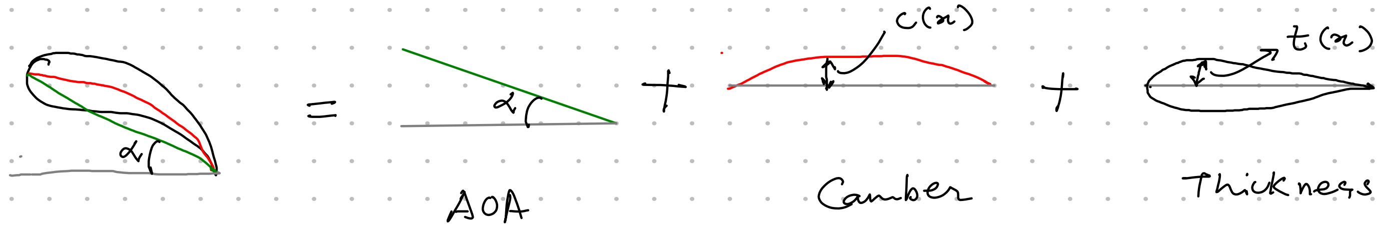
Thin airfoil theory

We have seen that $C_p = \frac{2\theta}{\sqrt{M^2-1}}$ under the assumption of θ being small.

This relationship is linear in θ.

Let us consider a general thin airfoil.



$$\therefore \quad C_p = C_{p_{AOA}} + C_{p_{camber}} + C_{p_{thickness}}$$

Another way at looking at the problem is

$$\theta = \frac{dy}{dx} \quad (\text{for small } \theta)$$

$$\therefore \quad C_p = \frac{2\frac{dy}{dx}}{\sqrt{M^2-1}}.$$

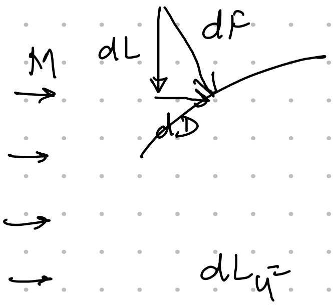
$$\therefore \quad C_{p_u} = \frac{2}{\sqrt{M^2-1}} \frac{dy_u}{dx}$$

$$C_{p_l} = \frac{-2}{\sqrt{M^2-1}} \frac{dy_l}{dx}$$

Clearly,
$$\frac{dy_u}{dx} = -\alpha + \frac{dt(x)}{dx} + \frac{dC(x)}{dx}$$

$$\frac{dy_l}{dx} = -\alpha - \frac{dt(x)}{dx} + \frac{dc(x)}{dx}$$

Consider small elements of upper surface



$$df = p(x)\,dx$$
$$= (p(x) - p_1)\,dx + p_1\,dx$$
$$= q_1 \cdot C_{p_u}\,dx + p_1\,dx$$

$$dL_u = -df\cos\theta \approx -df = -q_1 C_{p_u}\,dx - p_1\,dx$$
$$dD_u = df\sin\theta \approx df\,\theta = df\cdot\frac{dy_u}{dx} = q_1 C_{p_u}\frac{dy_u}{dx}\cdot dx + p_1\frac{dy_u}{dx}\,dx$$

Similarly, lower surface

$$dL_l = df = q_1 C_{p_l}\,dx + p_1\,dx$$
$$dD_l = -df\,\theta = -C_{p_l}\frac{dy_l}{dx}\,dx - p_1\frac{dy_l}{dx}\,dx$$

Total lift

$$L = \int_0^c (dL_u + dL_l)$$
$$= q_1\int_0^c (C_{p_l} - C_{p_u})\,dx$$

$$D = q_1\int_0^c \left(C_{p_u}\frac{dy_u}{dx} - C_{p_l}\frac{dy_l}{dx}\right)dx + p_1\int_0^c\left(\frac{dy_u}{dx} - \frac{dy_l}{dx}\right)dx$$

After putting in the values of C_{p_u} & C_{p_l}, we get

$$L = \frac{4\alpha q_1\cdot c}{\sqrt{M^2-1}} \Rightarrow C_L = \frac{L}{q_1 c} = \frac{4\alpha}{\sqrt{M^2-1}}$$

Note :- Here an assumption of $t(x=0) = 0$, $C(x=0) = 0$, $t(x=c) = 0$, $C(x=c) = 0$ has been made.

For a supersonic thin airfoil, AOA is the only deciding factor for C_L.

Similarly, after a lot of algebra (which you should carry out)

$$D = q_1\int_0^c \frac{2}{\sqrt{M^2-1}}\left[\left(\frac{dy_u}{dx}\right)^2 + \left(\frac{dy_l}{dx}\right)^2\right]dx + p_1\int_0^c \underbrace{\left[\frac{dy_u}{dx} - \frac{dy_l}{dx}\right]}_{=0 \text{ as } t(x=0)=t(x=c)=0}dx$$

$$\therefore D = \frac{2q_1}{\sqrt{M^2-1}}\int_0^c\left\{\left[-\alpha + \frac{dt(x)}{dx} + \frac{dC(x)}{dx}\right]^2 + \left[-\alpha - \frac{dt(x)}{dx} + \frac{dC(x)}{dx}\right]^2\right\}dx$$

$$= \frac{2q_1}{\sqrt{M^2-1}} \left\{ \int_0^c \left[2\alpha^2 + 2\left(\frac{dt(x)}{dx}\right)^2 + 2\left(\frac{dC(x)}{dx}\right)^2 \right] dx - \int_0^c 4\alpha \frac{dC(x)}{dx} dx \right\}$$

Now, $\displaystyle\int_0^c \left(\frac{dt}{dx}\right)^2 dx = \underbrace{\overline{\left(\frac{dt}{dx}\right)^2}}_{\text{Average value}} c$, $\displaystyle\int_0^c \left(\frac{dC}{dx}\right)^2 dx = \overline{\left(\frac{dC}{dx}\right)^2} C$

$$\Rightarrow C_D = \frac{4}{\sqrt{M^2-1}} \left[\alpha^2 + \overline{\left(\frac{dt}{dx}\right)^2} + \overline{\left(\frac{dC}{dx}\right)^2} \right]$$

Induced drag

Form drag.

Reference :- Section 13.6 of John & Keith

& " 6.15 of Rathakrishnan.