Problem

A subset of the nodes of a graph G is a ***dominating set*** if every other node of G is adjacent to some node in the subset. Let

$$\textit{DOMINATING-SET} = \{\langle G, k\rangle \mid G \text{ has a dominating set with } k \text{ nodes}\}.$$

Show that it is NP-complete by giving a reduction from *VERTEX-COVER*.

Step-by-step solution

Step 1 of 3

NP-Complete:

A language *B* is NP-Complete if it satisfies 2 conditions.

1. *B* is in NP
2. Every *A* in NP is polynomial time reducible to *B*.

Comment

Step 2 of 3

1. *DOMINATING – SET* is in NP:

• Consider an instance $\langle G,k\rangle$ of the *DOMINATING-SET* and a covering *D*.

• Check that each node of *G* is adjacent to some node in *D*.

• This can be done in polynomial time.

• Therefore, *DOMINATING-SET* is in NP.

Comment

Step 3 of 3

2. $\textbf{\textit{VERTEX-COVER}} \le_P \textbf{\textit{DOMINATING-SET}}$

Now show that *VERTEX-COVER* reduces to *DOMINATING-SET*.

• Consider an instance $\langle (V,E),k\rangle$ of *VERTEX-COVER*.

• Construct an instance $\langle ((V-S)\cup V', E\cup E'),k\rangle$ of *DOMINATING-SET* where $S\subseteq V$ are nodes of degree 0.

• For each edge $(u,v)\in E$, there are edges (u,w) and (w,v) in E' where $w\in v'$ in new vertex corresponding to (u,v).

• Let $G=\langle V,E\rangle$ and $G'=\langle (V-S)\cup V', E\cup E'\rangle$.

• Suppose $(\langle V,E\rangle,k)$ is in *VERTEX-COVER*.

• There exits $C\subseteq V$ of size *k* where each edge $(u,v)\in E$ has either $u\in C$ or $v\in C$.

• If $v\in(V-S)$ then the degree of v is one or more then there exists a node u such that $(u,v)\in E$ which implies that at least one of *u* or *v* is in *C*. Thus, v is covered.

• If $w\in V'$ then *w* is adjacent to both *u* and *v* where $(u,v)\in E$ which implies that at least one of *u* or *v* is in *C*. Thus, *w* is covered.

In other direction, suppose that $\langle ((V-S)\cup V', E\cup E'),k\rangle$ is in *DOMINATING-SET*.

• Then there exists $C\subseteq((V-S)\cup V')$ of size k.

• In such cases in which multiple such C exist, it can be said that at least one includes no vertices in V'.

• This is always exists since $w \in (C \cap V')$ that corresponds to edge (u,v) covers only nodes $u, v, w,$ but using u instead of w covers $u, v, w,$ and possibly more.

• Therefore, $C \subseteq (V - S)$ and C is a vertex cover for G.

• This is because C is a *DOMINATING-SET* for G' implying that all nodes of V' are covered. Thus, every edge $(u,v) \in E$ has at least one of u, v in c.

Therefore, $VERTEX\text{-}COVER \leq_P DOMINATING\text{-}SET$.

From (1) and (2), *DOMINATING-SET* is NP-complete.

Comment