Problem

Show that the set of incompressible strings contains no infinite subset that is Turing-recognizable.

Step-by-step solution

Step 1 of 2

Incompressible strings:

Let w_i be a string. If w_i doesn't have any description shorter than itself then w_i is incompressible.

Now we have to show that set of incompressible strings contains no infinite subset that is Turing recognizable.

Let *A* be the set of incompressible strings and assume the contradiction *A* contains infinite subset and Turing recognizable.

We construct an enumeration function $f : N \rightarrow A$, *A* is recognized by machine *M*.

Enumeration: $f : N \rightarrow A$ such that $f(1) = w1, f(2) = w2, f(3) = w3...$ where first, second, and third enumerated strings are respectively *w*1, *w*2, *w*3, etc.

Since *A* reaches infinite there is a string $w_i \in A$.

Comment

Step 2 of 2

Turing machine *N* which computes an incompressible string of length at least *n*.

$N =$" On input *n* (an integer in binary notation)

1. Call *M* to enumerate incompressible strings w_i.

2. If $|w_i| >= n$, output w_i and halt"

Now the length of $\langle N \rangle n$ is

$$|\langle N \rangle n| = |\langle N \rangle| + \log(n)$$

$= \log(n) + c$ where *c* is a constant

Now $|w_i| = n$

$n > \log n + c$ for large value of *n*.

$n > |\langle N \rangle n| \qquad [\text{As} |\langle N \rangle| n = \log n + c]$

Therefore $|w_i| > n$

This contradicts the incompressibility of string w_i, therefore our assumption that *M* enumerates an infinite subset of incompressible strings w_i is wrong.

Hence set of incompressible strings contains no infinite subset is Turing – recognizable.

Comment