Problem

Let S(n) = 1 + 2 + · · · + n be the sum of the first n natural numbers and let C(n) = $1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that C(n) = $S^2(n)$ for every n.

a. $S(n) = \frac{1}{2}n(n+1)$.

b. $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$.

Step-by-step solution

Step 1 of 3

a) The sum of the first n natural numbers $(S_n = 1+2+3+...+n)$ is given by:

$$S_n = \frac{1}{2}n(n+1).$$

Here, induction method is used to prove the above equality. Let's write $S_n = 1+2+3+...+n$ in shorter form like:

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1).$$

• For n=1, it is true that

$$1 = \frac{1}{2}1(1+1).$$

• For n=2, it is true that

$$1+2 = \frac{1}{2}2(2+1).$$

• In the same way it is true for n

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1).$$

So, finally it has to prove that for n+1 or $S_{n+1} = \sum_{i=1}^{n+1} i = \frac{1}{2}(n+1)(n+2)$.

• If n+1 is added to each side of the next identity:

$$\begin{aligned} S_{n+1} &= 1+2+...+n+n+1 \\ &= S_n + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

• Which is equivalent to: $\sum_{i=1}^{n+1} i = \frac{1}{2}n(n+1) + \frac{2(n+1)}{2}$ or $S_{n+1} = \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

So, the above equality proves that the given equality for the sum of n natural or $S_n = \frac{1}{2}n(n+1)$ is also true for $(n+1)$. Hence the given equality is correct.

Comment

Step 2 of 3

b) The sum of the cube of the first n natural numbers or $C_n = 1^3 + 2^3 + ... + n^3$ is given by:

$$C_n = \frac{1}{4}\left(n^4 + 2n^3 + n^2\right) = \frac{1}{4}n^2\left(n+1\right)^2$$

The above equality can be proved by induction method. Let's write $C_n = \frac{1}{4}n^2\left(n+1\right)^2$ in shorter form like:

$$\sum_{i=1}^{k=n} i^3 = 1^3 + 2^3 + ... + n^3 = \frac{1}{4}n^2\left(n+1\right)^2$$

• For n=1, it is true that:

$1^3 = \frac{1}{4}1^2\left(1+1\right)^2$ or 1=1.

• In the same way it is true for n:

$$\sum_{i=1}^{k=n} i^3 = 1^3 + 2^3 + ... + n^3 = \frac{1}{4}n^2\left(n+1\right)^2.$$

So, finally it has to prove that for n+1 or $C_n = 1^3 + 2^3 + ... + n^3 + \left(n+1\right)^3 = \frac{\left(n+1\right)^2\left(n+2\right)^2}{4}$

$$\begin{aligned} C_{n+1} &= 1^3 + 2^3 + 3^3 + ... + n^3 + (n+1)^3 \\ &= \frac{1}{4}n^2(n+1)^2 + (n+1)^3 \\ &= (n+1)^2\left(\frac{n^2 + 4n + 4}{4}\right) \\ &= \frac{(n+1)^2(n+2)^2}{4} \end{aligned}$$

Therefore,

$$C_{n+1} = \frac{\left(n+1\right)^2\left(n+2\right)^2}{4}$$

So, the above equality proves that the given equality for the sum of the cube of n natural numbers is also true for n+1. Hence, the given equality is correct.

Comment

Step 3 of 3

From the above explanation:

$$C_n = \frac{1}{4}\left(n^4 + 2n^3 + n^2\right) = \frac{1}{4}n^2\left(n+1\right)^2 = \left[\frac{n\left(n+1\right)}{2}\right]^2 = \left(S_n\right)^2$$

$$C_n = \left(S_n\right)^2.$$

It is concluded that "the sum of the cube of the first n natural number is equal to the square of the sum of first n natural number".

Comment