Problem

Consider the algorithm *MINIMIZE*, which takes a DFA M as input and outputs DFA M'.

$MINIMIZE =$ "On input $\langle M \rangle$, where $M = (Q, \Sigma, \delta, q_0, A)$ is a DFA:

1. Remove all states of M that are unreachable from the start state.
2. Construct the following undirected graph G whose nodes are the states of M.
3. Place an edge in G connecting every accept state with every nonaccept state. Add additional edges as follows.
4. Repeat until no new edges are added to G:
5. For every pair of distinct states q and r of M and every $a \in \Sigma$:
6. Add the edge (q, r) to G if $(\delta(q, a), \delta(r, a))$ is an edge of G.
7. For each state q, let $[q]$ be the collection of states $[q] = \{r \in Q \mid \text{ no edge joins } q \text{ and } r \text{ in } G\}$.
8. Form a new DFA $M' = (Q', \Sigma, \delta', q_0', A')$ where
   $Q' = \{[q] \mid q \in Q\}$ (if $[q] = [r]$, only one of them is in Q'),
   $\delta'([q], a) = [\delta(q, a)]$ for every $q \in Q$ and $a \in \Sigma$,
   $q_0' = [q_0]$, and
   $A' = \{[q] \mid q \in A\}$.
9. Output $\langle M' \rangle$."

a. Show that M and M' are equivalent.

b. Show that M' is minimal—that is, no DFA with fewer states recognizes the same language. You may use the result of Problem 1.52 without proof.

c. Show that *MINIMIZE* operates in polynomial time.

Step-by-step solution

Step 1 of 3

• Consider if M accepts the string $s = s_1 s_2 s_3 \ldots s_l$ of the length l then there must be some states in sequence $\left\{ \begin{array}{l} (q_{j_0}, q_{j_1}, q_{j_2} \ldots\ldots, q_{j_l}) : q_{j_0} = q_0 \\ \text{and } q_{j_i} = \delta(q_{j_{i-1}}, s_i) \end{array} \right\}$ and $[q_{j_l}] \in F$. That shows M' accept s depends on states sequence $\left\{ \begin{array}{l} [q_{j_0}], [q_{j_1}], [q_{j_2}] \ldots\ldots, [q_{j_l}] : [q_{j_0}] = q_0 \\ \text{and } q_{j_i} = \delta'(q_{j_{i-1}}, s_i) \end{array} \right\}$ and $[q_{j_l}] \in F'$.

Hence, $L(M)$ is subset of $L(M')$.

• Now if some other string $u = u_1 u_2 u_3 \ldots u_l$ of l length accepted with M', consider $\{[q_{j_0}], [q_{j_1}], [q_{j_2}] \ldots\ldots, [q_{j_l}] : [q_{j_0}] = q_0 \text{ and } q_{j_i} = \delta'(q_{j_{i-1}}, s_i)\}$ and $[q_{j_l}] \in F'$. By induction when u is input to M, now the corresponding sequence state which are visited by M, say $p_0, p_1, \ldots, p_l$ such that $p_0 = q_0$ and $r_i \in |q_{j_i}| \forall i$.

As $[q_{j_l}] \in F'$ and $[q_{j_l}] \in F$, hence $p_l \in F$ therefore $L(M') \subseteq L(M)$

Comment

Step 2 of 3

• Consider $\delta(q_0,s)$ is state of M after reading s after start from q_0. Two different states p and q in graph which is undirected is connected by edge iff there exist s and u strings. Such that $\delta(q_0,s)=q$ and $\delta(q_0,u)=p$.

Based on given case $|q|$ will store states q' so that s and u with $\delta(q_0,s)=q$ and $\delta(q_0,u)=q'$ such that s and u indistinguishable. For $q' \in [q]$ hence $q'=[q]$

• As statement given by theorem Myhill Nerode Deterministic Finite Automata recognize $L(M)$ must be $|Q'|$ number of states.

Hence, M' also have $|Q'|$ number of states and $L(M)$ will be equal to $L(M')$ and M' will minimal.

Comment

Step 3 of 3

• Consider $|Q|=n_1$. In given algorithm Step 1 will take $O(n_1^2|\Sigma|+n_1^3)$, by Brute force algorithm. Also 3rd 5th and 6th Step will take $O(n_1^2)$ time and each repetition will take $O(n_1^2\Sigma)$ time.

• 10th Step will complete in $O(n_1^3)$. In 8th step it will check that either $|q|=|r|$ or not that takes $O(n_1^2)$

• When construct final Deterministic Finite Automata M', will additional take additional $O(n_1^2\Sigma)$ time.

Comment