Homework 2 — Due: Tuesday, September 13, 2022

Please submit your work on Brightspace, in PDF format only.

1. (a) Prove that, if $f : A \to B$ and $g : B \to C$ are injective, then $g \circ f : A \to C$ is also injective.

   (b) Prove that, if $f : A \to B$ and $g : B \to C$ are surjective, then $g \circ f : A \to C$ is also surjective.

   (c) Use the previous two facts to prove a similar result about bijective functions.

   These results may be summarized by saying that the classes of injective, surjective, and bijective functions are *closed under function composition.*

2. If R is a binary relation on a set A, then the *transitive closure* of R is the binary relation R^* on A defined as follows:

   > For all $a, b \in A$, $(a, b) \in R^*$ if and only if there exists $n \geq 0$ and a sequence $a_0, a_1, a_2, \ldots a_n$ of elements of A such that $a_0 = a$, $a_n = b$, and $(a_k, a_{k+1}) \in R$ for all $k \in \{0, 1, \ldots, n-1\}$.

   Let R be a binary relation on A.

   (a) Prove that R^* is a transitive relation.

   (b) Prove that, if T is any transitive relation on A that contains R (*i.e.* $R \subseteq T$), then T contains R^*; *i.e.* R^* is the $\subseteq$-smallest transitive relation on A that contains R.

   (c) Prove that, if $\mathcal{T}$ is the set of *all* transitive relations on A that contain R,

      i. $\bigcap \mathcal{T}$ is a transitive relation that contains R.

      ii. R^* contains $\bigcap \mathcal{T}$.

      Conclude that $\bigcap \mathcal{T} = R^*$.

3. Let $\Sigma = a_1, a_2, \ldots, a_k$ be an alphabet. The *Parikh vector* of $w \in \Sigma^*$ is the vector

   $$\psi_w = (\#_{a_1}(w), \#_{a_2}(w), \ldots, \#_{a_k}(w))$$

   where $\#_{a_i}(w)$ denotes the number of occurrences of a_i in w. Let R be the binary relation on Σ^* such that $(w, x) \in R$ if and only if either $w = x$ or else there exist $u, v \in \Sigma^*$ and $a, b \in \Sigma$, such that $w = uabv$ and $x = ubav$; that is, w can be transformed into x by the interchange of two adjacent letters.

   (a) Prove that $(w, x) \in R^*$ if and only if $\psi_w = \psi_x$.