

Problem

Show that if G is a CFG in Chomsky normal form, then for any string $w \in L(G)$ of length $n \geq 1$, exactly $2n - 1$ steps are required for any derivation of w .

Step-by-step solution

Step 1 of 1

Given that G is a CFG in Chomsky Normal Form (CNF).

The length of the string $w \in L(G)$ is $n \geq 1$ for the string w .

It is required to show that exactly $2n-1$ steps are required for the derivation of string w .

It can be proved applying the **induction method** by on the string w of length n .

For $n = 1$: Consider a string "a" of length 1 in Chomsky normal form, so the valid derivation for this will be $S \rightarrow a$, where $a \in \Sigma$ and S is starting symbol.

The number of steps can be obtained as follows:

$$\begin{aligned} 2n - 1 &= 2(1) - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

Hence it is true that $2n-1$ (for $n=1$) steps are required to derive a string a .

For $n = k$: Take a string of length $k \geq n$ in Chomsky normal form, so valid derivation for this will take $2k - 1$ steps.

The number of steps can be obtained as follows:

$$\begin{aligned} 2n - 1 &= 2(k) - 1 \\ &= 2k - 1 \end{aligned}$$

Assume a string of length at most $k \geq 1$ terminal symbols and it has a string of length

$n = k + 1$ is in Chomsky Normal Form

Since $n > 1$, Consider a language as follows in CNF where derivation starts with start symbols S :

$$\begin{aligned} S &\rightarrow BC \\ B &\rightarrow *x \\ C &\rightarrow *y \end{aligned}$$

So length of the string starting with start symbol S is $|w| = xy$ where $|x| > 0$ and $|y| > 0$.

Using the inductive hypothesis, for the above language in CNF the length of any derivation of string w must be

$$1 + (2|x| - 1) + (2|y| - 1) = 2|x| + 2|y| + 1 - 1 - 1 = 2(|x| + |y|) - 1$$

Here $n = |x| + |y|$.

Since $B \rightarrow *x$ has a length of $|x|$ and $C \rightarrow *y$ has a length of $|y|$.

Hence, it is proved that it requires $2n-1$ steps required for the derivation of string

$w \in L(G)$ in Chomsky Normal Form(CNF).

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