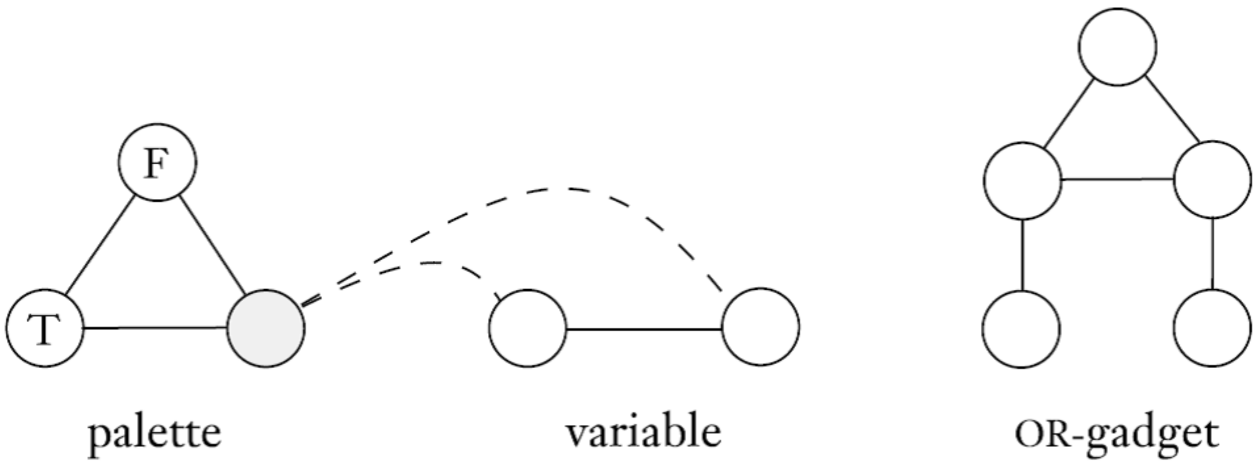
Problem

A ***coloring*** of a graph is an assignment of colors to its nodes so that no two adjacent nodes are assigned the same color. Let

$$3COLOR = \{\langle G \rangle \mid G \text{ is colorable with 3 colors}\}.$$

Show that *3COLOR* is NP-complete. (Hint: Use the following three subgraphs.)



Step-by-step solution

Step 1 of 2

NP **– complete:**

A language *B* is *NP* – complete if it satisfies following two conditions:

1. *B* is in *NP*

2. Every *A* in *NP* is polynomial time reducible to *B*.

Comment

Step 2 of 2

1.3 COLOR is in *NP* because a coloring can be verified in polynomial time.

2. $3SAT \le_p 3COLOR$:

• " $3SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable } 3cnf - \text{formula}\}$ " and "$3cnf$ -formula is the one in which all the clauses have three literals"

• Let $\phi = c_1 \wedge c_2 \wedge \ldots \wedge c_l$ be a $3cnf$ formula over the variable $x_1, \ldots, x_n$.

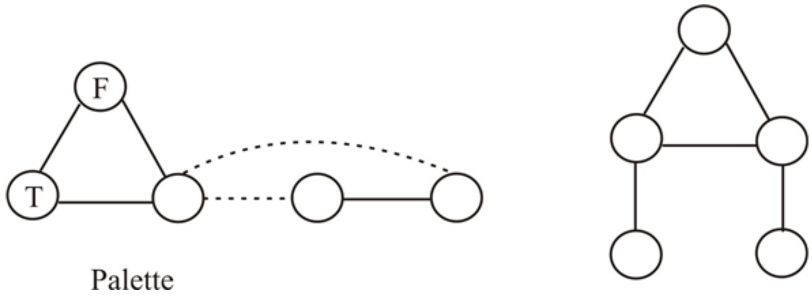
• To build a graph *G* with $2n + 6l + 3$ nodes, containing a variable gadget for each variable x_i, one clause gadget for each clause and one palette gadget as follows.

• Label the nodes of the palette gadget *T, F* and *R.*

• Label the node since each variable gadget + and – and cannot reach to the *R* node in the palette gadget.

• For each clause, create a gadget.

• Given three sub graphs.

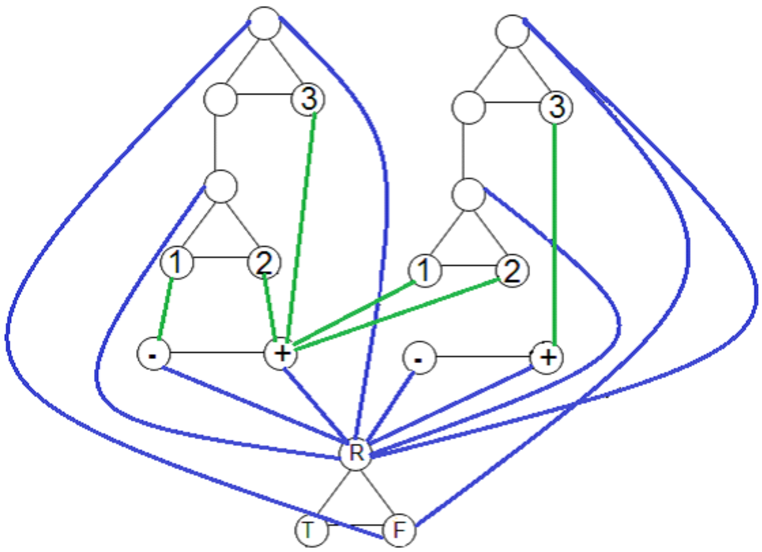


• Connect the *F* and *R* nodes to the top of the clause gadget in the palette.

• Also, connect the top of its bottom triangle to the *R* node.

• For every clause c_j, connect the $i^{th}\ (1 \le i \le 3)$ bottom node of its clause gadget to the literal node that appears in its i^{th} location.

• An example is shown below.



• To show that the construction is correct, we first demonstrate that if ϕ is satisfiable, the graph is 3- colored.

• The three colors are called T, F and R.

• Color the palette with its labels.

• For each variable, color the + node T and – node F if the variable is true in a satisfying assignment: otherwise reverse the colors.

• Because each clause has one True literal in the assignment, we can color the nodes of that clause so that the node connected to the F node in the palette is not colored F.

• Hence we have proper 3-coloring.

• Similarly, if we are given a 3-coloring, we can obtain a satisfying assignment by taking the colors assigned to the + nodes of each variable.

• Observe that neither node of the variable gadget can be can be colored R, because all variable nodes are connected to the R node in the palette.

• Furthermore, if both bottom nodes of the clause gadget are colored F, the top node must be colored F, and hence, each clause must contain a true literal.

Hence, $3\text{SAT} \le_P 3$COLOR

Therefore, from (1) and (2) 3 COLOR is NP- complete.

Comment