## Problem

Consider the language B = L(G), where G is the grammar given in Exercise 2.13. The pumping lemma for context free languages, Theorem 2.34, states the existence of a pumping length p for B. What is the minimum value of p that works in the pumping lemma? Justify your answer.

THEOREM **2.34**

**Pumping lemma for context-free languages** If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces s = uvxyz satisfying the conditions

**1.** for each $i \geq 0$, $uv^ixy^iz \in A$,

**2.** $|vy| > 0$, and

**3.** $|vxy| \leq p$.

When s is being divided into uvxyz, condition 2 says that either v or y is not the empty string. Otherwise the theorem would be trivially true. Condition 3 states that the pieces v, x, and y together have length at most p. This technical condition sometimes is useful in proving that certain languages are not context free.

## Step-by-step solution

**Step 1** of 3

**Given:**

The minimum value of the pumping length $p$ for the language $B = L(G)$

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**Step 2** of 3

**Finding minimum length of $\mathbf{p}$** :

The language $B = L(G)$ is defined by the

Grammar $G = (V, \Sigma, R, S)$ where $V = \{S, T, U\}$; $\Sigma = \{0, \#\}$ and the given set of rules $R$ :

$$S \to TT \mid U$$
$$T \to 0T \mid T0 \mid \#$$
$$U \to 0U00 \mid \#$$

Theorem 2.34 states that the pumping lemma $p$, for a context-free language $A$, is the minimum length of any string $s$ in $A$ such that it may be split into five pieces $s = uvxyz$. The string $s$ will also fulfill the following three conditions:

1. For all $i \geq 0$, the string $uv^ixy^iz$ is a part of the context-free language $A$.

2. The strings which are pumped, which are $v$ and $y$, cannot both be the empty string $\varepsilon$, that is $|vy| > 0$.

3. The combined length of the strings lying inside $u$ and $z$ must not be greater that the pumping length $p$. In other words $|vxy| \leq p$.

The string $s = uvxyz$ can be taken as $\#\#0$, with the substrings being as follows:

$$u = v = z = \varepsilon$$
$$x = \#\#$$
$$y = 0$$

Thus the pumping length $p$ is $|vxy| = |\#\#0| = 3$

Since, $|vy| = |\varepsilon 0| = |0| = 1 > 0$, condition 2 of the theorem holds.

Condition 3 of Theorem 2.34 is also valid as $|vxy| = |\varepsilon\#\#0| = |\#\#0| = 3 \leq p = 3$.

To meet condition 1 it has to be proven that $uv^ixy^iz \in B$ for $i \geq 0$.

The string $uv^i xy^i z$, where $u=\varepsilon, v=\varepsilon, x=\#\#, y=0, z=\varepsilon$, can be expressed as the regular expression $\#\#0^i$.

The derivation for the case when $i=0$ is:

$S \Rightarrow TT \Rightarrow \#T \Rightarrow \#\#$

So the string $s=uv^0xy^0z$ lies in the language $B$.

When $i>0$ the string $s=uv^ixy^iz$ will also lie in $B$ as the derivation will be:

$S \Rightarrow TT \Rightarrow \#T \Rightarrow \#T0 \overset{*}{\Rightarrow} \#T0^+ \Rightarrow \#\#0^+$

Condition 1 has been proven true as $uv^ixy^iz \in B \forall i \geq 0$.

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**Step 3** of 3

**Conclusion:**

Thus the string satisfies all three conditions of Theorem 2.34 for a context-free language. Therefore, **the minimum value is 3** for the pumping lemma $p$.

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