

BENCHMARKS GUIDE

The Standard NAFEMS Benchmarks:
Linear Elastic Tests

Converged StressCheck® results are provided for the Linear Elastic Test benchmark models referenced in “The Standard NAFEMS Benchmarks”, Rev. 3, October 1990



STRESSCHECK® BENCHMARKS GUIDE

THE STANDARD NAFEMS BENCHMARKS - LINEAR ELASTIC TESTS

| | |
|---|----|
| About The StressCheck® Results..... | 3 |
| StressCheck® Minimum Mesh vs. Dense Mesh..... | 3 |
| StressCheck® Convergence Graphs | 3 |
| StressCheck® Results Tables..... | 3 |
| NAFEMS LE1: Plane Stress– Elliptic Membrane | 4 |
| Model Description..... | 4 |
| NAFEMS Reference Solution | 4 |
| StressCheck® Results..... | 5 |
| NAFEMS LE2: Cylindrical Shell Bending Patch Test..... | 8 |
| Model Description..... | 8 |
| NAFEMS Reference Solution | 8 |
| StressCheck® Results..... | 9 |
| NAFEMS LE3: Hemispherical shell with point loads | 12 |
| Model Description..... | 12 |
| NAFEMS Reference Solution | 12 |
| StressCheck® Results..... | 13 |
| NAFEMS LE5: Z-section Cantilever..... | 15 |
| Model Description..... | 15 |
| NAFEMS Reference Solution | 15 |
| StressCheck® Results..... | 16 |
| NAFEMS LE6: Skew Plate Under Normal Pressure | 19 |
| Model Description..... | 19 |
| NAFEMS Reference Solution | 19 |
| StressCheck® Results..... | 20 |
| NAFEMS LE7: Axisymmetric Cylinder/Sphere Under Pressure | 23 |
| Model Description..... | 23 |
| NAFEMS Reference Solution | 23 |
| StressCheck® Results..... | 24 |
| NAFEMS LE8: Axisymmetric Shell Under Pressure | 26 |
| Model Description..... | 26 |

| | |
|--|----|
| NAFEMS Reference Solution | 26 |
| StressCheck® Results..... | 27 |
| NAFEMS LE10: Thick Plate Under Pressure | 29 |
| Model Description..... | 29 |
| NAFEMS Reference Solution | 29 |
| StressCheck® Results..... | 30 |
| NAFEMS LE11: Solid Cylinder/Taper/Sphere – Temperature Loading | 32 |
| Model Description..... | 32 |
| NAFEMS Reference Solution | 32 |
| StressCheck® Results..... | 33 |

ABOUT THE STRESSCHECK® RESULTS

The following StressCheck® results were generated using StressCheck® Professional v10.3 (Build 40322). [Learn more about StressCheck's Numerical Simulation Technology](#), including how the degrees of freedom (DOF) are increased on a fixed mesh until convergence is achieved.

STRESSCHECK® MINIMUM MESH VS. DENSE MESH

The authors define “Minimum Mesh” as the least-refined mesh required to achieve numerical convergence within 1% of the target extraction, and “Dense Mesh” as an overly-refined mesh relative to the “Minimum Mesh” to demonstrate that adding more elements produced insignificant changes in the target extraction.

- The “Minimum Mesh” for each benchmark was determined by first generating the minimum number of elements (either manually or using our automatic mesh generator) required to fill the geometric domain. [We then increased the DOF on this fixed mesh by p-extension](#) and determined if convergence within 1% of the target extraction was achieved.
 - If not, the mesh was refined accordingly until convergence was achieved with the fewest possible elements. *Note: in most cases only one iteration of mesh refinement was needed.*
- The “Dense Mesh” is provided for comparative purposes to show that the approximation of the solution was already achieved for the Minimum Mesh within the required error tolerance.

STRESSCHECK® CONVERGENCE GRAPHS

Numerical convergence was evaluated independently of the NAFEMS reference solutions, hence StressCheck® results for each fixed mesh included graphical convergence evidence to automatically quantify the discretization error in the target extraction. *Note: graph numbering (e.g. Graph10) is inconsequential to this report.*

Converged StressCheck® results for the solution with highest DOF are extracted from each graph and reported in the StressCheck® results table. *Note: the estimated limit in each graph is for reference only and should not be reported for the target extraction.*

Remark: the discretization error was reported to be **< 1% for all StressCheck® results in all benchmark models.**

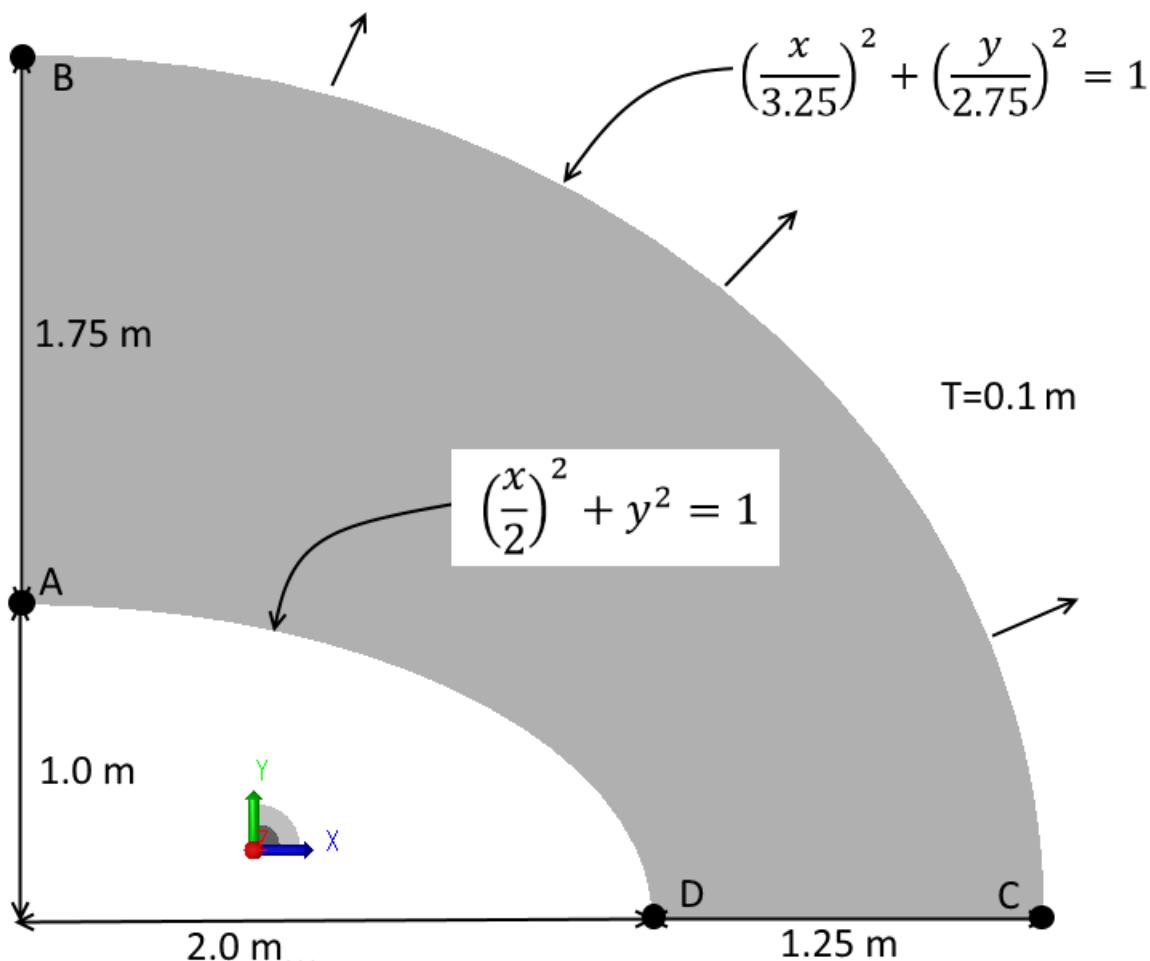
STRESSCHECK® RESULTS TABLES

Each StressCheck® result table row includes the following information:

- Element topology
- Number of elements for the Minimum Mesh of the element topology
- Converged StressCheck® result for the Minimum Mesh, with the relative % difference between the converged StressCheck® result and the NAFEMS reference benchmark solution.
- Number of elements for the Dense Mesh of the element topology
- Converged StressCheck® result for the Dense Mesh, with the relative % difference between the converged StressCheck® result and the NAFEMS reference benchmark solution.

Remark: the StressCheck® results and the NAFEMS reference benchmark solutions differed by **< 3% for all benchmarks.**

NAFEMS LE1: PLANE STRESS– ELLIPTIC MEMBRANE



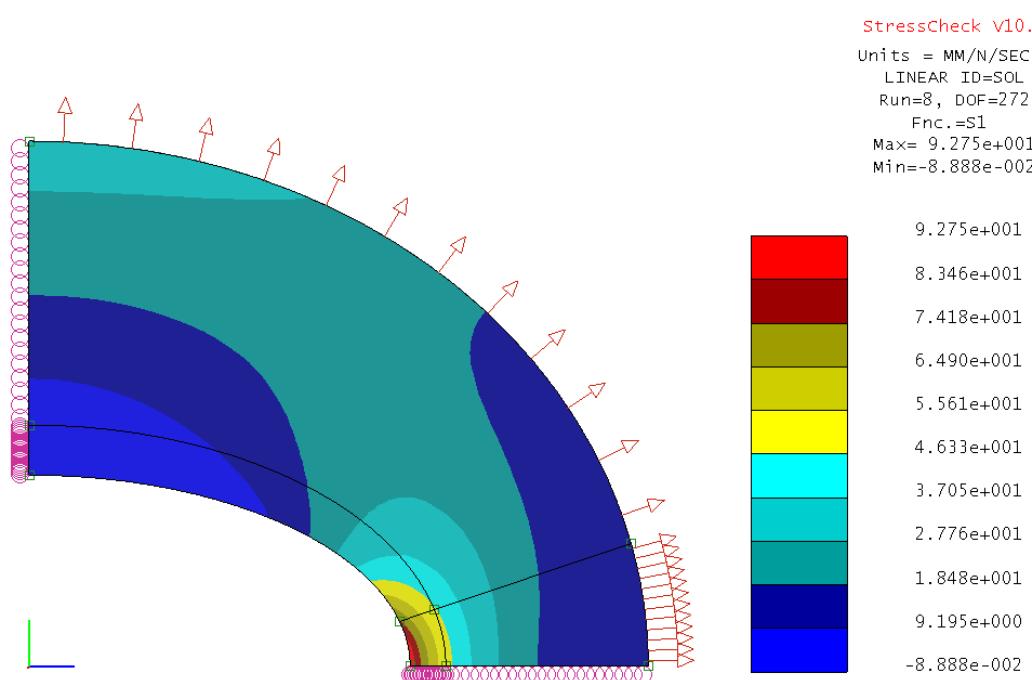
MODEL DESCRIPTION

- Plane stress problem with elliptic boundaries defined by ABCD.
 - Functions defining the shape of the ellipses BC and AD are given in the figure above.
- Linear elastic analysis, Young's modulus = 210 GPa, Poisson's ratio = 0.3.
- Symmetry boundary conditions along AB and DC.
- Uniform outward pressure of 10 MPa at outer edge BC.
- Objective of the analysis is to compute the tangential edge stress (σ_y) at point D.

NAFEMS REFERENCE SOLUTION

- Tangential edge stress at point D is **92.7 MPa**.

STRESSCHECK® RESULTS



| Element | Minimum Mesh | Solution | Dense Mesh | Solution |
|---------|--------------|----------------------|--------------|-----------------------|
| Quad | 4 elements | 92.75 MPa (0.05%) | 144 elements | 92.70 MPa (0.00%) |
| Tri | 6 elements | 92.84 MPa (0.15%) | 216 elements | 92.66 MPa (-0.04%) |

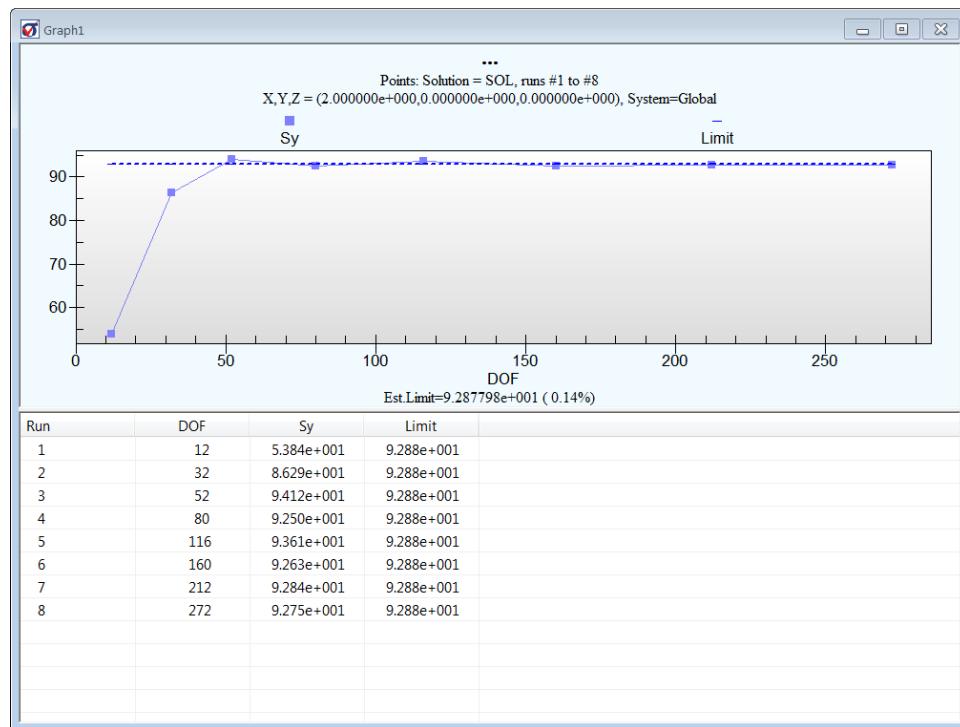


Figure 1: Convergence evidence for minimum quad mesh.

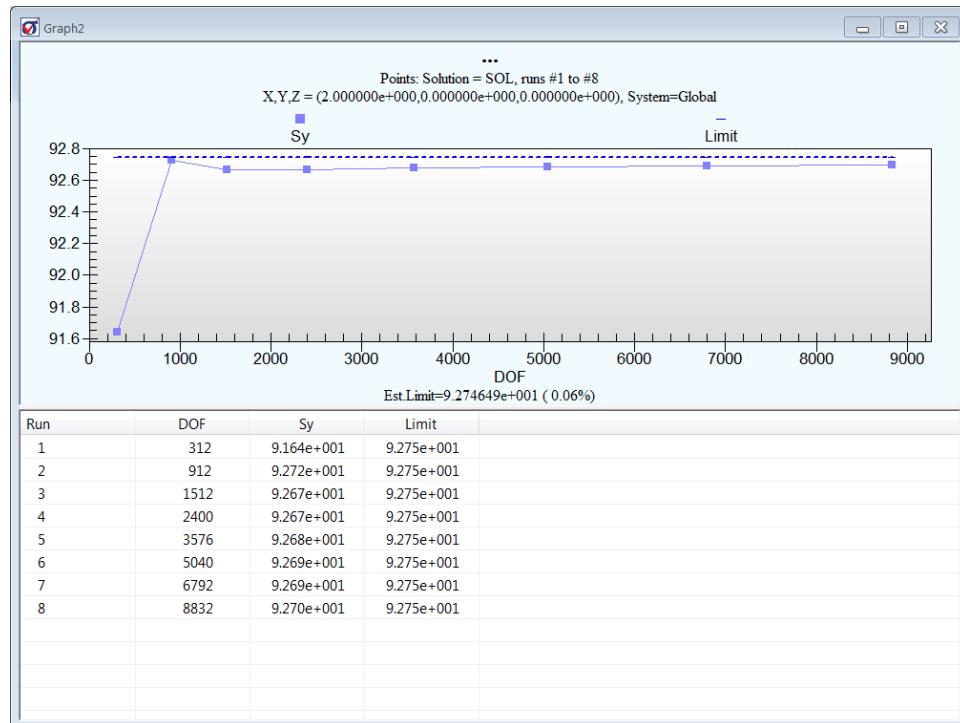


Figure 2. Convergence evidence for dense quad mesh.

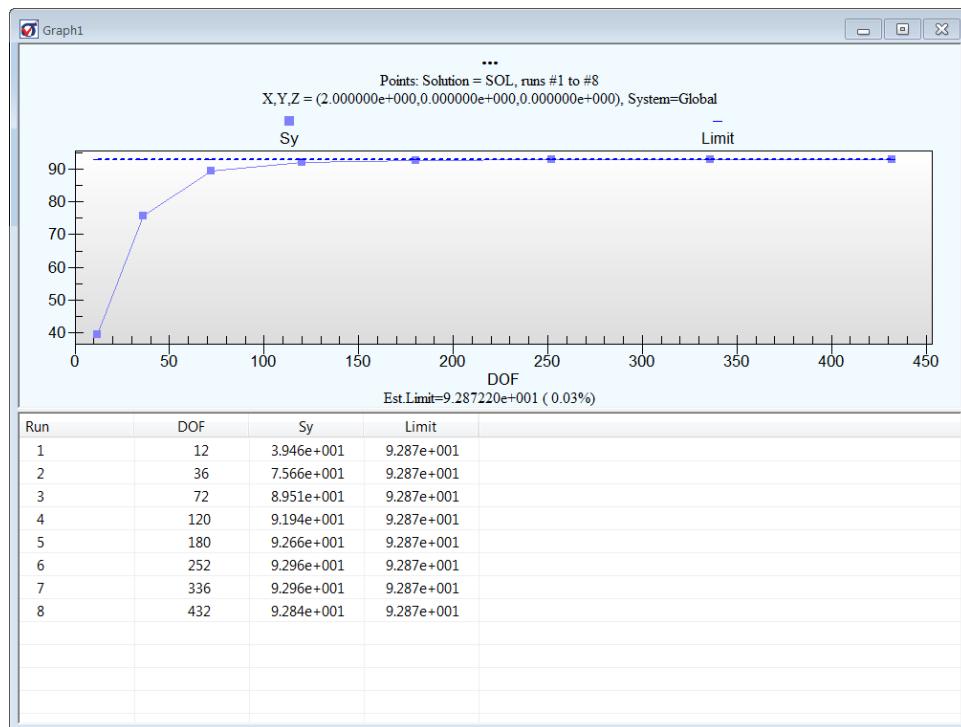


Figure 3. Convergence evidence for minimum tri mesh

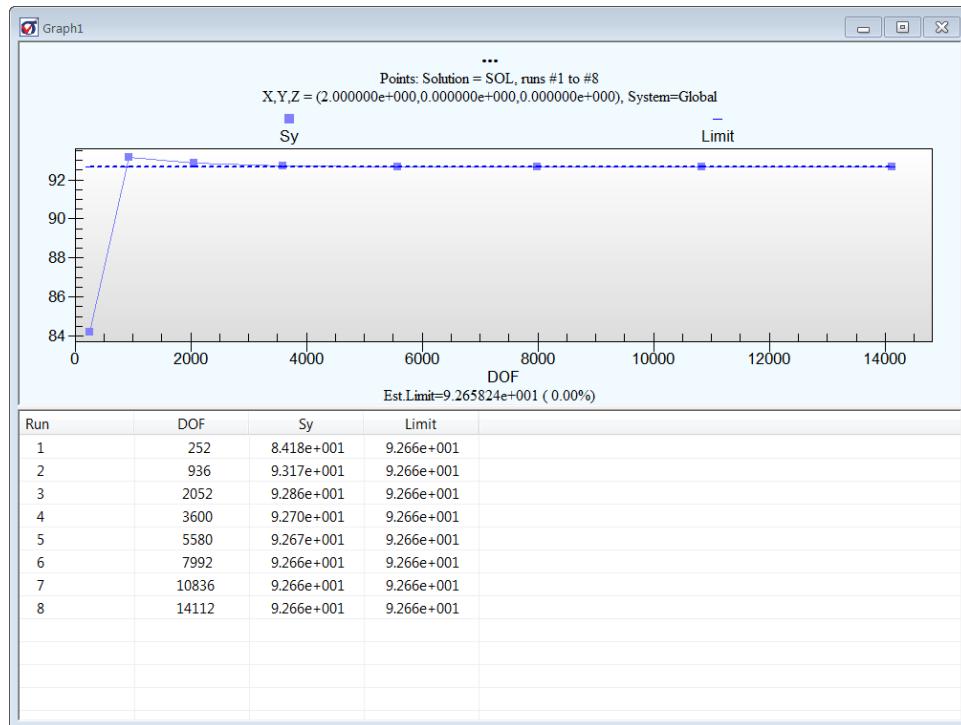
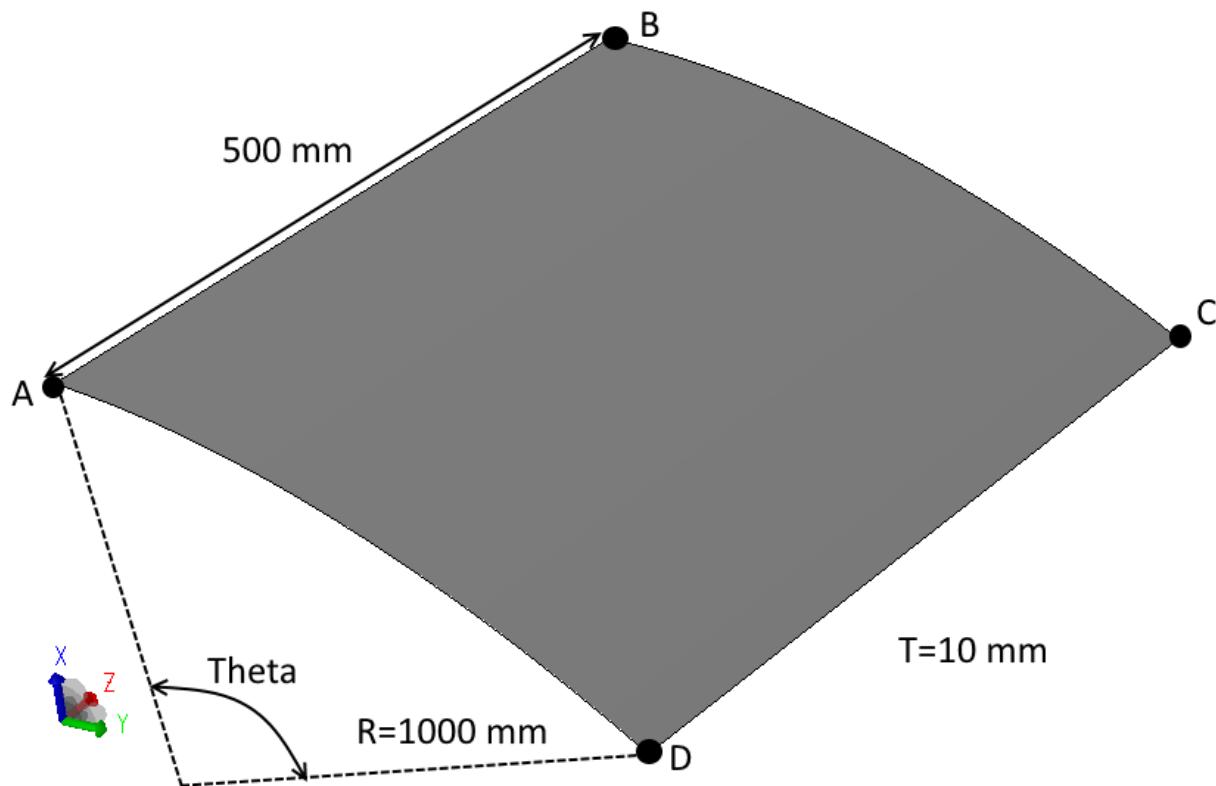


Figure 4. Convergence evidence for dense tri mesh

NAFEMS LE2: CYLINDRICAL SHELL BENDING PATCH TEST



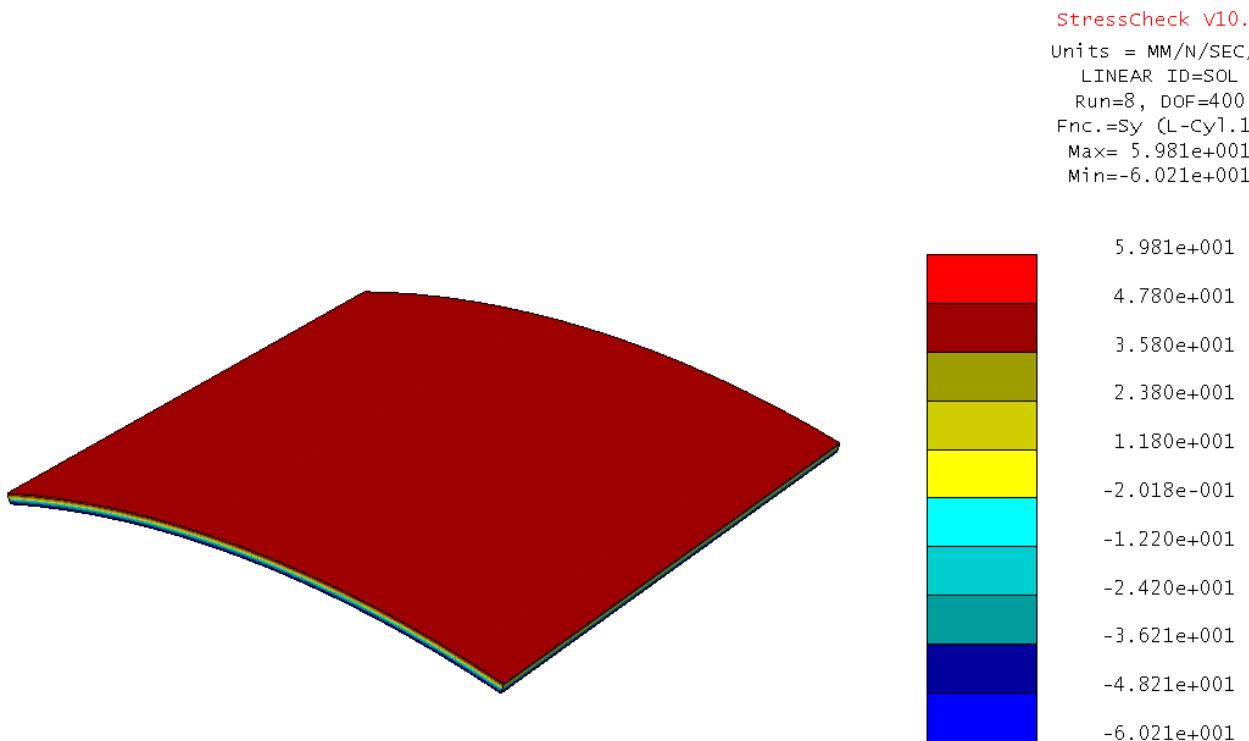
MODEL DESCRIPTION

- $\Theta = 30^\circ$ sector of cylindrical shell with a constant thickness $T = 10 \text{ mm}$.
- Linear elastic analysis, Young's modulus = 210 GPa, Poisson's ratio = 0.3.
- Fixed (clamped) boundary conditions along edge AB.
- Symmetry boundary conditions along edges AD and BC.
- Uniform bending moment of 1000 N-mm/unit length along edge DC.
- Objective of the analysis is the outer surface tangential stress.

NAFEMS REFERENCE SOLUTION

- Outer surface tangential stress is **60 MPa**.

STRESSCHECK® RESULTS



| Element | Minimum Mesh | Solution | Dense Mesh | Solution |
|---------|--------------|-------------------|--------------|-------------------|
| Quad | 1 element | 59.81 MPa (-0.3%) | 121 elements | 59.80 MPa (-0.3%) |
| Tri | 2 elements | 59.81 MPa (-0.3%) | 200 elements | 59.82 MPa (-0.3%) |

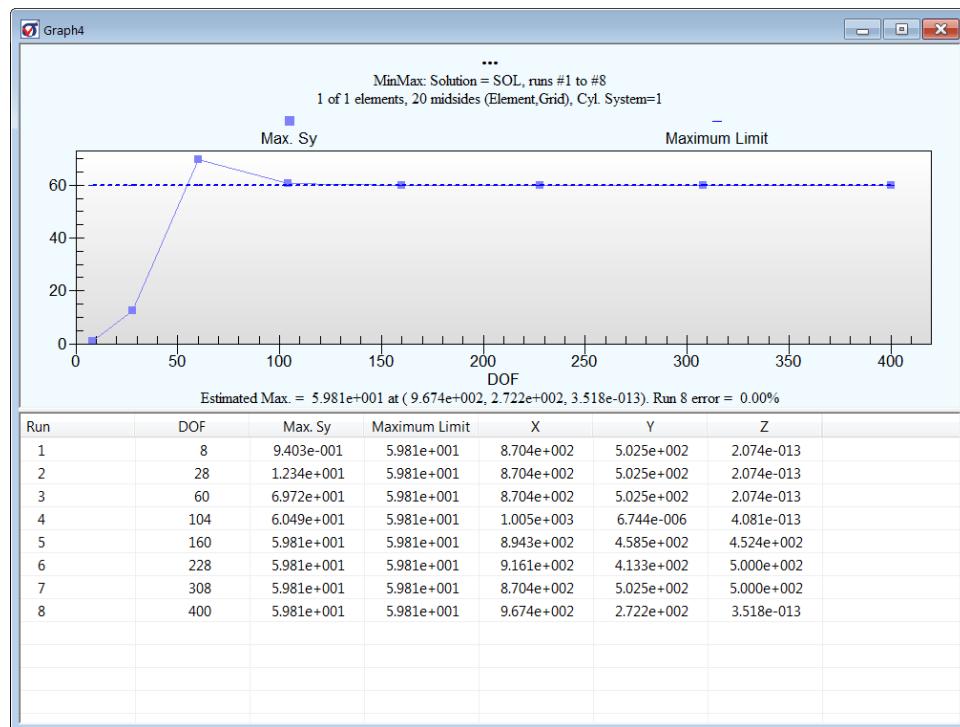


Figure 5. Convergence evidence for minimum quad mesh

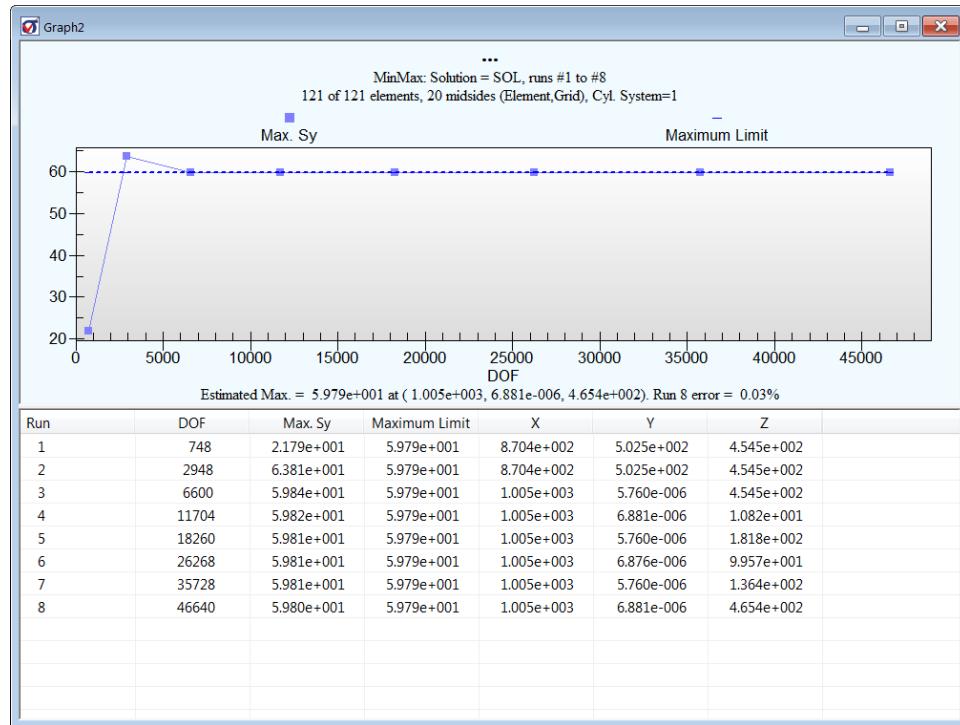


Figure 6. Convergence evidence for dense quad mesh

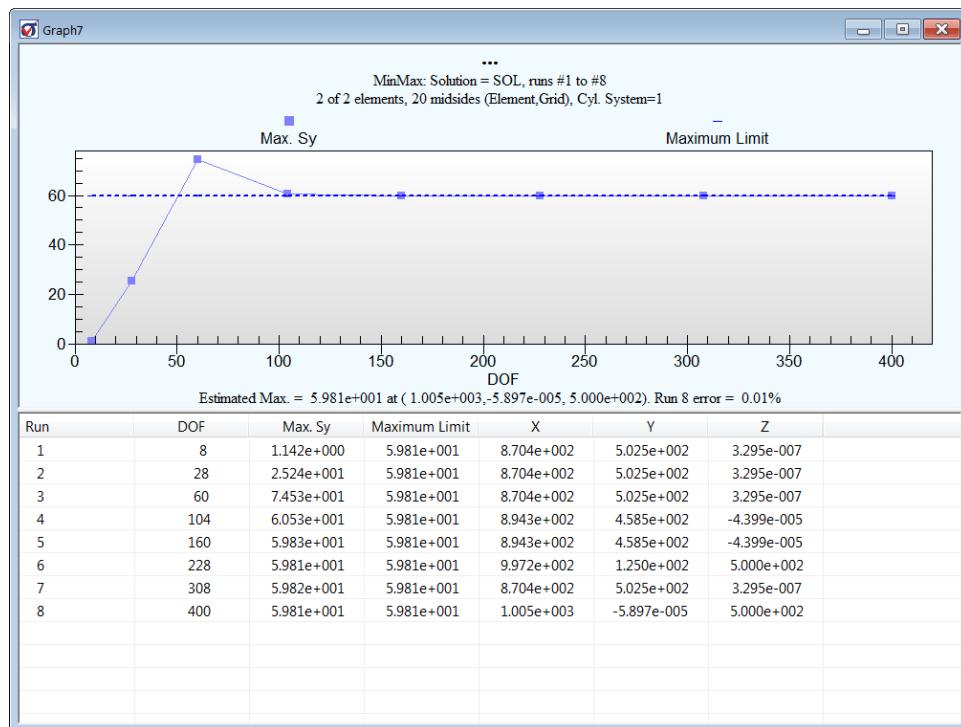


Figure 7. Convergence evidence for minimum tri mesh

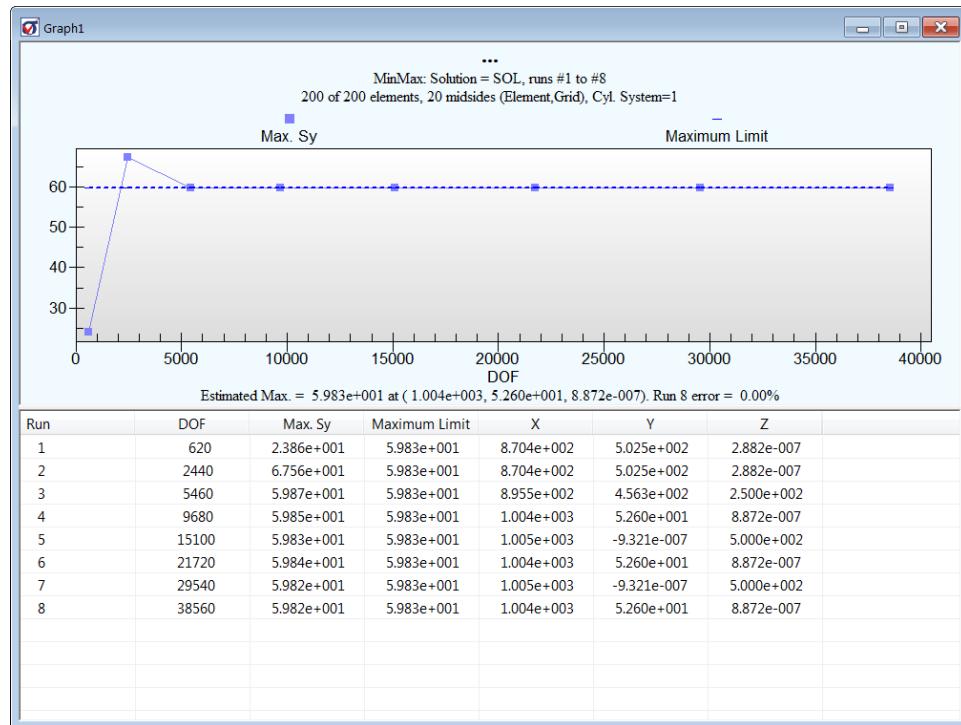
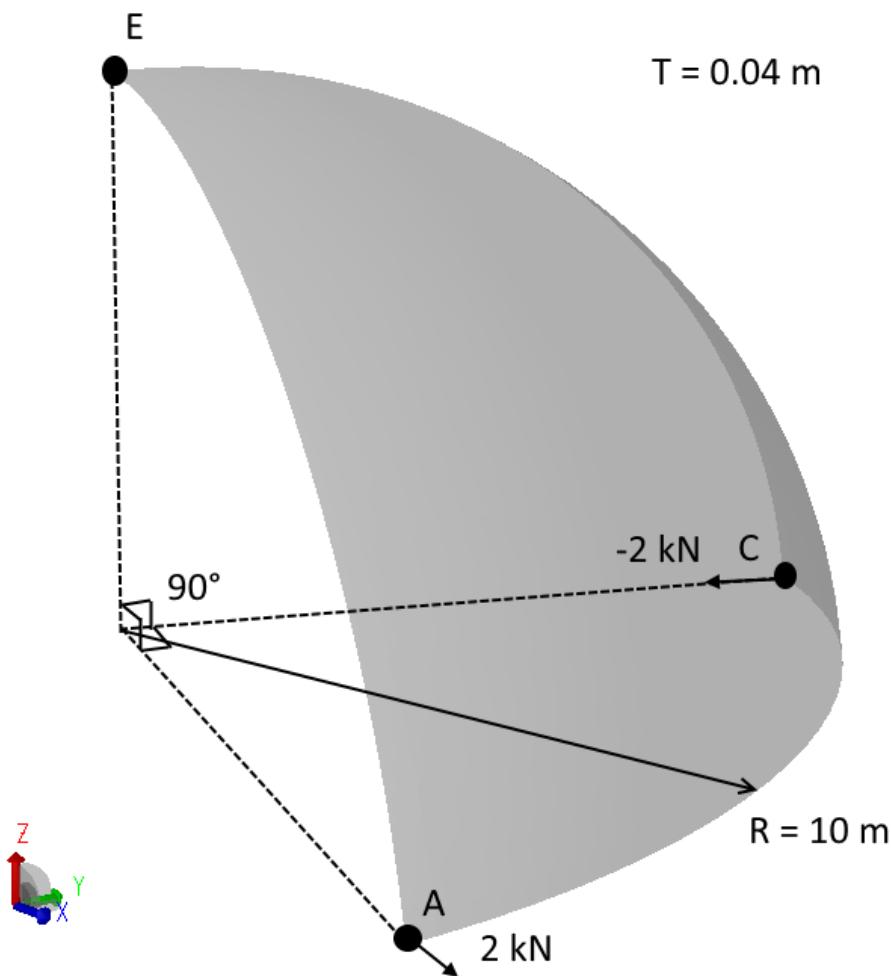


Figure 8. Convergence evidence for dense tri mesh

NAFEMS LE3: HEMISPHERICAL SHELL WITH POINT LOADS



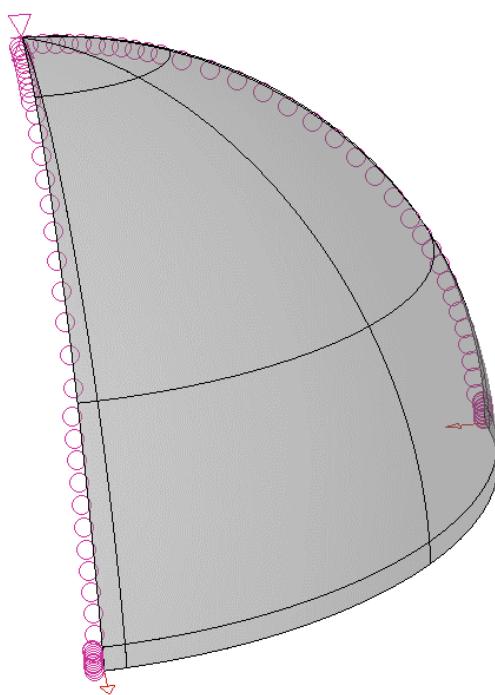
MODEL DESCRIPTION

- 90° sector of hemispherical shell of $R = 10 \text{ m}$ with a constant thickness $T = 0.04 \text{ m}$.
- Linear elastic analysis, Young's modulus = 68.25 GPa, Poisson's ratio = 0.3.
- $U_z = 0$ at point E.
- Symmetry boundary conditions along edges AE and CE.
- Concentrated point loads of $F_x = 2 \text{ kN}$ at point A, $F_y = -2 \text{ kN}$ at point C.
- Objective of the analysis is to compute the radial displacement at point A.

NAFEMS REFERENCE SOLUTION

- Radial displacement at point A is **185 mm**.

STRESSCHECK® RESULTS



| Element | Minimum Mesh | Solution | Dense Mesh | Solution |
|----------|--------------|---------------------|-------------|---------------------|
| Quad/Tri | 16 elements | 184.3 mm (-0.4%) | 64 elements | 184.4 mm (-0.3%) |

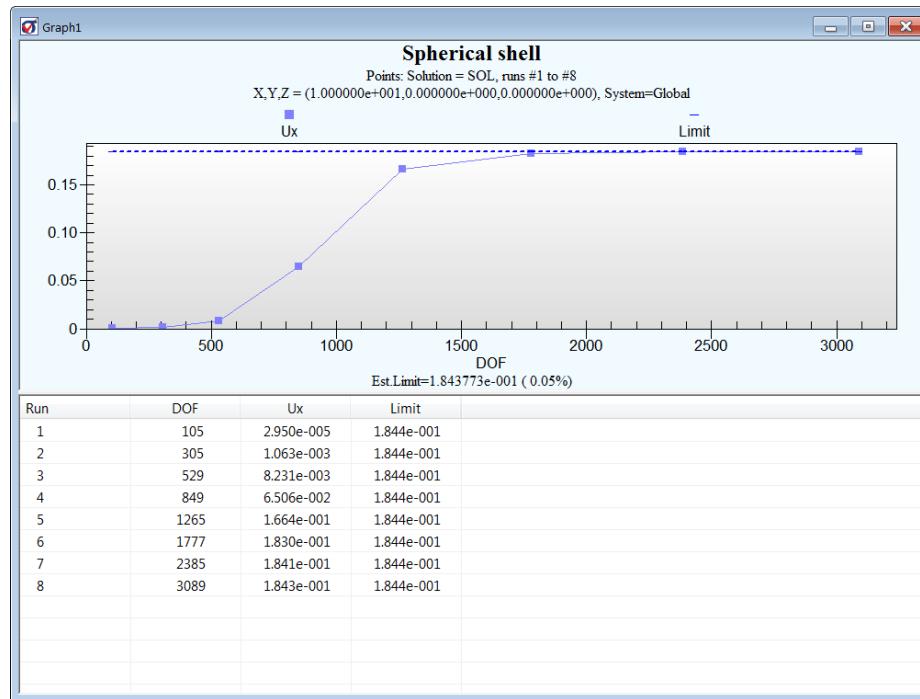


Figure 9. Convergence evidence for minimum quad/tri mesh

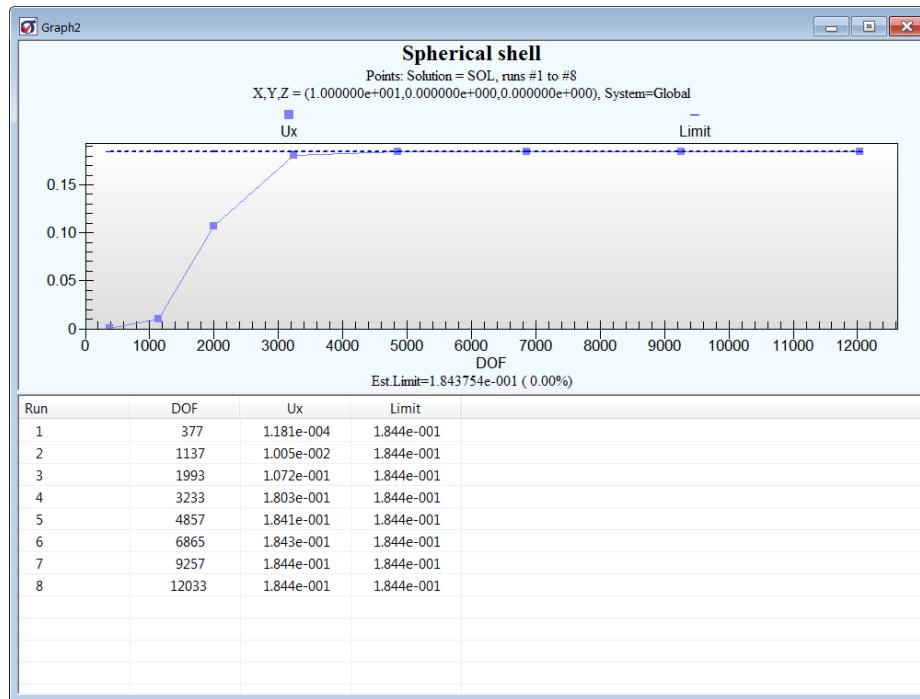
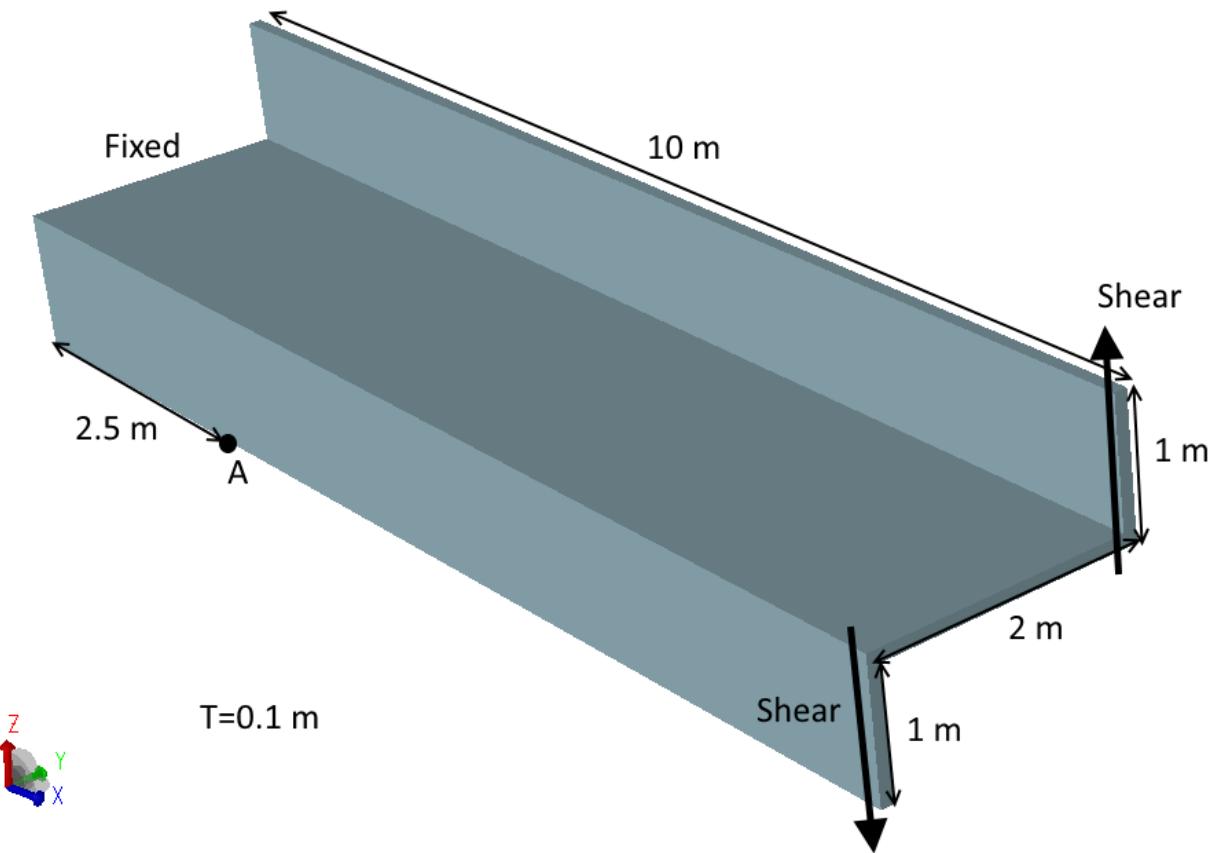


Figure 10. Convergence evidence for dense quad/tri mesh

Note: Point loads are inadmissible input data for hierachic shell models because the strain energy associated with a point load is not finite and therefore the corresponding displacements cannot be finite. However when point loads are used for computing displacements (as in this benchmark problem) the divergence in the data of interest is extremely slow and the reported results compare well with the reference solution.

NAFEMS LE5: Z-SECTION CANTILEVER



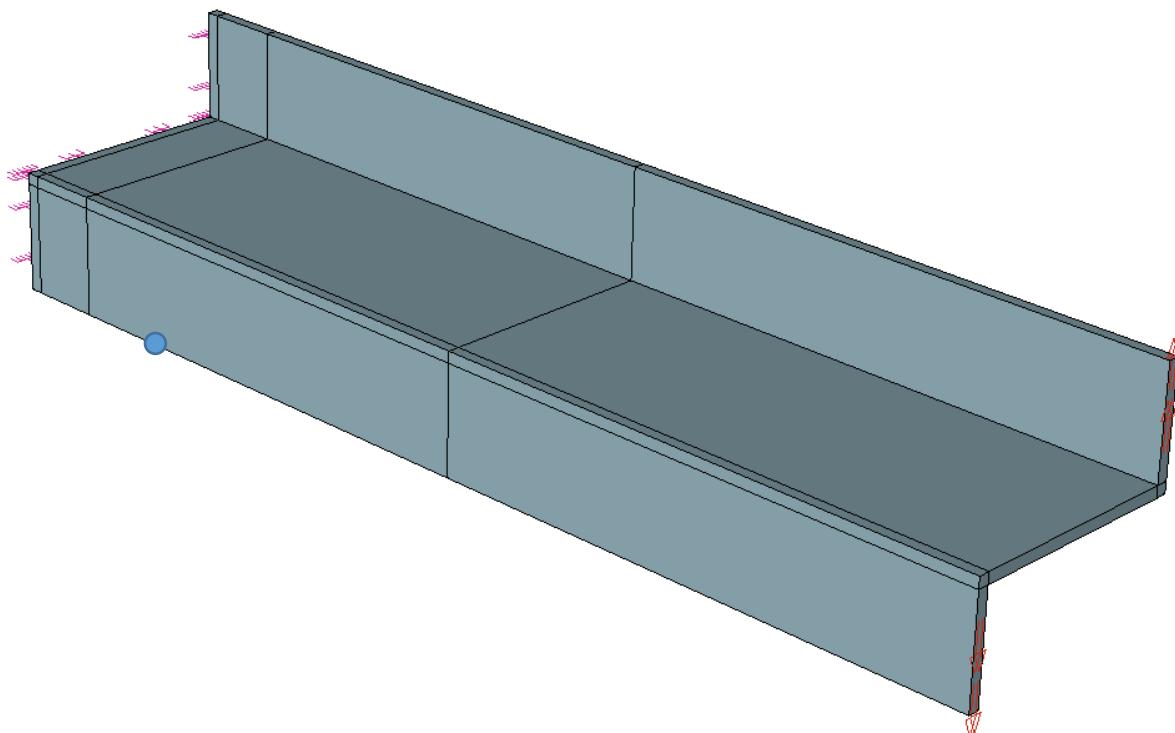
MODEL DESCRIPTION

- Z-section cantilever under torsional loading.
- Linear elastic analysis, Young's modulus = 210 GPa, Poisson's ratio = 0.3.
- All displacements are fixed at X=0.
- Torque of 1.2 MN-m applied at X=10.
 - The torque is applied by two uniformly distributed shear loads of 0.6 MN at each flange surface.
- Objective of the analysis is to compute the axial stress at X = 2.5 from fixed end.

NAFEMS REFERENCE SOLUTION

- Axial stress at X = 2.5 from fixed end (point A) at the midsurface is **-108 MPa**.

STRESSCHECK® RESULTS



| Element | Minimum Mesh | Solution | Dense Mesh | Solution |
|---------|--------------|----------------------|---------------|----------------------|
| Hexa | 20 elements | -109.8 MPa (1.7%) | 78 elements | -109.5 MPa (1.4%) |
| Tetra | 136 elements | -109.3 MPa (1.2%) | 1186 elements | -109.5 MPa (1.4%) |

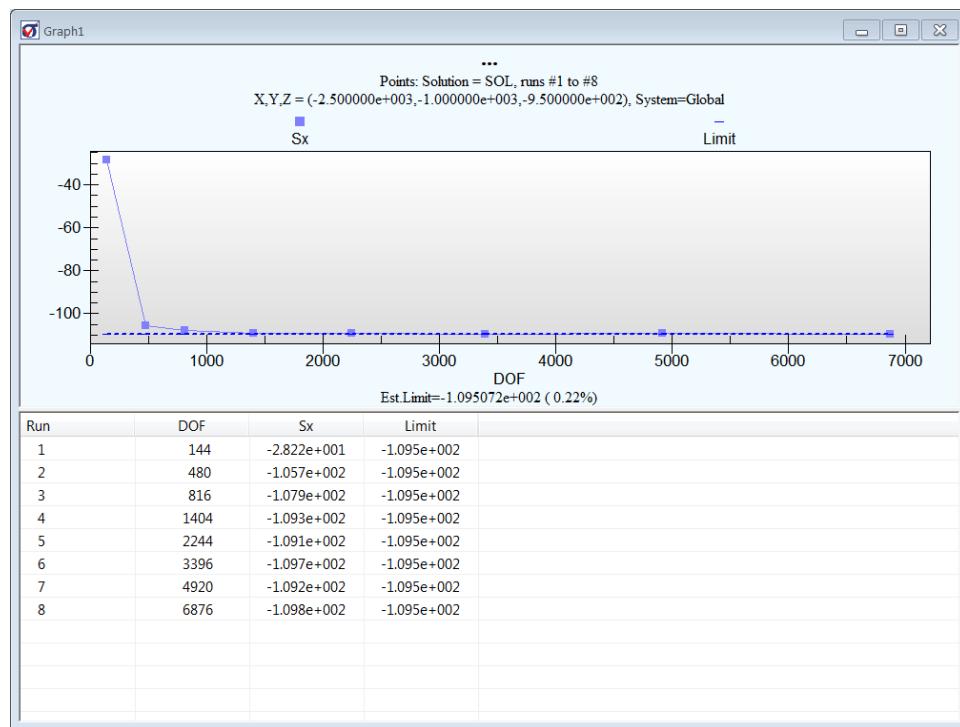


Figure 11. Convergence evidence for minimum hexa mesh

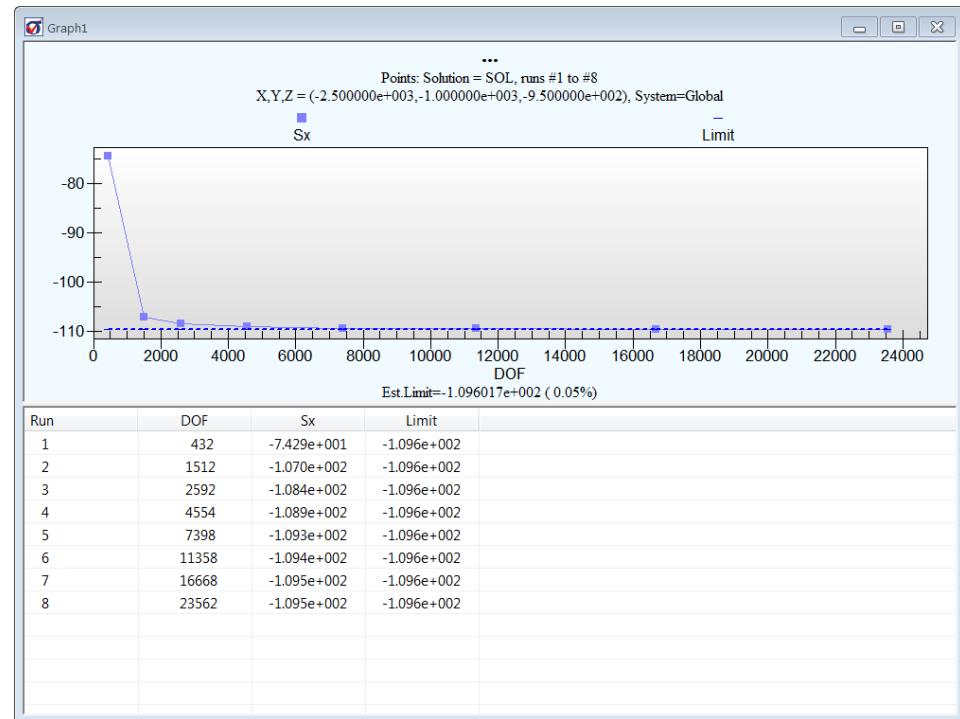


Figure 12. Convergence evidence for a dense hexa mesh

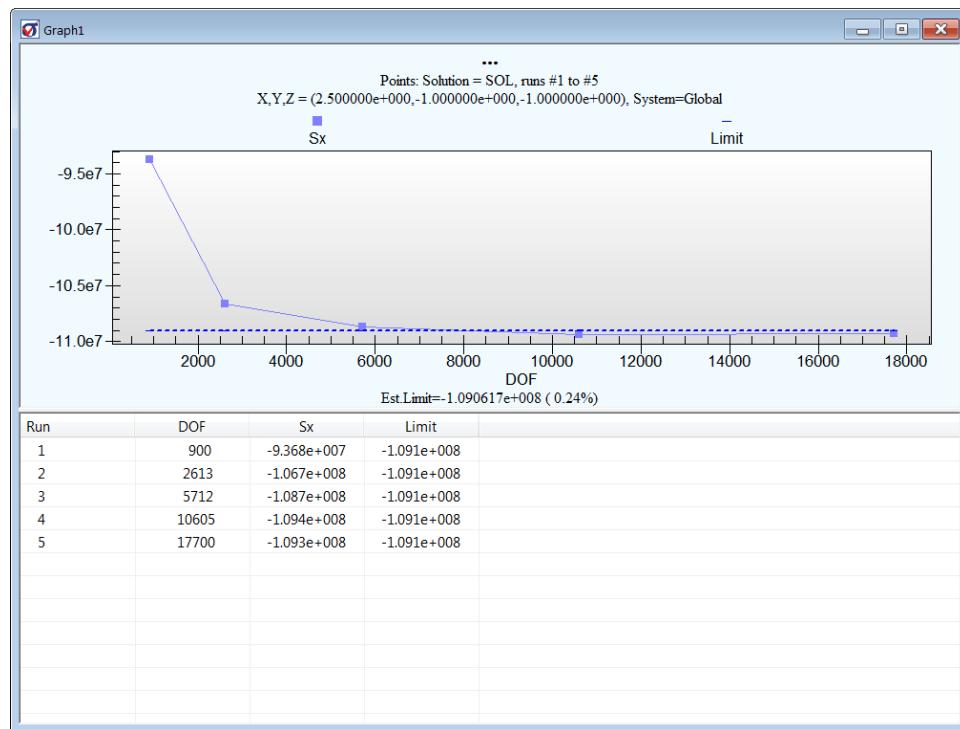


Figure 13. Convergence evidence for minimum tetra mesh

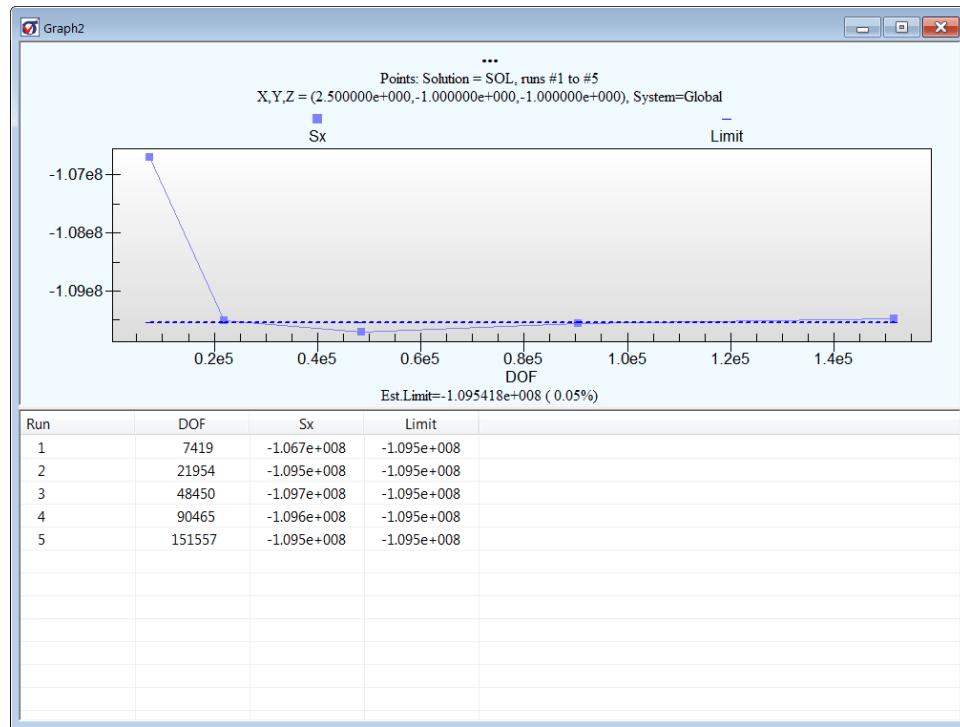
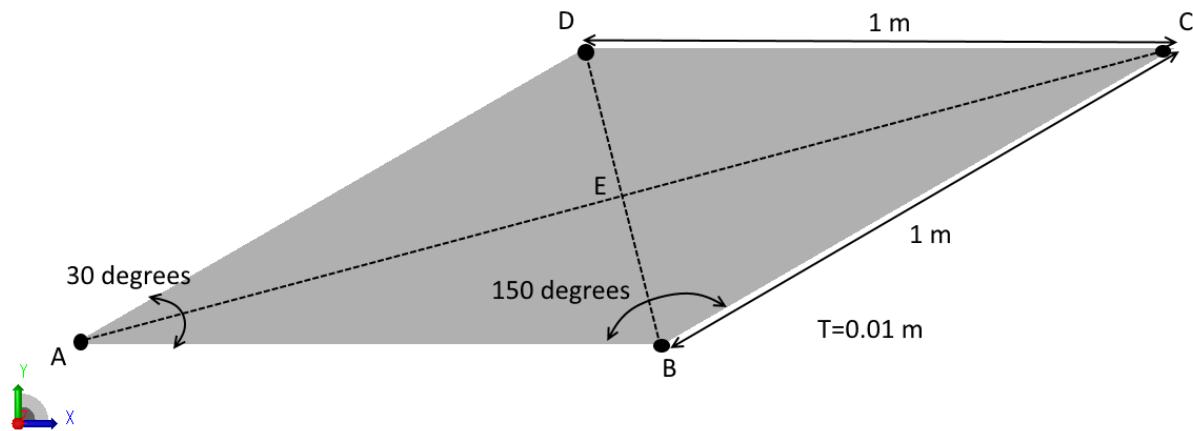


Figure 14. Convergence evidence for dense tetra mesh

Note: Hexas/tetras were used for this benchmark as StressCheck® does not implement slope-discontinuous shells.

NAFEMS LE6: SKEW PLATE UNDER NORMAL PRESSURE



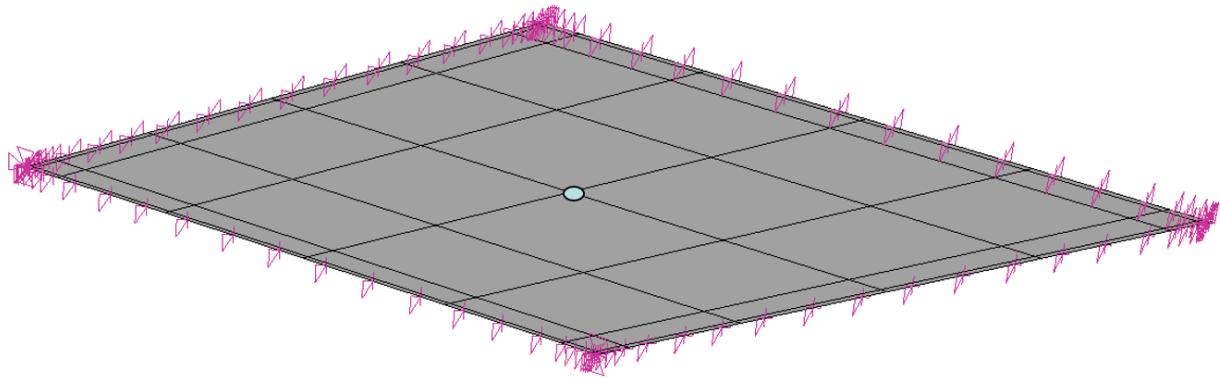
MODEL DESCRIPTION

- Skew plate under normal pressure.
- Linear elastic analysis, Young's modulus = 210 GPa, Poisson's ratio = 0.3.
- $U_z = 0$ along edges AB, BC, CD, and AD.
- $U_x = U_y = 0$ at point A and $U_y = 0$ at point B to prevent rigid body motion.
- Uniform pressure of -0.7 kPa in the vertical z-direction.
- Objective of the analysis is to compute the maximum principal stress on the lower surface at point E.

NAFEMS REFERENCE SOLUTION

- Maximum first principal stress on lower surface in the middle of the plate is **0.802 MPa**.

STRESSCHECK® RESULTS



| Element | Minimum Mesh | Solution | Dense Mesh | Solution |
|---------|--------------|---------------------|--------------|---------------------|
| Quad | 64 elements | 0.820 MPa (2.2%) | 169 elements | 0.820 MPa (2.2%) |
| Tri | 128 elements | 0.826 MPa (2.9%) | 254 elements | 0.822 MPa (2.5%) |

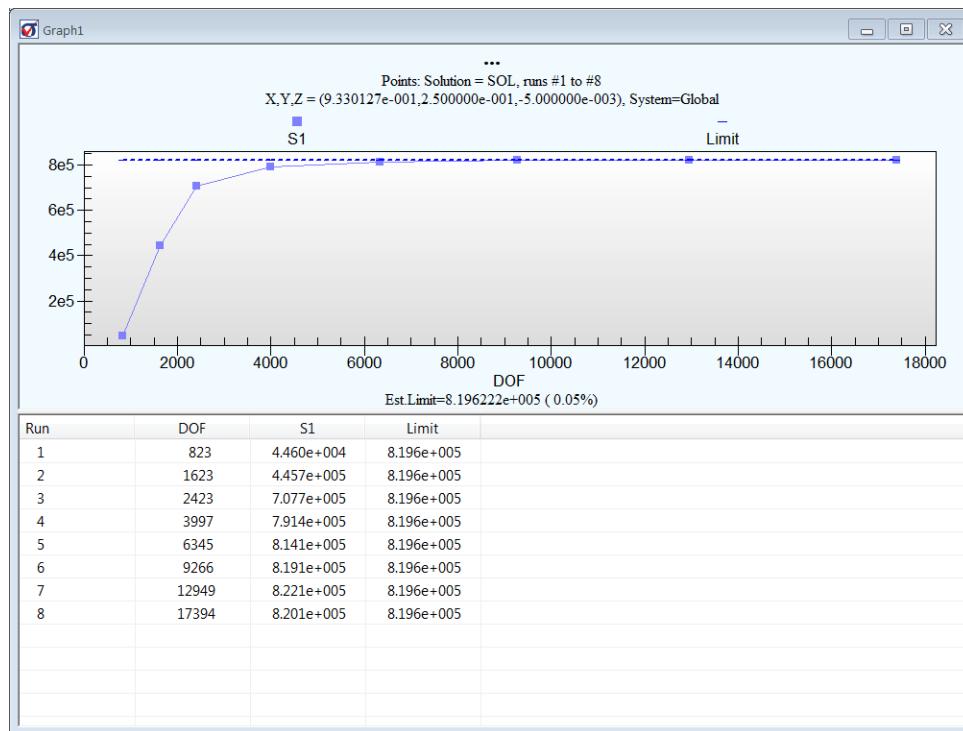


Figure 15. Convergence evidence for minimum quad mesh

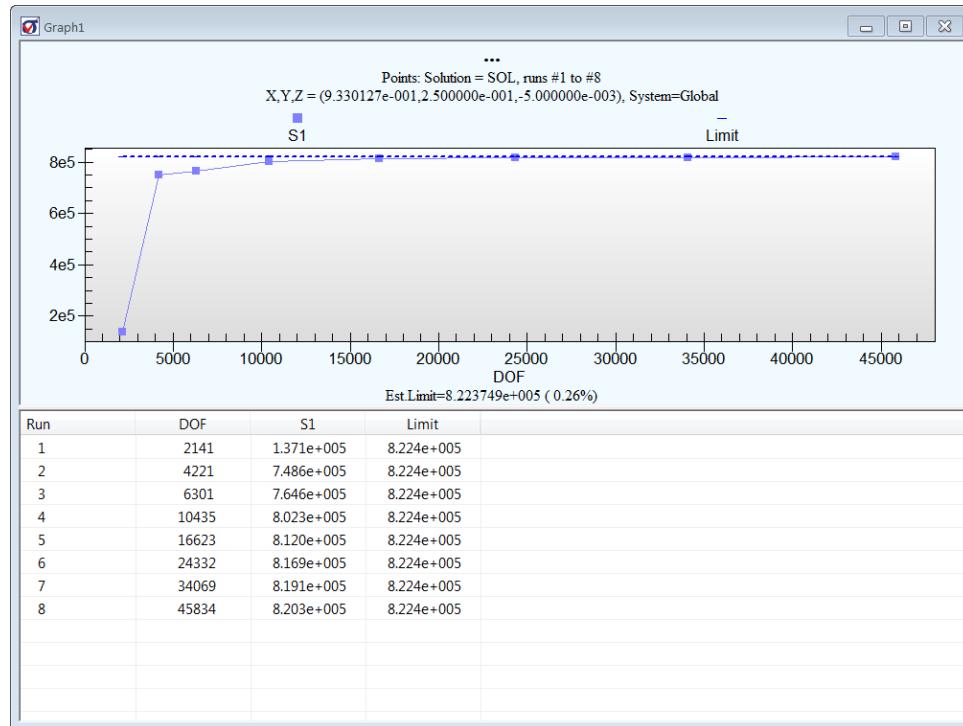


Figure 16. Convergence evidence for dense quad mesh

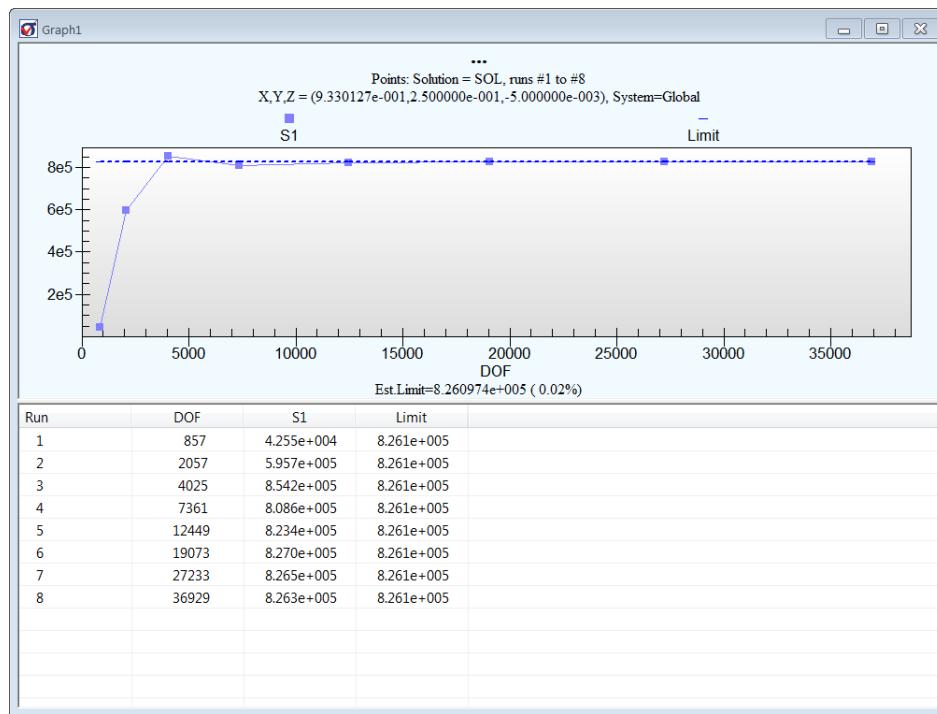


Figure 17. Convergence evidence for minimum tri mesh

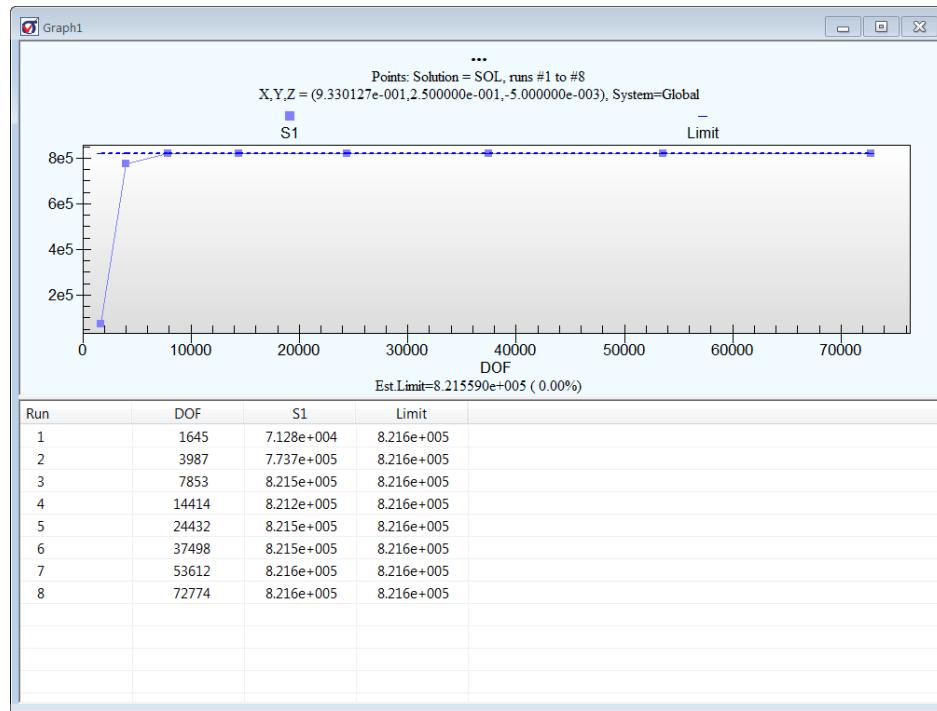
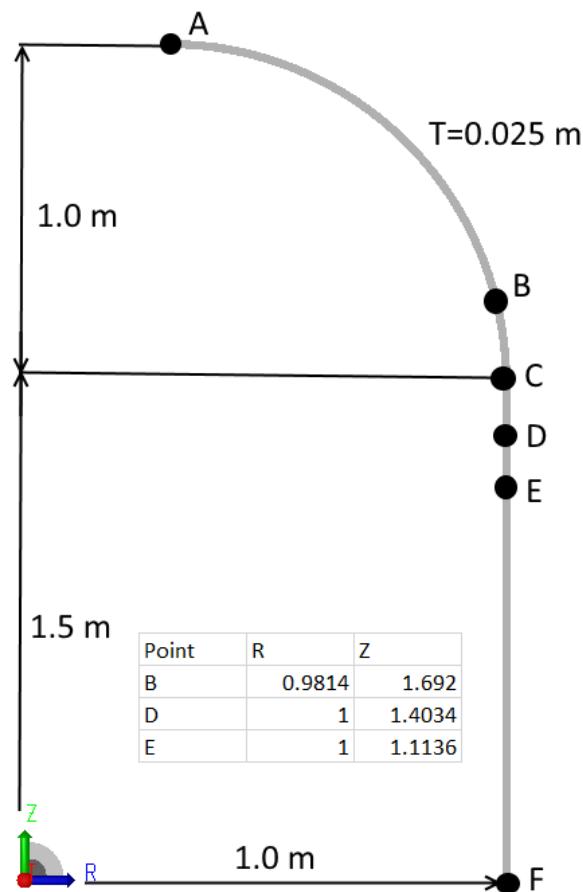


Figure 18. Convergence evidence for dense tri mesh

Note: There are multiple corner singularities in the problem description that required graded meshing techniques for convergence. This benchmark problem was solved using a 3D thin-solid formulation which may be different from the plate model from which the analytical solution was obtained.

NAFEMS LE7: AXISYMMETRIC CYLINDER/SPHERE UNDER PRESSURE



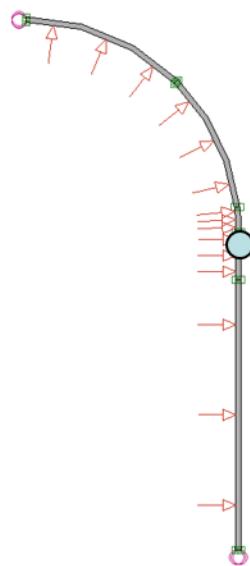
MODEL DESCRIPTION

- Axisymmetric thin-walled pressure vessel.
- Linear elastic analysis, Young's modulus = 210 GPa, Poisson's ratio = 0.3.
- $U_r = 0$ at point A.
- $U_z = 0$ at point F.
- Uniform internal pressure of 1.0 MPa.
- Objective of the analysis is to compute the axial stress on the outer surface at point D.

NAFEMS REFERENCE SOLUTION

- Axial stress on outer surface at $R = 1.0125$, $Z = 1.4034$ is **25.86 MPa**.

STRESSCHECK® RESULTS



| Element | Minimum Mesh | Solution | Dense Mesh | Solution |
|---------|--------------|----------------------|--------------|----------------------|
| Quad | 5 elements | 25.27 MPa (-2.3%) | 605 elements | 25.27 MPa (-2.3%) |

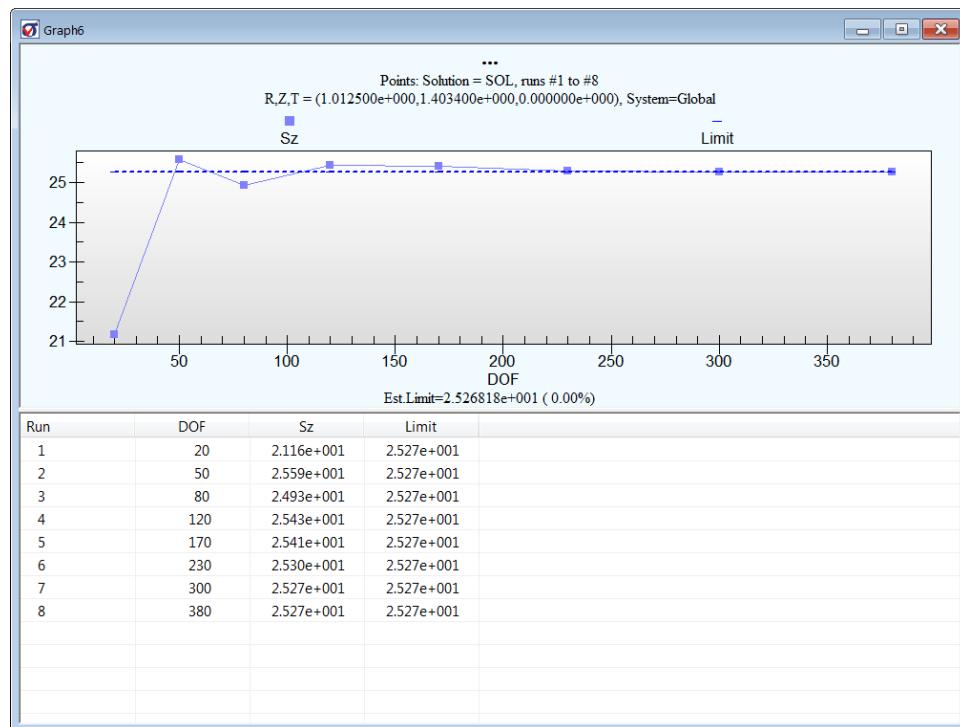


Figure 19. Convergence evidence for minimum quad mesh

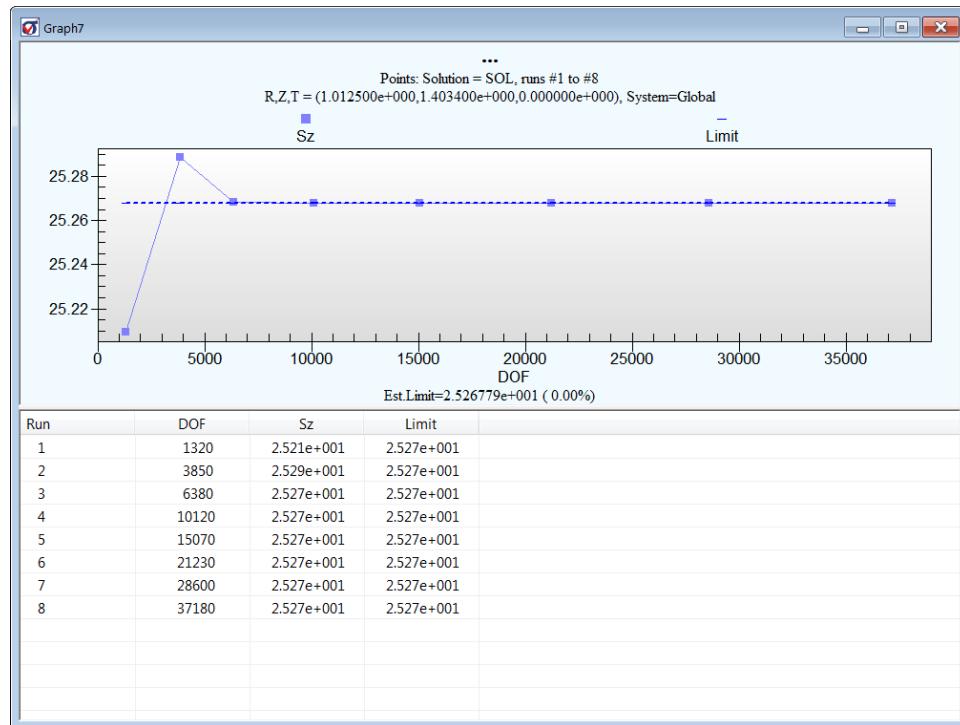
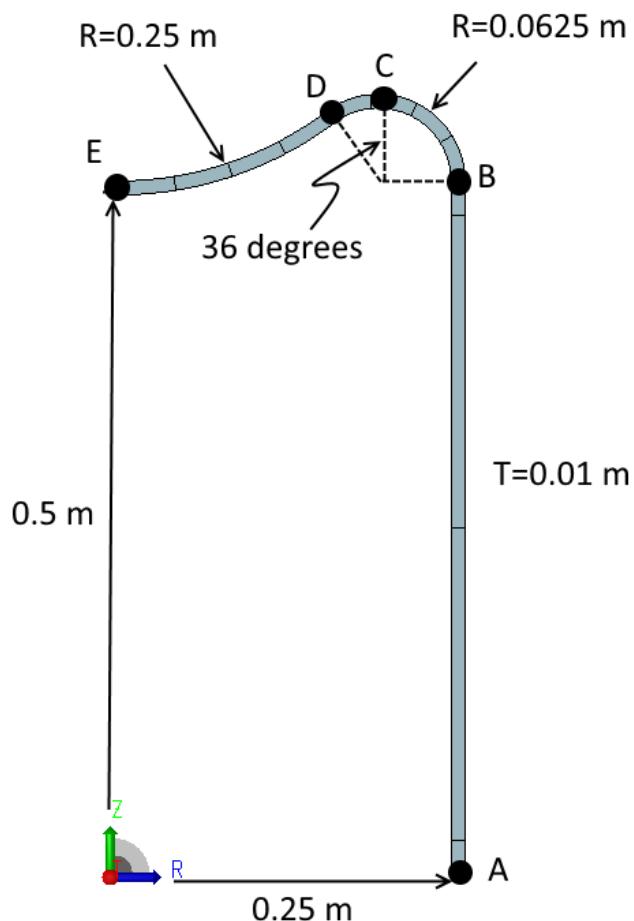


Figure 20. Convergence evidence for dense quad mesh

Note: Axisymmetric shell theory was used in the original NAFEMS benchmark test. In StressCheck®, axisymmetric solids were used to represent the model.

NAFEMS LE8: AXISYMMETRIC SHELL UNDER PRESSURE



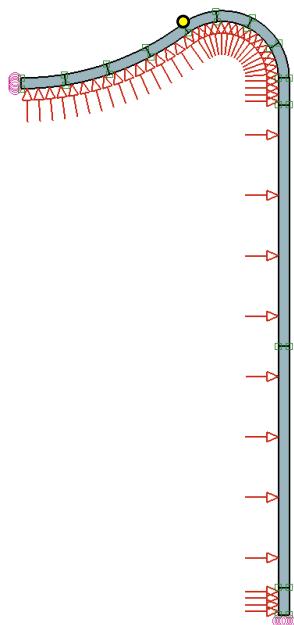
MODEL DESCRIPTION

- Axisymmetric shell under pressure.
- Linear elastic analysis, Young's modulus = 210 GPa, Poisson's ratio = 0.3.
- $U_z = 0$ at point A.
- $U_r = 0$ at point E.
- Uniform internal pressure of 1.0 MPa.
- Objective of the analysis is to compute the hoop stress on the outer surface at point D.

NAFEMS REFERENCE SOLUTION

- Hoop stress on the outer surface at 36 degrees from circle center is **94.55 MPa**.

STRESSCHECK® RESULTS



| Element | Minimum Mesh | Solution | Dense Mesh | Solution |
|---------|--------------|----------------------|-------------|----------------------|
| Quad | 12 elements | 91.93 MPa (-2.8%) | 48 elements | 92.05 MPa (-2.6%) |

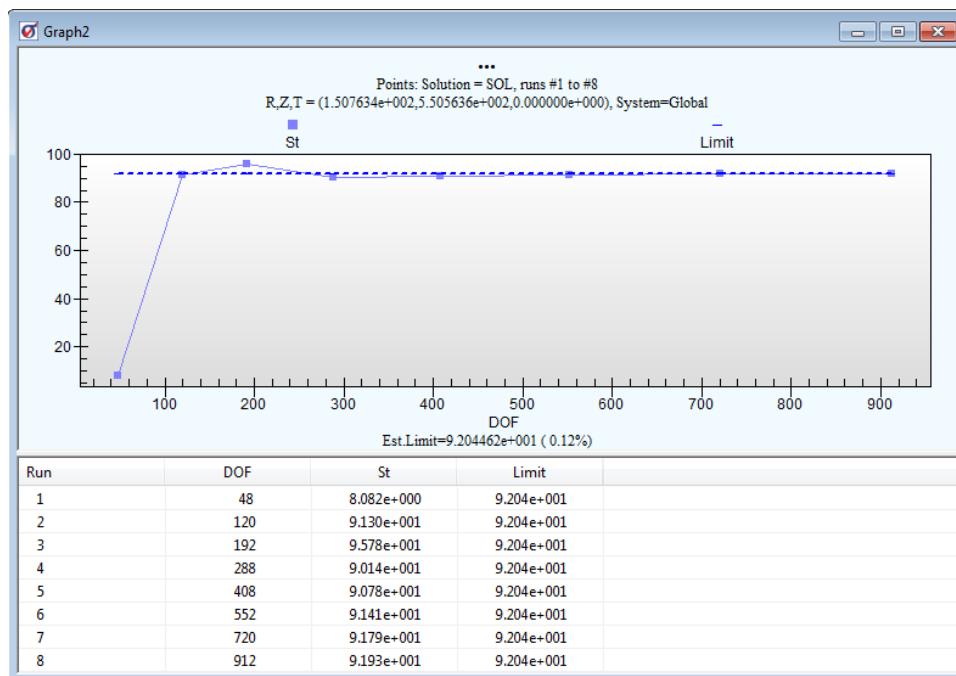


Figure 21. Convergence evidence for minimum quad mesh

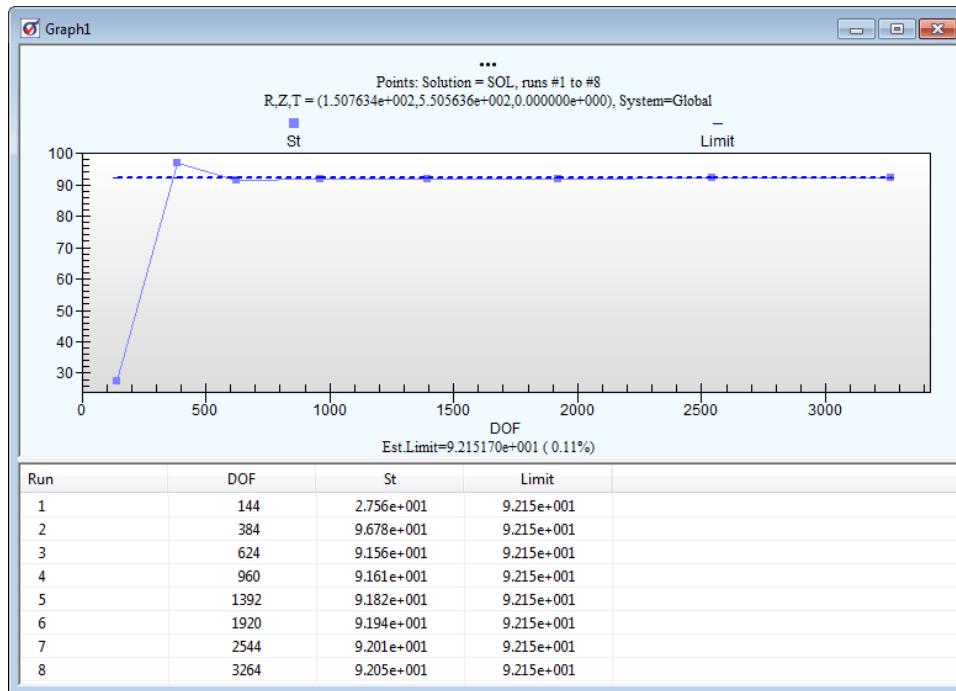
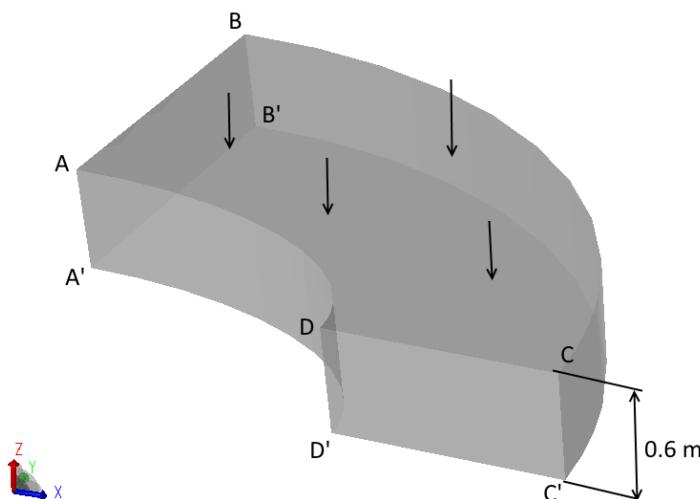
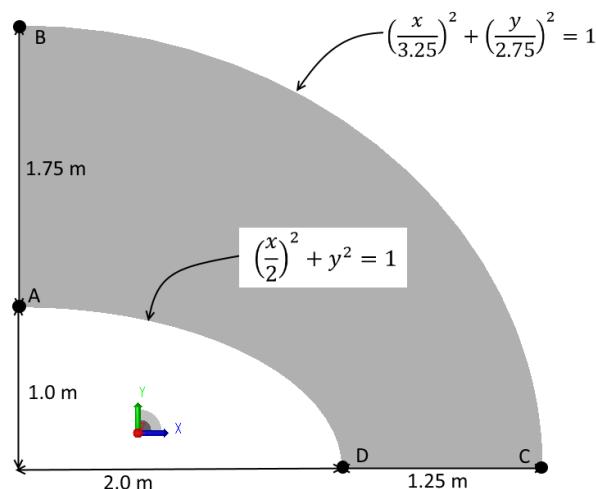


Figure 22. Convergence evidence for dense quad mesh

Note: Axisymmetric shell theory was used in the original NAFEMS benchmark test. In StressCheck®, axisymmetric solids were used to represent the model.

NAFEMS LE10: THICK PLATE UNDER PRESSURE



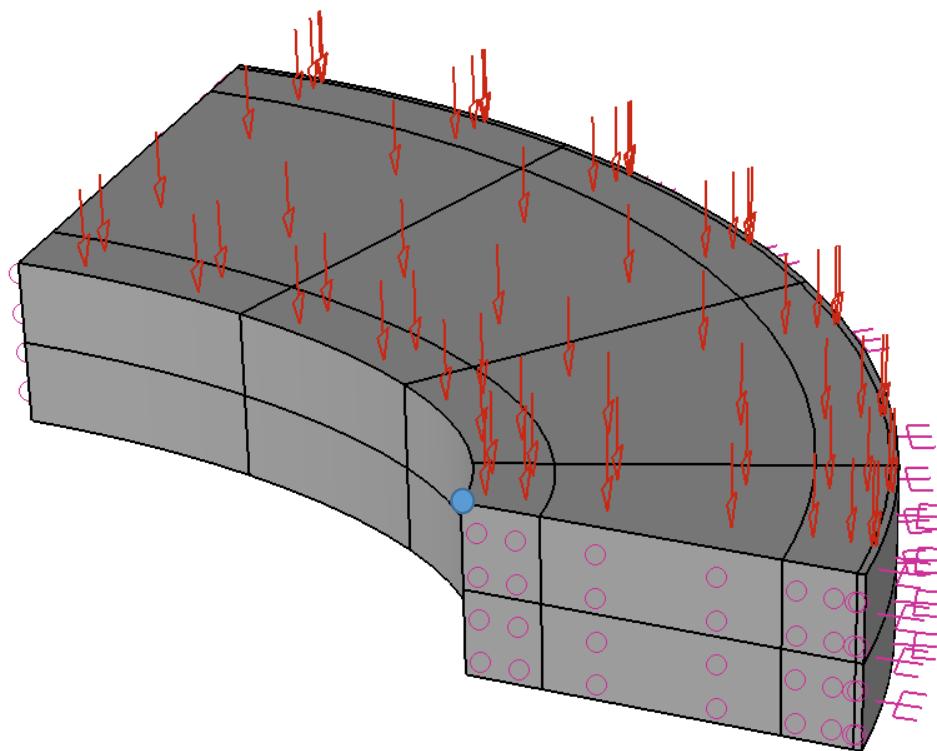
MODEL DESCRIPTION

- Thick plate under uniform pressure.
- Linear elastic analysis, Young's modulus = 210 GPa, Poisson's ratio = 0.3.
- Symmetry on faces DCD'C' and ABA'B'.
- $U_x = U_y = 0$ on face BCB'C'.
- Z-displacement fixed along mid-plane of face BCB'C'
- Uniform normal pressure of 1.0 MPa on the upper surface of the plate.
- Objective of the analysis is to compute the direct stress at point D.

NAFEMS REFERENCE SOLUTION

- Direct stress in y-direction at point D is **-5.38 MPa**.

STRESSCHECK® RESULTS



| Element | Minimum Mesh | Solution | Dense Mesh | Solution |
|---------|--------------|----------------------|-------------|----------------------|
| Hexa | 32 elements | -5.24 MPa (-2.6%) | 60 elements | -5.25 MPa (-2.4%) |

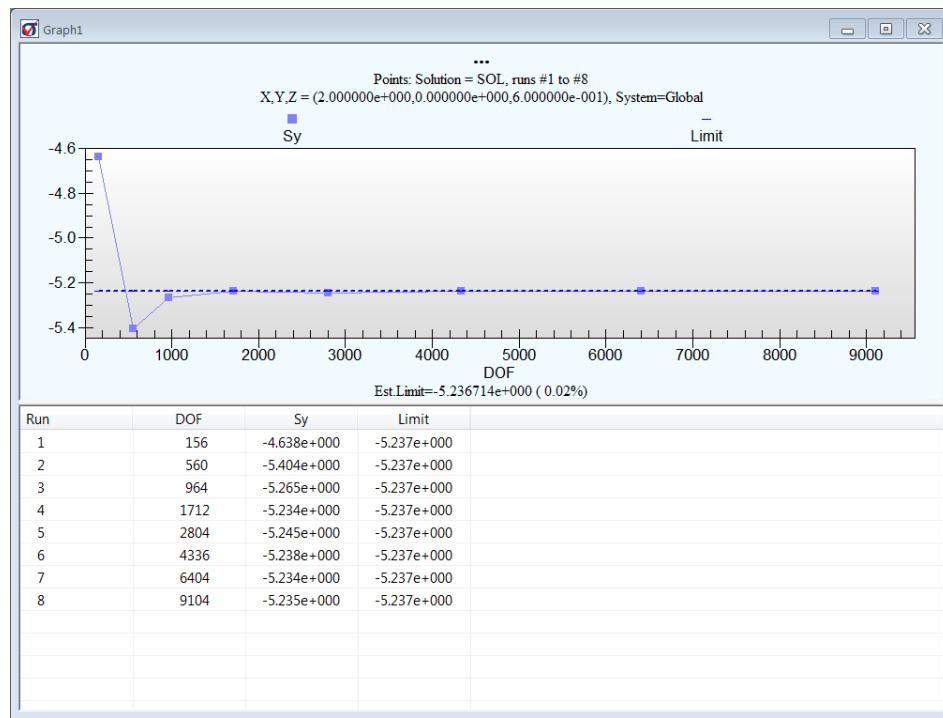


Figure 23. Convergence evidence for minimum hexa mesh

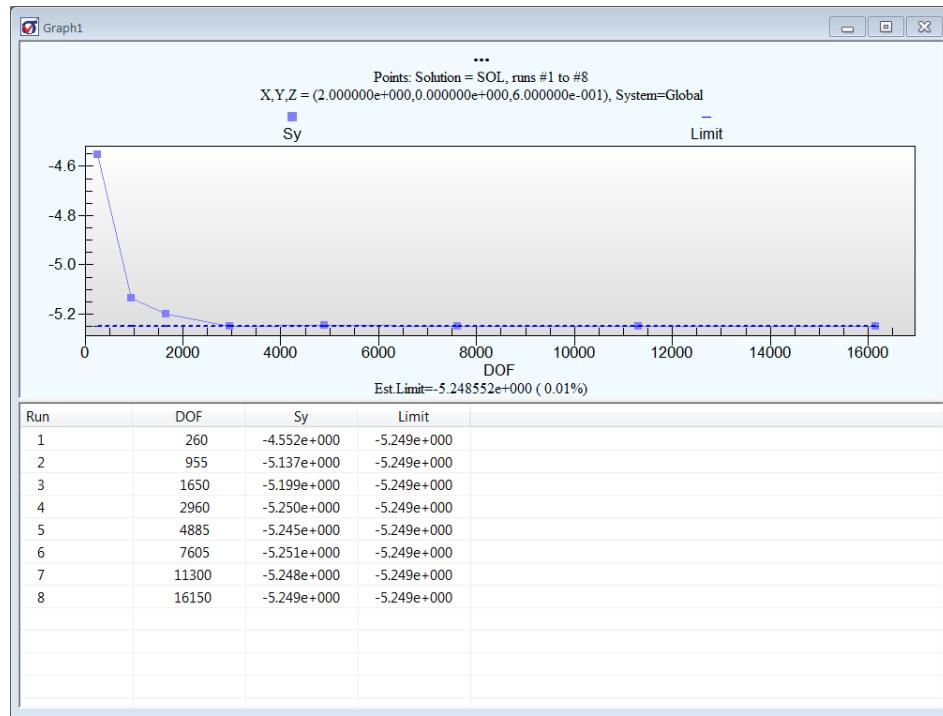
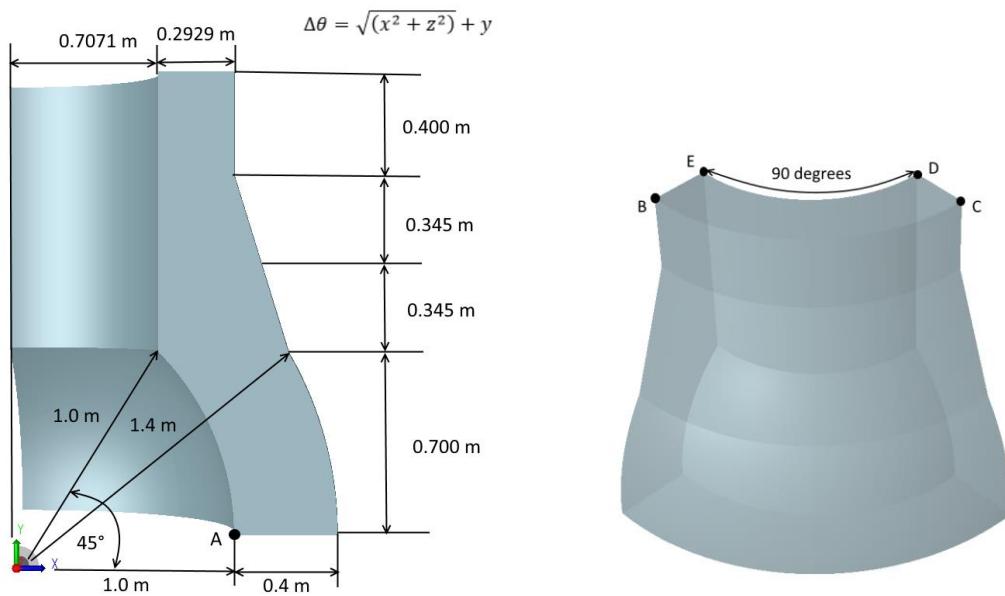


Figure 24. Convergence evidence for a dense hexa mesh

Note: Since constraints along a line are incompatible with 3D-elasticity, the StressCheck® results were obtained by fixing the z-displacement of the face BCB'C'.

NAFEMS LE11: SOLID CYLINDER/TAPER/SPHERE – TEMPERATURE LOADING



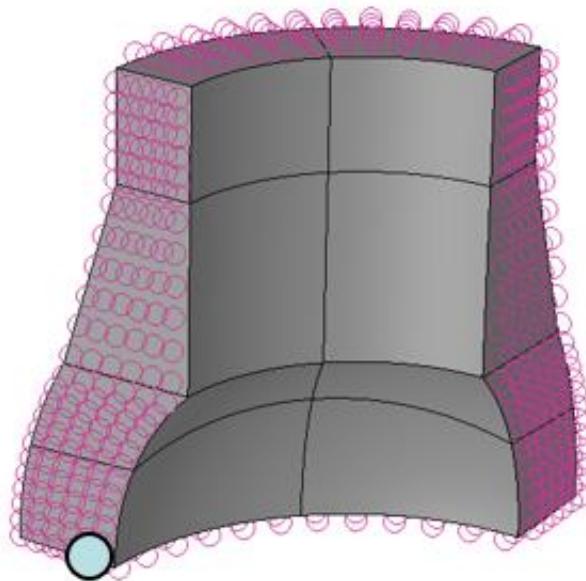
MODEL DESCRIPTION

- Solid cylinder/taper/sphere with applied temperature loading $\Delta\theta$.
- Linear elastic analysis, Young's modulus = 210 GPa, Poisson's ratio = 0.3, coefficient of thermal expansion = $2.3E-4/\text{°C}$.
- $U_z = 0$ on the plane $Z = 0$.
- $U_x = 0$ on the plane $X = 0$.
- $U_y = 0$ on the plane $Y = 0$ and the face BCDE.
- Linear temperature gradient in the radial and axial directions is given by $\Delta\theta$ in the above figure.
- The objective of the analysis is to compute the direct stress σ_y at point A.

NAFEMS REFERENCE SOLUTION

- Direct stress in y-direction at point A is **-105 MPa**.

STRESSCHECK® RESULTS



| Element | Minimum Mesh | Solution | Dense Mesh | Solution |
|---------|--------------|-----------------------|---------------|-----------------------|
| Hexa | 8 elements | -105.2 MPa (0.19%) | 216 elements | -105.4 MPa (0.38%) |
| Tetra | 317 elements | -105.5 MPa (0.48%) | 3531 elements | -105.4 MPa (0.38%) |

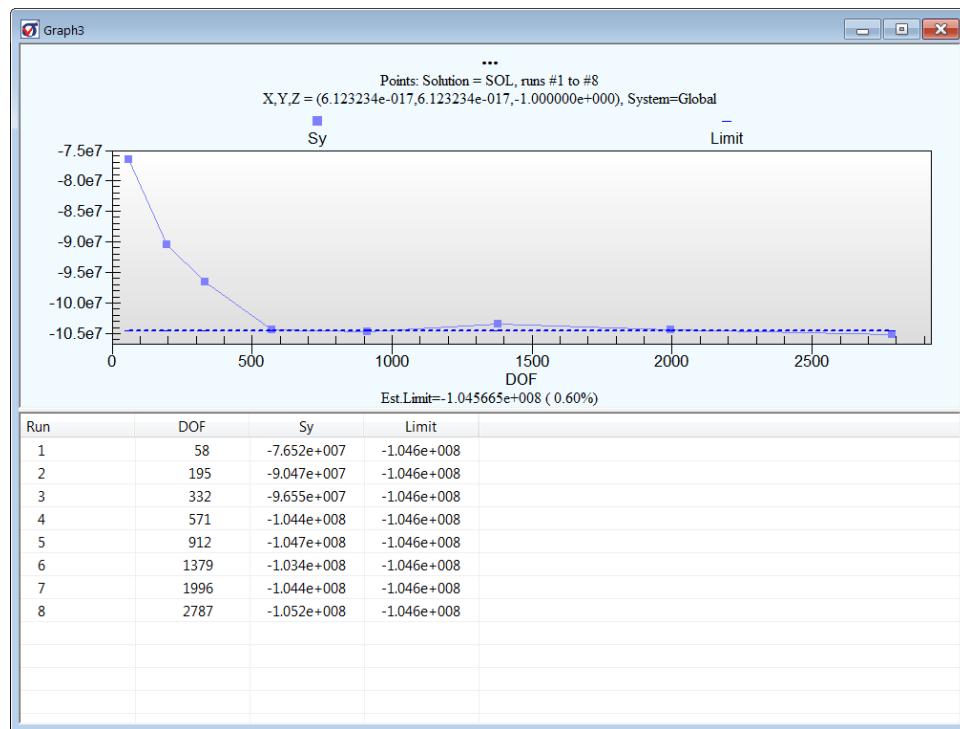


Figure 25. Convergence evidence for minimum hexa mesh

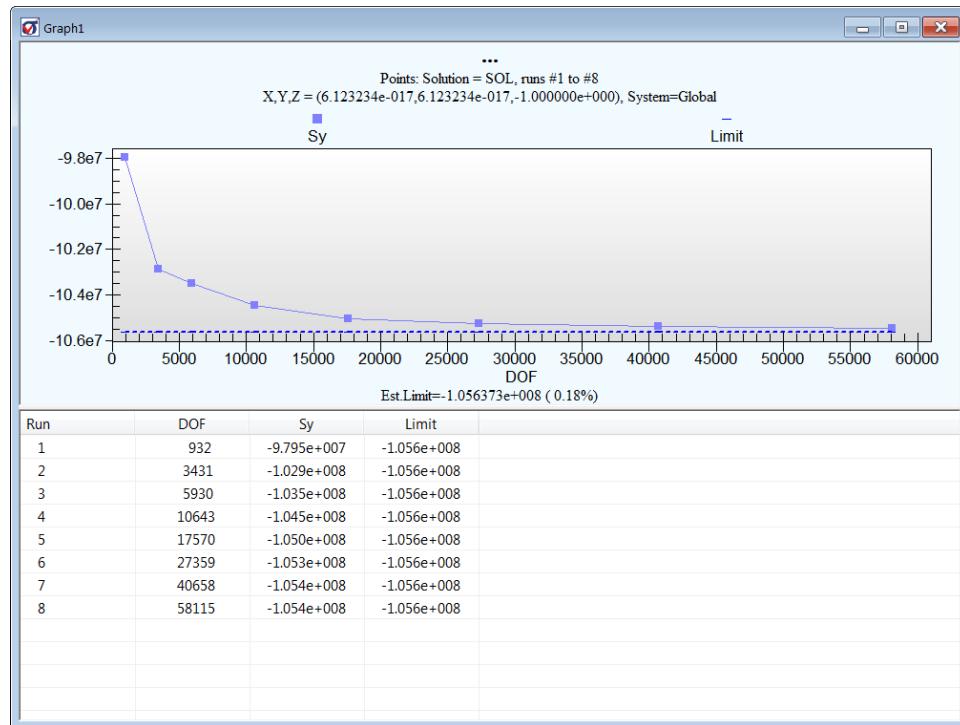


Figure 26. Convergence evidence for dense hexa mesh

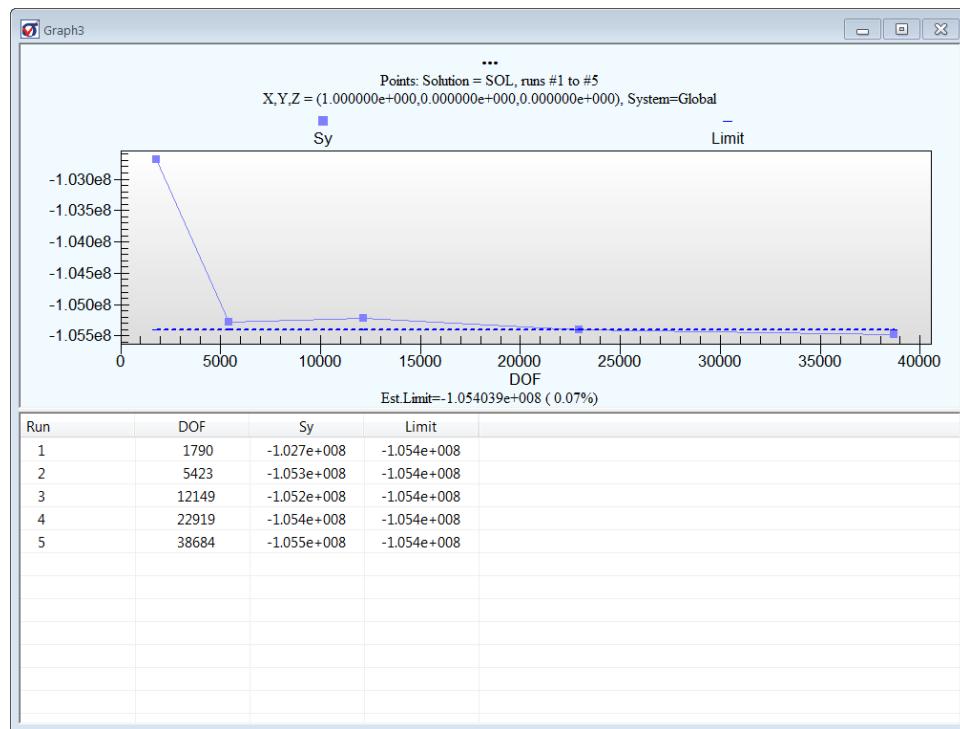


Figure 27. Convergence evidence for minimum tetra mesh

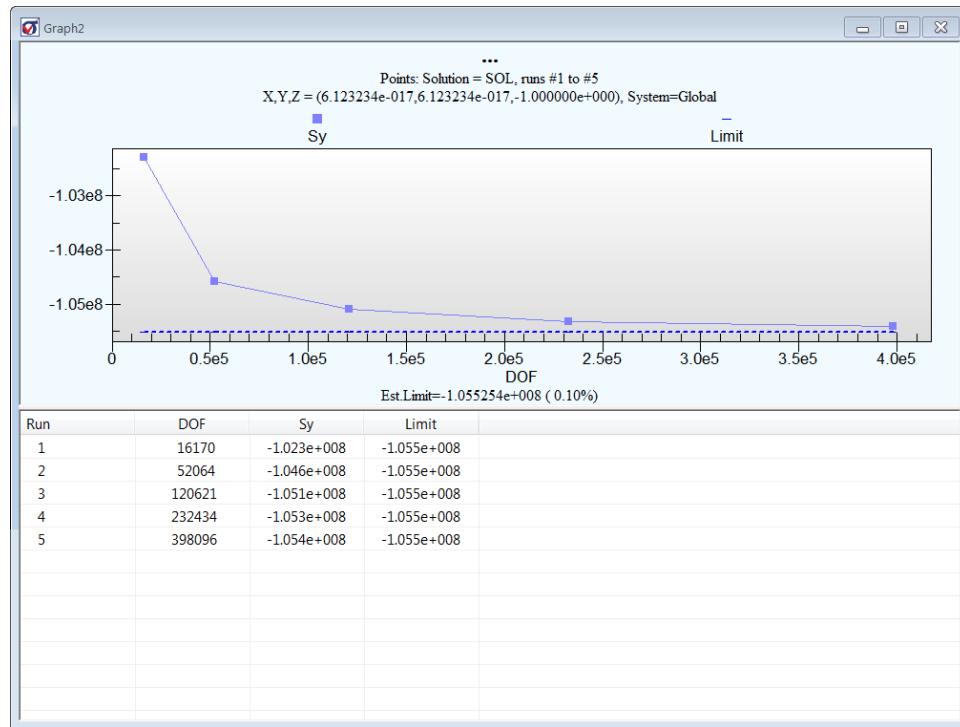


Figure 28. Convergence evidence for dense tetra mesh