Problem

Let $\Sigma = \{0,1\}$. Show that the problem of determining whether a CFG generates some string in 1^* is decidable. In other words, show that

$$\{\langle G\rangle \mid G \text{ is a CFG over } \{0,1\} \text{ and } 1^* \cap L(G) \neq \emptyset\}$$

is a decidable language.

Step-by-step solution

Step 1 of 3

• Given a CFG over $(0,1)$ that generates **some** string in 1^*. The language generated by this CFG, denoted by Language A can be rephrased in the following way: $A=\{<G>|G \text{ is a CFG over } (0,1)^* \text{ and } 1^* \cap L(G) \neq \varnothing\}$.

• **To test decidability, the fact that intersection of a Regular Language (RL) & Context Free Language (CFL) is a CFL shall be harnessed.**

Comment

Step 2 of 3

• Note that 1^* is a Regular Language & $L(G)$ is a CFL (since G is a CFG), therefore, $1^* \cap L(G)$ is clearly a CFL.

• A Turing Machine (TM) has the task to test whether the problem at hand is decidable or undecidable.

• If for an input, TM culminates in either ACCEPT or REJECT state, the problem is referred to as **decidable**.

Comment

Step 3 of 3

Let H be the TM that decides A. Using the Theorem 4.8, the decidability is employed by the following algorithm:

H="on input <G>, where G is a CFG":

Construct a CFG B, such that $L(B)=1^* \cap L(G)$ (remember that $1^* \cap L(G)$ is CFL so the statement is valid).

1. Run the TM R that decides E_{CFG} on $<B>$. It may be elaborated here that E_{CFG} denotes the problem of determining whether a CFG (here B) generates any strings at all is decidable. Formally,

$$E_{CFG}=\{<B>|\, B \text{ is a CFG and } L(B)=\varnothing\}$$

2. **If R accepts, reject**. It means that the language generated by the CFG B is empty. Therefore, TM H culminates in REJECT state.

3. **If R rejects, accept**. In other words, the problem at hand is Decidable since language generated by the CFG *B* is NOT empty.

Elaboration: If TM R accepts, it would imply that,

$$\begin{aligned} L(B) &= 1^* \cap L(G) \\ &= \varnothing \end{aligned}$$

That, is for some string $w \in 1^*$, $w \notin L(G)$, hence R should reject.

Comment