~ Mathematics 2 ~

Notes By
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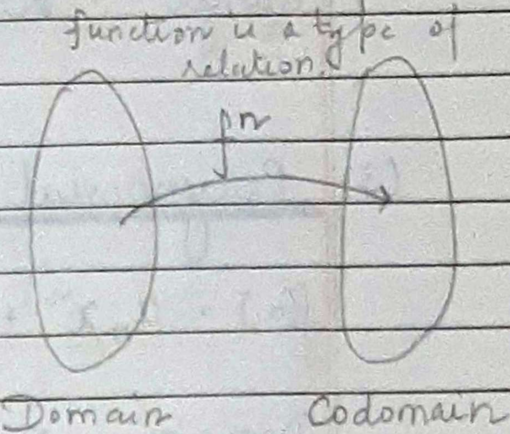
WEEK 1

SOME TOPICS FROM MATHS 1

1. What is a function?

→ One input can't give 2 different outputs

→ Each element of the domain must have an output



\# $f : X \longrightarrow Y$

Domain $= X$

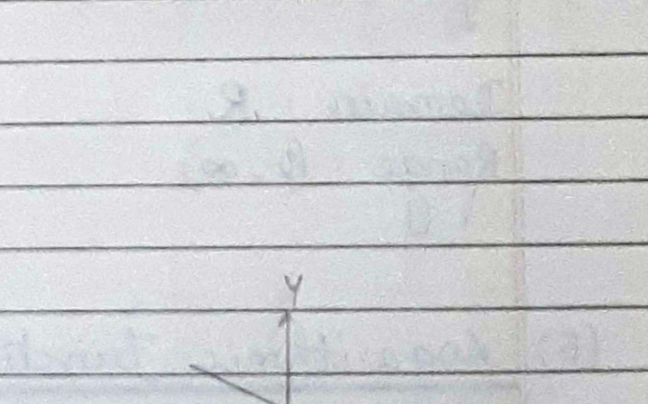
Codomain $= Y$

Range $\subseteq$ Codomain

\# Range $= \{ f(x) \mid x \in X \}$

\# Examples :

(1) Linear Functions



$f : \mathbb{R} \longrightarrow \mathbb{R}$

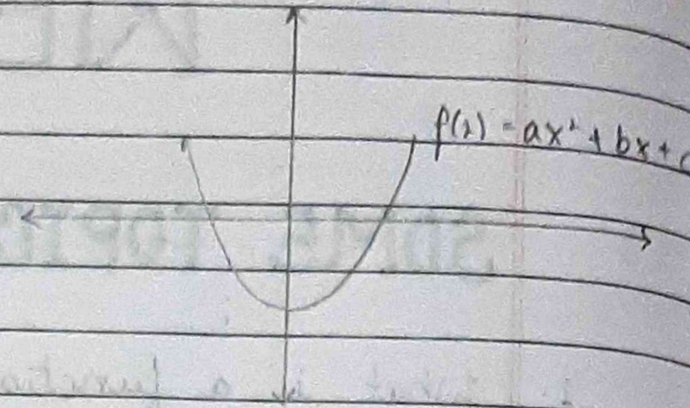
$f(x) = mx + c$ graph : straight line

Domain $= \mathbb{R}$; Range $= \mathbb{R}$

(2) Quadratic Functions

$f(x) = ax^2 + bx + c$

graph = parabola



Domain $= \mathbb{R}$

Range $= \mathbb{R}^{+}$

(3) Polynomial Functions

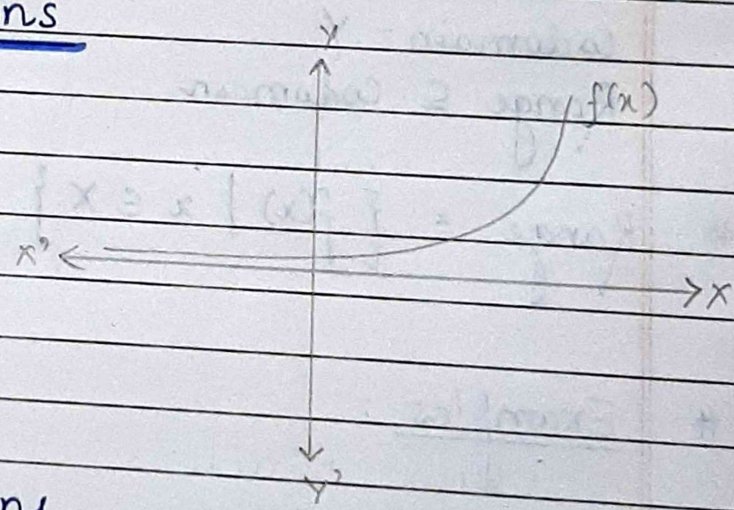
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$

Domain : $\mathbb{R}$

Range : depends on degree of highest x

(4) Exponential Functions

$f(x) = a^x$

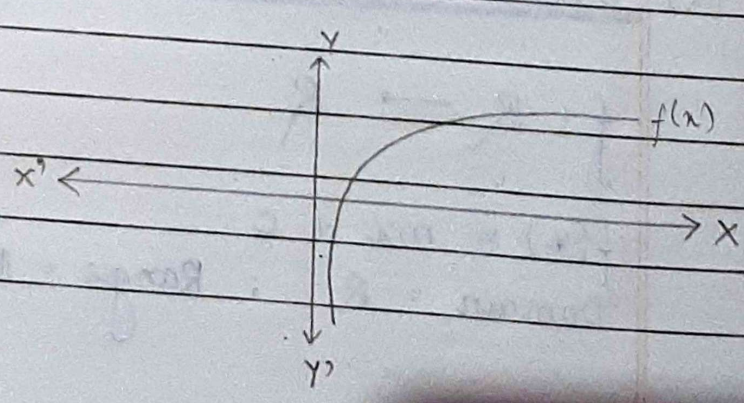


Domain : $\mathbb{R}$

Range : $(0, \infty)$

(5) Logarithmic Functions

$f(x) = \log_a(x)$

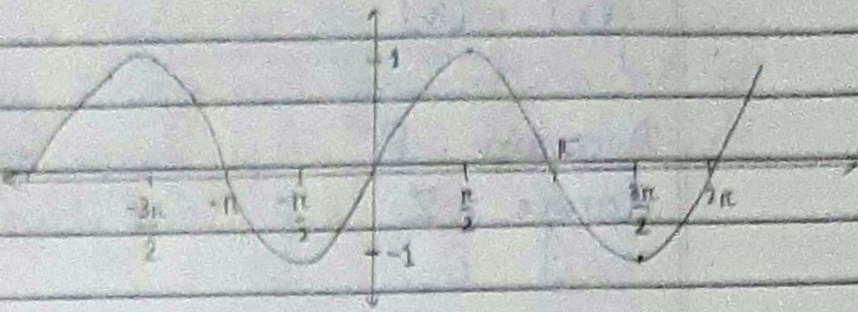


Domain : $(0, \infty)$

Range : $\mathbb{R}$

(6) Trignometric functions

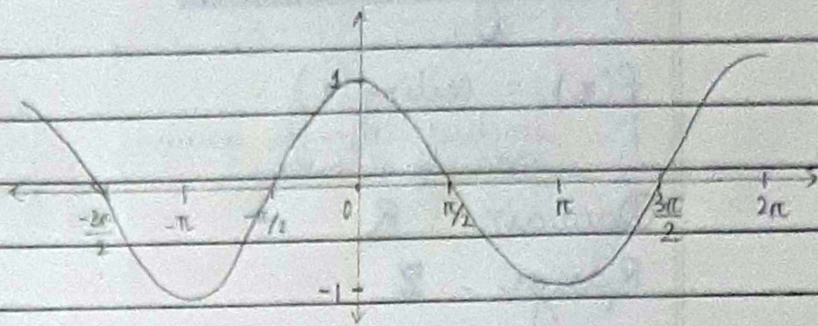
(i) sine function



$f(x) = \sin(x)$
Domain : R
Range : $[-1, 1]$

It is a period function.

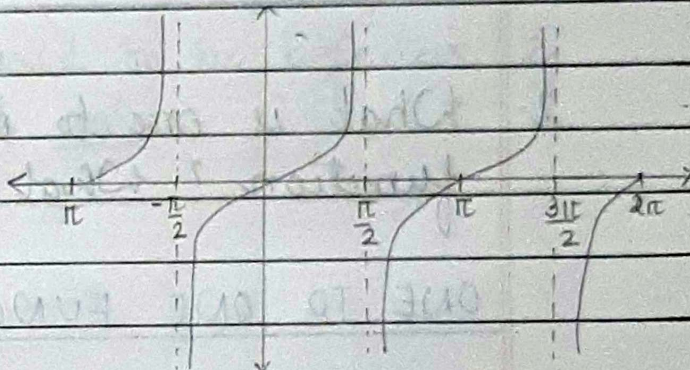
(ii) cosine function



$f(x) = \cos(x)$
Domain : R
Range : $[-1, 1]$

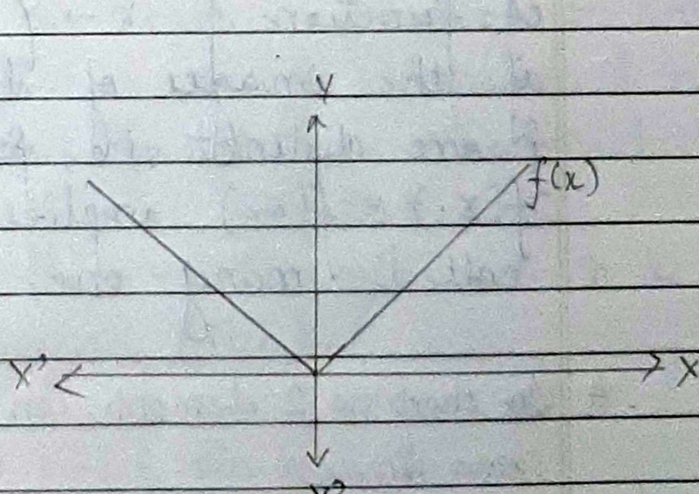
It is a period function

(iii) tangent function



$f(x) = \tan(x)$
Domain : $R - \{\frac{\pi}{2} + n^* \pi\}$
Range : R

(7) Modulus Function



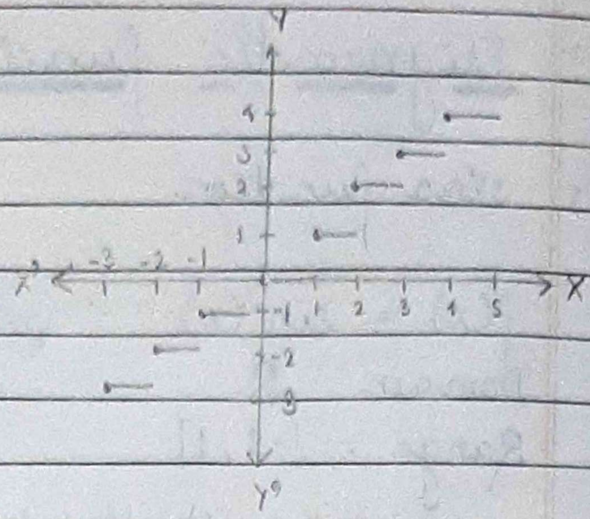
$f(x) = |x|$

Domain : R
Range : $[0, \infty)$

(8) **Floor function**

$f(x) = [x]$

greatest integer smaller than equal to x.



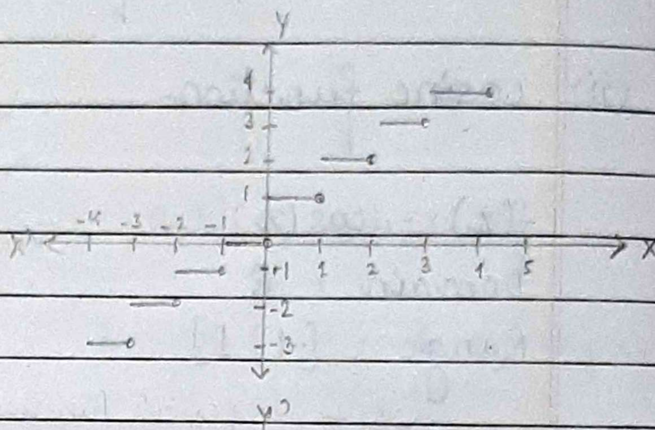
Domain : $\mathbb{R}$

Range : $\mathbb{Z}$

(9) **Ceiling Function**

$f(x) = \text{ceiling}(x)$

smallest integer greater than equal to x.



Domain : $\mathbb{R}$

Range : $\mathbb{Z}$

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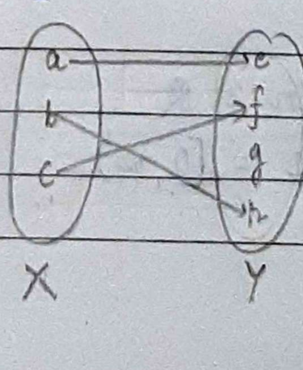
2. What is one to one function? What is onto function? What is bijective function?

ONE TO ONE FUNCTION (Injective)

A function $f: X \to Y$ is defined to be one-one, if the images of distinct elements of X under f are distinct, i.e, for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called many-one.



Example:
$f(x) = ax + b$

In short no 2 elements can have same image.

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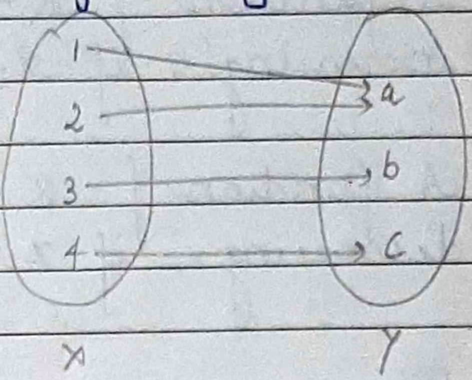
ONTO FUNCTION (SURjective)

A function $f: X \to Y$ is said to be onto, if every element of Y is the image of some element of X under f, i.e, for every $y \in Y$, there exists an element in X such that $f(x) = y$.

$f: X \to Y$ is onto if and only if Range = Codomain



BIJECTIVE FUNCTION

A function $f: X \to Y$ is said to be bijective if f is both one-one and onto.

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3. What is bounded function?

The functions that have at least 1 pair of m and n such that $m \leq f(x) \leq n$, where m & n $\in R$, are called bounded functions. The greatest such value of 'm' is known as Greatest Least Bound (glb) and smallest value of such n is known as Least Upper Bound (lub).

Example: $f(x) = \sin x \Rightarrow -1 \leq \sin x \leq 1$

lower bound

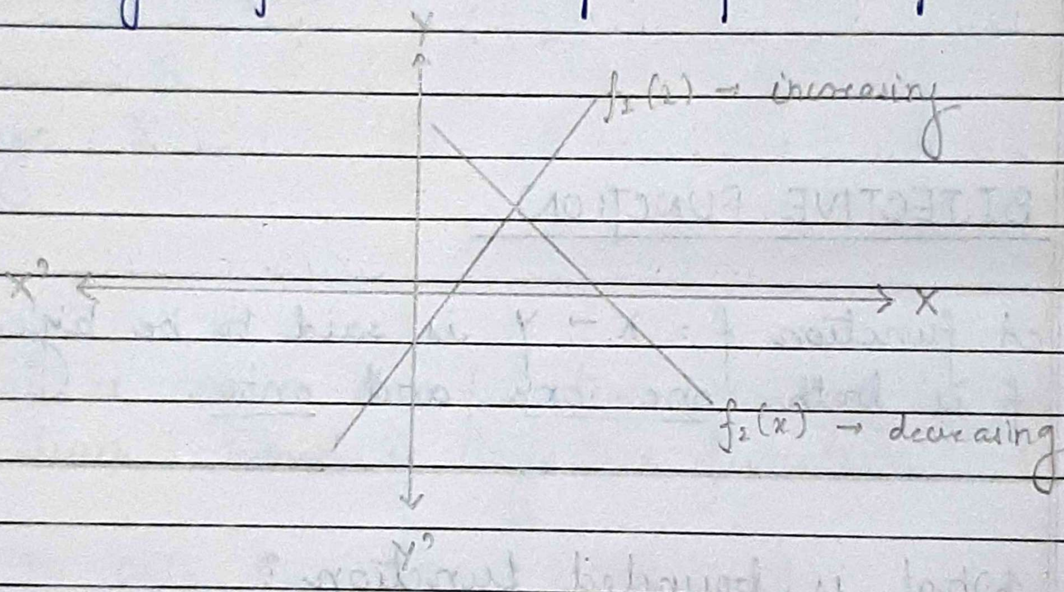
$\text{Range}(f) \subseteq [m, n]$

upper bound

4. Monotonicity of functions

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be monotone increasing, if $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$.

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be monotone decreasing, if $x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$.

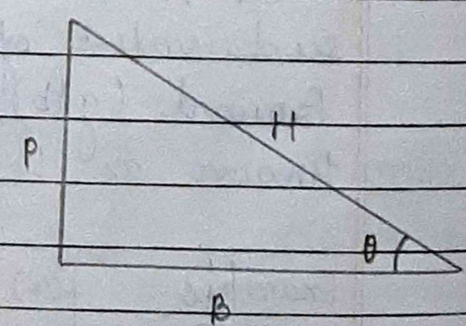


5. Properties of Basic Trignometric Functions

$\sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{P}{H}$

$\cos\theta = \frac{1}{\sec\theta} = \frac{B}{H}$

$\tan\theta = \frac{1}{\cot\theta} = \frac{P}{B}$



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$\sin^2\theta + \cos^2\theta = 1$

$\frac{\sin\theta}{\cos\theta} = \tan\theta$

$\sin(-\theta) = -\sin(\theta)$ $\quad$ $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$
$\cos(-\theta) = \cos\theta$ $\quad$ $\sec(-\theta) = +\sec\theta$
$\tan(-\theta) = -\tan\theta$ $\quad$ $\cot(-\theta) = -\cot\theta$

sin x and cos x are period functions.

Period fn : $f(x+a) = f(x)$

Example: $f(x) = \sin x$ $\quad$ Range $\in [-1, 1]$

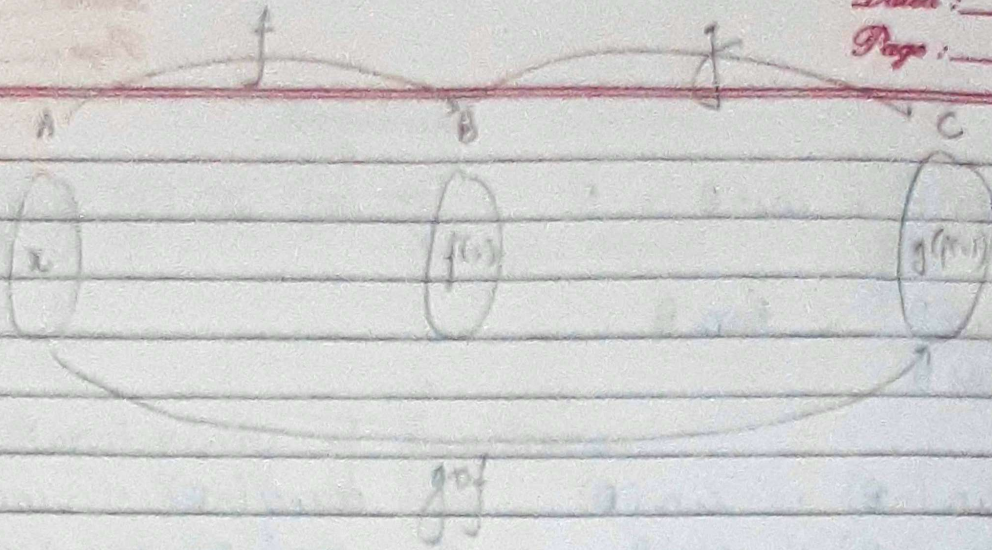
$\sin\frac{\pi}{2} = 1$, $\sin\left(\frac{\pi}{2} + 2\pi\right) = 1$

$\sin\left(\frac{\pi}{2} + 4\pi\right) = 1$

6. Composition of Two Functions

Let $f: A \to B$ and $g: B \to C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof: A \to C$ given by

$$gof(x) = g(f(x)), \forall x \in A$$



7. Invertible Function

A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}.

If 'f' is invertible, the f must be one-one & onto and conversely, if f is one-one and onto, then f must be invertible.

8. Tangents

A tangent line to a curve C at a point p (on C) often has a property that is passes through the point p but does not intersect the curve C_1, in any other point close to the point p.

If there is a sharp change in the slope of the curve then tangent does not exist.

9. CURVE : A curve is a figure that is obtained as the path of a moving point. A curve can be thought of as a figure obtained by bending a line at various places.

LIMITS FOR SEQUENCES

WHAT IS A SEQUENCE ?

A sequence is an arrangement of any objects or a set of numbers in a particular order followed by some rule. If $a_1, a_2, a_3, a_4, \ldots$ etc. denote the terms of a sequence, then 1, 2, 3, 4 ... denotes the position of the term.

A sequence can be defined based on the no. of terms, i.e, either finite sequence or infinite sequence.

Example: Arithmetic sequence, Geometric sequence etc. Also, $\left\{\frac{1}{n}\right\}$, $\{n\}$, $\{(-1)^n\}$, etc are sequences.

(1) $\left\{\frac{1}{n}\right\} \rightarrow \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

(2) $\{n\} \rightarrow \{1, 2, 3, 4, \ldots\}$

(3) $\{(-1)^n\} \rightarrow \{-1, 1, -1, 1, \ldots\}$

→ Sequence is a function that is defined from $\mathbb{N}$ to $\mathbb{R}$.

$$f : \mathbb{N} - \{0\} \longrightarrow \mathbb{R}$$

WHAT IS THE LIMIT OF A SEQUENCE?

Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ has ~~a~~ limit $a \in \mathbb{R}$ if as n increases, the numbers a_n come closer and closer to a.

Other equivalent terminology:

1. $\{a_n\}$ tends to a.
2. $\{a_n\}$ converges to a.
3. $\lim\limits_{n \to \infty} a_n = a$
4. $a_n \longrightarrow a$
5. $a_n \xrightarrow{n \to \infty} a$
6. $\lim a_n = a$
7. $\lim\limits_{n \to \infty} \{a_n\} = a$
8. $\lim \{a_n\} = a$

CONVERGENT & DIVERGENT SEQUENCES

→ A sequence $\{a_n\}$ is called <u>convergent</u> if it converges to some limit (i.e a real no.)

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Example : the sequence $\{\frac{1}{n}\}$ is convergent and has limit 0.

-> A sequence $\{a_n\}$ is called divergent, if it does not have any limit.

Example : the sequence $\{(-1)^n\}$ is divergent.

SUBSEQUENCE

A subsequence of a sequence is a new sequence formed by (possibly) excluding some entries of a sequence.

Example : sequence : $-1, 1, -1, 1, -1, 1, \ldots$
subsequence : $1, 1, 1, 1, 1, 1, \ldots$

MORE EXAMPLES OF CONVERGENT AND DIVERGENT SEQUENCES

1. The sequence $\{n\}$ is divergent.
2. The sequence $\{-n\}$ is divergent.
3. Let $x \in \mathbb{R}$. Then the sequence $\left\{\sum_{k=0}^{n} \frac{x^k}{k!}\right\}$ is convergent and converges to e^x.

4. Let $x \in \mathbb{R}$. Then $\left\{ \left(1 + \frac{x}{n}\right)^n \right\}$ coverges to e^x.

5. The sequence $\left\{ n \left(\frac{\sqrt{2\pi n}}{n!} \right)^{1/n} \right\}$ converges to e.

6. The sequence $\left\{ \frac{n}{\sqrt[n]{n!}} \right\}$ converges to e.

USEFUL RULES REGARDING CONVERGENCE OF SEQUENCES

1. If $a_n \to a$, then every subsequence of $\{a_n\}$ also converges to a.

2. If $a_n \to a$ and $b_n \to a$, then $a_n + b_n \to a+b$

3. If $a_n \to a$ and $c \in \mathbb{R}$, then $c \cdot a_n \to c \cdot a$

4. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.

5. If $a_n \to a$ and $b_n \to b$, then $a_n - b_n \to a - b$

6. If $a_n \to a$ and f is a polynomial function in one variable, then $f(a_n) \to f(a)$

7. If $a_n \to a$ and $b_n \to b$ & $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$

8. If $a_n \to a$ & $c \in \mathbb{R}$, then $c^{a_n} \to c^a$

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9. If $a_n \to a$ and $c \in \mathbb{R}$ such that $a_n > 0 \ \forall n$ and $a, c > 0$, then $\log_c(a_n) \to \log_c(a)$

10. <u>The Sandwich Principle</u>: If $a_n \to a$ and $b_n \to a$ and $\{c_n\}$ is a sequence such that $a_n \leq c_n \leq b_n$, then $c_n \to a$.

Limits for functions in one variable

LIMIT OF A FUNCTION AT A POINT FROM THE LEFT

Let f be a function and 'a' be a point such that $a_n \to a$ where a_n belongs to the domain of definition of f.

If there is a real no. L such that $f(a_n) \to L$ for all sequences a_n such that $a_n \to a$ and $a_n < a$, then we say that the limit of f at a from the left exists and equals L.

We denote this by $\lim\limits_{x \to a^-} f(x) = L$.

If there is no such number L then we say that the limit of f at 'a' from the left does not exist.

An equivalent way of thinking of $\lim_{x \to a^-} f(x) = L$ is that as x comes closer and closer to 'a' from the left, f(x) eventually comes closer & closer to L.

LIMIT OF A FUNCTION AT A POINT FROM THE RIGHT

Similarly, if there is a real number R such that $f(a_n) \xrightarrow{L} R$ for all sequences a_n such that $a_n \longrightarrow a$ and $a_n > a$, then we say that the limit of f at a from the right exists and equals R.

We denote this by $\lim_{x \to a^+} f(x) = L$.

If there is no such number R then we say that the limit of f at a from the right does not exist.

An equivalent way of thinking of $\lim_{x \to a^+} f(x) = R$ is that as x comes closer & closer to a from the right, f(x) eventually comes closer and closer to R.

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LIMIT OF A FUNCTION AT A POINT

Let f be a function and a be a point such that $a_n \longrightarrow a$ where a_n belongs to the domain of definition of f.

Suppose the limit of f at a from both sides (i.e left and right) exist and are equal, i.e, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$. Then we say that the limit of f at a exists and equals the left (& right) limit.

We denote it by $\lim_{x \to a} f(x)$.

THE LIMIT AS $x \longrightarrow (\pm)\infty$

- Let f be a function such that there is an M such that it is defined for all $x > M$.

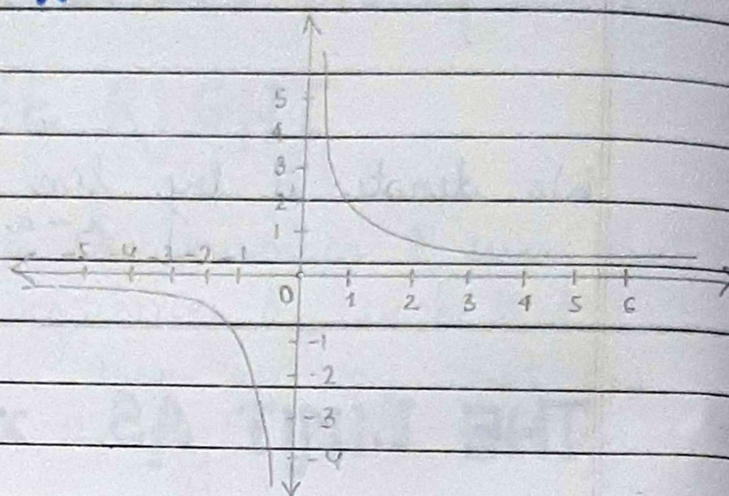
  Suppose for all sequences x_n diverging to ∞, there exists L such that $f(x_n)$ converges to L (i.e, as x becomes larger and larger, $f(x)$ eventually gets closer and closer to L). Then we say that $\lim_{x \to \infty} f(x)$ exists and equals L.

- Similarly, let f be a function such that there is an N such that it is defined for all $x < N$.

Suppose for all sequences x_n diverging to $-\infty$, there exists L such that $f(x_n)$ converges to L (i.e. as x becomes smaller and smaller $f(x)$ eventually gets closer & closer to L). Then we say that $\lim_{x \to -\infty} f(x)$ exists & equals L.

THE FUNCTION $\frac{1}{x}$

$$\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$$



Both,

$\lim_{x \to 0^-} \frac{1}{x}$ and $\lim_{x \to 0^+} \frac{1}{x}$

do not exist.

SOME KNOWN LIMITS

1. $$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

2. $$\lim_{x \to 0} \frac{1 + \log_e(x)}{x} = 1$$

3. $$\lim_{x \to \infty} \frac{a + be^x}{c + de^x} = \frac{b}{d}$$

CONTINUITY OF A F^N AT A POINT

Let f be a function and a be a point such that $a_n \to a$ where a_n and a belong to the domain of f.

Then the function f is said to be continuous at the point a if $\lim_{x \to a} f(x) = f(a)$.

i.e, continuity means that limit at 'a' can be obtained by evaluating at function at a.

Useful Rules Regarding Continuity of a function at a point

If $\lim_{x \to a} f(x) = F$, $\lim_{x \to a} g(x) = G$, then (and $c \in \mathbb{R}$)

(1) $\lim_{x \to a} (f+g)(x) = F + G$

(2) $\lim_{x \to a} (cf)(x) = cF$

(3) $\lim_{x \to a} (f-g)(x) = F - G$

(4) $\lim_{x \to a} (fg)(x) = FG$

(5) $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{F}{G}$

(6) The Sandwich Principle: if $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = L$ and $h(x)$ is a f^n such that $f(x) \leq h(x) \leq g(x)$ then $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$