Problem

Let *SET-SPLITTING* = $\{\langle S, C\rangle |\ S$ is a finite set and $C = \{C_1, \ldots, C_k\}$ collection of subsets of S, for some k > 0, such that elements of S can be colored *red* or *blue* so that no C_i has all its elements colored with the same color}. Show that *SET-SPLITTING* is NP-complete.

Step-by-step solution

Step 1 of 3

NP –Complete:

A language *B* is *NP*-complete if it satisfies 2 conditions

1. *B* is in *NP*

2. Every A in *NP* is polynomial time reducible to B.

Comment

Step 2 of 3

1. **SET – SPLITING is in NP :**

SET – SPLITING is in NP because we can verify in polynomial time that no subset C_i is monochromatic.

2. $3\ SAT \leq_P SET - SPLITING$:

To prove that the problem is NP complete, we give a polynomial time reduction from 3SAT to SET-SPLITING.

Given an instance of 3SAT ϕ , set $S = \left\{x_1, \bar{x}_1, \ldots x_n, \bar{x}_n, y\right\}$, where x_is are the variables and y is a special color variable.

Comment

Step 3 of 3

The splitting is done as follows:

For every clause C_i in ϕ, Let C_i be a subset of S containing the elements corresponding to the literally, in C_i and the special elements $y \in s$, Then $C = C_1, \ldots, C_k$

If ϕ is satisfiable, consider a satisfying assignment.

If we color all the true literals red, all the false ones are blue, and y blue, then every subset C_i of S has at least one red element (because it is satisfiable and it also contain one blue element y.

In addition, for a given splitting $\langle S, C\rangle$, we can able to set the literals that are colored differently from y to true.

In the same way, we can able to set the literals that have the same color as y to false.

This concludes that satisfying assignment for ϕ.

Thus, SET – SPLITTING is NP-Complete.

Comment