2D Linearized Potential Flow equations for compressible flow

We start with 2D - unsteady - inviscid equations in the conservative form

$$\frac{\partial \bar{Q}}{\partial t} + \frac{\partial \bar{E}}{\partial x} + \frac{\partial \bar{F}}{\partial y} = \vec{H}$$

where $\bar{Q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e_t \end{bmatrix}$, $\bar{E} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u e_t + p u \end{bmatrix}$, $\bar{F} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v e_t + p v \end{bmatrix}$, $H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \rho \dot{q} \end{bmatrix}$

Note :- $u \rightarrow x$ component of the velocity

$v \rightarrow y$ " " " "

$v \neq 1/\rho$ as we have used throughout the course.

Also, $e_t = e + \dfrac{u^2 + v^2}{2}$

e = Internal energy

$= h - p/\rho$

h = enthalpy

$\Rightarrow$ Clearly, from this we see that

$$h_0 = h + \frac{u^2 + v^2}{2}$$
$$= e + p/\rho + \frac{u^2 + v^2}{2}$$

$$h_0 = e_t + p/\rho$$

Exercise :- Try to apply the assumptions of 1D - steady - inviscid - constant area - adiabatic flow conditions and derive the three conservation eq^n's that derived earlier in the course.

Note :- Governing eq^n's for Quasi-1D - CD nozzle cannot be derived directly from above general equations.

This is termed as the conservative form of equations because we are using $[\rho, \rho u, \rho v, \rho e_t]$ as the thermodynamic variables instead of $[\rho, u, v, p]$.

→ Direct solution of these equations is difficult.

→ Also, these equations do not assume isentropic conditions (as we have assumed throughout in the 1D flow).

So the question is, How to impose isentropicity on these eq^n's ?

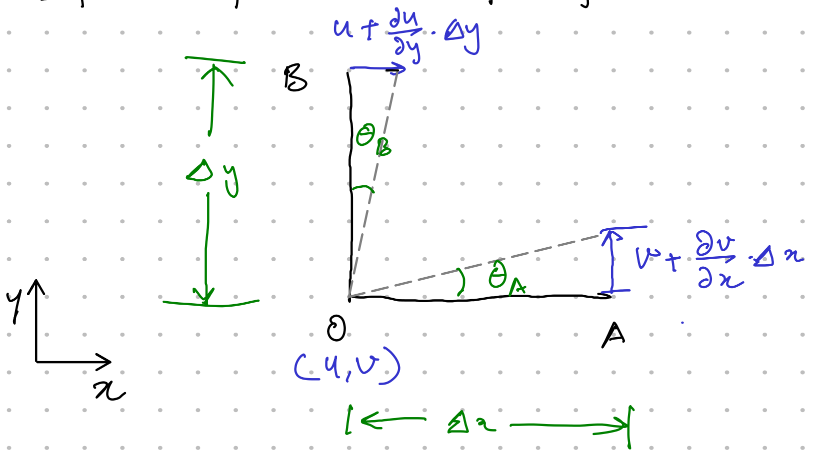
→ Answer, comes in the form of Crocco's theorem.

If a flow is irrotational, then it is also isentropic.

So in order to derive the governing equations under isentropic conditions, all I have to do is impose irrotationality.

What is Irrotational flow?

Definition of rotation of a fluid element.



At time t,
point O has velocity (u, v)
point A
$\left(u+\frac{\partial u}{\partial x}\cdot\Delta x,\ v+\frac{\partial v}{\partial x}\cdot\Delta x\right)$
point B
$\left(u+\frac{\partial u}{\partial y}\Delta y,\ v+\frac{\partial v}{\partial y}\Delta y\right)$

Rotation of a fluid element is defined as the average of the rotations of two fluid elements placed perpendicular to each other.

In time Δt, elements (A) & (B) will move. For rotation, only $v|_A$ & $u|_B$ are of interest.

$$\therefore \text{Rotation } \omega|_O = \frac{\omega|_A+\omega|_B}{2}$$

$$= \frac{\theta|_A+\theta|_B}{2\Delta t}$$

Now, $\theta|_A = \left(v+\frac{\partial v}{\partial x}\cdot\Delta x - v\right)\cdot\frac{\Delta t}{\Delta x} = \frac{\partial v}{\partial x}\cdot\Delta t$

Also, $\theta|_B = \left(u+\frac{\partial u}{\partial y}\cdot\Delta y - u\right)\cdot\frac{\Delta t}{\Delta y} = \frac{\partial u}{\partial y}\Delta t$

Also by right hand rule, $\theta|_A$ is positive & $\theta|_B$ is negative.

$$\therefore \theta|_B = -\frac{\partial u}{\partial y}\Delta t$$

$$\therefore \omega|_O = \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

It can be shown for a general 3D case that,

$$\underbrace{\bar{\omega}}_{\text{Rotation vector}} = \frac{1}{2}\underbrace{(\bar{\nabla}\times\bar{V})}_{\text{Curl}},$$

where $\bar{\omega} = \hat{i}\omega_x+\hat{j}\omega_y+\hat{k}\omega_z$

$\bar{V} = u\hat{i}+v\hat{j}+w\hat{k}$

Also, vorticity (ξ) is defined as

$$\bar{\xi} = 2\bar{\omega}$$

So, now have a definition of rotation of a fluid element.

$\therefore$ Clearly, in a 2D flow, if

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad \text{everywhere}$$

then I have a irrotational flow-

Crocco tells me that such a flow will also be isentropic.

So now the question is :-

Can I derive simpler set of equations under these assumptions of steady, 2D, adiabatic, isentropic flow.

I can follow the lead from incompressible flow, and define a velocity potential. Just an idea.

[Note:- Please remember the potential flow equation $\nabla^2\phi = 0$ that you derived in Aerodynamics. We wish to derive something similar for compressible flow.]

Velocity potential

Now, under the assumption of isentropic flow,

$\Big\downarrow$ Crocco's theorem

Irrotational flow

$\Big\downarrow$

Potential flow

Since I know the flow is irrotational,

I have $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$ everywhere.

So clearly if I define a function $\phi(x,y)$ which is smooth enough and also for which

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \text{ then the flow}$$

represented by such a function we always be irrotational.

$$\therefore \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \underbrace{\frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial x \partial y}}_{\phi \text{ is sufficiently smooth}} = 0$$

So a potential flow is always irrotational and Crocco's theorem tells us that an irrotational flow is isentropic.

So all I have to do is get a equation for ϕ, solve it and I will know the flow properties everywhere.

This equation is called Potential equation.

we will replace, 3 governing equations with a single one.

So by considering the mass conservation, we have

$$(\rho \phi_x)_x + (\rho \phi_y)_y = 0$$

$$\therefore \quad \rho_x \phi_x + \rho_y \phi_y + \rho(\phi_{xx} + \phi_{yy}) = 0 \quad \text{———} \quad \alpha$$

Momentum:-

$$(\rho u^2 + P)_x + (\rho u v)_y = 0 \quad - \text{C}$$

$$(\rho u v)_x + (\rho v^2 + P)_y = 0 \quad - \text{D}$$

Adding, $\text{C}\, dx + \text{D}\, dy$

$$P_x dx + P_y dy + \left[\rho_x (u^2) + \rho(u^2)_x + (\rho v)_y u + u_y (\rho v)\right] dx + \left[\rho_y v^2 + \rho (v^2)_y + (\rho u)_x \cdot v + v_x \cdot (\rho u)\right] dy = 0$$

Clearly, $\underset{\text{Total derivative}}{dp} = P_x dx + P_y dy$, / $u_y = v_x$ / $(\rho v)_y = -(\rho u)_x$

$$\therefore \; dp + \rho\left[(u^2)_x \cdot dx + (v^2)_y dy\right] - \left[(\rho u)_x u\, dx + (\rho v)_y v\, dy\right] + \rho_x u^2 dx + \rho_y v^2 dy + u_y (\rho v) dx + v_x (\rho u) dy = 0$$

Now, the blue part can

$$dp + \rho\left[(u^2)_x dx + (v^2)_y dy\right] - \cancel{\rho_x u^2 dx} - \rho u_x u\, dx - \cancel{\rho_y v^2 dy} - \rho v_y v\, dy$$
$$+ \cancel{\rho_x u^2 dx} + \cancel{\rho_y v^2 dx} + v_x(\rho v)\, dx + u_y(\rho u)\, dy = 0$$

* Irrotational

$$\therefore dp + \rho\left[\qquad\qquad\right] - \rho\frac{(u^2)_x}{2} dx - \rho\frac{(v^2)_y}{2} dy$$
$$+ \rho\frac{(v^2)_x}{2} dx + \rho\frac{(u^2)_y}{2} dy = 0$$

Gives,
$$\frac{dp}{\rho} + \frac{d(u^2+v^2)}{2} = 0$$

In terms of velocity potential,

$$dp = \frac{-\rho}{2} d\left[(\phi_x)^2 + (\phi_y)^2\right]$$

Since for isentropic flow,

$$a^2 = \frac{dp}{d\rho}$$

$$dp = a^2 d\rho = -\frac{\rho}{2} d\left[(\phi_x)^2 + (\phi_y)^2\right]$$

Since $d\rho = \rho_x \cdot dx + \rho_y dy$, we can write

$$\left.\begin{aligned} \rho_x &= \frac{\rho}{a^2}\left(\phi_x\phi_{xx} + \phi_y\phi_{xy}\right) \\ \rho_y &= \frac{\rho}{a^2}\left(\phi_x\phi_{xy} + \phi_y\phi_{yy}\right) \end{aligned}\right\} \text{——} \beta$$

From α & β,

$$\phi_{xx} + \phi_{yy} - \frac{1}{a^2}\left[(\phi_x)^2\phi_{xx} + (\phi_y)^2\phi_{yy}\right] - \frac{2}{a^2}\left[\phi_x\phi_y\phi_{xy}\right] = 0 \quad \text{——} \gamma$$

Q:- What type of PDE is this?

Q:- For incompressible flow, what will this reduce to?

Now, still we do not have a PDE for ϕ as, "a" is not known.
we need to get "a" in terms of ϕ.
we use energy equation for that.

Now $h_0 = \text{const.}$

$$\therefore h_0 = C_p T_0 = C_p T + \frac{u^2+v^2}{2}$$

For a perfect gas, $C_p = \frac{\gamma R}{\gamma - 1}$, $a^2 = \gamma R T$,

$$T = T_0 - \frac{u^2+v^2}{2C_p}$$

$$\therefore \gamma R T = \gamma R T_0 - \frac{\gamma R(\gamma-1)(u^2+v^2)}{2\gamma R}$$

$$\therefore a^2 = a_0^2 - \frac{\gamma-1}{2}(u^2+v^2)$$

$$\therefore a^2 = a_0^2 - \frac{\gamma-1}{2}\left((\phi_x)^2 + (\phi_y)^2\right) \quad \text{——} \quad (\delta)$$

Speed of sound at stagnation condition remains constant as T_0 is constant.
Hence this not a variable. But a constant.

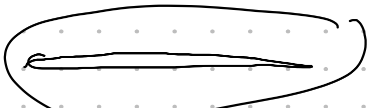
Putting (δ) in (γ), we get the complete potential flow equation.

Difficult to solve in all general cases. But analytical sol^{n}s exist for some simple problems.

Linearise the eq^{n}s about a given condition.

$u = U_\infty + u_p$

$v = 0 + v_p$



→ Linearised potential flow eq^n.

$$(1-M^2)\,\phi_{xx} + \phi_{yy} = 0$$

→ Similarity rules

⇓

L, D, Thin airfoil theory