Problem

If A is a set of natural numbers and k is a natural number greater than 1, let

$B_k(A)$ = {wl w is the representation in base k of some number in A}.

Here, we do not allow leading 0s in the representation of a number. For example, $B_2(\{3, 5\}) = \{11, 101\}$ and $B_3(\{3, 5\}) = \{10, 12\}$. Give an example of a set A for which $B_2(A)$ is regular but $B_3(A)$ is not regular. Prove that your example works.

Step-by-step solution

Step 1 of 1

Regular and Non Regular Expression

Assume $\begin{aligned} A &= \{2n \mid \text{n is a natural number}\} \\ &= \{2,\ 4,\ 8,\ 16,\ 18,\ 20\ \ldots\} \end{aligned}$

So $B_2(A) = \{10,\ 100,\ 1000, 10000\ \ldots\}$ should be regular, because $B_2(A)$ is recognized by regular expression $10*$.

Now $B_3(A) = \{2,\ 11,\ 22,\ 121\ldots\}$

$B_3(A)$ is non regular.

It can be proved by contradiction that $B_3(A)$ is non regular.

Take on the contrary that $B_3(A)$ is regular.

Now p will be pumping length according to pumping lemma.

Choose u as element of $B_3(A)$ and the length of u should at least $p+1$.

Since $u \in B_3(A)$ and $i > 1$, by pumping lemma u can be divided into three part, $u = xyz$ where $\forall i \geq 0$ the string $xy^iz \in B_3(A)$

According to condition three of pumping lemma, $|xy| \leq p$, and $|z| > 0$.

If rightmost digit of z is 0, then u will be the power of 3. But u is power of 2, hence it is impossible. Hence the rightmost digit of 0 will must be 1 or may be 2.

For $i > 1$, $u' = xy^iz$ is power of 2 which is greater than u, so it is easy to generate it after u is added to itself few numbers of times. That is, if u is added at least 3 times to itself, there must be carry from the right to left column of z. When i will increase, the carries will affect more columns. For very large i, the carries will bleed on y, as pumping lemma condition two says that y should not be empty.

Power of 2 will be generated for very large value of i which cannot be generated by copying y to i times. The carries will force y as well as z to change. Hence, it is showing that $B_3(A)$ does not satisfy pumping lemma. Hence, the initial assumption that $B_3(A)$ is regular, is wrong.

Hence, $B_3(A)$ is non-regular.

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