Problem

Refer to Problem 1.42 for the definition of the shuffle operation. Show that the class of context-free languages is not closed under shuffle.

Problem 1.42

For languages A and B, let the ***shuffle*** of A and B be the language

$$\{w \mid w = a_1 b_1 \cdots a_k b_k, \text{ where } a_1 \cdots a_k \in A \text{ and } b_1 \cdots b_k \in B, \text{ each } a_i, b_i \in \Sigma^*\}.$$

Show that the class of regular languages is closed under shuffle.

Step-by-step solution

Step 1 of 2

Context free language is the language which is generated by CFG or the context free grammar. It is possible to get different types of context free languages from different types of context free grammar.

Consider two languages as *P* and *Q* defined over the alphabet and let the shuffle operation defined for these two languages is:

$$\left\{\mathrm{w} \;\middle|\; \mathrm{w} = p_1 q_{1......} p_k q_{k,} \text{where } p_{1....} p_k \in P, q_{1....} q_k \in q \text{ and each } a_j b_j \in \Sigma^*\right\}$$

Now, the two languages under consideration can be defined as follows:

$$P = \left\{w \in \{0,1\}^* \;\middle|\; n_0(w) = n_1(w)\right\}$$

$$Q = \left\{w \in \{p,q\}^* \;\middle|\; n_p(w) = n_q(w)\right\}$$

Where, in the above grammar $n_1(w)$ is the number of one's in *w* and $n_0(w)$ is the number of zeros present in the string *w*. Also, $n_p(w)$ and $n_q(w)$ denotes the number of *p's* and number of *q's* in *w* respectively.

Comment

Step 2 of 2

The language belonging to the shuffle of *P* and *Q* can be given as:

$$S = \left\{w \in \{0,1,p,q\}^* \;\middle|\; n_0(w) = n_1(w) \text{ and } n_p(w) = n_q(w)\right\}$$

The language shown above for the shuffle of *P* and *Q* is not a context free language and it can be proved by using the proof by contradiction.

Therefore, suppose that the language defined for the shuffle of P and Q is context free and let *t* be the length of its pumping lemma.

- Now, also suppose that $ab^2de^2f \neq S$ and by using the approach of pumping lemma shuffle can be represented or written as *abdef* where $|ae| > 1$ and $|\mathrm{abd}| \leq r$. Now, $|\mathrm{abd}|$ cannot contain both zeros and ones and both *p's* and *q's* as $|\mathrm{abd}| \leq r$.
- Thus, if the principle of pumping lemma is applied to ab^2de^2f once then the resultant string that will be obtained will have unequal number of zeros and ones or unbalanced ratio of *p's* and *q's*.
- This implies that $ab^2de^2f \neq S$ and thus this also signifies that the language S belonging to the shuffle of *P* and *Q* is not a context free language or CFL.

Now, the two languages *P* and *Q* which are under consideration are context free but the shuffle of these two languages is not context free.

Thus, it can be concluded and deduced that context free languages or CFL's are not closed under shuffle operation.

Comments (2)