Problem

Use Rice's theorem, which appears in Problem 5.28, to prove the undecidability of each of the following languages.

A**a.** $INFINITE_{\mathsf{TM}} = \{\langle M\rangle |\ M \text{ is a } \mathsf{TM} \text{ and } L(M) \text{ is an infinite language}\}.$

b. $\{\langle M\rangle |\ M \text{ is a } \mathsf{TM} \text{ and } \mathtt{1011} \in L(M)\}.$

c. $ALL_{\mathsf{TM}} = \{\langle M\rangle |\ M \text{ is a } \mathsf{TM} \text{ and } L(M) = \Sigma^*\}.$

Problem 5.28

Rice's theorem. Let P be any nontrivial property of the language of a Turing machine. Prove that the problem of determining whether a given Turing machine's language has property P is undecidable.

In more formal terms, let P be a language consisting of Turing machine descriptions where P fulfills two conditions. First, P is nontrivial—it contains some, but not all, TM descriptions. Second, P is a property of the TM's language whenever $L(M_1) = L(M_2)$, we have $\langle M_1\rangle \in P$ iff $\langle M_2\rangle \in P$. Here, M_1 and M_2 are any TMs. Prove that P is an undecidable language.

Step-by-step solution

Step 1 of 3

Un-decidability using Rice Theorem

a. $INFINITE_{TM}$ is a language of TM descriptions. It satisfies the two conditions of Rice's Theorem. First, is that it is non-trivial because some TMs have infinite languages and others have not. Second, is that it depends only on language. If two Turing Machine recognize same language, then either both should have descriptions in $INFINITE_{TM}$ or neither does. Consequently, Rice's theorem implies that $INFINITE_{TM}$ is un-decidable.

Comment

Step 2 of 3

b. Consider $\{\langle M\rangle |\ M \text{ is a Turing Machine and } 1011 \in L(M)\}$. P is language of Turing Machine descriptions. It satisfies the two conditions of Rice's Theorem. First, it is non-trivial because some TMs contain the string 1011 in their language and others do not. Second, it only depends on the language. If two TMs recognize same language then either both should have descriptions in P (because they both accept 1011), or neither do. Thus, Rice's theorem implies that P is un-decidable.

Comments (1)

Step 3 of 3

c. ALL_{TM} is a language of TM descriptions. It satisfies the two conditions of Rice's Theorem. First, it is non-trivial because some TMs accept all possible strings of an alphabet Σ and others do not. Second, is that it depends only on language. If two TMs recognize same language, then either both should have descriptions in ALL_{TM} or neither does. Therefore, Rice's theorem implies that ALL_{TM} is un-decidable.

Comment