Problem

Let

$$\Sigma_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Here, Σ_2 contains all columns of 0s and 1s of height two. A string of symbols in Σ_2 gives two rows of 0s and 1s. Consider each row to be a binary number and let

$$C = \{w \in \Sigma_2^* | \text{ the bottom row of } w \text{ is three times the top row}\}.$$

For example, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in C$, but $\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin C$. Show that C is regular. (You may assume the result claimed in Problem 1.31.)

Step-by-step solution

Step 1 of 5

Consider the language,

$C = \{w \in \Sigma_2^* | \text{ the bottom row of } w \text{ is three times the top row}\}$

Over the alphabet $\Sigma_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Comment

Step 2 of 5

Here each row is binary number.

- The regular languages are closed under reversal. Use this property to prove the language *C* is regular.
- Scan the input in reverse order.
- Begin with the lower order bits and multiply it by 3 in binary format to add the multiplicand with the result of shifting the multiplicand itself by 1 bit.
- Consider a binary number 110 (6). Three times of 110(6) is 18 (10010) which is obtained by adding 110(6) with 1100(12).
- The top row can be represented as $X_n\ X_{n-1}...X_1\ X_0$.
- The bottom row can be represented as $P_n\ P_{n-1}...P_1\ P_0$.
- Add $X_{n-1}\ X_{n-2}...X_0\ 0$ to the top row to obtain the bottom row. It can be represented as follows:

$$\begin{array}{l} X_n\ X_{n-1}......X_1\ X_0 \\ X_{n-1}\ X_{n-2}...X_0\ 0 \\ \hline P_n\ P_{n-1}.......P_1\ P_0 \end{array}$$

- The value of P_i depends on the values of X_i and X_{i-1}.
- The bit at the top of the column is the carry-in (C_i^{in}). For the first column the carry-in will be zero.
- If two 1 bits are added, the carry will be generated. It is called carry-out (C_{i-1}^{out}).
- The P_i can be obtained by performing the XOR operation on X_i, X_{i-1} and C_i^{in}. The carry-out of the currently working column will be given as the carry-in for the next column.

Comment

Step 3 of 5

The language is said to be regular if there exists a finite automaton (DFA or NFA) for that language.

• To prove that the language C is regular, construct a DFA.

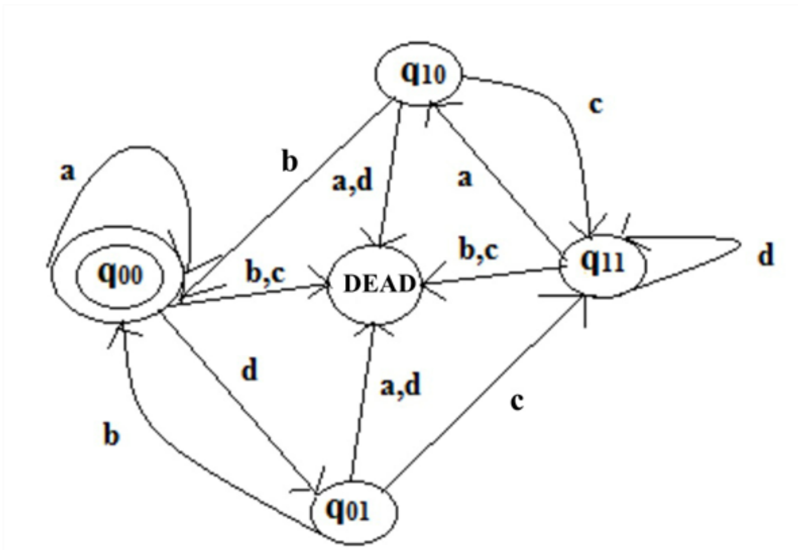
• It is necessary with 4 states without counting sink state to keep track of all the possible combinations of C_{i+1}^{in} *and* X_{i-1} *and* for each of these states, there will be exactly two possible valid output transitions depending on whether X_i *is* 0 or 1.

• Assume that q_{ij} represents the state for which $C^{in}(Carry-in)$ is equal to *i* and the preceding symbol is observed at the top is equal to *j*.

Comment

Step 4 of 5

The state diagram of DFA is as follows:



Where $a=\begin{bmatrix}0\\0\end{bmatrix}, b=\begin{bmatrix}0\\1\end{bmatrix}, c=\begin{bmatrix}1\\0\end{bmatrix}, d=\begin{bmatrix}1\\1\end{bmatrix}$

Consider an example string $\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}$.

Read the string in reverse order. Here, the initial state is $q00$. The input $\begin{bmatrix}0\\0\end{bmatrix}$ to the $q00$, stays in the same state. The input $\begin{bmatrix}1\\1\end{bmatrix}$ to $q00$, moves to the state $q01$. The input $\begin{bmatrix}0\\1\end{bmatrix}$ to $q01$, moves to the state $q00$. The input $\begin{bmatrix}0\\0\end{bmatrix}$ to the $q00$, stays in the same state. The string $\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}$ is accepted in reverse order.

Comment

Step 5 of 5

Since a DFA that recognizes C^R is built. Therefore C^R is regular. Hence, it is proved that *C* is regular.

Comment