CSE303: Theory of Computation, Fall 2021 **Date:** Oct 7, 2021

Midterm Exam (Solutions)

(9:45 AM – 11:00 AM : 75 Minutes)

- This exam will account for 30% of your overall grade.
- There are six (6) questions, worth 75 points in total. Please answer all of them.
- This is a *closed book*, *closed notes* exam. *No cheat sheets* are allowed.
- You are allowed to *use scratch papers* for your calculations.
- You are *not allowed to use your own calculator*. A scientific calculator will be available inside the Respondus Lockdown Browser.

Good Luck!

Question	Parts	Points
1. DFA Construction	(i), (ii)	$10 + 10 = 20$
2. DFA Composition	–	10
3. Regular Expressions	(i), (ii)	$5 + 5 = 10$
4. NFA to DFA	–	15
5. Non-regularity	–	15
6. Context-free Grammar	–	5
Total		75

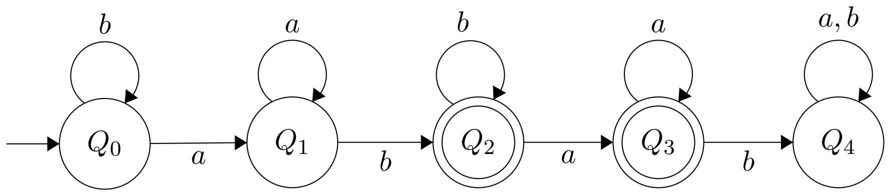
Question 1. [**20 Points**] **DFA Construction.** Write down a DFA in the 5-tuple form to accept each of the following two regular languages.

Assume that $\Sigma = \{a, b\}$.

Your answers do not need to include DFA diagrams (though you may draw them on your scratch papers if you like).

1. [**10 Points**] $L = \{w|\ ab$ **appears exactly once in** $w\}$

   Solution:



5-Tuple: $M = (Q, \Sigma, \delta, q_0, F)$, where,

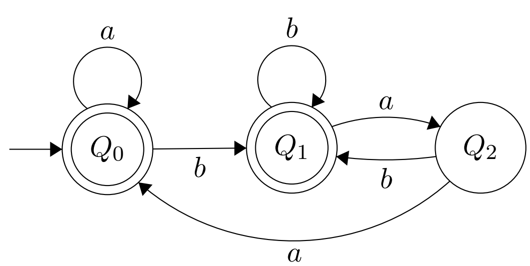
Set of states is	$Q = \{Q_0, Q_1, Q_2, Q_3, Q_4\}$
Set of symbols is	$\Sigma = \{a, b\}$
Start state is	$q_0 = Q_0$
Set of accept states is	$F = \{Q_2, Q_3\}$
Transition function is	

δ:

	a	b
Q_0	Q_1	Q_0
Q_1	Q_1	Q_2
Q_2	Q_3	Q_2
Q_3	Q_3	Q_4
Q_4	Q_4	Q_4

2. [**10 Points**] $L = \{w|\ w$ **does not end with** $ba\}$

   Solution:

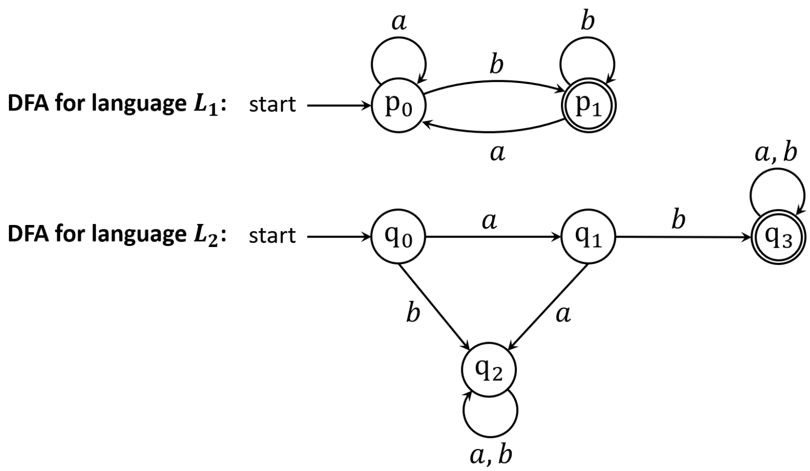


5-Tuple: $M = (Q, \Sigma, \delta, q_0, F)$, where,

$$\begin{aligned} \text{Set of states is} \qquad & Q = \{Q_0, Q_1, Q_2\} \\ \text{Set of symbols is} \qquad & \Sigma = \{a, b\} \\ \text{Start state is} \qquad & q_0 = Q_0 \\ \text{Set of accept states is} \qquad & F = \{Q_0, Q_1\} \\ \text{Transition function is} \qquad & \end{aligned}$$

$$\delta: \begin{array}{c|c|c} & a & b \\ \hline Q_0 & Q_0 & Q_1 \\ Q_1 & Q_2 & Q_1 \\ Q_2 & Q_0 & Q_1 \end{array}$$

Question 2. [**10 Points**] **DFA Composition.** Consider the following two DFAs.



Write down a DFA in the 5-tuple form that accepts the language $L_1 \cap L_2$.

Your answer does not need to include the DFA diagram (though you may draw it on your scratch papers if you like).

Solution:

5-Tuple: $M = (Q, \Sigma, \delta, q_0, F)$, where,

Set of states is $Q = \{p_0q_0, p_0q_1, p_0q_2, p_0q_3, p_1q_0, p_1q_1, p_1q_2, p_1q_3\}$

Set of symbols is $\Sigma = \{a, b\}$

Start state is $q_0 = p_0q_0$

Set of accept states is $F = \{p_1q_3\}$

Transition function is

δ:

	a	b
p_0q_0	p_0q_1	p_1q_2
p_0q_1	p_0q_2	p_1q_3
p_0q_2	p_0q_2	p_1q_2
p_0q_3	p_0q_3	p_1q_3
p_1q_0	p_0q_1	p_1q_2
p_1q_1	p_0q_2	p_1q_3
p_1q_2	p_0q_2	p_1q_2
p_1q_3	p_0q_3	p_1q_3

Question 3. [10 Points] Regular Expressions. Answer the following questions.

1. [**5 Points**] Write down a regular expression for the following language.

$$L = \{w| \textbf{ every } a \textbf{ in } w \textbf{ is followed by an even number of } b\textbf{'s}\}, \quad \Sigma = \{a, b, c\}$$

Solution:

$$(b \cup (a(bb)^+)^*c)^*(a(bb)^+)^* \qquad \textbf{assuming 0 is not an even number}$$

$$(b \cup (a(bb)^*)^*c)^*(a(bb)^*)^* \qquad \textbf{assuming 0 is an even number}$$

Some other possible answers (assuming 0 is an even number) :

$$b^*((c^+b^*)^*(a(bb)^*)^*)^*$$

$$b^*(a(bb)^* \cup c^+b^*)^*$$

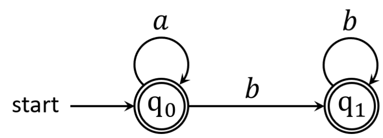
$$(b \cup c)^*(a(bb)^*(c^+b^*)^*)^*$$

$$(b \cup c)^*(a(bb)^* \cup cb^*)^*$$

2. **[5 Points] For each of the following pairs of regular expressions write down 'True' if the pair represent the same language and 'false' otherwise. Proofs are not needed.**

   (a) $(a \cup b)^*$ **and** $(a^*b^*)^*$ — **Solution: True**
   (b) $(a \cup b)^+$ **and** $(a^+b^+)^+$ — **Solution: False**
   (c) $a^*(ab^*)^*$ **and** $a^*(b^*ab^*)^*$ — **Solution: False**
   (d) $b^*(ab^*)^*$ **and** $b^*(b^*ab^*)^*$ — **Solution: True**
   (e) $b^*(ab^+)^*$ **and** $(b \cup ab)^+$ — **Solution: False**

Question 4. [**15 Points**] **NFA to DFA.** Consider the following NFA.



Convert this NFA into a DFA and write it down in the 5-tuple form.

Your answer does not need to include the DFA diagram (though you may draw it on your scratch papers if you like).

Solution:

5-Tuple: $M = (Q, \Sigma, \delta, q_0, F)$, where,

Set of states is	$Q = \{\{q_0\}, \{q_1\}, \phi\}$
Set of symbols is	$\Sigma = \{a, b\}$
Start state is	$q_0 = \{q_0\}$
Set of accept states is	$F = \{\{q_0\}, \{q_1\}\}$
Transition function is	

$$\delta: \begin{array}{c|c|c} & a & b \\ \hline \{q_0\} & \{q_0\} & \{q_1\} \\ \{q_1\} & \phi & \{q_1\} \\ \phi & \phi & \phi \end{array}$$

Question 5. [15 Points] Non-regularity. Use the pumping lemma to prove that the following language is not regular.

$$L = \{w|\ w = a^m b^{2^n},\ m, n \geq 0\}, \quad \Sigma = \{a, b\}$$

Solution:

- Assume L is regular. Then it must satisfy the pumping property.
- Let $s =$ number of states
- Let $w = a^0 b^{2^s}$.
- Let $w = xyz$, $x = b^p$, $y = b^q$, $z = b^r b^{2^s - s}$, where $p + q + r = s$, $p + q \leq s$, and $q \geq 1$ so, $|xy| \leq s$ and $|y| \geq 1$
- Then $xy^i z$ should belong to L for all integer $i \geq 0$.
- However, $xz = b^p b^r b^{2^s - s} = a^0 b^{2^s - q}$, which is not in L because:

$$\begin{aligned}
& s < 2^{s-1} \qquad where:\ s > 2 \\
\Longrightarrow\ & \frac{s}{2^{s-1}} < 1 \\
\Longrightarrow\ & 0 < 1 - \frac{s}{2^{s-1}} \\
\Longrightarrow\ & 1 < 2 - \frac{s}{2^{s-1}} \\
\Longrightarrow\ & 2^{s-1} < 2^s - s \\
\Longrightarrow\ & 2^{s-1} < 2^s - q \\
\Longrightarrow\ & 2^{s-1} < 2^s - q < 2^s
\end{aligned}$$

Therefore, $2^s - q$ is between two consecutive powers of 2. This means that $(2^s - q)$ is not a power of 2. Hence $xz = a^0 b^{2^s - q}$ is not in L.
This is a contradiction to our assumption that L is regular! Hence, L is not regular.

Question 6. [5 Points] Context-free Grammar. Write down a context-free grammar to accept the following language:

$$L = \{w|\ w = a^m b^{m+n+1} c^n,\ m, n \geq 0\},\quad \Sigma = \{a, b, c\}$$

Solution:

$G = (V, \Sigma, R, S)$ *where* :

The set of variables is $V = \{S, A, C\}$

The set of terminals is $\Sigma = \{a, b, c\}$

The set of rules is $R =$

$$S \rightarrow AbC$$
$$A \rightarrow aAb \mid \epsilon$$
$$C \rightarrow bCc \mid \epsilon$$

The start variable is S