Problem

Say that a variable A in CFG G is ***necessary*** if it appears in every derivation of some string w ? G. Let *NECESSARY*$_{\text{CFG}}$ = { $\langle G, A\rangle |$ A is a necessary variable in G}.

a. Show that *NECESSARY*$_{\text{CFG}}$ is Turing-recognizable.

b. Show that *NECESSARY*$_{\text{CFG}}$ is undecidable.

Step-by-step solution

Step 1 of 2

Consider a CFG *G* which contains variables *A* in his each production of some string *w* which belongs to *G.*

Consider a Turing machine *D* which works as follow:

- $D = \text{On input} \langle G, A\rangle, \text{where } G \text{ is CFG and } A \text{ is non terminal}$

1. Create context free grammar $\frac{G}{A}$ by eliminating variable A from the derivations of *G.*

2. Create list of strings *w* generated by grammar G. Create a decider for A_{CFG} and then check each string of *w* can also be generated by $\frac{G}{A}$.

3. If *w* strings cannot generated by $\frac{G}{A}$ then **accept**, else **continue**.

• *D*= On other input instead of $\langle G, A\rangle$, **reject**

The $L\left(\frac{G}{A}\right)$ language contains all and only that strings of $w \in L(G)$ which does not require non terminal *A* for their derivation. If variable *A* is necessary for grammar *G* then few strings of $w \in L(G)$ cannot produce without use of non-terminal *A.*

The Turing machine *D* finds out those strings of *w* which cannot derived without use of *A.*

On the other hand, when variable A is not necessary for grammar G, which means $L(G) = L\left(\frac{G}{A}\right)$, then *D* continue move in a loop.

Hence, NECESSARY_{CFG} recognize by Turing machine *D.*

Comment

Step 2 of 2

It is already known that ALL_{CFG} is un-decidable, this shows that NECESSARY_{CFG} is also un-decidable. Therefore, NECESSARY_{CFG} must also un-decidable.

When any language complement is un-decidable then that language will also become un-decidable.

This can be proved by reduction R. The computation of reduction R is as follow:

$\text{R} = \text{On input} \langle G\rangle :$

• Create production of *G* after adding variable A and their productions in G.

$S \to A$

$A \to \in$

And

$A \to aA$ for each $a \in \Sigma$

• The output $\langle G, A\rangle$

The Grammar *G* produce with the help of reduction *R* is always $L(G) = \Sigma_*$. Thus, if $L(G) = \Sigma_*$, then there is no need of variable *A* in *G* because each string of $w \in \Sigma_*$ can derived by G without using *A.*

Also, if $L(G)=\Sigma_*$ then variable *A* is necessary for production of *G* because if $w \in L(G)$ then production of *G* is only possible with the help of *A*.

Thus, $\langle G \rangle \in ALL_{CFG}$, then $\langle G, A \rangle \in \text{NECESSARY}_{CFG}$

Hence, *R* reduces ALL_{CFG} to NECESSARY_{CFG}

Comments (6)