Problem

Let $c_1x^n + c_2x^{n-1} + \cdots + c_nx + c_{n+1}$ be a polynomial with a root at $x = x_0$. Let c_{max} be the largest absolute value of a ci. Show that

$$|x_0| < (n+1)\frac{c_{\max}}{|c_1|}.$$

Step-by-step solution

Step 1 of 3

Consider a polynomial $f(x) = c_1x^n + c_2x^{n-1} + \cdots + c_nx + c_{n+1}$.

Since above polynomial has a root at

$x = x_0$

So $f(x_0) = 0$ implies that

$c_1x_0^n + c_2x_0^{n-1} + \cdots + c_nx_0 + c_{n+1} = 0$

Rearrange the above equation as:

$c_1x_0^n = -(c_2x_0^{n-1} + \ldots + c_nx_0 + c_{n+1})$

Now, take modulus on both the sides of the equation as:

$$\left|c_1x_0^n\right| = \left|-(c_2x_0^{n-1} + \cdots + c_nx_0 + c_{n+1})\right|$$
$$|c_1|x_0^n = \left|(c_2x_0^{n-1} + \cdots + c_nx_0 + c_{n+1})\right|$$

Comments (4)

Step 2 of 3

Use sub-additive property $|a+b| \le |a| + |b|$ of modulus function,

$\left|c_1x_0^n\right| \le \left|c_2x_0^{n-1}\right| + \ldots + |c_nx_0| + |c_{n+1}|$

Also, as $c_{\max}$ is the largest absolute value of c_i then for each $i = 1, 2, \cdots, (n+1)$,

$c_{\max} = |c_{n+1}|$

So, from above equation,

$\left|c_1x_0^n\right| \le c_{\max}\left(1 + |x_0| + \ldots + \left|x_0^{n-1}\right|\right)$

Comments (2)

Step 3 of 3

Substitute $n.x_0^{n-1}$ for $1 + |x_0| + \ldots + \left|x_0^{n-1}\right|$ where, x_0^{n-1} is the largest one if $x_0 > 1$:

$$\left|c_1 x_0^{\,n}\right| \le c_{\max} \cdot n\left|x_0^{\,n-1}\right|$$

$$\left|\frac{x_0^{\,n}}{x_0^{\,n-1}}\right| \le n.\frac{c_{\max}}{|c_1|}$$

$$\left|x_0^{\,n-(n-1)}\right| \le n.\frac{c_{\max}}{|c_1|}$$

$$|x_0| \le n.\frac{|c_{\max}|}{|c_1|}$$

It also can be written as:

$$|x_0| \le n.\frac{c_{\max}}{|c_1|}$$

To make the term $n.\frac{c_{\max}}{|c_1|}$ strictly greater than $|x_0|$, we can re-write as:

$$|x_0| < (n+1)\frac{c_{\max}}{|c_1|} \quad (\text{since, } n < n+1 \text{ always holds})$$

Comment