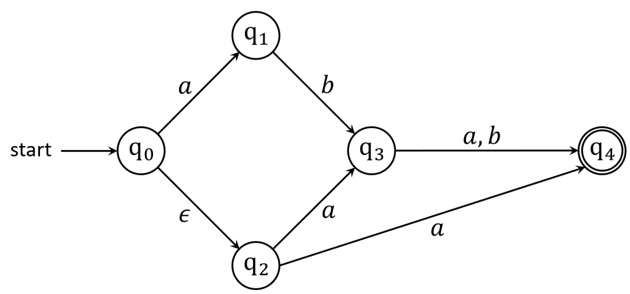
Homework #2

(Due: Oct 5)

Task 1. [40 Points] NFA to DFA

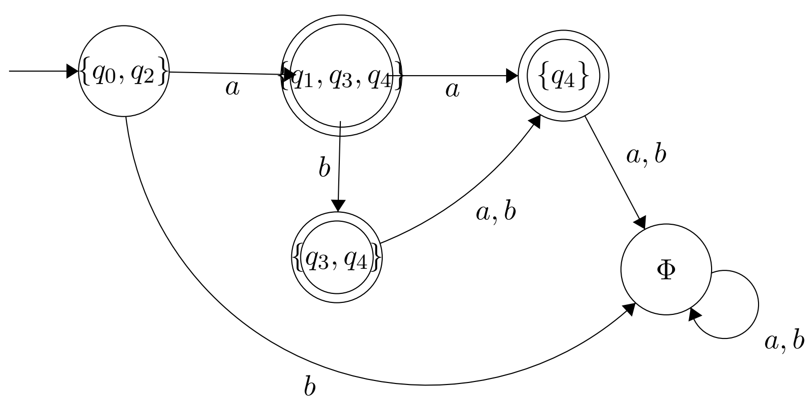
Convert each of the following NFAs to an equivalent DFA.

(a) [**10 Points**] Alphabet, $\Sigma = \{a, b\}$.



Solution:

DFA Diagram:

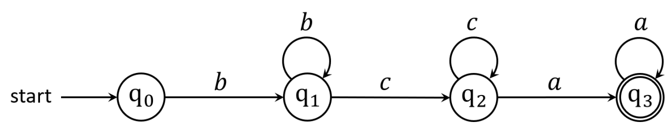


5-Tuple: $M = (Q, \Sigma, \delta, q_0, F)$, where,

Set of states is	$Q = \{\{q_0, q_2\}, \{q_1, q_3, q_4\}, \{q_4\}, \{q_3, q_4\}, \phi\}$
Set of symbols is	$\Sigma = \{a, b\}$
Start state is	$q_0 = \{q_0, q_2\}$
Set of accept states is	$F = \{\{q_1, q_3, q_4\}, \{q_3, q_4\}\}$
Transition function is	

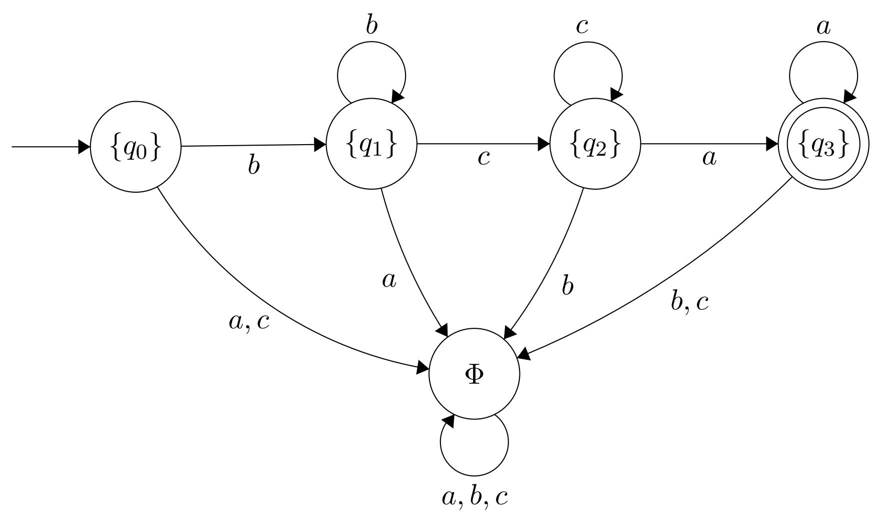
		a	b
δ:	$\{q_0, q_2\}$	$\{q_1, q_3, q_4\}$	ϕ
	$\{q_1, q_3, q_4\}$	$\{q_4\}$	$\{q_3, q_4\}$
	$\{q_4\}$	ϕ	ϕ
	$\{q_3, q_4\}$	$\{q_4\}$	$\{q_4\}$
	ϕ	ϕ	ϕ

(b) [**10 Points**] Alphabet, $\Sigma = \{a, b, c\}$.



Solution:

DFA Diagram:



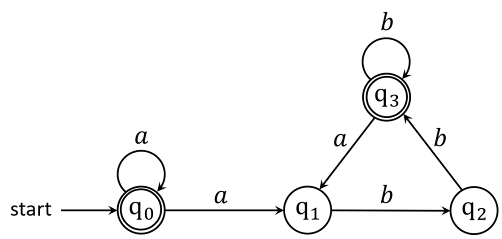
5-Tuple: $M = (Q, \Sigma, \delta, q_0, F)$, where,

Set of states is	$Q = \{\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \phi\}$
Set of symbols is	$\Sigma = \{a, b, c\}$
Start state is	$q_0 = \{q_0\}$
Set of accept states is	$F = \{\{q_3\}\}$
Transition function is	

δ:

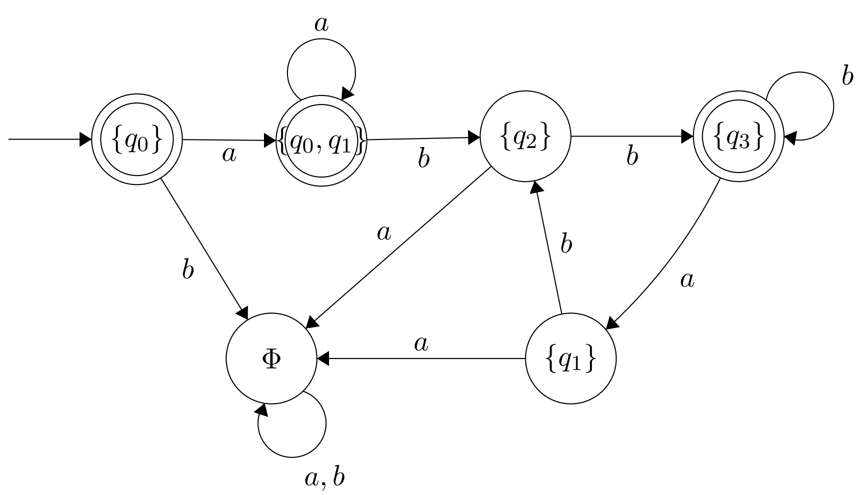
	a	b	c
$\{q_0\}$	ϕ	$\{q_1\}$	ϕ
$\{q_1\}$	ϕ	$\{q_1\}$	$\{q_2\}$
$\{q_2\}$	$\{q_3\}$	ϕ	$\{q_2\}$
$\{q_3\}$	$\{q_3\}$	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ

(c) [**10 Points**] Alphabet, $\Sigma = \{a, b\}$.



Solution:

DFA Diagram:

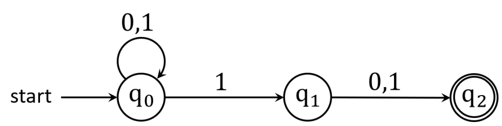


5-Tuple: $M = (Q, \Sigma, \delta, q_0, F)$, where,

Set of states is	$Q = \{\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \{q_0, q_1\}, \phi\}$
Set of symbols is	$\Sigma = \{a, b\}$
Start state is	$q_0 = \{q_0\}$
Set of accept states is	$F = \{\{q_0\}, \{q_0, q_1\}\}$
Transition function is	

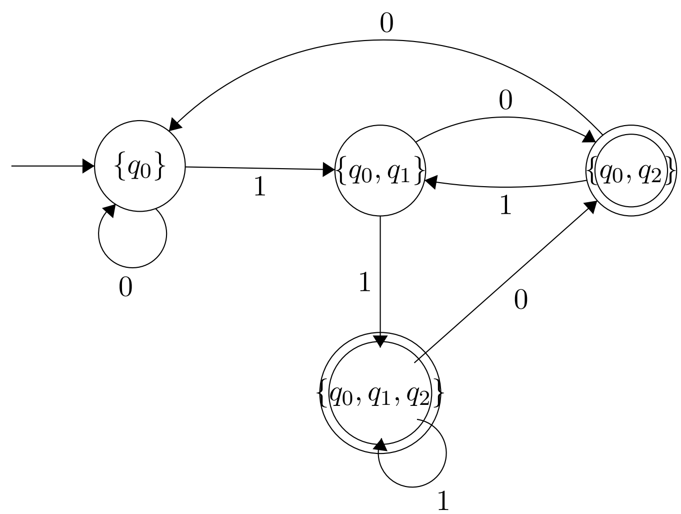
		a	b
	$\{q_0\}$	$\{q_0, q_1\}$	ϕ
	$\{q_1\}$	ϕ	$\{q_2\}$
δ:	$\{q_2\}$	ϕ	$\{q_3\}$
	$\{q_3\}$	$\{q_1\}$	$\{q_3\}$
	$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_2\}$
	ϕ	ϕ	ϕ

(d) [**10 Points**] Alphabet, $\Sigma = \{0, 1\}$.



Solution:

DFA Diagram:



5-Tuple: $M = (Q, \Sigma, \delta, q_0, F)$, where,

Set of states is	$Q = \{\{q_0\}, \{q_0, q_1\}, \{q_0, q_2\}, \{q_0, q_1, q_3\}\}$
Set of symbols is	$\Sigma = \{0, 1\}$
Start state is	$q_0 = \{q_0\}$
Set of accept states is	$F = \{\{q_0, q_2\}\ \{q_0, q_1, q_2\}\}$
Transition function is	

δ:

	0	1
$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
$\{q_0, q_1\}$	$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$
$\{q_0, q_2\}$	$\{q_0\}$	$\{q_0, q_1\}$
$\{q_0, q_1, q_2\}$	$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$

Task 2. [**20 Points**] **Regular expressions**

Find regular expressions to describe the following languages.

(a) [**5 Points**] $L = \{w|$ binary number w is divisible by 3$\}$, $\Sigma = \{0, 1\}$

Regular Expression: $(0 \cup 1(01^*0)^*1)^*$

(b) [**5 Points**] The language represented by the NFA in Task 1(c).

Regular Expression: $\epsilon \cup a^+ \cup a^+bb^+(abb^+)^*$

(c) [**5 Points**] The language represented by the NFA in Task 1(a).

Regular Expression: $a(\epsilon \cup a \cup b(a \cup b \cup \epsilon))$

(d) [**5 Points**] $L = \{w|$ length of w is not divisible by 6$\}$, $\Sigma = \{a, b\}$

Regular Expression: $((a \cup b)^6)^*(a \cup b)(a \cup b \cup \epsilon)^4$

Task 3. [**30 Points**] **Non-regular languages**

Use the pumping lemma to show that the following languages are not regular.

(a) [**10 Points**] $L = \{a^m b^n c^{m+n}|\ m, n \geq 0\}$, $\Sigma = \{a, b, c\}$

Solution:
_ Assume L is regular. Then it must satisfy the pumping property.
_ Let $s =$ number of states
_ Let $w = a^s b^s c^{2s}$
_ Let $w = xyz$, $x = a^p$, $y = a^q$, $z = a^{s-p-q} b^s c^{2s}$
_ where $|xy| \leq s$ and $|y| \geq 1$
_ Then all xy^iz should belong to L.
_ However, $xyyz = a^{s+q} b^s c^{2s}$, which is not in L, as $s + q + s \neq 2s$.
_ This is a contradiction to our assumption! Hence, L is not regular.

(b) [**10 Points**] $L = \{a^f|\ f$ is a Fibonacci number$\}$, $\Sigma = \{a\}$

Solution:
_ Assume L is regular. Then it must satisfy the pumping property.
_ Let $s =$ number of states
_ Let $w = a^{f_n}$, where f_n is the n^{th} Fibonacci number for some $n \geq 0$ and $f_n \geq \max\{3, 2s\}$.
_ We know that for all $f_n \geq 3$ the following holds: $\frac{f_{n+1}}{f_n} > 1.6$
(indeed, $\frac{f_{n+1}}{f_n}$ approaches $\frac{\sqrt{5}+1}{2} = 1.61803398874...$ (*Golden ratio*) as n approaches ∞)
_ Now let $w = xyz$, $x = a^p$, $y = a^q$, $z = a^r a^{f_n - s}$, where $p + q + r = s$, $p + q \leq s$ and $q \geq 1$

- Then xy^iz should belong to L for all integer $i \geq 0$.
- However, $xyyz = a^p a^q a^q a^r a^{f_n - s} = a^{f_n + q}$ does not belong to L. This is because:

$$\frac{f_n + q}{f_n} = 1 + \frac{q}{f_n} \leq 1 + \frac{s}{2s} = 1.5 < 1.6 < \frac{f_{n+1}}{f_n},$$

which means that $f_n + q < f_{n+1}$ and thus is not a Fibonacci number.
- This is a contradiction to our assumption that L is regular! Hence, L is not regular.

(c) [**10 Points**] $L = \{a^m b^{n^3} |\ m, n \geq 0\}$, $\Sigma = \{a, b\}$

Solution:
- Assume L is regular. Then it must satisfy the pumping property.
- Let s = number of states
- Let $w = a^0 b^{s^3}$.
- Let $w = xyz$, $x = b^p$, $y = b^q$, $z = b^r b^{s^3 - s}$, where $p + q + r = s$, $p + q \leq s$, and $q \geq 1$.
- Then xy^iz should belong to L for all integer $i \geq 0$.
- However, $xyyz = b^p b^q b^q b^r b^{s^3 - s} = a^0 b^{s^3 + q}$, which is not in L because:

$$\begin{aligned} & 1 \leq q \leq s \\ \Longrightarrow\ & 0 < q < 3s^2 + 3s + 1 \\ \Longrightarrow\ & s^3 < s^3 + q < s^3 + 3s^2 + 3s + 1 \\ \Longrightarrow\ & s^3 < s^3 + q < (s+1)^3 \end{aligned}$$

Therefore, $s^3 + q$ is between two consecutive perfect cube numbers. This means that $(s^3 + q)$ is not a perfect cube. Hence $xyyz = a^0 b^{s^3 + q}$ is not in L.
This is a contradiction to our assumption that L is regular! Hence, L is not regular.

Task 4. [**10 Points**] **More non-regular languages**

Prove that the following languages are not regular. You are not required to use the pumping lemma but you can build on results we have already proved in the class.

(a) [**5 Points**] $L = \{a^m |\ m \text{ is a composite number}\}$, $\Sigma = \{a\}$

Solution:
- Assume L is regular.
- We also know that $L_1 = \{a^m |\ m \geq 2\}$ is regular.
- Let $L_2 = \{a^m |\ m \text{ is a prime number}\}$
- As regular languages are closed under intersection and complementation, $L_2 = L_1 - L = L_1 \cap \overline{L}$ is a regular language.
- But, the language L_2 was earlier proved to be non-regular.
- This is a contradiction from our assumption! Hence L is not regular.

(*b*) [**5 Points**] $L = \{a^m |\ m \text{ is neither a prime nor divisible by } 3\}$, $\Sigma = \{a\}$

Solution:
_ Assume L is regular.
_ We also know that $L_1 = \{a^m |\ m \text{ is divisible by } 3\}$ is regular. Because we can draw a DFA for it.
_ Let $L_2 = \{a^m |\ m \text{ is not a prime number}\} = \{a^m |\ m \text{ is a composite number}\} \cup \{a\}$
_ As regular languages are closed under Union, $L_3 = L \cup L_1$ is a regular language.
_ And we know that $L_3 = (L_2 \cup \{a^3\})$
_ And because $\{a, a^3\}$ is a regular language, $L_4 = L_3 - \{a, a^3\}$ is a regular language
_ But, the language $L_4 = \{a^m |\ m \text{ is a composite number}\}$ was earlier (in 4_a) proved to be non-regular.
_ This is a contradiction from our assumption! Hence L is not regular.