Problem

LetM be a probabilistic polynomial time Turing machine, and let C be a language where for some fixed $0 < \epsilon_1 < \epsilon_2 < 1$,

a. $w \notin C$ implies $\Pr[M \text{ accepts } w] \leq \epsilon_1$, and

b. $w \in C$ implies $\Pr[M \text{ accepts } w] \geq \epsilon_2$.

Show that $C \in$ BPP. (Hint: Use the result of Lemma 10.5.)

Step-by-step solution

Step 1 of 1

Given M be probabilistic Turing Machine and C be a language where for some fixed $0 < \varepsilon_1 < \varepsilon_2 < 1$,

1. $w \notin C$ implies $\Pr[M \text{ accepts } w] \leq \varepsilon_1$.
2. $w \in C$ implies $\Pr[M \text{ accepts } w] \leq \varepsilon_2$

It is required to show $C \in BPP$.

• Between any two distinct real numbers $\varepsilon_1 < \varepsilon_2$ there exists another real number that lies strictly between them. Thus to choose c such that $\varepsilon_1 < c < \varepsilon_2$.

• Consider another machine S which repeatedly runs M. Now S accepts if the proportion of $M's$ acceptance is greater or equal to c, and S rejects if the proportion of $M's$ acceptance is less than c. Now to show S decides in BPP.

• Consider the variable S_k be the total number of acceptances by machine M after k runs on input w. Hence, for $w \in C$, S_k is the sum of k $0-1$ random variables with common mean $\mu_2 > \varepsilon_2$, and for $w \notin C$, S_k is sum of k $0-1$ random variable with common mean $\mu_1 > \varepsilon_1$. The error probabilities can then be expressed as follows:

$$1.\text{For } w \in C, \Pr[S \text{ rejects } w] = \Pr\left[\frac{S_k}{k} < c\right] \leq \Pr\left[\left|\frac{S_k}{k} - \mu_2\right| > \mu_2 - c\right]$$
$$2.\text{For } w \in C, \Pr[S \text{ accepts } w] = \Pr\left[\frac{S_k}{k} \geq c\right] \leq \Pr\left[\left|\frac{S_k}{k} - \mu_1\right| \geq c - \mu_1\right]$$

By the weak law of large numbers (or various other bounds from probability theory), there exist k that will make those probabilities on the right as small as desired, and in particular, there exist k that will make them both strictly less than $\frac{1}{2}$.

• By using "Amplification lemma" this shows $C \in BPP$.

Comment