Problem

Next

Let β be the set of all infinite sequences over {0,1}. Show that β is uncountable using a proof by diagonalization.

Step-by-step solution

Step 1 of 3

Let $\mathcal{B}$ be the set of all infinite sequences over {0,1}. Every element in $\mathcal{B}$ is an infinite sequence $(a_1, a_2, a_3, \ldots)$ where $a_i \in \{0,1\}$. Assume $\mathcal{B}$ is countable. Define a correspondence *f* between $\mathcal{B}$ and $N = \{1,2,3,\ldots\}$.

Comment

Step 2 of 3

Suppose, for $z \in N$, $f(z) = (a_{z1}, a_{z2}, a_{z3}, \ldots)$ where a_{zi} is defined as the i^{th} bit in the z^{th} sequence.

In other words,

z	$f(z)$
1	$(a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \ldots)$
2	$(a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, \ldots)$
3	$(a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, \ldots)$
4	$(a_{41}, a_{42}, a_{43}, a_{41}, a_{45}, \ldots)$
$\vdots$	$\vdots$

Comment

Step 3 of 3

Now, a sequence b is defined in such a way that $b = (b_1, b_2, b_3, \ldots)$ belongs to $\mathcal{B}$ over $\{0,1\}$ where $b_i = 1 - a_{ii}$ for $i \in N$.

Consider the following example,

z	$f(z)$
1	$(0,1,1,0,0,\ldots)$
2	$(1,0,1,0,1,\ldots)$
3	$(1,1,1,1,1,\ldots)$
4	$(1,0,0,1,0,\ldots)$
$\vdots$	$\vdots$

The sequence *b* is computed as $b = (1-a_{11}, 1-a_{22}, 1-a_{33}, 1-a_{44}, \ldots) = (1,1,0,0,\ldots)$. Therefore, $b \in \mathcal{B}$ is different from every sequence by minimum one bit. Thus, b is not equal to $f(z)$ for any z. It is a contradiction that $\mathcal{B}$ is uncountable.

Therefore, $\mathcal{B}$ is uncountable.

Comment