Problem

The game of ***Nim*** is played with a collection of piles of sticks. In one move, a player may remove any nonzero number of sticks from a single pile. The players alternately take turns making moves. The player who removes the very last stick loses. Say that we have a game position in Nim with k piles containing $s_1, \ldots, s_k$ sticks. Call the position ***balanced*** if each column of bits contains an even number of 1s when each of the numbers s_i is written in binary, and the binary numbers are written as rows of a matrix aligned at the low order bits. Prove the following two facts.

a. Starting in an unbalanced position, a single move exists that changes the position into a balanced one.

b. Starting in a balanced position, every single move changes the position into an unbalanced one.

Let *NIM* = { $\langle s_1, \ldots, s_k \rangle$ I each s_i is a binary number and Player I has a winning strategy in the Nim game starting at this position}. Use the preceding facts about balanced positions to show that *NIM* ? L.

Step-by-step solution

Step 1 of 5

In a game of **NIM** that consists two players, there are k heaps containing $s_1, s_2, \ldots, s_k$ numbers of stacks. Now, the binary equivalent of each of the numbers (s_i) is taken and arrange them row wise and aligned them at the low order bits.

• The arrangement is said to be "balanced" if there are even no of one's in each column. An XOR operation is performed on the stick numbers in each heaps by the column values. **This is called "NIM-sum" of the numbers**.

• Hence, from the definition of balanced position, it can be said that if **zero value is acquired by the NIM-sum, then the arrangement is balanced otherwise not**.

Comment

Step 2 of 5

Now to prove the given facts, consider that $s_1, s_2, \ldots, s_k$ be the heaps size before any move is made and $s_1', s_2', \ldots, s_k'$ be the size of heaps after a move is made.

• Assume NIM_s be the NIM-sum before any move (that is, $NIM_s = s_1 \oplus s_2 \oplus \cdots \oplus s_k$) and NIM_s' be the NIM-sum after any move (that is, $NIM_s' = s_1' \oplus s_2' \oplus \cdots \oplus s_k'$).

• If the move is in the heap l then it has $s_i = s_i'$ for all $i \neq l$, and $s_l > s_l'$. It is considered that, the XOR function or more precisely in this case.

• The NIM-sum function $(\oplus)$ follow the simple associative and communicative laws and also follow one more property, that is, $s \oplus s = 0$.

Therefore, by using these properties, which are discussed above:

$$\begin{aligned}
NIM_s' &= 0 \oplus NIM_s' \\
&= NIM_s \oplus NIM_s \oplus NIM_s' \\
&= NIM_s \oplus (s_1 \oplus s_2 \oplus \cdots \oplus s_k) \oplus (s_1' \oplus s_2' \oplus \cdots \oplus s_k')
\end{aligned}$$

$$\begin{aligned}
&= NIM_s \oplus (s_1 \oplus s_1') \oplus \cdots \oplus (s_k \oplus s_k') \\
&= NIM_s \oplus 0 \oplus 0 \oplus \cdots \oplus (s_l \oplus s_l') \oplus \cdots \oplus 0 \\
&= NIM_s \oplus s_l \oplus s_l'
\end{aligned}$$

Comment

Step 3 of 5

Consider an unbalanced position of piles (that is when $NIM_s \neq 0$), Suppose d be the most significant position of non-zero bit in the binary format of NIM_s.

• Now, select a l in such a way that a non-zero value is acquired by dth bit of s_l. The null or zero value acquired by the dth bit of NIM_s, when no such l exists.

• Now consider, $s_l' = NIM_s \oplus s_l$ and $s_l' < s_l$ because all bits to the left of the dth bit is are same in s_l and s_l' and bit d decreases from 1 to 0 or the value decreased by 2^d.

• So the first player by the winning strategy makes a move by choosing $s_l - s_l'$ objects from heap l. Then

$$\begin{aligned} NIM_s' &= NIM_s \oplus s_l \oplus s_l' \\ &= NIM_s \oplus s_l \oplus (NIM_s \oplus s_l) \\ &= 0 \end{aligned}$$

Therefore, there exists a single move that changes the position from unbalanced to a balanced one.

Comment

Step 4 of 5

Now, initialize from a balanced position, that is $NIM_s = 0$, then $NIM_s' = s_l \oplus s_l'$. Suppose, the first move is made on heap l making $s_l \neq s_l'$ then NIM_s' acquires a non-zero value**. Hence, every single move changes the position balanced to an unbalanced one.**

Comment

Step 5 of 5

In a normal play condition, after considering the fact (that is, none of the players makes any mistakes) it is easy to find a winning strategy. It is clear from the above mathematical deduction, that the strategy is to make the NIM-sum zero after each move.

• Consider the winning strategy, if the game is in balanced condition (that is NIM-sum is already zero). User has to move second letting Player II to move first and conversely when the game condition is unbalanced one (that is, the NIM-sum is non-zero). Then player I's has to move first.

• Now considering an algorithm to design a TM for Player I's winning strategy, where each input is already in its binary equivalent. These easy steps can be followed:

Suppose, M = Step

Cross off the 1's on finding of two 1s.

a. If no 1 left uncrossed after the entire scan of input, Player-I has to move second.

b. Else, Player-I has to move first.

So, the only space tradeoff is to store the binary numbers, which requires $O(\log(k))$ space. Therefore, $NIM \in L$. □

Comment