Problem

Let $M = (Q, \Sigma, \delta, q_0, F)$ (q, s) equals the state where M ends up when M starts at state q and reads input s.) Say that M is ***synchronizable*** if it has a synchronizing sequence for some state h. Prove that if M is a k-state synchronizable DFA, then it has a synchronizing sequence of length at most k^3. Can you improve upon this bound?

Step-by-step solution

Step 1 of 1

Given:

A DFA $M=(Q,\Sigma,\delta,q_0,F)$ and suppose h is a state of DFA M known as its home state.

Proof:

The at most length of synchronizing sequences is k^3 0 for a k-state synchronizable DFA. In the year 1964, a Slovak scientist named Jan Cerny first tried to solve the problem of synchronizing automata in real time. This problem is sometimes referred as *Cerny's Conjecture.*

To prove the upper bound on the synchronizing sequence we try to device a greedy algorithm.

Algorithm:

1. Let a synchronizing DFA be $M=(Q,\Sigma,\delta,q_0,F)$. Initialize the synchronizing sequence $\omega \leftarrow 1$ (empty word) and a set of states $P \leftarrow Q$.

2. ***while*** $|P|>1$

a. Find a word υ which is belongs to $\Sigma^*$and it has minimum length $|\delta(P,\upsilon)|<|P|$.

b. If none exists, ***return*** failure.

c. $\omega \leftarrow \omega\upsilon$

$P \leftarrow \delta(P,\upsilon)$

3. ***return*** ω

• Now suppose that M is a k-state DFA, that is, $|Q|=k$ then clearly the main loop of the algorithm runs at most $k-1$ times. In order to get the length of the output word ω user has to estimate the length of each word υ derived at each loop.

• Consider a generic step at which $|P|=n>1$ and let $\upsilon=a_1\cdots a_l$ with $a_i \in \Sigma,\ i=1\cdots l$. Then it is quite simple to see that the sets, $P_1=P,\ P_2=\delta(P_1,a_1),\ \ldots,\ P_l=\delta(P_{l-1},a_{l-1})$ are n-element subsets of Q.

• Furthermore, since $|\delta(P_l,a_l)|<|P_l|$, there exists two states $q_l,q_l' \in P_l$ such that $\delta(q_l,a_l)=\delta(q_l',a_l)$.

• Now define two element subsets $R_i=\{q_i,q_i'\}\subseteq P_i,\ i=1,\cdots,l-1.$ then the condition that υ is a word that has minimum length $|\delta(P,\upsilon)|<|P|$ which implies that $R_i \not\subset P_j$ for $1\le j<i<l$. Now by the Peter Frankl inequality, the tight bound over l be $\binom{k-n+2}{2}$.

• Summing up these inequalities from $n=k$ to $n=2$, user can get the upper bound over the synchronizing sequence $|\omega|\le\frac{k^3-k}{6}$.

Conclusion:

Therefore, at most length of synchronizing sequences is k^3 0 for a k-state synchronizable DFA.

Comment