Problem

Let $AMBIG_{CFG}$ = $\{\langle G\rangle \mid G \text{ is an ambiguous CFG}\}.$ Show that *AMBIG*CFG is undecidable. (Hint: Use a reduction from *PCP*. Given an instance

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\}$$

of the Post Correspondence Problem, construct a CFG G with the rules

$$\begin{aligned} S &\to T \mid B \\ T &\to t_1 T \mathtt{a}_1 \mid \cdots \mid t_k T \mathtt{a}_k \mid t_1 \mathtt{a}_1 \mid \cdots \mid t_k \mathtt{a}_k \\ B &\to b_1 B \mathtt{a}_1 \mid \cdots \mid b_k B \mathtt{a}_k \mid b_1 \mathtt{a}_1 \mid \cdots \mid b_k \mathtt{a}_k , \end{aligned}$$

where $a_1, \ldots, a_k$ are new terminal symbols. Prove that this reduction works.)

Step-by-step solution

Step 1 of 1

Un-decidability

1. If P has match with $t_{i1}t_{i2}\cdots t_{il} = b_{i1}b_{i2}\cdots b_{il}$ then it can be observed that string $t_{i1}t_{i2}\cdots t_{il}a_{il}\cdots a_{i2}a_{i1}$ has minimum two derivations, first from T and other one from B.

2. If the Context free grammar G is ambiguous, then some string s should have multiple derivations. As G generate s, s can be written as $wa_{j1}a_{j2}\cdots a_{jm}$ for some w that do not have symbols from $a_i's$.

After checking the grammar G, It can be observe that the derivation of B and derivation of T can each generate maximum one strings of same form as s. The multiple derivations of s as follows:

$$\begin{aligned} &S \Rightarrow T \overset{*}{\Rightarrow} s = t_{jm}t_{jm-1}\cdots t_{j1}a_{j1}a_{j2}\cdots a_{jm} \\ &S \Rightarrow B \overset{*}{\Rightarrow} s = b_{jm}b_{jm-1}\cdots b_{j1}a_{j1}a_{j2}\cdots a_{jm} \end{aligned}$$

Thus, $t_{jm}t_{jm-1}\cdots t_{j1} = b_{jm}b_{jm-1}\cdots b_{j1}$

By combining (1) and (2), P has a match iff G is ambiguous.

So, the reduction from PCP to $AMBIG_{CFG}$ works. Thus, $AMBIG_{CFG}$ is un-decidable.

Comment