CSE 303 ToC. 115060128.
Assignment: 2.

Problem 1:

a) Given:- $f: A \to B$ & $g: B \to C$ are injective.
TPT: $g \circ f: A \to C$ is also injective

We will use proof by contradiction:
Suppose $g \circ f: A \to C$ is not injective
$\Rightarrow \exists a, a' \in A.\ g\circ f(a) = g\circ f(a')$ — (1)

Since f is injective $\Rightarrow f(a) \neq f(a')$ $\because a \neq a'$
Let $f(a) = b$ & $f(a') = b'$ $\Rightarrow b \neq b'$ — (2)

By (1), $g(b) = g(b')$.
Since g is injective, $g(b) = g(b') \Rightarrow b = b'$ which contradicts (2).
$\therefore g \circ f: A \to C$ is injective. Hence proved

b) Given $f: A \to B$ & $g: B \to C$ are surjective.
TPT: $g \circ f: A \to C$ is surjective.

We will prove $\forall c \in C.\ \exists a \in A$ s.t $g\circ f(a) = c$.
This by definition will prove $g \circ f$ is surjective.
Let c be an arbitary but fixed element in C.
~~We have~~ $\cancel{g(f(a)) = c}$.
Since g is surjective $\exists b \in B.\ g(b) = c$.

Since f is surjective $\exists a \in A.\ f(a) = b$.

Thus $g(f(a)) = g(b) = c$

$\Rightarrow$ $g \circ f$ is surjective. Hence proved

c. Given : $f: A \to B$ and $g: B \to C$ are bijective.
TPT $g \circ f: A \to C$ is bijective.

→ Since f and g are injective.
$\Rightarrow$ $g \circ f$ is injective ..... Using (a)

Since f and g are surjective
$\Rightarrow$ $g \circ f$ is surjective .... Using (b)

Since $g \circ f$ is injective and surjective
$\Rightarrow$ $g \circ f$ is bijective. Hence proved.

Problem 2:-

a). Let (a,b) and $(b,c) \in R^*$
we need to prove $(a,c) \in R^*$

→ $(a,b) \in R^* \Rightarrow \exists\, n \geq 0$ and sequence $a_0, a_1, \ldots a_n \in A$
s.t $a_0 = a$, $a_n = b$ & $(a_k, a_{k+1}) \in R$ $\forall k \in \{0, 1, \ldots, n-1\}$ — (1)

$(b,c) \in R^*$ $\exists\, m \geq 0$ and sequence $b_0, b_1, \ldots b_m \in A$ s.t
$b_0 = b$, $b_m = c$ & $(b_k, b_{k+1}) \in R$ $\forall k \in \{0, 1, \ldots, m-1\}$ — (2)

We will prove $\exists$ a sequence, from a to c, i.e
$c_0, c_1, \ldots c_l$.

Take ~~$a_0 = c_0$~~, $c_0 = a_0$, $c_1 = a_1, \ldots$, $c_n = a_n = b$
$c_{n+1} = b_1$, $c_{n+2} = b_2, \ldots$, $c_l = b_m = c$

~~We~~ o Take $l = m + n$

We have ~~$\circledast$~~ $(c_k, c_{k+1}) \in R \ \forall k \in \{0, 1, \dots, l\}$.
using 1 & 2.

$\therefore$ By definition $(a, c) \in R^*$
$\Rightarrow R^*$ is transitive relation.

b). We have $R \subseteq T$.
Let T be any transitive relation on A that contains R.
we need to prove $R^* \subseteq T$.

Let $(a, b) \in R^*$ be arbitary but fixed elements.
We will show $(a, b) \in T$.
Thus we will get $R^* \subseteq T$
i.e every element in R^* also belongs to T.

$(a, b) \in R^* \Rightarrow \exists n \geq 0$ & sequence $a_0, a_1, \dots, a_n \in A$
s.t $a_0 = a$, $a_n = b$ & $(a_k, a_{k+1}) \in R \ \forall k \in \{0, 1, \dots, n-1\}$

Since $R \subseteq T$; $(a_k, a_{k+1}) \in T \ \forall k \in \{0, 1, \dots, n-1\}$

We have $(a_0, a_1) \in T$ & $(a_1, a_2) \in T \Rightarrow (a_0, a_2) \in T$.
$(a_0, a_2) \in T$ & $(a_2, a_3) \in T \Rightarrow (a_0, a_3) \in T$.

$(a_0, a_k) \in T$ & $(a_k, a_{k+1}) \in T \Rightarrow (a_0, a_{k+1}) \in T$
$k \in \{1, 2, \dots n-1\}$.

$\Rightarrow (a_0, a_n) \in T \Rightarrow (a, b) \in T$.

Thus every element (a, b) in R^* exists in T

$\Rightarrow R^* \subseteq T$.

c) i) We have from b, any Transitive relation on R will contain R^*.
So if we take many transitive relations, their intersection will contain R^* because all of them contain R^*.
Thus $\cap T$ will contain R^*.

ii) R^* contains $\cap T$.
i.e $\cap T \subseteq R^*$

$\cap T$ is the intersection ~~of~~ of all transitive relations on A that contain R.
R^* is one of them.
So by definition if $C = A \cap B$
then $C \subseteq A$
$\Rightarrow \cap T \subseteq R^*$.

Problem 3:

a) We will need to prove this in two parts.

Part I: Given $(w, x) \in R^* \Rightarrow \Psi_x = \Psi_w$

$\Rightarrow$ either $w = x$ which means $\Psi_x = \Psi_w$ Proved
or $w = uabv$ and $x = ubav$.

Since Parikh vector is count of occurrences.
if $c = ab$ then $\Psi_c = \Psi_a + \Psi_b$.

Thus $w = u\,ab\,v \Rightarrow \psi_w = \psi_u + \psi_a + \psi_b + \psi_v$

$x = u\,ba\,v \Rightarrow \psi_x = \psi_u + \psi_b + \psi_a + \psi_v$.

$\Rightarrow \psi_w = \psi_x$ Hence proved.

Part II: Given $\psi_x = \psi_w$.

Because w and x have same occurences of $a_1, a_2, \ldots a_k$ they are a permutation of same letters appearing same no. of times.

Let y be a sorted element on Σ^* s.t $\psi_y = \psi_x = \psi_w$.
by sorted we mean lexicographic order

We will check first two elements of x, If the are sorted we will move to 2nd & 3rd element. if they are also sorted we will move ahead. If not, we will swap them. Thus process will be ~~read~~ repeated multifold to get intermediate $x_1, x_2, \ldots x_n$. which only one adjacent element swapped bet x_k, x_{k+1}, $k \in \{1, 2, \ldots n\}$
Hence $\psi_{x_k} = \psi_{x_{k+1}}$, $k \in \{1, 2, \ldots n-1\}$

Since x_n is lexicographically sorted, $x_n = y$.

We can follow same process for w & bring it to y through $w_1, w_2, \ldots, w_m$ intermediate strings
Then we can transform x to w through
$x \to x_1 \to x_2 \to \ldots \to x_n = y = w_m \to w_{m-1} \to w_{m-2}$
$\ldots \to w_2 \to w_1 \to w$
Since each of these involves one adjacent ~~element~~ element swap all their Parikh vectors are same

And $(x, x_1) \in R^*$
and $(x_1, x_2) \in R^*$
$\vdots$
$(w_i, w) \in R^*$

$\Rightarrow (x, w) \in R^*$ by transitivity Rule applied $n+m$ times.

Thus $(x, w) \in R^*$.
$\Rightarrow (w, x) \in R^*$ by taking exact opposite path.