Problem

Prove that for each n > 0, a language B_n exists where

a. B_n is recognizable by an NFA that has n states, and

b. if $B_n = A_1 \cup \cdots \cup Ak$, for regular languages A_i, then at least one of the A_i requires a DFA with exponentially many states.

Step-by-step solution

Step 1 of 2

a.

Proving that a language is recognizable by an NFA

Suppose B_n be a language where $n > 0$. User has to prove that the language is recognizable by an NFA with n states.

BASIS: Let $n = 1$ hence $B_n = \{\varepsilon, 0, 1\}$. Therefore formally we can design an NFA $N = (\{q_o\}, \Sigma, \delta, q_0, \{q_0\})$ with a single state that accepts all the given language as $\delta(q_0, \varepsilon \mid 0 \mid 1) = q_0$.

Proof by induction: suppose one can divide B_n in two regular expressions say E and F of length $n_1, n_2 < n$ and $n_1 + n_2 = n$.

Now by inductive hypothesis it can easily concluded that the NFA's accepting E and F are consisting of at least n_1 and n_2 states.

But it is already known to us that the set of regular expression is closure under Union, Concatenation and Star operation.

Therefore the language B_n is recognizable by an NFA with n states.

Comment

Step 2 of 2

From the above part it is proved that a language B_n where $n > 0$ is recognizable by an NFA with n states.

Now for $B_n = A_1 \bigcup A_2 \bigcup \cdots \bigcup A_k$ where A_i's are regular.

If a DFA is constructed which is equivalent to the DFA of the given NFA.

There could be at least n and at most 2^n states in the resultant equivalent DFA. Every regular language is recognized by a DFA so there is a corresponding DFA for all the A_is. Now, by the pigeon hole principle, one can state that there is at least one DFA which requires 2^i states to recognize a language among all the A_i.

Comment