Problem

Show that PSPACE is closed under the operations union, complementation, and star.

Step-by-step solution

Step 1 of 4

PSPACE is the class of languages that are decidable in polynomial space on a deterministic Turing machine i.e. $\text{PSPACE} = \bigcup_k \text{SPACE}(n^k)$.

Consider the two languages L_1 and L_2 that are decided by PSPACE Turing machines M_1 and M_2.

M_1 decides L_1 in deterministic time $O(n^k)$ and M_2 decides L_2 in deterministic time $O(n^L)$.

If any polynomial solvable in polynomial time, then it is solvable in polynomial space.

Comment

Step 2 of 4

UNION:

$M =$"on input w:

1. Run M_1 on w, if M_1 accepted then accept

2. Else run M_2 on w, if M_2 accepted then accept

3. Else reject"

Clearly, the longest branch in any computation tree on input w of length *n* is $O\left(n^{\max\{k,L\}}\right)$. Thus, M is a polynomial time deterministic decider for $L_1 \cup L_2$. If any polynomial solvable in polynomial time, then it is solvable in polynomial space. Therefore, PSPACE is closed under union.

Comment

Step 3 of 4

COMPLEMENTATION:

$M =$"on input w:

1. Run M_1 on w, if M_1 accepted then reject.

2. Else accept"

Clearly, the longest branch in any computation tree on input w of length n is $O\left(n^{\{k\}}\right)$. Thus, M is a polynomial time deterministic decider for $\overline{L_1}$. If any polynomial solvable in polynomial time, then it is solvable in polynomial space. Therefore, PSPACE is closed under complementation.

Comment

Step 4 of 4

STAR:

$M =$"on input w:

1. If $w = \varepsilon$ then accept

2. Deterministically select a number m such that $1 \le m \le |w|$

3. Deterministically split w into m pieces such that $w = w_1 w_2 \ldots w_m$.

4. For all *i,* $1 \le i \le m$: run M_1 on w_i, if M_1 rejected then reject.

5. Else (M_1 accepted all w_i, $1 \le i \le m$), accept".

The steps 1 and 2 takes $O(m)$ time. Step 3 also possible in polynomial time. In step 4, the for loop is run at most m times and every run takes almost $O(m^k)$. The total times is $O(m^{k+1})$. This means that M is a polynomial time decides for $L_1{}^*$. If any polynomial solvable in polynomial time, then it is solvable in polynomial space. Therefore, PSPACE is closed under star.

Therefore, PSPACE is closed under Union, Complementation and star.

Comment