## Problem

Use the recursion theorem to give an alternative proof of Rice's theorem in Problem 5.28.

Problem 5.28

**Rice's theorem**. Let P be any nontrivial property of the language of a Turing machine. Prove that the problem of determining whether a given Turing machine's language has property P is undecidable.

In more formal terms, let P be a language consisting of Turing machine descriptions where P fulfills two conditions. First, P is nontrivial—it contains some, but not all, TM descriptions. Second, P is a property of the TM's language—whenever $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$. Here, $M_1$ and $M_2$ are any TMs. Prove that P is an undecidable language.

## Step-by-step solution

**Step 1** of 2

Rice's theorem: Let *P* be any nontrivial property of the language of a Turing machine.

In more formal terms, Let *P* be a Language consisting of Turing machine descriptions where *P* fulfills two conditions.

• First, *P* is nontrivial – It contains some, but not all, *TM* descriptions.

• Second, *P* is a property of the *TM's* language- whenever $L(M_1)=L(M_2)$, we have $\langle M_1\rangle \in P$ iff $\langle M_2\rangle \in P$. Here $M_1$ and $M_2$ are any $TMs$.

Then we have to prove that *P* is an undecidable Language.

Assume that *P* is a decidable Language satisfying the Properties of Rice theorem.

Comment

**Step 2** of 2

Let the Turing machine *T* decides the language *P*. According to the first condition of Rice theorem, *P* contains some, but not all, *TM* descriptions.

From above properties we assume $TM_1 \& TM_2$ be the Turing machines and $\langle TM_1\rangle \in P$ and $\langle TM_2\rangle \notin P$.

Now construct the Turing machine $TM_3$ using $TM_1 \& TM_2$.

$TM_3$"on input *w*:

1. By using recursion theorem, construct our own description for $\langle TM_3\rangle$

2. Run *T* on Turing machine $\langle TM_3\rangle$

3. If *T* accepts $\langle TM_3\rangle$, simulate $\langle TM_2\rangle$ on *w.*

If *T* rejects $\langle TM_3\rangle$, simulate $\langle TM_1\rangle$ on *w.*

• Suppose $\langle TM_3\rangle \in P$

Then *T* accepts $\langle TM_3\rangle$ and $L(TM_3)=L(TM_2)$

According to second condition of Rice's theorem $\langle TM_2\rangle \in P$ but our condition is $\langle TM_2\rangle \notin P$. It is a contradiction here.

• So, we will get similar contradiction if $\langle TM_3\rangle \notin P$.

As a result of that our assumption is wrong for *P* is a decidable language. Hence every property satisfying the conditions of Rice's theorem is undecidable.

Comment