## Problem

Let N be an NFA with k states that recognizes some language A.

**a.** Show that if A is nonempty, A contains some string of length at most k.

**b.** Show, by giving an example, that part (a) is not necessarily true if you replace both A's by $\overline{A}$.

**c.** Show that if $\overline{A}$ contains some string of length at most $2^k$.

**d.** Show that the bound given in part (c) is nearly tight; that is, for each k, demonstrate an NFA recognizing a language $A_k$ where $\overline{A_k}$ shortest member strings are of length exponential in k. Come as close to the bound in (c) as you can.

## Step-by-step solution

**Step 1** of 5

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA with *K* states that recognizes some language *A*

(a) Suppose *A* is non empty

• Then there must be an accept state $q \in F$ that can be reached from the start state $q_0$.

• Let *w* be the string that can be accepted by *N* when traveling along the shortest path $q_0$ to $q$.

• Let *n* be the length of *w.*

• Then, the sequence of state $q_0, q_1 ... q$ in the shortest path from $q_0$ to *q* has length $n+1$.

• Note that all the states from $q_1$ to $q$ in this sequence must be distinct; otherwise we would find a shorter path from $q_0$ to *q* by removing the repeated states.

• Since there are only *K* states in N and there are *n* distinct states in the shortest path from $q_0$ to $q$, we have $n \le K$.

• Clearly, *w* is accepted by *N* because *q* is an accept state

• So *A* contains a string of length at most *K.*

Comment

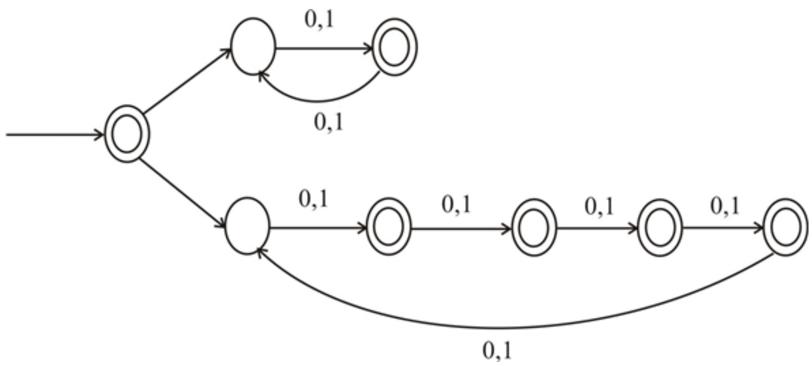
**Step 2** of 5

(b) <u>Example</u>:

Suppose $\Sigma = \{0,1\}$ and N be the NFA with $K = \delta$ states.

Let *A* be the language recognized by *N.*

The State diagram of *N* is as follows




Comment

**Step 3** of 5

- Clearly *N* accepts the empty string.
- For any nonempty string *w*, *N* will reject *w* if and only if the length of *w* is divisible by 2 and 5.
- Thus $\overline{A}$ consists of all non empty strings of length divisible by 10
- So $\overline{A}$ is non-empty and the shortest string in $\overline{A}$ has length $10 > K$
- Hence we got the contradiction of part (a) when we replace *A* by $\overline{A}$.

Comment

**Step 4** of 5

(c) We know that " Every non deterministic finite automaton has an equivalent

deterministic finite automaton"

- So we convert *N* into a DFA *M* that also recognizes *A*, where the set of states in *M* is the set of subsets of *Q*.
- Then we swap the accept and non accept states of *M* to obtain a DFA $\overline{M}$ that recognizes $\overline{A}$.
- Note that $\overline{M}$ has $2^K$ states.
- Applying part (a) by replacing *N* with $\overline{M}$, we can conclude that if $\overline{A}$ is non-empties, then $\overline{A}$ contains a string of length at most $2^K$.

Comment

**Step 5** of 5

(d) The idea used in part (b) can be generalized to obtain a bound close to an exponential form.

- Let $2 \le P_1 < P_2 < ... < P_n \le \sqrt{K}$ be all the primes in $\left[1, \sqrt{K}\right]$
- For each $P_i$, construct a DFA $M_i$ of $P_i$ states that rejects only strings of length divisible of $P_i$.
- Finally, construct an NFA *M* by union all these $M_i$ machines in the following ways:

→ create a separate starting state $q_0$ add an $\in$ transition from $q_0$ to the starting states of each $M_i$, and also designate $q_0$ as an accepting state.

→ This machine M will have $1 + P_1 + P_2 + ... + P_m \le K$ states.

→ On the other hand, *M* rejects a string *w* if and only if a is non empty and $|w|$ is divisible by $P_1 P_2 ... P_m$.

→Hence, the shortest string is rejected by *M* has length $P_1 P_2 ... P_m$.

→ Now coming to the analysis of the bound,

By the prime Number theorem, there are approximately $\ln$ *n* prime numbers in $[0, n]$ for *n* sufficiently large.

→ Hence, $m \approx \frac{1}{2} \ln K$.

→ Since there are about $\frac{1}{4} \ln K$ primes in $\left[0, \sqrt[4]{K}\right]$, there are about $m - \frac{1}{4} \ln K \approx \frac{1}{4} \ln K$ primes in $\left[\sqrt[4]{K}, \sqrt{K}\right]$

Thus $P_1 P_2 ... P_m \ge \left(\sqrt[4]{K}\right)^{\frac{1}{4} \ln K} = K^{\frac{1}{16} \ln K}$