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High Performance Computing 2

SoSe 2017

Sheet 2

Introduction

In todays exercise, we want to consider the implementation of the finite element method. For this, we want to consider the presentation in *Remarks around 50 lines of Matlab: short finite element implementation* by C. Carstensen and S. Funken, which can be downloaded from the homepage. Moreover, the implementation of the FEM can be found in the material.

Exercise

1) Download the material from the homepage.

2) Read and try to understand the above mentioned paper. Try to understand, how the local stiffness matrix is assembled. How is the right-hand-side assembled?

3) Try to run and understand the code.

4) Try to solve the poisson problem from the introduction on Exercise Sheet 1 (see last week). Compare it to your FDM solution.

5) Compute the largest and smallest eigenavalues of the stiffness matrix. What do you observe?

6) Try to solve the same poisson problem now on the L-shape. What do you observe? Try to vary Neumann and Dirichlet boundary conditions in a meaningful manner.

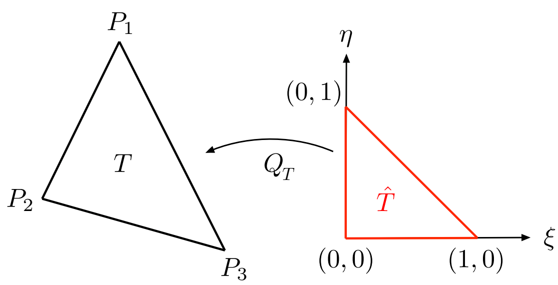
7) Show, that for an equidistant grid, the stiffness matrix from the FEM discretization is equal to that, from the FDM discretization.

8) When working with FEM, the domain $\Omega \subset \mathbb{R}^2$ is usually partitioned into triangles $\mathcal{T}_h$. For

$$T := \mathrm{conv}\{P_1, P_2, P_3\} \in \mathcal{T}_h,$$

we define the affine mapping Q_T, which maps the reference element $\hat{T}$ onto T by

$$Q_T : \hat{T} \to T$$



with $Q_T(0,0) = P_1$, $Q_T(1,0) = P_2$ und $Q_T(0,1) = P_3$ (the mapping Q_T is well defined and unique).
Further, the basis functions $\hat{\varphi}_i$ on the reference element $\hat{T}$ are given as

$$\hat{\varphi}_1(\xi,\eta) := 1 - \xi - \eta$$
$$\hat{\varphi}_2(\xi,\eta) := \xi$$
$$\hat{\varphi}_3(\xi,\eta) := \eta$$

and the basis functions φ_i on the triangle T are given by $\varphi_i := (\hat{\varphi}_i \circ Q_T^{-1})$ $(i = 1, 2, 3)$. It holds $\varphi_i(P_j) = \delta_{ij}$. Compute the entries for $i, j = 1, 2, 3$:

$$(i) \quad \int_T \varphi_i(x,y)\varphi_j(x,y)\, d(x,y)$$

$$(ii) \quad \int_T \nabla\varphi_i(x,y)^T \nabla\varphi_j(x,y)\, d(x,y)$$

$$(iii) \quad \int_T \frac{\partial}{\partial x}\varphi_i(x,y) d(x,y) \quad \text{und} \quad \int_T \frac{\partial}{\partial y}\varphi_i(x,y) d(x,y)$$

$$(iv) \quad \int_T \varphi_i(x,y) d(x,y)$$

For this exercise you can compute the integrals directly or use the transformation onto the reference element.

Hints

For computational reasons, it is helpful to transform any arbitrary triangle $T \in \mathcal{T}$ with nodes $P_1 = (x_1, y_1)^T$, $P_2 = (x_2, y_2)^T$ and $P_3 = (x_3, y_3)^T$ onto a reference element $\hat{T}$, where

$$\hat{T} := \{(\xi, \eta)^T \in \mathbb{R}^2 : \xi, \eta > 0,\ 0 < \xi + \eta < 1\}.$$

There exists an affine transformation $Q_T : \hat{T} \to T$ with

$$Q_T(\xi, \eta) = b + B\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \hat{T}, \qquad Q_T(\hat{T}) = T.$$

The matrix $B \in \mathbb{R}^{2\times 2}$ and the vector $b \in \mathbb{R}^2$ are given by

$$b = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad B = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}.$$

The inverse matrix B^{-1} is the inverse mapping $Q_T^{-1} : T \to \hat{T}$ with

$$Q_T^{-1}(x, y) = B^{-1}\begin{pmatrix} x \\ y \end{pmatrix} - B^{-1}b, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \in T, \qquad Q_T^{-1}(T) = \hat{T}$$

and it holds

$$B^{-1} = \frac{1}{\det(B)}\begin{pmatrix} y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{pmatrix} \quad \text{with} \quad \det(B) = x_1y_2 - x_1y_3 + x_2y_3 - x_2y_1 + x_3y_1 - x_3y_2.$$

More precise

$$Q_T^{-1}(x, y) = \frac{1}{\det(B)}\begin{pmatrix} (y_3 - y_1)x + (x_1 - x_3)y - x_1y_3 + x_3y_1 \\ (y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1 \end{pmatrix}.$$

For the basis functions φ_i, $1 \le i \le 3$ on T we receive

$$\varphi_1(x, y) = \left(\hat{\varphi}_1 \circ Q_T^{-1}\right)(x, y) = \frac{(y_2 - y_3)x - (x_2 - x_3)y + x_2y_3 - x_3y_2}{\det(B)},$$

$$\varphi_2(x, y) = \left(\hat{\varphi}_2 \circ Q_T^{-1}\right)(x, y) = \frac{(y_3 - y_1)x + (x_1 - x_3)y - x_1y_3 + x_3y_1}{\det(B)},$$

$$\varphi_3(x, y) = \left(\hat{\varphi}_3 \circ Q_T^{-1}\right)(x, y) = \frac{(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1}{\det(B)}.$$

and analogously for the gradient

$$\nabla\varphi_1(x,y) = \frac{1}{\det(B)}\begin{pmatrix} y_2 - y_3 \\ x_3 - x_2 \end{pmatrix},$$
$$\nabla\varphi_2(x,y) = \frac{1}{\det(B)}\begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix},$$
$$\nabla\varphi_3(x,y) = \frac{1}{\det(B)}\begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix},$$

We will compute now examplary Exercise 8 (i). (Rest is then shown analogously).

(i) With the help of the transformation we receive for $1 \leq i, j \leq 3$:

$$\int_T \varphi_i(x,y)\varphi_j(x,y)\, d(x,y) = \int_{\hat{T}} \hat{\varphi}_i(\xi,\eta)\hat{\varphi}_j(\xi,\eta) \cdot |\det(B)|\, d(\xi,\eta)$$

Due to $|\det(B)| = 2 \cdot |T|$ it holds:

$$\int_T \varphi_i(x,y)\varphi_j(x,y)\, d(x,y) = 2|T| \cdot \int_{\hat{T}} \hat{\varphi}_i(\xi,\eta)\hat{\varphi}_j(\xi,\eta)\, d(\xi,\eta) = \begin{cases} \frac{|T|}{6}, & \text{for } i = j \\ \frac{|T|}{12}, & \text{for } i \neq j \end{cases}.$$