Problem

Let $\Sigma = \{1, \#\}$ and let

$$Y = \{w \mid w = x_1\#x_2\#\cdots\#x_k \text{ for } k \geq 0, \text{ each } x_i \in 1^*, \text{ and } x_i \neq x_j \text{ for } i \neq j\}.$$

Prove that Y is not regular.

Step-by-step solution

Step 1 of 4

Consider the following language over the alphabet $\Sigma = \{1, \#\}$.

$Y = \{w \mid w = x_1 \# x_2 \# ... \# x_k \text{ for } k \geq 0, \text{ each } x_i \in 1^*, \text{ and } x_i \neq x_j \text{ for } i \neq j\}$. The language *Y* accepts the words of the form $x_1 \# x_2 \# ... \# x_k$ where $x_1, x_2, \ldots x_k$ are the substrings that are formed with any number of 1s. Here, x_i can be any number of 1s.

The words that are accepted by the language *Y* contains only 1s and #s because $\{1, \#\}$ are the input alphabet. Any two substrings cannot be equal i.e., $x_i \neq x_j$. Every two substrings are separated by the input alphabet #.

The language is said to be regular if it is satisfied by the pumping lemma. Otherwise the language is not regular.

Comment

Step 2 of 4

Pumping lemma:

If *A* is a regular language, then there is a number *p* (the pumping length) where *S* is any string that belongs to *A* of length at least *p*, then *S* may be divided into three pieces, $S = uvw$, satisfying the following conditions.

1. For each $i \geq 0, uv^i w \in A$

2. $|v| > 0$, and

3. $|uv| \leq p$

Comment

Step 3 of 4

Assume that Y is a regular language.

Let p be the pumping length for Y. The strings of the language Y are of the form $w = x_1 \# x_2 \# ... \# x_k$.

Consider a string $S = x_1 \# x_2$ for $k = 2$ and $x_1 \neq x_2$. Here, x_1 and x_2 can be formed with only 1s but both cannot be equal. Any two different strings can be taken for x_1 and x_2.

Assume $x_1 = 1^p 1$ and $x_2 = 111^p$. Then the string $S = 1^p 1 \# 111^p$. Here, x_1 and x_2 are two different substrings and the value of x_1 and x_2 depends on the *p* value. For example, if *p*=2 then the values of x_1 and x_2 are 111 and 1111.

Clearly, the length of *S* is greater than *p* and $S \in Y$.

Comment

Step 4 of 4

Let $111\#1111$ be the string that belongs to Y. The pumping length of the string is 2.

To satisfy the conditions of the pumping lemma, divide the string $111\#1111$ into three parts *u*, *v* and *w*. Here *u* is equal to 1, *v* is equal to 1, *w* is equal to $1\#1111$ (the remaining part of the string).

$$S = 111\#1111$$
$$= \frac{1}{u}\ \frac{1}{v}\ \frac{1\#1111}{w}$$

Pump the middle part such that $uv^iw\ (i \geq 0)$. For *i*=2, the *v* becomes 11.

$$S = (1)\ (1)^i\ (1\#1111)$$
$$= \frac{1}{u}\ \frac{11}{v}\ \frac{1\#1111}{w} \qquad [when\ i{=}2]$$

The string after pumping is $1111\#1111$.

The string $1111\#1111 \notin Y$ because, the substring x_1 is equal to x_2 which violates the condition of the language *Y*. It is a contradiction. Thus, the pumping lemma is violated.

Therefore, Y is not a regular language.

Comment