Problem

Show that *2SAT* is NL-complete.

Step-by-step solution

Step 1 of 4

<u>NL – completeness:-</u> A language *B* is *NL*- complete if

1. $B \in NL$, and

2. Every *A* in *NL* is log space reducible to *B,* that is *B* is $NL-$hard.

Now we have to prove that 2*SAT* is *NL* –complete.

We know that $2SAT =\{ \langle\phi\rangle \mid \phi$ is a satisfiable Boolean formula in $2cnf$ }.

We know the fact that $NL = CONL$

Thus if we prove that $\overline{2SAT}$ is $NL-$complete, then that implies that 2*SAT* is $NL-$complete.

Comment

Step 2 of 4

To prove $\overline{2SAT}$ is *NL*- complete, we have to prove the two conditions of $NL-$completeness.

(i) $\overline{2SAT} \in NL$:

In order to prove $\overline{2SAT} \in NL$ construct the graph NTM M as follows:

$M =$ "On input ϕ:

(1) Construct graph *G* such that ϕ is not satisfiable if and only if there two vertices u, v the paths uv and vu are in *G*.

(2) For each variable *x* create a node labeled *x* and another labeled $\bar{x}$.

(3) Now non-deterministically select a vertex *x* in *G*.

(4) Check if *G* contain both *x* $x\bar{x}$ and $\bar{x}x$ paths using $NL-$algorithm for $PATH$.

(5) If both paths exist, then accept

(6) Otherwise reject ".

The entire construction is done in log – space and *M* decides $\overline{2SAT}$ in log space only.

Thus $\overline{2SAT} \in NL$

Comment

Step 3 of 4

(ii) $\underline{\overline{2SAT} \text{ is } NL\_\text{hard:}}$

We do this by reducing path to $\overline{2SAT}$.

<u>Construction of</u> ϕ:

• Given G, s, t we have to construct a $2-cnf$ formula ϕ such that *G* has an $s, t-$path if and only if ϕ is unsatisfiable.

• Let $v(G) = \{s, t, y_1, \ldots y_n\}$

• The resulting formula ϕ consists of $m+1$ variables $x, y_1, \ldots y_m$

• The vertex *s* is identified with *x* and the vertex *t* with $\overline{x}$.

• The asses of ϕ will be $x \vee x$ and $\overline{u} \vee u$ every edge (u,v) of *G*.

Comment

Step 4 of 4

→ In order to show that this construction works, first assume that *G* has an s,t path. Let this path be $s,u_1,...u_k t$. Then ϕ contains the clause $(\overline{x} \vee u_1),(\overline{u}_1 \vee u_2)...(\overline{u}_k,\overline{x})$.

If we designs $x=1$, at least one of these clauses is not satisfied on the other hand, if $x=0$, then the clause $(x \vee x)$ of ϕ is not satisfied. So ϕ is zero for any assignment of $x,y_1,...y_m$.

→ Conversely, suppose that *G* contains no $s-t$ path.

• Let *U* be the set of vertices in *G* reachable from s,V the set of vertices from which *t* is reachable and $w:V(G)(U \vee V)$.

By hypothesis and construction U,V,W are pair wise disjoint, and there are no edges from *U* to $V \cup W$ or from $U \cup W$ to V. let us assign the rue value to every variable in $U \cup W$ and the false value to every variable in *V*. it is now simple to check that ϕ is satisfied for this truth assignment.

Comment