Problem

Let $A = \{wtw^{\mathcal{R}} \mid w, t \in \{0,1\}^* \text{ and } |w| = |t|\}$. Prove that A is not a CFL.

Step-by-step solution

Step 1 of 2

In the given function $A = \left\{wtw^R \middle| w, t \in \{0,1\}^* \text{ and } |w| = |t|\right\}$, every string $s = wtw^R \in A$ where $|w| = |t| = |w^R|$ and $|s|$ is a multiple of three. Let us assume that *A* is context free and reach to a contradiction.

• Let *p* be the pumping length for A that is guaranteed to exit by pumping lemma.

• Select string $s = 0^{2p}0^p1^p0^{2p} \in A$ with $|s| > p$.

• Therefore, there exits $uvxyz$ such that

1) $uv^ixy^iz \in B$ for all $i \geq 0,$

2) $|uy| > 0$,

3) $|vxy| \leq p$.

• Consider these cases for pumping lemma:

Case 1: $|vy|$ is not a multiple of 3. Then $s' = uv^2xy^2z \notin A$ since $|s'|$ is no longer a multiple of 3.

Case 2: vxy consist of only 0s from the prefix set of 0s and $|vy| = 3r$ for some *r*. Then, $uv^2xy^2z = 0^{3p+3r}1^p0^{2p} = 0^{2p+r}0^{p+2r}1^{p-r}1^r0^{2p} \notin A$, since $w = 0^{2p+r}$ and $w^R \neq 1^r0^{2p}$, the w^R of the string s.

Case 3: uxy consists of only 1s and $|vy| = 3r$ for some *r*. Then, the string $uv^2xy^2z = 0^{3p}1^{p+3r}0^{2p} = 0^{2p+r}0^{p-r}1^{p+2r}1^r0^{2p} \notin A$, since $w = 0^{2p+r}$ and $w^R \neq 1^r0^{2p}$, the w^R of the string s.

Case 4: vxy consist of only 0s from the suffix set of 0s and $|vy| = 3r$ from some *r*. Then $uv^0xy^0z = 0^{3p}1^p0^{2p-2r} \notin A,$ since $w = 0^{2p-r}$ and $w^R \neq 1^{2r}0^{2p-3r}$, the w^R of the string s.

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Case 5: $uy = 0^m1^n$ with $m, n > 0$ and $m + n = 3r$ from some *r*. Then, the string $uv^2xy^2z = 0^{3p+m}1^{p+n}0^{2p} = 0^{2p+r}0^{p+m-r}1^{p+n-r}1^r0^{2p} \notin A,$ since $w = 0^{2p+r}$ and $w^R \neq 1^r0^{2p}$, the w^R of the string s.

Case 6: $vy = 1^m0^n$ with $m, n > 0$ and $m + n = 3r$ for some *r*.

• **Sub-case 6.1**: $n < r$. Then $uv^2xy^2z = 0^{3p}1^{p+m}0^{2p+n} = 0^{2p+r}0^{p-r}1^{p+m+n-r}1^{r-n}0^{2p+n} \notin A$, since $w = 0^{2p+r}$ and $w^R \neq 1^{r-n}0^{2p+n}$, the w^R of the string s.

• **Sub-case 6.2**: $n > r$. Then $uv^0xy^0z = 0^{3p}1^{p-m}0^{2p-n} = 0^{2p-r}0^{p+r}1^{p+r-m-n}1^{n-r}0^{2p-n} \notin A$, since $w = 0^{2p-r}$ and $w^R \neq 1^{n-r}0^{2p-n}$, the w^R of the string s.

• **Sub-case 6.3**: $n = r$. Then $uv^{p+2}xy^{p+2}z = 0^{3p}1^{p+2r(p+2-1)}0^{2p+r(p+2-1)n} = 0^{3p}1^{rp+r-p}1^{2r+rp+r}0^{2p+rp+r} \notin A$, since $w = 0^{2p+r}$ and $w \neq 0^{3p}1^{rp+r-p}$ and $w^R \neq 0^{2p+rp+r}$, the w^R of the string s.

If $i < p+2,$ then take $r = 1$, $uv^ixy^iz = 0^{3p}1^{p+2(i-1)}0^{2p+(i-1)}$, $= 0^{2p+(i-1)}0^{p-(i-1)}1^{p+2(i-1)}0^{2p+(i-1)} \in A$. As there are not enough 1's to push or pump into *w*, $p+2$ is the first time that is guaranteed.

Comment

Step 2 of 2

Thus, in all the cases, the 1) of pumping lemma results in a contradiction. Therefore, the assumption that *A* is context free language, is false.

Comment