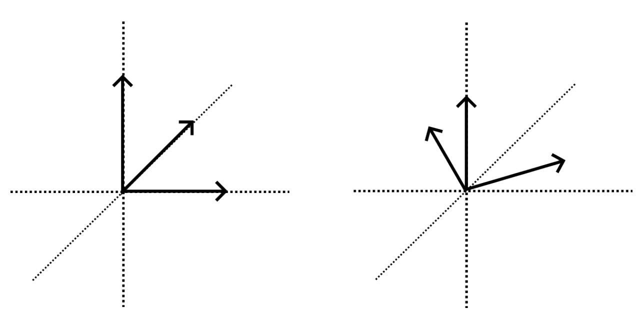
$$\sum_{ij=1}^{3} A_{ij}(x,...)\partial_i\partial_j u + \sum_{k=1}^{3} b_k(x,...)\partial_k u + c(x,y,z)u = 0$$

is *elliptic* if

$$A = \begin{pmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{pmatrix}$$

is *diagonal* with strictly positive entries.

However, from linear algebra we know matrices representing the same linear transformation, can appear different in different bases, so A being diagonal is not a good definition.



We need a rigorous definition to be independent of coordinates. From linear algebra we also know the eigenbasis *is* a diagonal basis. Therefore,

Definition: A second order linear PDE

$$\sum_{ij=1}^{3} A_{ij}(x,...)\partial_i\partial_j u + \sum_{k=1}^{3} b_k(x,...)\partial_k u + c(x,y,z)u = 0$$

if elliptic if $A(x,y,z)$ has all eigenvalues $\lambda_1(x,y,z), ..., \lambda_n(x,y,z) > 0$ strictly positive for all x, y, z. Then A is diagonal in the eigenbasis $\{\lambda_1, \lambda_2, ...\lambda_n\}$.

Example: Some examples of elliptic ODEs:

$$\triangle u + \Lambda u = 0 \qquad \Lambda \le 0 \qquad \text{Helmholtz equation}$$

$$-\triangle u + V(x,y,z)u = 0 \qquad V \text{ potential} \qquad \text{time-indep. Schrodinger}$$

2.2 Parabolic

A prototypical example is the heat equation:

$$\frac{\partial u}{\partial t} - \triangle u = 0.$$

Remark: The diffusion equation, which describes how the concentration u of solute in water changes over time:

$$\frac{\partial u}{\partial t} - \alpha \triangle u = 0$$

is the same as the heat equation but with a constant α. The Poincaré principle then tells us we can import intuition from the one to the other, because their underlying PDE is mathematically the same.

Definition: A second order linear PDE on $\mathbb{R}_t \times \mathbb{R}^3$ is *parabolic* if it has the form

$$\frac{\partial u}{\partial t} - \{\text{any time ind. second order linear elliptial}\}u = 0.$$

2.3 Hyperbolic

A prototypical example is the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \triangle u = 0 \qquad c = \text{propagation speed.}$$

Definition: A second order PDE on $\mathbb{R}_t \times \mathbb{R}^3$ is *hyperbolic* if for

$$\sum_{ij=0}^{3} A_{ij}(t,...)\partial_i\partial_j u + \sum_{k=0}^{3} B_k(t,...)\partial_k u + C(t,...)u = 0,$$

A has *exactly one strictly negative* eigenvalue with the rest remaining strictly positive (for all t, x, y, z).

Exercise I: Is this second order equation elliptic, parabolic, hyperbolic, or none?

$$\partial_t^2 u + 4\partial_t\partial_x u + 3\partial_x^2 u + 8 \underbrace{\partial_t\partial_y u}_{4\partial_t\partial_y u + 4\partial_y\partial_t u} + 2\partial_y^2 u - 2\partial_x\partial_y u = 0$$

Solution: From this PDE, we have

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 2 \end{pmatrix},$$

which has eigenvalues $\{-3.279, 3.592, 5.686\}$. Therefore this is a hyperbolic PDE.

If an arbitrary PDE is none of the above types, it is typically not well-behaved, even if it has a solution, and most do not. Fortunately God made these equations rarely observed in nature.

Remark: The PDEs are called elliptical, parabolic, and hyperbolic from their algebraic structure. Replacing derivatives $\frac{\partial}{\partial t} \mapsto T$, $\frac{\partial}{\partial x} \mapsto X$, we find

$$a\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad aX^2 + bY^2 \qquad \text{(ellipse)}$$

$$\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \qquad T - X^2 \qquad \text{(parabola)}$$

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \qquad T^2 - X^2 \qquad \text{(hyperbola)}$$