Problem

A ***cut*** in an undirected graph is a separation of the vertices V into two disjoint subsets S and T . The size of a cut is the number of edges that have one endpoint in S and the other in T . Let

$$MAX\text{-}CUT = \{\langle G, k\rangle \mid G \text{ has a cut of size } k \text{ or more}\}.$$

Show that *MAX-CUT* is NP-complete. You may assume the result of Problem 7.26. (Hint: Show that $\neq SAT \leq_P$ *MAX CUT*. The variable gadget for variable x is a collection of 3c nodes labeled with x and another 3c nodes labeled with $\overline{x}$. The clause gadget is a triangle of three edges connecting three nodes labeled with the literals appearing in the clause. Do not use the same node in more than one clause gadget. Prove that this reduction works.)

Step-by-step solution

Step 1 of 2

***NP* – complete definition:**

A language *B* is *NP* – complete if it satisfies two conditions

1. *B* is in *NP*

2. Every *A* in *NP* is polynomial time reducible to *B.*

Comment

Step 2 of 2

MAX-CUT* is in *NP. We can guess the partition of the graph into two parts and verify that the number of edges cut is at least *k*.

We can show that *MAX-CUT* is in *NP* by showing that $\neq SAT \leq_p MAX-CUT$.

- Let *n* be the number of variables and *c* be the number of clauses in the $\neq SAT$ instance ϕ.
- We know that "$\neq SAT$ is the collection of $3cnf$-formula that have an $\neq -$assignment" and
- "An $\neq -$assignment to a variable ϕ is one where each clause contains two literals with unequal truth values".
- Let *G* be the resulting graph.
- For every literal z, G contain 2*c* nodes each labeled as *z* (let's call this block" of nodes corresponding to *z*)
- Add all $(3c)^2$ edges between the block *z* and block $\overline{z}$.
- For every clause, there is a triangle between three nodes that are labeled by the three literals that appear in that clause.
- Same node in a block cannot be used for more than one clause triangles.
- Now G has G nodes and $(3c)^2 n + 3c$ edges set $k = (3c)^2 n + 2c$
- We show that $\neq SAT$ has a $\neq -$assignment iff *G* has a cut of size at least *k.*

For forward direction,

assume that $a \neq -$assignment

- Place all nodes labeled by a TRUE literal on one side of the cut and all nodes labeled by a FALSE literal on the other side of the cut.
- This cuts all $(3c)^2 n$ edges between the blocks.
- Also Since every clause gets a TRUE and a FALSE literal, for every triangle. Two of the three edges are cut.
- Thus overall $(3c)^2 n + 2c$ edges are cut.

For the backward direction,

proof for any partition that cuts at least *k* edges must

1. Place every block on one side of the partition entirely.

2. Place blocks corresponding to complementary literals on opposite sides.

3. Therefore the partition defines an assignment to literals.

4. And then every clause must have a TRUE as well as a FALSE literal. So, two edges in that clause triangle get cut.

Thus, $MAX-CUT$ is in *NP*–complete.

Comment