**Problem**

Prove the following stronger form of the pumping lemma, wherein *both* pieces v and y must be nonempty when the string s is broken up.

If A is a context-free language, then there is a number k where, if s is any string in A of length at least k, then s may be divided into five pieces, s = uvxyz, satisfying the conditions:

**a.** for each $i \geq 0$, $uv^ixy^iz \in A$,

**b.** $v \neq \varepsilon$ and $y \neq \varepsilon$, and

**c.** $|vxy| \leq k$.

**Step-by-step solution**

**Step 1** of 8

It has to be proven that if $A$ is a context-free language, there is a number $k$ where is a string $s$ such that $s \in A$ and $|s| \geq k$, then the string $s$ can be split into five pieces $s = uvxyz$ such that the following conditions hold:

a. For each $i \geq 0, uv^ixy^iz \in A$,

b. $v \neq \varepsilon$ and $y \neq \varepsilon$, and

c. $|vxy| \leq k$.

Comment

**Step 2** of 8

Now, define a context-free grammar $G = (V, \Sigma, R, T)$ for the context-free language $A$ such that the right-hand side of a rule $b$ has maximum number of symbols, with $b$ being at least 3 symbols large.

• When the height of parse tree is $k$ then the length of the string $s$ generated is at least $b^k$. This is as there will be at most $b$ leaves 1 step from $S$, at most $b^2$ at a 2 steps and at-most $b^k$ leaves at a depth of $k$ steps.

• Alternatively if the length of string is $b^k$ then all smallest possible parse trees for the string must be at least $k$ deep. Thus any path in a parse tree for the string $s$ will have a path that is at least $k$ terminals long. The path will start with $k-1$ variables and terminate with a single terminal.

Comment

**Step 3** of 8

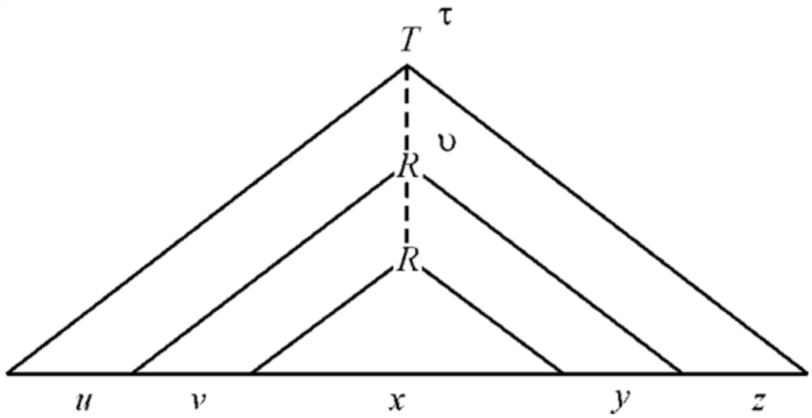
The number of variables in the grammar $G$ is $|V|$. A string $s$ is taken whose length is at least $|V|+1$. Hence any of the smallest possible parse trees will have a depth that is at least $b^{|V|+1}$ steps. Or in other words all paths will be at least $|V|+1$ long, which will start with $|V|+1$ variables and end with a terminal.

• There are only $|V|$ variables in the grammar $G$. The pigeonhole principle is applied with the variables (or non-terminal nodes) being the holes and with the pigeons being the variables encountered on any path in the tree.

• There are $|V|+1$ pigeons going into $|V|$ holes, so at least one hole must contain more than one pigeon. That is at least one variable occurs more once in the longest path of the smallest possible parse tree $\tau$.

Comment

Let this variable be $R$. Use the parse tree $\tau$ from the start symbol $T$ to divide the string $s$ into five pieces, of the form $uvxyz$. Consider the figure which is given below:

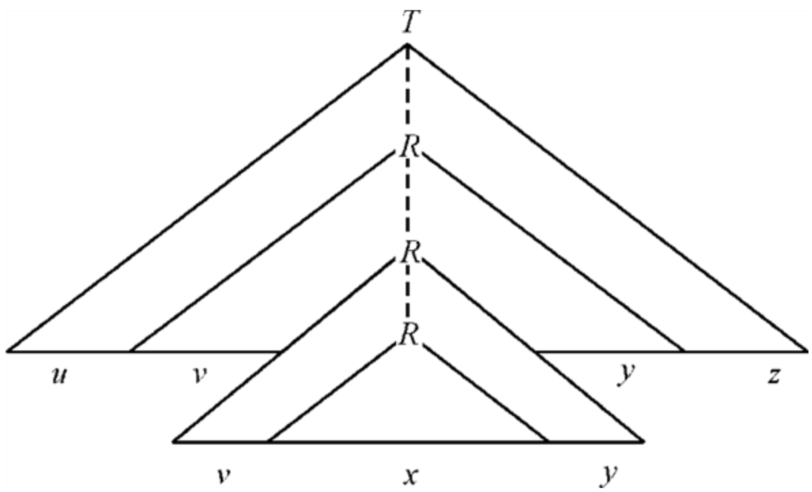



The parse tree $\upsilon$ at the first $R$ vertex hit by the start symbol $T$ breaks down the substring $vxy$ into three strings of terminals. Therefore this tree $\upsilon$ can be repeated any number of times. This is shown by considering a couple of examples.

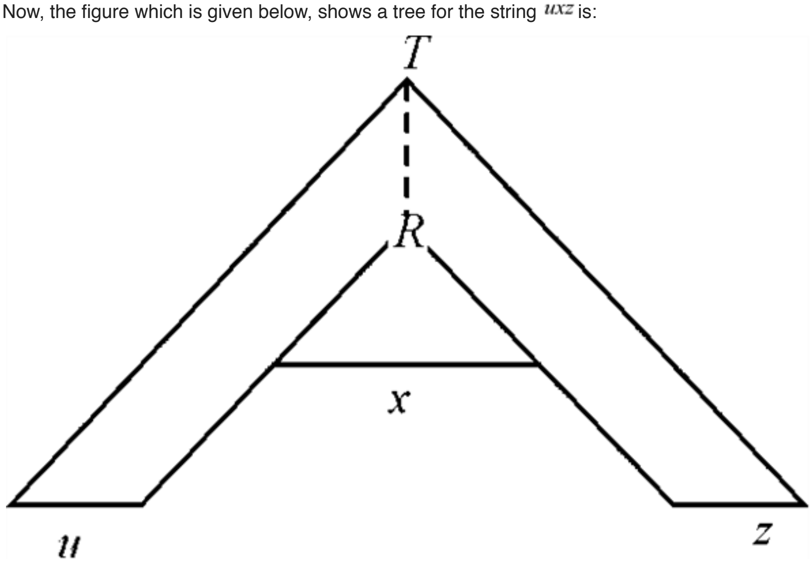
Comment



Consider the figure which is given below. It shows a tree for the string $uv^2xy^2z$ is:




Now, the figure which is given below, shows a tree for the string $uxz$ is:




**In this way, it can be said that "The first condition $uv^ixy^iz \in A, i \geq 0$ has been shown to be true".**

Comment

**Step 6** of 8

**In the second condition, it is** stated that **either of** $v$ **and** $y$ **cannot be the empty string** $\varepsilon$. The parse tree $\tau$ is the smallest possible tree.

- If $v = \varepsilon$ or if $y = \varepsilon$, then one or more vertices can be removed from the sub-trees ending in $\varepsilon$ in the parse tree $\tau$.
- 

Comments (1)

**Step 7** of 8

This is a contradiction as it is not possible to remove vertices from the tree $\tau$ as it is the smallest possible parse tree.

**It has been shown that** $v \neq \varepsilon$ **and** $y \neq \varepsilon$ **.**

Comment

**Step 8** of 8

**Lastly it has to be proven that** $|vxy| \leq k$. Alternatively, the depth of the tree $\upsilon$ has to be at least $k$. Since the root node $R$ repeats and the path has $|V|+1$ variables. **Thus depth of the tree** $\upsilon$ **is at least** $|V|+1$ **and its yield will be at most** $b^{|V|+1} = k$ **.**

Comment