Problem

A ***homomorphism*** is a function $f: \Sigma \longrightarrow \Gamma^*$ for any language A.

a. Show, by giving a formal construction, that the class of regular languages is closed under homomorphism. In other words, given a DFA M that recognizes B and a homomorphism f, construct a finite automaton M` that recognizes f(B). Consider the machine M` that you constructed. Is it a DFA in every case?

b. Show, by giving an example, that the class of non-regular languages is not closed under homomorphism.

Step-by-step solution

Step 1 of 2

a.

Class of Regular Languages is closed under Homomorphism:

In general by the definition of homomorphism if R be a regular expression where each symbol a in Σ, Assume, $f(R)$ be the expression this is obtained by replacing each symbol a in R by $f(a)$.

Now assume a language $B = B(R)$ for some regular expression R. Here $f(R)$ defines the language $f(B)$.

Therefore it is needed to proof that if user take a sub-expression E of R and apply homomorphism f to it and get $f(E)$, the language of $f(E)$ is the same language if user apply f to the language $L(E)$. It implies $L(f(E)) = f(L(E))$.

BASIS: Suppose, $E = \Phi$ that is $L(E)$ does not contain any strings, then $f(L(E)) = L(f(E))$. Again suppose $E = \{\varepsilon\}$ that is $L(E)$ contains a string with no symbols, then also $f(L(E)) = L(f(E))$

Now assume $E = \{a\}$ for some symbol $a \in \Sigma$. Here, in this case $L(E) = \{a\}$ then $f(L(E)) = \{f(a)\}$. Again, $f(E)$ is the regular expression that is the string of symbols $f(a)$,

Therefore, it can be said that $L(f(E)) = \{f(a)\}$.

It implies $L(f(E)) = f(L(E))$.

INDUCTION: By the rule of union over regular expression. Assume, $E = E_1 + E_2$.

By the application of homomorphism over regular expression user may say that, $f(E) = f(E_1) + f(E_2)$. By the grammar of regular language

$L(E) = L(E_1) \cup L(E_2)$.

Therefore,

$$\begin{aligned} L(f(E)) &= L(f(E_1) + f(E_2)) \\ &= L(f(E_1)) \bigcup L(f(E_2)) \end{aligned}$$

Again, though f is applied to all the strings of a language separately and individually,

$$\begin{aligned} f(L(E)) &= f(L(E_1) \bigcup L(E_2)) \\ &= L(f(E_1)) + L(f(E_2)) \end{aligned}$$

Here by the Inductive Hypothesis we may assert that, $L(f(E_1)) = f(L(E_1))$ and $L(f(E_2)) = f(L(E_2))$. It implies $L(f(E)) = f(L(E))$.

Hence, it can be said that the class of regular languages is closed under homomorphism.

Comment

Class of non-regular language is not assumed to be closed under homomorphism:

Consider a given language $B=\{0^n1^n|n\geq 0\}$ is defined over the alphabet $\Sigma=\{0,1\}$. Now by pumping lemma of regular languages it can easily be said that the language B is non-regular.

Here another alphabet set is $\Gamma=\{a,b\}$ also defined.

The homomorphism function $f:\Sigma\rightarrow\Gamma^*$ is defined as $f(0)=ab$ and $f(1)=\varepsilon$. Therefore the language B' over alphabet Γ could be described as $B'=\{(ab)^n|n\geq 0\}$.

If user apply pumping lemma of regular languages over B' and divide it in xyz where x is ε, $y=ab$ and z is also ε then we can say that B' is a regular language.

Therefore user can see from this that applying homomorphism over a non-regular language yields a regular language.

Hence homomorphism is not closed over the class of non-regular languages.

Comment