Problem

Let $R \subseteq \mathcal{N}^k$

A
a. $R_0 = \{0\}$
b. $R_1 = \{1\}$
c. $R_= = \{(a,a) \mid a \in \mathcal{N}\}$
d. $R_< = \{(a,b) \mid a, b \in \mathcal{N} \text{ and } a < b\}$

Step-by-step solution

Step 1 of 5

The k-ary relation $R \subseteq N^k$ is ***definable*** in $Th(N,+)$, if a formula ϕ can be given with k free variables $x_1,\dots,x_k$, such that for all $a_1,\dots,a_k \in N$, the formula $\phi(a_1,\dots,a_k)$ is true only when $a_1,\dots,a_k \in R$.

• The theory $Th(M)$ of a model M is the collection of true sentences in the language of M.

• The theory for this problem is $Th(N,+)$, so the model will be $(N,+)$. Thus, only need to find formulas that are true for the given relations over the $(N,+)$ model.

Comment

Step 2 of 5

a)

$R_0 = \{0\}$

• There is only one value 0 to be considered. Adding this value to the variable $y \in N$ will produce no change in the value of y as $y + 0 = y$.

• So, in the formula ϕ_0 to make R_0 definable in $Th(N,+)$ for all y, it has; $\forall x \in R_0 \forall y[x + y \to 0 + y]$.

From this, the formula can be expressed as:

$\phi_0(x) = \forall y[x + y = y]$

Comment

Step 3 of 5

b)

$R_1 = \{1\}$

• Similarly, in this case only one value is to be considered. So adding any value from $R_1 = \{1\}$ to the variable $y \in N$ will increment the value of y by one as:

$\forall x \in R_1 \forall y[x + y \to 1 + y]$

Therefore, the formula ϕ_1 to make R_1 definable in $Th(N,+)$ will be:

$\phi_1(x) = \forall y[x + y = y + 1]$

Comment

Step 4 of 5

c)

$R_= = \{(a,a) \mid a \in N\}$

• The $R_=$ relation is a 2-ary relation. So, the formula to make it definable will be a binary function $\phi_=(x,y)$. This function has to be true if and only if $x = y$.

• For it be defined for $\forall z \in N$, the only possible value of z is 0. This is true when $\phi_0(z)$ is true, which can be written as $\forall z[\phi_0(z) \to x + z = y]$.

Hence, the function is given by:

$\phi_=(x,y) = \forall z[\phi_0(z) \to z + x = y]$

Comment

Step 5 of 5

d)

$R_< = \{(a,b) \mid a,b \in N \text{ and } a < b\}$

• This relation is also 2-ary, so the formula will be a binary function $\phi_<(x,y)$. It needs to be true when $x < y$ and for all the values of the variable $\forall z \in N$.

• The only possible value of z is 0. Since all the other non-zero values will affect the magnitude of the quantity $x + z$, which will make it difficult to compare it against the value of y.

• As $\phi_0(z)$ is true when $z = 0$, it can be incorporated into $\phi_<$.

$\forall z[\phi_0(z) \to x + z = y]$.

This is the same as the definition of the function:

$\phi_<(x,y) = \forall z[\phi_0(z) \to z + x < y]$

Comment