CSE303: Theory of Computation, Fall 2021 **Date:** Oct 29

Homework #3

(Due: Nov 12)

Task 1. [**25 Points**] **Construct CFGs**

Construct CFG for each of the following languages.

(a) [**5 Points**] $L = \{a^i b^j c^k |\ i \geq 0,\ j \geq 0,\ k = 2i + 3j\}$, $\Sigma = \{a, b, c\}$

Solution:

$G = (V, \Sigma, R, S)\ \ where:$

The set of variables is $V = \{S, B\}$

The set of terminals is $\Sigma = \{a, b\}$

The set of rules is $R =$

$$S \rightarrow aScc \mid B \mid \epsilon$$

$$B \rightarrow bBccc \mid \epsilon$$

The start variable is S

(b) [**5 Points**] $L = \{a^i b^j c^k |\ i \neq j$ **or** $j \neq k\}$**,** $\Sigma = \{a, b, c\}$

Solution:

$G = (V, \Sigma, R, S)\ \ where:$

The set of variables is $V = \{S, S_1, S_2, A, B, C, D, E\}$

The set of terminals is $\Sigma = \{a, b, c\}$

The set of rules is $R =$

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$$S \rightarrow S_1 \mid S_2$$

$$S_1 \rightarrow S_1 c \mid A$$

$$A \rightarrow aAb \mid D \mid E$$

$$D \rightarrow aD \mid a$$

$$E \rightarrow bE \mid b$$

$$S_2 \to aS_2 \mid B$$
$$B \to bBc \mid E \mid C$$
$$C \to cC \mid c$$

The start variable is S

(c) [**5 Points**] $L = \{a^i b^j c^k d^l \mid\ i + j = k + l\}$**,** $\Sigma = \{a, b, c, d\}$

Solution:

$G = (V, \Sigma, R, S)\ \ where:$

The set of variables is $V = \{S, S_1, S_2, A, B, C\}$

The set of terminals is $\Sigma = \{a, b, c\}$

The set of rules is $R =$

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$$S \to S_1 \mid S_2$$
$$S_1 \to aS_1d \mid A$$
$$A \to bAd \mid B$$
$$B \to bBc \mid \epsilon$$
$$S_2 \to aS_2d \mid C$$
$$C \to aCc \mid B$$

The start variable is S

(d) [**5 Points**] $L = \{ucv \mid\ u^R \in \{a, b\}^*$ **is a substring of** $v\}$**,** $\Sigma = \{a, b, c\}$

Solution:

$G = (V, \Sigma, R, S)\ \ where:$

The set of variables is $V = \{S, M, N\}$

The set of terminals is $\Sigma = \{a, b, c\}$

The set of rules is $R =$

$$S \to MN$$
$$M \to aMa \mid bMb \mid cN$$
$$N \to Na \mid Nb \mid Nc \mid \epsilon$$

The start variable is S

(e) [**5 Points**] $L = \{ucv|\ u^R \in \{a,b\}^*$ **is a subsequence of** $v\}$**,** $\Sigma = \{a,b,c\}$

Solution:

$G = (V, \Sigma, R, S)\ \ where:$

The set of variables is $V = \{S, M, N\}$

The set of terminals is $\Sigma = \{a, b, c\}$

The set of rules is $R =$

$$S \to MN$$
$$M \to aMNa \mid bMNb \mid cN$$
$$N \to Na \mid Nb \mid Nc \mid \epsilon$$

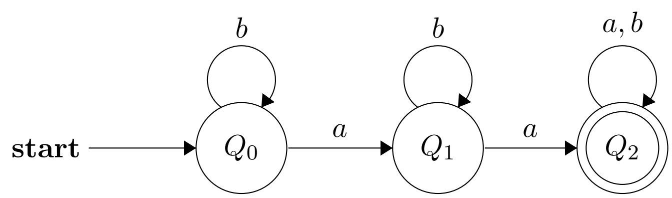
The start variable is S

Task 2. [**25 Points**] **Construct CFGs from DFAs**

You constructed a DFA for each of the following languages in HW1. This task asks you to convert each of those DFAs to an equivalent CFG. Assume that $\Sigma = \{a, b\}$ unless specified otherwise.

(a) [**5 Points**] $L = \{w|\ n_a(w) \geq 2\}$

Solution:
DFA Diagram:



$G = (V, \Sigma, R, S)\ \ where:$

The set of variables is $V = \{S_0, S_1, S_2\}$

The set of terminals is $\Sigma = \{a, b\}$

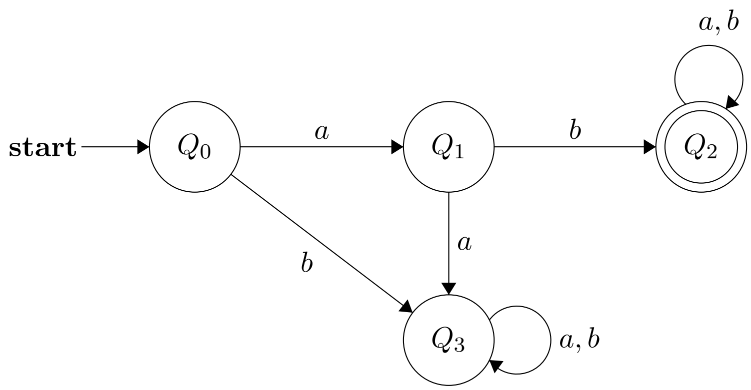
The set of rules is $R =$

$$S_0 \to aS_1 \mid bS_0$$
$$S_1 \to aS_2 \mid bS_1$$
$$S_2 \to aS_2 \mid bS_2 \mid \epsilon$$

The start variable is S_0

(b) [**5 Points**] $L = \{w | \ w$ **starts with** $ab\}$

Solution:
DFA Diagram:



$G = (V, \Sigma, R, S)$ *where* :

The set of variables is $V = \{S_0, S_1, S_2\}$

The set of terminals is $\Sigma = \{a, b\}$

The set of rules is $R =$

$$S_0 \to aS_1$$

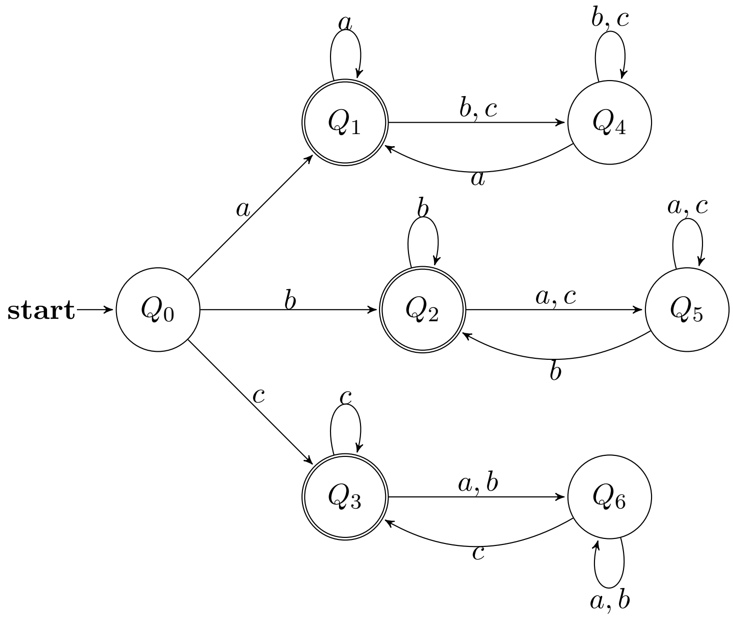
$$S_1 \to bS_2$$

$$S_2 \to aS_2 \mid bS_2 \mid \epsilon$$

The start variable is S_0

(c) [**5 Points**] $L = \{w | \ w$ **starts and ends with the same symbol**$\}$ **for** $\Sigma = \{a, b, c\}$

Solution:
DFA Diagram:



$G = (V, \Sigma, R, S)$ *where* :

The set of variables is $V = \{S_0, S_1, S_2, S_3, S_4, S_5, S_6\}$

The set of terminals is $\Sigma = \{a, b, c\}$

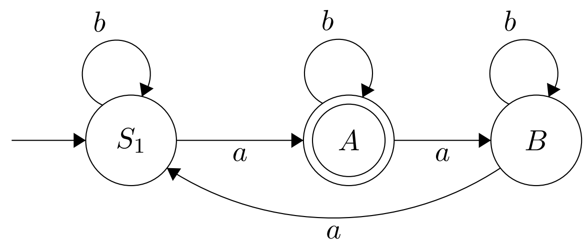
The set of rules is $R =$

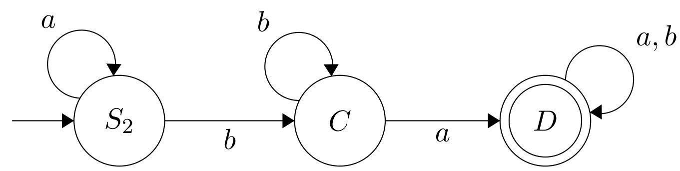
$$S_0 \rightarrow aS_1 \mid bS_2 \mid cS_3$$
$$S_1 \rightarrow aS_1 \mid bS_4 \mid cS_4 \mid \epsilon$$
$$S_2 \rightarrow aS_5 \mid bS_2 \mid cS_5 \mid \epsilon$$
$$S_3 \rightarrow aS_6 \mid bS_6 \mid cS_3 \mid \epsilon$$
$$S_4 \rightarrow aS_1 \mid bS_4 \mid cS_4$$
$$S_5 \rightarrow aS_5 \mid bS_2 \mid cS_5$$
$$S_6 \rightarrow aS_6 \mid bS_6 \mid cS_3$$

The start variable is S_0

(d) **[5 Points]** $L = \{w \mid n_a(w) \bmod 3 = 1$ **or** w **contains** $ba\}$

Solution:
DFA Diagram:





$G = (V, \Sigma, R, S)$ *where* :

The set of variables is $V = \{S, S_1, S_2, A, B, C, D\}$

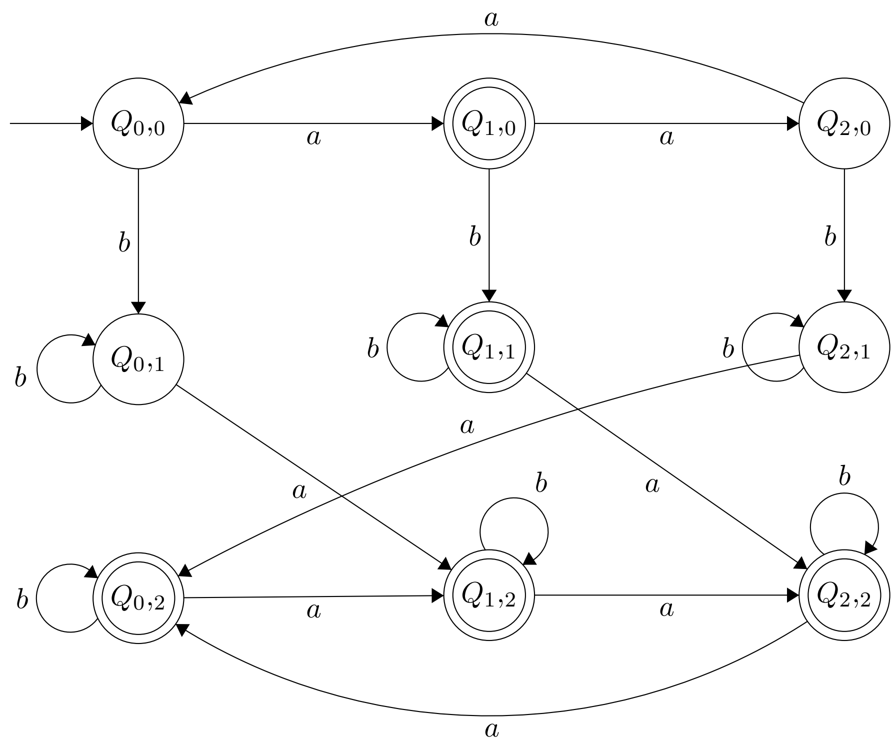
The set of terminals is $\Sigma = \{a, b\}$

The set of rules is $R =$

$$S \to S_1 \mid S_2$$
$$S_1 \to aA \mid bS_1$$
$$A \to aB \mid bA \mid \epsilon$$
$$B \to aS_1 \mid bB$$
$$S_2 \to aS_2 \mid bC$$
$$C \to aD \mid bC$$
$$D \to aD \mid bD \mid \epsilon$$

The start variable is S

Another Solution:
DFA Diagram:



$G = (V, \Sigma, R, S)$ *where* :

The set of variables is $V = \{S_{00}, S_{10}, S_{20}, S_{01}, S_{11}, S_{21}, S0_2, S_{12}, S_{22}\}$

The set of terminals is $\Sigma = \{a, b\}$

The set of rules is $R =$

$$S_{00} \to aS_{10} \mid bS_{01}$$

$$S_{10} \to aS_{20} \mid bS_{11} \mid \epsilon$$

$$S_{20} \to aS_{00} \mid bS_{21}$$

$$S_{01} \to aS_{12} \mid bS_{01}$$

$$S_{11} \to aS_{22} \mid bS_{11} \mid \epsilon$$

$$S_{21} \to aS_{02} \mid bS_{21}$$

$$S_{02} \to aS_{12} \mid bS_{02} \mid \epsilon$$

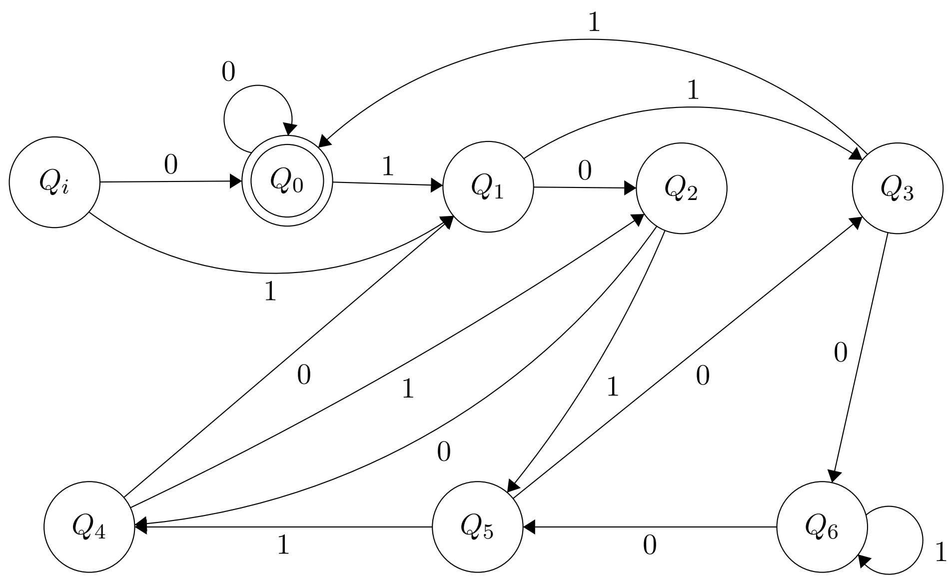
$$S_{12} \to aS_{22} \mid bS_{12} \mid \epsilon$$

$$S_{22} \to aS_{02} \mid bS_{22} \mid \epsilon$$

The start variable is S_{00}

(e) [**5 Points**] $L = \{w|$ **binary number** w **is divisible by** $7\}$ **for** $\Sigma = \{0, 1\}$

Solution:
DFA Diagram:



$G = (V, \Sigma, R, S)$ *where :*

The set of variables is $V = \{S_i, S_0, S_1, S_2, S_3, S_4, S_5, S_6\}$

The set of terminals is $\Sigma = \{0, 1\}$

The set of rules is $R =$

$$S_i \to 0S_0 \mid 1S_1$$
$$S_0 \to 0S_0 \mid 1S_1 \mid \epsilon$$
$$S_1 \to 0S_2 \mid 1S_3$$
$$S_2 \to 0S_4 \mid 1S_5$$
$$S_3 \to 0S_6 \mid 1S_0$$
$$S_4 \to 0S_1 \mid 1S_2$$
$$S_5 \to 0S_3 \mid 1S_4$$
$$S_6 \to 0S_5 \mid 1S_6$$

The start variable is S_i

Task 3. [20 Points] Regular expressions to CFGs

For each of the following regular expressions construct a CFG to accept the language it represents.

(a) [**5 Points**] $(0 \cup 1(01^*0)^*1)^*$

Solution:

$G = (V, \Sigma, R, S)$ *where* :

The set of variables is $V = \{S, A, B, C, D, E\}$

The set of terminals is $\Sigma = \{0, 1\}$

The set of rules is $R =$

$$S \to SA \mid \epsilon$$
$$A \to B \mid 0$$
$$B \to 1C1$$
$$C \to CD \mid \epsilon$$
$$D \to 0E0$$
$$E \to E1 \mid \epsilon$$

The start variable is S

(b) [**5 Points**] $\epsilon \cup a^+ \cup a^+bb^+(abb^+)^*$

Solution:

$G = (V, \Sigma, R, S)$ *where* :

The set of variables is $V = \{S, A, B, C, D, E\}$

The set of terminals is $\Sigma = \{a, b\}$

The set of rules is $R =$

$$S \to A \mid B \mid \epsilon$$
$$A \to aA \mid a$$
$$B \to AbCD$$
$$C \to Cb \mid b$$
$$D \to DE \mid \epsilon$$
$$E \to abC$$

The start variable is S

(c) **[5 Points]** $a(\epsilon \cup a \cup b(a \cup b \cup \epsilon))$

Solution:

$G = (V, \Sigma, R, S)$ *where* :

The set of variables is $V = \{S, A, B, C\}$

The set of terminals is $\Sigma = \{a, b\}$

The set of rules is $R =$

$$S \rightarrow aA$$
$$A \rightarrow B \mid a \mid \epsilon$$
$$B \rightarrow bC$$
$$C \rightarrow a \mid b \mid \epsilon$$

The start variable is S

(d) **[5 Points]** $((a \cup b)^6)^*(a \cup b)(a \cup b \cup \epsilon)^4$

Solution:

$G = (V, \Sigma, R, S)$ *where* :

The set of variables is $V = \{S, A, B, C\}$

The set of terminals is $\Sigma = \{a, b\}$

The set of rules is $R =$

$$S \rightarrow ABCCCC$$
$$A \rightarrow ABBBBBB \mid \epsilon$$
$$B \rightarrow a \mid b$$
$$C \rightarrow a \mid b \mid \epsilon$$

The start variable is S

Task 4. [30 Points] Non-CFLs

Use the pumping lemma to show that the following languages are not context-free.

(a) [**10 Points**] $L = \{a^n b^{2n} a^n |\ n \geq 0\}$, $\Sigma = \{a, b\}$

Solution:
_ Assume L is CFL. Then it must satisfy the pumping property.
_ Let p = Pumping Length
_ Let $w = a^p b^{2p} a^p$
_ Let $w = uvxyz$, $u = a^p$, $v = b^p$, $x = \epsilon$, $y = \epsilon$, $z = b^p a^p$
_ $|vy| = p \geq 1$ and $|vxy| = p$
_ Then $uv^i xy^i z$ must belong to L for all integer $i \geq 0$
_ However$uvvxyyz = a^p b^{3p} a^p$, which is not in L

This is a contradiction to our assumption that L is CFL! Hence, L is not CFL.

(b) [**10 Points**] $L = \{a^i b^j c^k |\ k > i > 0,\ k > j > 0\}$, $\Sigma = \{a, b, c\}$

Solution:
_ Assume L is CFL. Then it must satisfy the pumping property.
_ Let p = Pumping Length
_ Let $w = a^p b^p c^{p+1}$
_ Let $w = uvxyz$, $u = a^p$, $v = b^r$, $x = b^{p-r}$, $y = \epsilon$, $z = c^{p+1}$ where $1 \leq r \leq p$
_ $|vy| = r \geq 1$ and $|vxy| = p \leq p$
_ Then $uv^i xy^i z$ must belong to L for all integer $i \geq 0$.
_ However$uvvxyyz = a^p b^{p+r} c^{p+1}$, which is not in L

This is a contradiction to our assumption that L is CFL! Hence, L is not CFL.

(c) [**10 Points**] $L = \{a^{n!} |\ \ n \geq 0,\ n! = 1 \times 2 \times \ldots \times n,\ 0! = 1\}$, $\Sigma = \{a\}$

Solution:
_ Assume L is CFL. Then it must satisfy the pumping property.
_ Let s = Pumping Length
_ Let $w = a^{s!}$
_ Let $w = uvxyz$, $u = \epsilon$, $v = a^q$, $x = a^r$, $y = \epsilon$, $z = a^{s!-s}$, where $q + r = s$, $r \geq 0$, and $q > 0$.
_ $|vy| = q \geq 1$ and $|vxy| = q + r = s \leq s$
_ Then $uv^i xy^i z$ must belong to L for all integer $i \geq 0$.
_ However, $uvvxyyz = a^q a^q a^r b^{s!-s} = a^{s!+q}$, which is not in L because:

$$
\begin{aligned}
& 0 \leq q \leq s \leq s! \\
\Longrightarrow\ & s! \leq s! + q < s + s! < s! + s! \\
\Longrightarrow\ & s! \leq s! + q < 2s! < (s+1)s! \\
\Longrightarrow\ & s! < s! + q < (s+1)!
\end{aligned}
$$

Therefore, $s! + q$ is between two consecutive factorials. This means that $(s! + q)$ is not a factorial. Hence $uvvxyyz = a^{s!+q}$ is not in L.
This is a contradiction to our assumption that L is CFL! Hence, L is not CFL.