Problem

Show that $EQ_{\mathrm{TM}} \not\leq_{\mathrm{m}} \overline{EQ_{\mathrm{TM}}}$.

Step-by-step solution

Step 1 of 3

TM equality:

The TM equality is represented as follows:

$EQ_{TM} = \{(\langle M\rangle, \langle N\rangle) \text{ where } M \text{ and } N \text{ are Turing machines and } L(M) = L(N)\}$.

$EQ'_{TM} = \{(\langle M'\rangle, \langle N'\rangle) \text{ where } M' \text{ and } N' \text{ are Turing machines and } L(M') = L(N')\}$

$EQ_{TM} \not\leq_m \overline{EQ_{TM}}$ means that EQ_{TM} is not mapping reducible to $\overline{EQ_{TM}}$. This means that EQ_{TM} is not mapping reducible to its complement.

Comment

Step 2 of 3

Proof:

In order prove that $EQ_{TM} \not\leq_m \overline{EQ_{TM}}$, first prove that EQ_{TM} is not Turing-recognizable.

According to Theorem 5.28 and Corollary 5.29, $A \leq_m B$ only if both A and B are Turing recognizable or not Turing recognizable.

$\overline{EQ_{TM}}$ is complement of EQ_{TM}. So, if EQ_{TM} is not Turing-recognizable then, $\overline{EQ_{TM}}$ is Turing-recognizable and vice-versa. This, result in not mapping reducibility between EQ_{TM} and $\overline{EQ_{TM}}$.

Comment

Step 3 of 3

Example:

Assume A_{TM} is a Turing machine and is mapping is mapping reducible to $\overline{EQ_{TM}}$ that is $A_{TM} \leq_m \overline{EQ_{TM}}$.

The function $f_2 : A_{TM} \to \overline{EQ_{TM}}$ is defined as follows:

$$
\begin{aligned}
&f_2 : On\ input\ \langle M, w\rangle \\
&\quad Construct\ machine\ M_3 : on\ any\ input,\ reject. \\
&\quad Construct\ machine\ M_4 : on\ any\ input\ x, run\ M\ on\ w. \\
&\qquad\qquad\qquad\qquad If\ it\ accepts,\ accept\ x. \\
&\quad Output\ \langle M_3, M_4\rangle
\end{aligned}
$$

Explanation:

- The machine M_1 accepts nothing.
- If M accepts w, then M_2 accepts everything. Otherwise it accepts nothing.
- So, $\langle M, w\rangle \in A_{TM} \Leftrightarrow \langle M_3, M_4\rangle \in \overline{EQ_{TM}}$ and f_2 is clearly computable. Thus, it is a reduction from A_{TM} to $\overline{EQ_{TM}}$.
- So, EQ_{TM} is not Turing-recognizable.

Thus, if EQ_{TM} is not Turing-recognizable and $\overline{EQ_{TM}}$ is Turing-recognizable then EQ_{TM} is not mapping reducible to $\overline{EQ_{TM}}$. That is, $EQ_{TM} \not\leq_m \overline{EQ_{TM}}$.