Problem

Show that P is closed under homomorphism iff P = NP.

Step-by-step solution

Step 1 of 2

A **homomorphism** is defined as a function f on strings with the property that $f(xy)=f(x)f(y)$. A **nonerasing homomorphism** is defined as a homomorphism f such that $f(c)$ is not an empty string, for any character c. As it is known that both NP and P are closed under other operations, except for the homomorphism operation.

• To see both of the class NP **and** P **are not closed under homomorphism**, first of all initialize with a very hard language L, which requires a time complexity of 2^{2^n}.

• It can be made easy by appending each word of length exactly 2^{2^n} c's, where c denotes a new symbol. That is, suppose $\mathrm{L}'=\mathrm{C}^{2^2}$ then it can be said that L' is surely in NP and P. But $\mathrm{L}=\mathrm{h}(\mathrm{L}')$, if h is defined as a homomorphism that send c to ε and is unique on all symbols of L.

• If NP and P are closed under homomorphism, then L would be in NP, which is not.

Comment

Step 2 of 2

Now, the above explanation can be used to show P **is closed under homomorphism if and only if** $\mathrm{P}=\mathrm{NP}$. Suppose $\mathrm{h}^{-1}(\mathrm{L})$: every homomorphism h can only expand the length of the string to which it is applied by a constant factor.

• To recognize $\mathrm{h}^{-1}(\mathrm{L})$, apply h to the taken input x and watch whether $\mathrm{h}(\mathrm{x})$ in L. Now, see if there is a **polynomial time or nondeterministic polynomial-time test for membership in** L**.**

• Now, the above test can be used in the same order for magnitude time complexity, which tells us whether, x is in $\mathrm{h}^{-1}(\mathrm{L})$.

Therefore, from the above explanation, it can be said that " P **is closed under homomorphism if and only if** $\mathrm{P}=\mathrm{NP}$ ".

Comment