## Problem

If A and B are languages, define $A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}.$

## Step-by-step solution

**Step 1** of 3

Consider the two regular languages *A* and *B* over the input alphabet $\Sigma$. The language $A \diamond B$ is defined as,

$A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}$. For the language $A \diamond B$, if PDA is constructed then it can be said that $A \diamond B$ is in CFL.

Comment

**Step 2** of 3

Consider the DFA $D_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$ and $D_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$ for the languages *A* and *B* respectively. The strings of the language $A \diamond B$ are formed by concatenating equal length strings from *A* and *B*. Construct the PDA $M = (Q, \Sigma, \Gamma, \delta, q_{start}, F)$ for the language $A \diamond B$ as,

- From the start state $q_{start}$, push the symbol $ into the stack to know the bottom of the stack.
- For every symbol from the language *A*, push 1 into the stack. It guesses the end of the string that belongs to *A* when it reaches to the final state of $D_A$.
- Then, for every symbol from the language *B*, pop 1 from the stack. When the string reaches the final state of $D_B$, it moves to the final state *F* if and only if the top of the stack is $.

The PDA can be constructed informally as above described.

Comment

**Step 3** of 3

The formal description of the PDA $M = (Q, \Sigma, \Gamma, \delta, q_{start}, F)$ is as follows:

- $Q = Q_A \cup Q_B \cup \{q_{start}, F\}$
- $\Sigma$ is the input alphabet for *A* and *B*
- $\Gamma = \{\$, 1\}$ where $ is the symbol used to know the bottom of the stack and 1 is pushed every time into the stack when the symbol read from the language *A*.

- The transition function is defined as,

$$
\begin{array}{ll}
\delta(q_{start}, \varepsilon, \varepsilon) = \{q_A, \$\} & \\
\delta(q, a, \varepsilon) = \{\delta_A(q, a), 1\} & \text{if } q \in Q_A,\ \ a \in \Sigma \\
\delta(q, \varepsilon, \varepsilon) = \{q_B, \varepsilon\} & \text{if } q \in F_A \\
\delta(q, a, 1) = \{\delta_B(q, a), \varepsilon\} & \text{if } q \in Q_B,\ \ a \in \Sigma \\
\delta(q, \varepsilon, \$) = \{F, \varepsilon\} & \text{if } q \in F_B
\end{array}
$$

Any other transitions apart from this will not be accepted.

- $q_{start}$ is the start state.
- *F* is the final state.

The PDA non deterministically guesses the end of the string from $D_A$ and transitions to the start symbol of $D_B$ if it is in a final state of $D_A$. The PDA *M* accepts the string when it is in accepting state of $D_B$ while hitting the empty stack. This shows that $L(M) = A \diamond B$.

Therefore, for any two regular languages *A* and *B*, $A \diamond B$ is in CFL.