Midterm Exam #1

INSTRUCTIONS:

- Put your NAME and SBU ID # on this exam booklet in the space provided.
- This is a CLOSED-BOOK exam, which TERMINATES AT 2:35PM (80 minutes). NO ELECTRONIC DEVICES, including calculators, may be used during the exam.
- Please place ALL ANSWERS IN THIS BOOKLET, on the sheet where the corresponding question is printed.
- THINK BEFORE YOU WRITE. A partial solution can get you partial credit, but too much extraneous information can prevent me from finding your correct solution.
- SOME QUESTIONS ARE HARDER THAN OTHERS, and you might not have time to answer all questions completely. LOOK OVER ALL THE QUESTIONS BEFORE STARTING, and work first on those that will get you the most credit fastest. Use the number of points listed for each question as a guide.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	9	5	5	5	5	5	15	10	69
Score:										

Note: Point values have been assigned so that you should expect to be answering roughly one point per minute.

NAME and SBU ID#: ______________________________

1. Give complete (with justifications) Fitch-style proofs for the following, using *only* the rules listed on the last pages of this exam.

   (a) (5 points)

   1. $\neg F \to G$
   2. $F \to H$

   $G \vee H$

Solution:

1. $\neg F \to G$	
2. $F \to H$	
3. $\neg F$	
4. G	$\to$ **E:1, 3**
5. $G \vee H$	$\vee$ **I:4**
6. F	
7. H	$\to$ **E:2, 6**
8. $G \vee H$	$\vee$ **I:7**
9. $G \vee H$	**LEM:**3–5,6–9

(b) (5 points)

1. $\exists x\ Dx$
2. $\forall x\ (x = p \leftrightarrow Dx)$

Dp

Solution:

1. $\exists x\ Dx$	
2. $\forall x\ (x = p \leftrightarrow Dx)$	
3. Dc	
4. $c = p \leftrightarrow Dc$	$\forall$ **E:2**
5. $c = p$	$\leftrightarrow$ **E:3, 4**
6. Dp	$=$ **E** 3, 5
7. Dp	$\exists$ **E:1, 3–6**

2. Let R be the binary relation on natural numbers defined as follows:

$$R = \{(m, n) \in \mathcal{N} \times \mathcal{N} \mid m < n\}.$$

(a) (2 points) Is R reflexive? Why or why not?

Solution: It is not reflexive, because, for example, $(0, 0)$ is not in R.

(b) (2 points) Is R symmetric? Why or why not?

Solution: It is not symmetric, because, for example, $(0, 1)$ is in R, but $(1, 0)$ is not in R.

(c) (2 points) Is R transitive? Why or why not?

Solution: It is transitive, because, for all m, n, p, if $m < n$ and $n < p$ then $m < p$.

(d) (3 points) Give a simple description of the reflexive, transitive closure of R.

Solution: The reflexive, transitive closure of R is defined to be the smallest reflexive and transitive relation that contains R. Since R is already transitive, to obtain a relation that is both reflexive and transitive it is only necessary to add to R all pairs of the form (m, m). A simple description of the resulting relation is the following:

$$\{(m, n) \in \mathcal{N} \times \mathcal{N} \mid m \leq n\}.$$

3. (5 points) Does the following statement hold for sets? Explain why or why not.

$$\forall A.\, \exists B.\, \forall X.\, X \in B \leftrightarrow X \notin A$$

Solution: It does not hold. Suppose it did, then taking $A = \{\}$, it would imply the existence of a set B such that for all sets X, X is an element of B if and only if X is not an element of $\{\}$. But $\{\}$ has no elements, so B would have every set X as an element, including B itself. The existence of a set B such that $B \in B$ leads directly to a contradiction via Russell's paradox, so such a set cannot exist (assuming, of course, that set theory is a consistent theory in which no contradiction can be proved).

This question did involve a bit of a "trick". However, if you were able to decode the quantifiers and come up with a clear statement that showed that you recognized that "complement" was somehow involved, then I typically gave you three points out of five. The subtle point is that in set theory, there is no absolute notion of complement; the complement of a set always has to be taken with respect to some larger enclosing set. It is not possible for this larger enclosing set to contain all sets, because the existence of a "set of all sets" leads to contradictions.

4. (5 points) Write a regular expression R over $\{a, b\}$, such that a string w is in $L(R)$ if and only if w has no more than three a's.

Solution:

$$b^* \cup b^*ab^* \cup b^*ab^*ab^* \cup b^*ab^*ab^*ab^*$$

Alternatively,

$$b^*(a \cup \epsilon)b^*(a \cup \epsilon)b^*(a \cup \epsilon)b^*$$

5. (5 points) For a nondeterministic finite automaton N, under exactly what conditions is the empty string in $L(N)$?

Solution: The empty string is in $L(N)$ exactly when there exists a path from the start state of N to an accepting state, such that all transitions along the path are ϵ-transitions.

6. (5 points) Prove that every infinite language has a subset that is a regular language.

Solution: The empty language $\{\}$ is regular and it is a subset of every language, not just every infinite language.

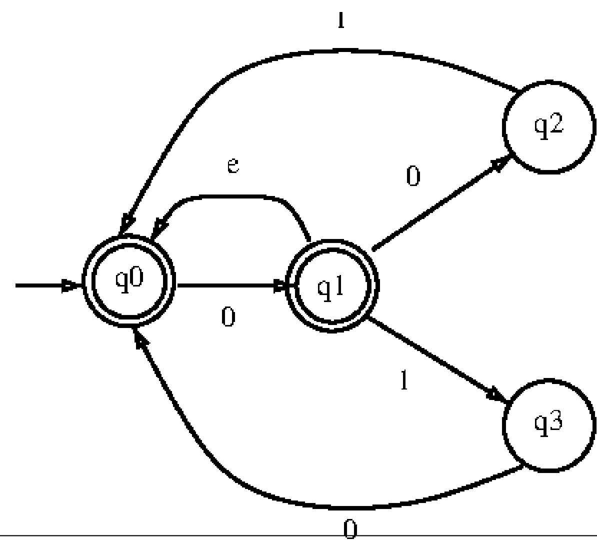
7. (5 points) Does the following equivalence hold for regular expressions? Give a proof or counterexample to justify your answer.

$$(R \cup S) \circ (T \cup V) = (R \circ T) \cup (S \circ V)$$

Solution: It does not hold. For example, assume $\Sigma = \{r, s, t, v\}$. If $R = r$, $S = s$, $T = t$, and $V = v$, then the language generated by the left-hand side is $\{rt, rv, st, sv\}$ and the language generated by the right-hand side is $\{rt, sv\}$, which are not equal.

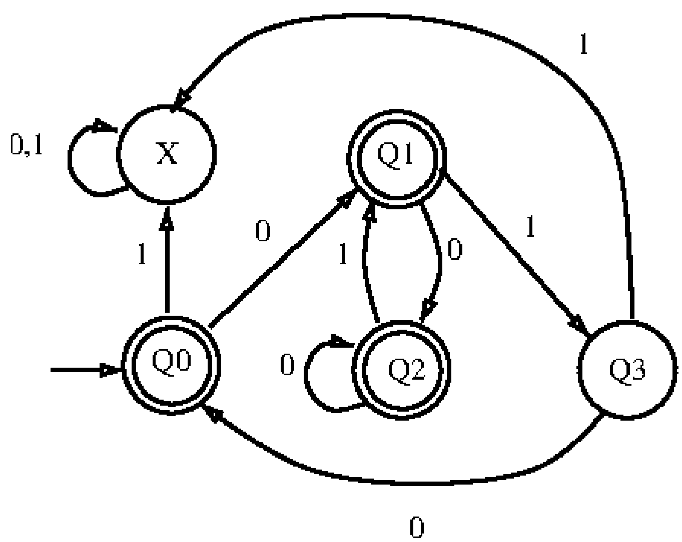
8. (a) (5 points) Give an NFA recognizing the language $(0 \cup 001 \cup 010)^*$ (a state diagram is sufficient). For the second part of the question it will be helpful if you use as few states as possible (it can be done with four states).

Solution:



(b) (10 points) Convert this NFA to an equivalent DFA (again, a state diagram is sufficient). Give only the portion of the DFA that is reachable from the start state.

Solution:



9. (10 points) Prove that every NFA can be converted to an equivalent one with a single accept state.

   (**Note:** "Prove" means not just to describe an idea but also to give an explicit formal construction and to use the definition of acceptance to show that the construction works.)

Solution: Let NFA $N = (Q, \Sigma, q_0, \delta, F)$ be given. Construct $N' = (Q', \Sigma, q_0, \delta', F')$ by defining $Q' = Q + \{q_{\text{accept}}\}$, $F' = \{q_{\text{accept}}\}$, $\delta'(r, x) = \delta(r, x)$ for $r \in Q$, $\delta'(r, x) = \{q_{\text{accept}}\}$ if $r \in F$, and $\delta'(r, x) = \{\}$ otherwise. Clearly, N' is an NFA with a single accept state. We claim that N' is equivalent to N; that is, for all strings $w \in \Sigma^*$, w is accepted by N if and only if w is accepted by N'. Let arbitrary $w \in \Sigma^*$ be fixed. We must show that w is accepted by N if and only if w is accepted by N'.

If w is accepted by N, then w can be written $w = w_1 \dots w_n$, with each w_k either in Σ or equal to ϵ, and there exist states $r_0, \dots r_n$, such that $r_0 = q_0$, $r_n \in F$, and $r_{k+1} \in \delta(r_k, w_{k+1})$ for $0 \leq k \leq n - 1$. Let $w_{n+1} = \epsilon$ and $r_{n+1} = q_{\text{accept}}$; then $w = w_1 \dots w_n, w_{n+1}$, $r_0 = q_0$, $r_{n+1} \in F'$ and $r_{k+1} \in \delta'(r_k, w_{k+1})$ for $0 \leq k \leq n$. That is, N' accepts w.

Conversely, if w is accepted by N', then w can be written $w = w_1 \dots w_n, w_{n+1}$, with each w_k either in Σ or equal to ϵ, and there exist states $r_0, \dots r_n, r_{n+1}$, such that $r_0 = q_0$, $r_{n+1} \in F'$ and $r_{k+1} \in \delta'(r_k, w_{k+1})$ for $0 \leq k \leq n$. Since q_{accept} is the only element of F', it must be that $r_{n+1} = q_{\text{accept}}$. Moreover, $r_n \in F$, because $\delta'(r_n, \epsilon) = \{\}$ unless $r_n \in F$. Then $w = w_1 \dots w_n$, $r_0 = q_0$, $r_n \in F$, and $r_{k+1} \in \delta(r_k, w_{k+1})$ for $0 \leq k \leq n - 1$. That is, N accepts w.

(scratch paper)

(scratch paper)

Basic rules

$$\begin{array}{l|ll} m & \mathscr{A} & \\ & \mathscr{A} & \text{R } m \end{array}$$

$$\begin{array}{l|ll} m & \mathscr{A} & \\ n & \mathscr{B} & \\ & \mathscr{A} \land \mathscr{B} & \land\text{I } m, n \end{array}$$

$$\begin{array}{l|ll} m & \mathscr{A} \land \mathscr{B} & \\ & \mathscr{A} & \land\text{E } m \end{array}$$

$$\begin{array}{l|ll} m & \mathscr{A} \land \mathscr{B} & \\ & \mathscr{B} & \land\text{E } m \end{array}$$

$$\begin{array}{l|ll} m & \mathscr{A} & \\ & \mathscr{A} \lor \mathscr{B} & \lor\text{I } m \end{array}$$

$$\begin{array}{l|ll} m & \mathscr{A} & \\ & \mathscr{B} \lor \mathscr{A} & \lor\text{I } m \end{array}$$

$$\begin{array}{l|ll} m & \mathscr{A} \lor \mathscr{B} & \\ i & \quad \begin{array}{|l} \underline{\mathscr{A}} \\ \mathscr{C} \end{array} & \\ j & & \\ k & \quad \begin{array}{|l} \underline{\mathscr{B}} \\ \mathscr{C} \end{array} & \\ l & & \\ & \mathscr{C} & \lor\text{E } m, i\text{–}j, k\text{–}l \end{array}$$

$$\begin{array}{l|ll} i & \quad \begin{array}{|l} \underline{\mathscr{A}} \\ \mathscr{B} \end{array} & \\ j & & \\ & \mathscr{A} \to \mathscr{B} & \to\text{I } i\text{–}j \end{array}$$

$$\begin{array}{l|ll} m & \mathscr{A} \to \mathscr{B} & \\ n & \mathscr{A} & \\ & \mathscr{B} & \to\text{E } m, n \end{array}$$

$$\begin{array}{l|ll} i & \quad \begin{array}{|l} \underline{\mathscr{A}} \\ \mathscr{B} \end{array} & \\ j & & \\ k & \quad \begin{array}{|l} \underline{\mathscr{B}} \\ \mathscr{A} \end{array} & \\ l & & \\ & \mathscr{A} \leftrightarrow \mathscr{B} & \leftrightarrow\text{I } i\text{–}j, k\text{–}l \end{array}$$

$$\begin{array}{l|ll} m & \mathscr{A} \leftrightarrow \mathscr{B} & \\ n & \mathscr{A} & \\ & \mathscr{B} & \leftrightarrow\text{E } m, n \end{array}$$

$$\begin{array}{l|ll} m & \mathscr{A} \leftrightarrow \mathscr{B} & \\ n & \mathscr{B} & \\ & \mathscr{A} & \leftrightarrow\text{E } m, n \end{array}$$

$$\begin{array}{l|ll} i & \quad \begin{array}{|l} \underline{\mathscr{A}} \\ \bot \end{array} & \\ j & & \\ & \neg\mathscr{A} & \neg\text{I } i\text{–}j \end{array}$$

$$\begin{array}{l|ll} m & \neg\mathscr{A} & \\ n & \mathscr{A} & \\ & \bot & \neg\text{E } m, n \end{array}$$

$$\begin{array}{l|ll} m & \bot & \\ & \mathscr{A} & \text{X } m \end{array}$$

$$\begin{array}{l|ll} i & \quad \begin{array}{|l} \underline{\neg\mathscr{A}} \\ \bot \end{array} & \\ j & & \\ & \mathscr{A} & \text{IP } i\text{–}j \end{array}$$

Figure 1: Rules for Fitch-Style Proofs

$$\begin{array}{r|ll} m & \mathscr{A}(\ldots c \ldots c \ldots) & \\ & \forall x \mathscr{A}(\ldots x \ldots x \ldots) & \forall\mathrm{I}\ m \end{array}$$

$$\begin{array}{r|ll} m & \forall x \mathscr{A}(\ldots x \ldots x \ldots) & \\ & \mathscr{A}(\ldots c \ldots c \ldots) & \forall\mathrm{E}\ m \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{A}(\ldots c \ldots c \ldots) & \\ & \exists x \mathscr{A}(\ldots x \ldots c \ldots) & \exists\mathrm{I}\ m \end{array}$$

$$\begin{array}{r|ll} m & \exists x \mathscr{A}(\ldots x \ldots x \ldots) & \\ i & \quad \mathscr{A}(\ldots c \ldots c \ldots) & \\ j & \quad \mathscr{B} & \\ & \mathscr{B} & \exists\mathrm{E}\ m,\ i\text{–}j \end{array}$$

$$\begin{array}{r|ll} & c = c & =\mathrm{I} \end{array}$$

$$\begin{array}{r|ll} m & a = b & \\ n & \mathscr{A}(\ldots a \ldots a \ldots) & \\ & \mathscr{A}(\ldots b \ldots a \ldots) & =\mathrm{E}\ m,\ n \end{array}$$

$$\begin{array}{r|ll} m & a = b & \\ n & \mathscr{A}(\ldots b \ldots b \ldots) & \\ & \mathscr{A}(\ldots a \ldots b \ldots) & =\mathrm{E}\ m,\ n \end{array}$$

Derived rules

$$\begin{array}{r|ll} m & \mathscr{A} \vee \mathscr{B} & \\ n & \neg\mathscr{A} & \\ & \mathscr{B} & \mathrm{DS}\ m,\ n \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{A} \vee \mathscr{B} & \\ n & \neg\mathscr{B} & \\ & \mathscr{A} & \mathrm{DS}\ m,\ n \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{A} \to \mathscr{B} & \\ n & \neg\mathscr{B} & \\ & \neg\mathscr{A} & \mathrm{MT}\ m,\ n \end{array}$$

$$\begin{array}{r|ll} m & \neg\neg\mathscr{A} & \\ & \mathscr{A} & \mathrm{DNE}\ m \end{array}$$

$$\begin{array}{r|ll} i & \quad \mathscr{A} & \\ j & \quad \mathscr{B} & \\ k & \quad \neg\mathscr{A} & \\ l & \quad \mathscr{B} & \\ & \mathscr{B} & \mathrm{LEM}\ i\text{–}j,\ k\text{–}l \end{array}$$

Figure 2: Rules for Fitch-Style Proofs (cont'd.)