Name: Solution User ID (NOT the 9 digit number):

CS4510 Automata and Complexity Spring 2020 Section A

Test 3 Solutions

Instructor: Richard Peng 11:00am - 1:15pm, Thursday, Apr 9, 2020

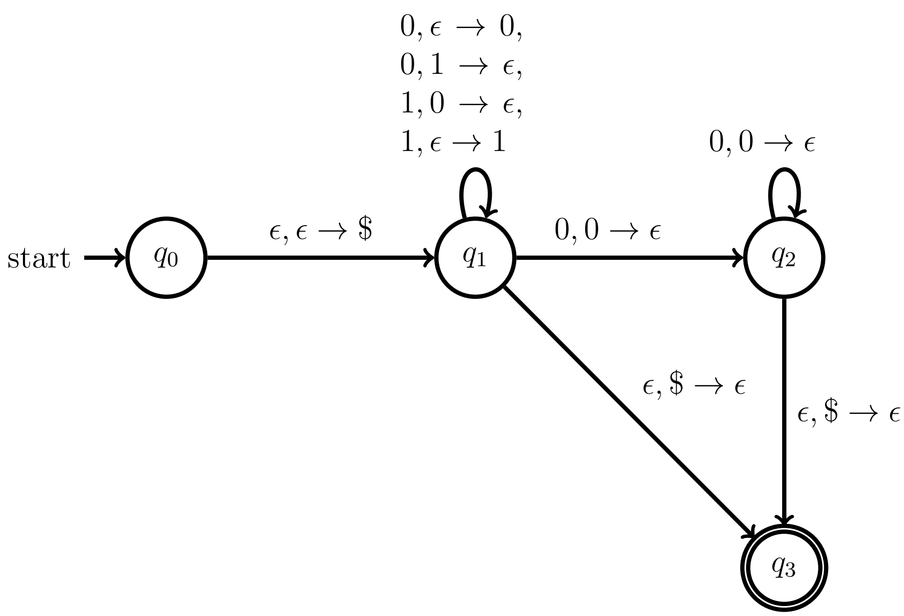
- This test is **online**, posted via Canvas and the course homepage, and handed in via GradeScope.
- The GradeScope submission page will close at 1:30pm (Eastern Time), but we suggest that you start wrapping up by around 1:15pm.
- You have 135 minutes to earn up to $1 + 4 + 4 + 8 + 6 + 3 = 26$ points, the test is graded out of 25.
- This booklet contains 5 **questions on 7 pages**, including this one.
- Write your solutions in the space provided. If you run out of space, continue your answer on the back of the last page, and make a notation on the front of the sheet.
- You may use any written or locally stored resources.
- However, the only internet resources that you could use during the test are: specifically:
  1. The BlueJeans Office Hours link at `https://gatech.bluejeans.com/242024735`, where clarifications will be posted in the chat.
  2. Piazza Test 3 clarification page: `https://piazza.com/class/k4xbfrttfnc687?cid=568`.
- You may use any of the theorems/facts/lemmas from the lecture notes, homeworks, or textbook without re-proving them unless explicitly stated otherwise. However, we may check with you about the citation / reference afterwards, possibly via regrades.
- If necessary make reasonable assumptions but please be sure to state them clearly
- Do not spend too much time on any one problem. Generally, a problem's point value is a good indication of how many minutes to spend on it.
- Good luck!

0. (/1 point) Submit your test through GradeScope before 1:30pm, under the right name/id, and with the pages corresponding to each problem clearly indicated.

1. (/4 points) Give a pushdown automata for the following language:

$$L = \{w \in \{0,1\}^* \mid \#0w \geq \#1w\},$$

that is, w is the set of binary strings with more 0s than 1s.



SOLUTION:

GRADING:

- 1 marks for accepting number of 0's = number of 1's
- 2 marks for accepting number of 0's > number of 1's
- 1 point for marking start and accepting states

2. (/4 points) Give a context free grammar for the language consisting of balanced parentheses in **both** () and []s. That is, the (can be paired up with the)s, and the [s can be paired up with the]s so that each substring between the corresponding pairs are also balanced parentheses. For example, [()()], ([[[]]()]) are balanced, while [(]) is **NOT** balanced.

SOLUTION:

$$
\begin{aligned}
s &\to \epsilon \\
s &\to (s) \\
s &\to [s] \\
s &\to ss
\end{aligned}
$$

GRADING:

- +1 for every correct rule (or equivalent)

3. (/8 points) Prove that the following languages are not context free using Pumping Lemma for context free languages.

(a) (/4 points)

$$L = \left\{a^i b^j c^k \mid 0 \le i < j < k\right\}.$$

SOLUTION: Take string $a^p b^{p+1} c^{p+2}$ for pumping length p. We have $s \in L$ and $|s| \ge p$. Now, by pumping lemma s must be partitioned into $uvxyz$, with $|vxy| \le p$. We can divide it into two cases.

If vxy consist of only one type of characters, we can divide it into two subcases. First if it consists of a or b, then after pumping up, the $i < j < k$ condition won't hold. Otherwise if it consists of c, then after pumping down, the $i < j < k$ condition won't hold.

If vxy consist of two types of characters, we can divide it into two subcases. First if it consists of a, b, then after pumping up, the $i < j < k$ condition won't hold. Otherwise if it consists of b, c, then after pumping down, the $i < j < k$ condition won't hold.

Because $|vxy| \le p$, it can't consist of three types of characters at the same time.

GRADING:

- -1 point: Incorrect choice of string
- -1 point: Does not consider all cases of vxy
- -1 point: Choice of i is incorrect/not specific enough
- -1 point: Insufficient reasoning as to why $uv^ixy^iz \notin L$

(b) (/4 points)

$$L = \left\{0^a 1^b 2^a 3^b \mid a, b \ge 0\right\}.$$

SOLUTION: Take string $s = 0^p 1^p 2^p 3^p$ for pumping length p. We have $s \in L$ and $|s| \ge p$. Now, by pumping lemma s must be partitioned into $uvxyz$, with $|vxy| \le p$. So there are two cases for vxy.

vxy consists of one type of digit: assume it contains all 1s, for other digits the proof is similar. In this case, since $|vy| > 0$, the string uv^ixy^iz is not in L for $i \ne 1$. Let $n = |vy|$, then $uv^ixy^iz = 0^a 1^{b+ni} 2^a 3^b$.

vxy spans at most two adjacent digit groups: assume the two groups are 0s and 1s, for other digits the proof is similar. Now at most 1 of v or y can contain both types of digits, assume it is v. In this case we have $v = 0^m 1^n, y = 1^q$. Since $|vy| > 0$, we have $m, n, q \ge 0, m+n+q > 0$. In this case uv^ixy^iz is not in L for $i \ne 1$, as $uv^ixy^iz = 0^{a+mi+ni} 1^{b+qi} 2^a 3^b$.

GRADING:

- -1 point: Incorrect choice of string
- -1 point: Does not consider all cases of vxy

- -1 point: Choice of i is incorrect/not specific enough
- -1 point: Insufficient reasoning as to why $uv^ixy^iz \notin L$

4. (/6 points) Select and solve **exactly two** of the following four questions.

Unless you clearly indicate which two to mark, only the first two in lexicographical order with work on them will be marked.

For proving a language is context free, you may provide either a CFG or a PDA.

(a) Let

$$L = \left\{0^i 1^j \mid i \neq j,\ i, j \geq 0\right\}.$$

Prove that this language is context free.

SOLUTION:

$$\begin{aligned} S &\to 0S1 \mid L \mid R \\ L &\to 0L \mid 0 \\ R &\to R1 \mid 1 \end{aligned}$$

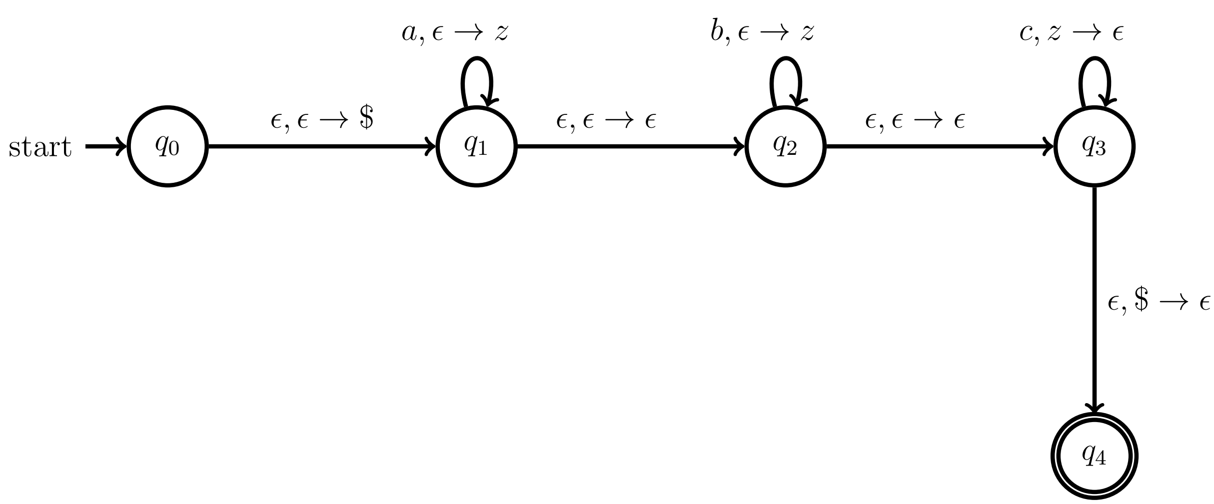
(b) Let

$$L = \left\{a^i b^j c^k \mid i, j, k \geq 0,\ i + j = k\right\}.$$

Prove that this language is context free.

SOLUTION:

$$\begin{aligned} S &\to aSc \mid M \\ M &\to bMc \mid \epsilon \end{aligned}$$



GRADING:

- 3 marks for correct CFG
- 3 marks for correct PDA, -1 for not mentioning start and accepting states

(c) Let
$$L = \left\{0^a 1^b 2^b 3^a \mid a, b \geq 0\right\}.$$
Prove that this language is context free.

SOLUTION:

$$S \to 0S3 \mid T$$
$$T \to 1T2 \mid \epsilon$$

GRADING:

- 3 marks for correct CFG
- 3 marks for correct PDA, -1 for not mentioning start and accepting states

(d) Consider the language over strings in a whose length is a factorial:
$$L = \left\{a^{n!} \mid n > 0\right\}.$$
Here $n!$ is the factorial of n, defined as the product of integers from 1 to n, $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$. Prove that this language is not context free.

SOLUTION: Let p be the length provided by the CFG pumping lemma. Consider the string
$$a^{p!},$$
and suppose it's written as
$$w = uv^i xy^i z \mid for \ \ i \geq 0$$
with $|vxy| <= p$ and $|vy| > 0$.
Let $|vy| = j$ where j is an integer between 1 and p. When the string is pumped, it can be represented as $a^{p!+j}$. Then the string is not in the language because $(p+1)! - p! = p(p!) > p \geq j$ so $p! + j < (p+1)!$. This suffices for when $p > 1$. Thus, the pumped string is not a factorial and not in the language L.

GRADING:

- -1 bad choice of w (not in language or fixed length)
- -1 doesn't consider all cases for vxy to arrive at $a^{p!+j}$
- -2 insufficient reasoning for why $uv^i xy^i z \notin L$ (**NOTE:** Saying "we will encounter a number of as between p! and (p+1)!" is NOT sufficient. You must prove this property and show that it is indeed never a factorial.)

5. (3 points) Provide a context free grammar for the following language

$$L = \left\{0^i 1^j 0^k \mid j \geq k + i\right\}.$$

SOLUTION:

$$\begin{aligned} S &\to LMR \\ L &\to 0L1 \mid \epsilon \\ R &\to 1R0 \mid \epsilon \\ M &\to M1 \mid \epsilon \end{aligned}$$

GRADING:

- -2 The rules will generate string with $j < k + i$ or some cases of $j >= k + i$ are not covered by the rules.
- -2 The format of the string is wrong, like generating "10101".
- -1 A simple modification to the rules can make the solution work. For example, some rules can't generate "1".