Homework Solution - Set 5
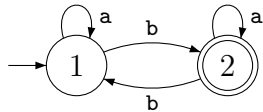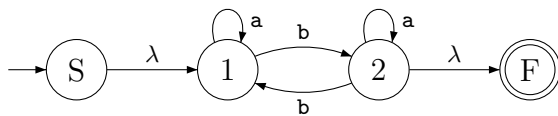Due: Friday 10/03/08

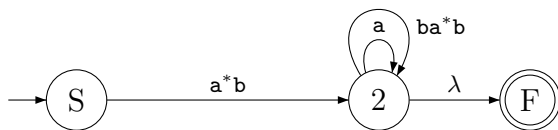1. Textbook, Page 86, Exercise 1.21.

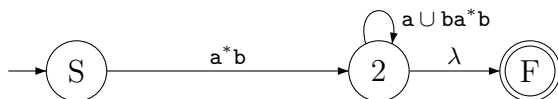   (a)

   Add new start state and final state. Make original final state non-final.
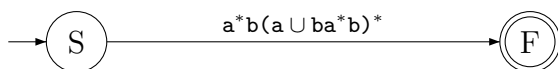

   No unions necessary, so eliminate state 1.

   Perform union on loops of state 2.

   Eliminate state 2. No final unions necessary.

(b)
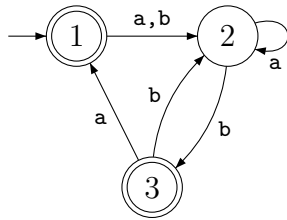
Add new start state and final state. Make original final states non-final.
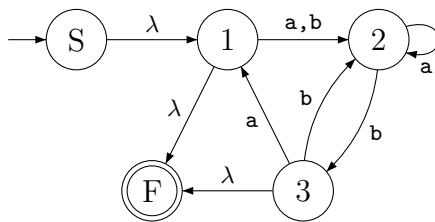
Perform union on edge from state 1 to state 2.
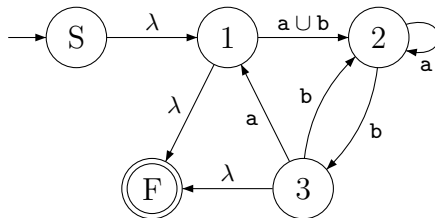
Eliminate state 1.
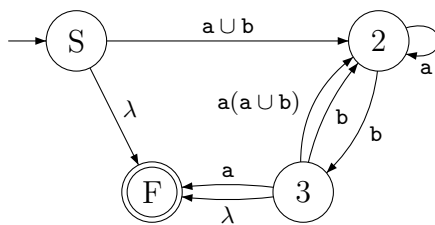
Perform unions on edges from state 3 to state 2 and from state 3 to the final state.

Eliminate state 2.

No unions necessary, so eliminate state 3.

Perform union on remaining edges.

2. Textbook, Page 88, Exercise 1.29.

   (a) $A_1 = \{0^n 1^n 2^n |\, n \geq 0\}$

     Condensed proof:

Proof. Suppose A_1 is regular. Let p be the pumping length given by the pumping lemma. Choose $s = 0^p1^p2^p$. By the lemma, $|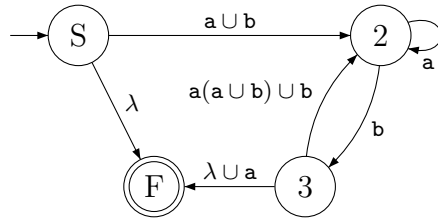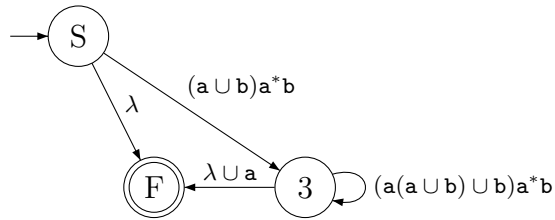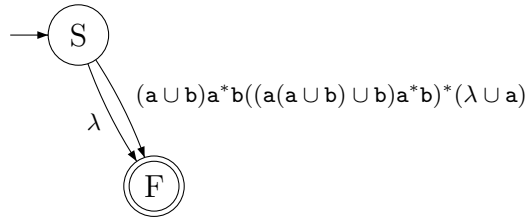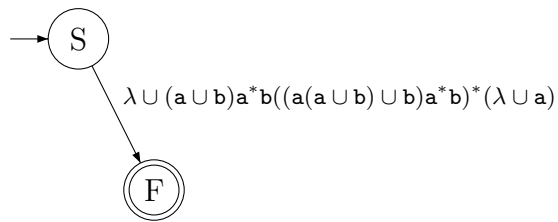xy| \leq p$ and $|y| > 0$ therefore $p \geq 0$ so $s \in A_1$. Clearly, $|s| \geq p$ thus $s = xyz$ for some x, y and z. Since $|xy| \leq p$, xy cannot extend beyond the first p symbols of s, meaning $xy = 0^k$ where $1 \leq k \leq p$. Let us write $x = 0^a$, $y = 0^b$, $z = 0^c1^p2^p$. The number of 0's, 1's and 2's in s are given by $a + b + c = p$. Let $i = 0$ such that $s' = xy^iz = xz$. The number of 1's in s' is p whereas the number of 0's in s' is $a + c$. For $s' \in A$, the number of 0's in s' must equal the number of 1's in s', namely $a + c = p$. Substituting for p, we have $a + c = a + b + c$ with equality holding when $b = 0$. Because $|y| > 0$ and $|y| = b$, $b > 0$, thus $s' \notin A$, a contradiction. Therefore A_1 is non-regular. □

Detailed proof:

Proof. We want to prove the language A_1 is non-regular. In order to use the pumping lemma, we must assume A_1 is regular, since the lemma only applies to regular languages. The goal is to show our assumption leads to a contradiction, meaning the assumption is false and therefore the opposite must be true. Since our assumption is that A_1 is regular, the opposite of this assumption is A_1 is non-regular, which is precisely what we want to show.

Once we assume A_1 is regular, the lemma provides us with the pumping length, p. We are now free to choose a word s which belongs to A_1 and has length $\geq p$. If we choose s appropriately, we should be able to "pump up" the size of s in the manner described by the pumping lemma and show the resulting word, s', does not belong to A_1. Since the lemma states all such words should also belong to the language, this would be a contradiction, leading us to our conclusion that A_1 is non-regular.

For s to be a word in A_1, it must follow the form given by the definition above. Due to the manner in which A_1 is defined, to obtain a unique word, we must fix a value for n. Given some careful thought, we can greatly reduce the number of cases we need to consider based on the value we choose for n. To understand the available choices, let us observe the effects "pumping" will have on the word s.

If we select s to be a large enough word from the language, the pumping lemma states s can be divided into three parts $s = xyz$. From this division, the lemma describes an infinite set of words of the form $s' = xy^iz$ where s' must also belong to the language for any $i \geq 0$. Depending on the size of y and likewise where y falls within the word s, we will have one of the following representative forms for y (where $1 \leq k \leq p$):

$$y = 0^k$$
$$y = 1^k$$
$$y = 2^k$$
$$y = 0^k 1^k$$
$$y = 1^k 2^k$$
$$y = 0^k 1^n 2^k$$

This means we have at least six different cases to consider if we allow the size and position of y to be arbitrary (with the exception $|y| > 0$).

However, we are not forced to allow this much variation in the structure of y. In fact, using the third condition of the pumping lemma, $|xy| \leq p$, we effectively limit both the size and position of y within the word s. Furthermore, depending on the word we choose for s (the value we choose for n), we can also limit the symbols which may appear in y and hence the relevance of each form of y in the above list.

Given the form of the words in A_1, setting n allows us to control the number of 0's in the prefix of s. If we set $n \geq p$, the entire string xy must consist entirely of 0's since xy consists of no more than the first p symbols of s, all of which are now 0. Hence, y takes on a single form, namely $y = 0^k$ where $1 \leq k \leq p$. So, by using condition three and choosing n appropriately (specifically, we will let $n = p$), we have narrowed the number of cases we need to consider to a single case!

Once we select the word $s = 0^p 1^p 2^p$, for the word to be useful in the context of the lemma, it must be evident that $s \in A_1$ and $|s| \geq p$. We have nearly proved that $s \in A_1$. What remains to be shown is $n \geq 0$. This follows from the conditions $|y| > 0$ and $|xy| \leq p$. We have $n = p \geq |xy| \geq |y| > 0$, thus $n > 0$ and, trivially, $n \geq 0$. Lastly, it is easy to show that $|s| \geq p$. Since $|s| = 3p > p$, $|s| \geq p$.

Now that it is clear $s \in A_1$ and $|s| \geq p$, the lemma allows us to divide s into $s = xyz$ for some x, y and z. We can write the representative forms of x, y and z as follows:

$$x = 0^a$$
$$y = 0^b$$
$$z = 0^c 1^p 2^p$$

We have already indicated that x and y consist entirely of 0's. As for the form of z, since $|xy| \leq p$ and there are p leading 0's in s, if $|xy| < p$ there will be some leftover 0's which carry over into z, hence the 0^c. (The rest of z is just the remainder of s.) Furthermore, since the 0's distributed across x, y and z are from the p leading 0's of s, $a + b + c$ must sum to p.

At this point, we focus our attention on the new word, $s' = xy^i z$, as provided by the lemma. The difference between s and s' is the number of times the substring y

is allowed to repeat. For s, y simply appears once but for s', y is allowed to repeat any number of times. A particular instance of s' can be chosen by fixing the value for i. Preferably, we would like a value for i other than 1, as $i = 1$ would make $s = s'$ and we are trying to construct a word which is not in A_1 (recall s needed to be a word in A_1). The simplest value we can choose for i is $i = 0$. In this case, $s' = xz$.

We must now show $xz \notin A_1$ in order to form a contradiction with the pumping lemma. (The first condition of the lemma states $xy^i z \in A_1$ for all $i \geq 0$.) Recall the representative forms of x and z. $x = 0^a$ and $z = 0^c 1^p 2^p$. One way to show $xz \notin A_1$ is by showing the number of 0's in xz does not equal the number of 1's in xz since the definition of A_1 requires these quantities to be equal. By the forms of x and z, it is apparent the number of 0's in xz is given by $a + c$ and the number of 1's is given by p. Remembering that $a + b + c = p$, we can determine when the two quantities are equal.

$$a + c = p$$
$$a + c = a + b + c$$
$$0 = b$$

If we can show that b cannot possibly be 0, then our proof is complete. Fortunately, we can demonstrate this fact using the condition $|y| > 0$. Replacing y with its representative form, we obtain $|0^b| > 0$. More or less by definition, $|0^b| = b$ and therefore $b > 0$. So, it follows b cannot be 0, meaning the number of 0's in xz cannot equal the number of 1's. Thus, $xz \notin A_1$ and since this forms a contradiction with the claims of the pumping lemma, our supposition that A_1 is regular must be incorrect. Hence, we conclude A_1 is non-regular. $\square$

(b) $A_2 = \{www|\ w \in \{\mathsf{a}, \mathsf{b}\}^*\}$

Proof. Suppose A_2 is regular. Let p be the pumping length given by the pumping lemma. Choose $s = www$ where $w = \mathsf{a}^p \mathsf{b}$. Clearly, $s \in A_2$ and $|s| \geq p$, thus $s = xyz$ for some x, y and z. Since $|xy| \leq p$, xy cannot extend beyond the first p symbols of s, meaning $xy = \mathsf{a}^k$ where $1 \leq k \leq p$. Let us write $x = \mathsf{a}^a$, $y = \mathsf{a}^b$, $z = \mathsf{a}^c \mathsf{ba}^p \mathsf{ba}^p \mathsf{b}$. The number of a's in each w is given by $a + b + c = p$. Let $i = 0$ and $s' = xy^i z = xz$. For reference, let $xz = w_1 w_2 w_3$ where $w_1 = \mathsf{a}^a \mathsf{a}^c \mathsf{b}$ and $w_2 = w_3 = \mathsf{a}^p \mathsf{b}$. The number of a's in w_2 and w_3 are each p whereas the number of a's in w_1 is $a + c$. For $s' \in A$, the number of a's in w_1 must equal the number of a's in w_2 and w_3, namely $a + c = p$. Substituting for p, we have $a + c = a + b + c$ with equality holding when $b = 0$. Because $|y| > 0$ and $|y| = b$, $b > 0$, thus $s' \notin A$, a contradiction. Therefore A_2 is non-regular. $\square$

(c) $A_3 = \{\mathsf{a}^{2^n} \mid n \geq 0\}$

NOTE: n is assumed to be an integer.

Proof. Suppose A_3 is regular. Let p be the pumping length given by the pumping lemma. Choose $s = \mathsf{a}^{2^n}$. By the lemma, $|xy| \leq p$ and $|y| > 0$ therefore $p \geq 0$ and $s \in A_3$. Clearly, $|s| = 2^p \geq p$, thus $s = xyz$ for some x, y and z. Let us write $x = \mathsf{a}^a$, $y = \mathsf{a}^b$, $z = \mathsf{a}^c$. The number of a's in s is $a + b + c = 2^p$. Let $i = 2$ and $s' = xy^i z = xyyz$. The number of a's in s', denoted $\#_{\mathsf{a}}(s')$, is $a + 2b + c = 2^p + b$. Since $|y| > 0$ and $|y| = b$, $b > 0$. From $2^p = a + b + c < a + 2b + c$, we conclude $2^p < \#_{\mathsf{a}}(s')$. Substituting for b on the right-hand side of $a + 2b + c = 2^p + b$, we find $a + 2b + c = 2^p + 2^p - a - c$. Since $|xy| \leq p$, $c = |xyz| - |xy| \geq 2^p - p > 0$, we have $a + 2b + c < 2^{p+1}$, thus $\#_{\mathsf{a}}(s') < 2^{p+1}$. Because $2^p < \#_{\mathsf{a}}(s') < 2^{p+1}$, $\#_{\mathsf{a}}(s')$ is not an even power of 2 and $s' \notin A_3$, a contradiction. Therefore A_3 is non-regular. $\square$