Problem

Show that a circuit family with depth O(log n) is also a polynomial size circuit family.

Step-by-step solution

Step 1 of 2

A circuit family of depth $O(\log n)$ can be obtained by taking an equivalent polynomial size family of formulas. **"To convert a formula μ with h leaves to a similar circuit of depth $O(\log h)$" is sufficient to show the above statement.** Here, it may be assumed that fan-in value of all the nodes is 2 and not gates are pushed to the leaves.

• Now, the proof of $h \geq 4$ (can be done using the induction hypothesis) that is a formula μ with h leaves is similar to a formula μ' with a maximum depth of $C \log_2 h$ .where the value of the constant C will be further determined.

• If $h \leq 4$, suppose $\mu' = \mu$. Otherwise apply the concept of tree which says that every tree with $m \geq 2$ leaves has a sub-tree with between $m/3$ and $2m/3$. By applying this concept, the tree structure of μ acquire a sub-formula β with between $h/3$ and $2h/3$.

Comment

Step 2 of 2

Suppose $\hat{\mu}(y)$ be μ with the sub-formula μ is replace by a new variable y. Thus, μ is similar to $\hat{\mu}(\beta)$ and μ is equivalent to μ_1 that is given by:

$$\mu_1 = \left(\beta \wedge \hat{\mu}(1)\right) \vee \left(\neg\beta \wedge \hat{\mu}(0)\right)$$

Here, $\hat{\mu}(1)$ and $\hat{\mu}(0)$ each contains maximum $2h/3$ leaves which is variable.

• Finally, suppose μ and μ' with the equivalent of the sub formulas β, $\hat{\mu}(1)$ and $\hat{\mu}(0)$ interchanged by similar small depth formula given by the induction hypothesis.

• **Thus, the depth of μ' is maximum** $C \log_2 \left((2/3)h\right) + 3$. This is maximum of $C \log_2 h$ provided $C \geq 3 / \log_2 \left(3/2\right)$.

• Thus, from the above explanation it can be said that "**A circuit family of depth $O(\log n)$ can be obtained by taking an equivalent polynomial size family of formulas**".

Comment