The Shape Matching Element Method for Meshless Animation of NURBs Models

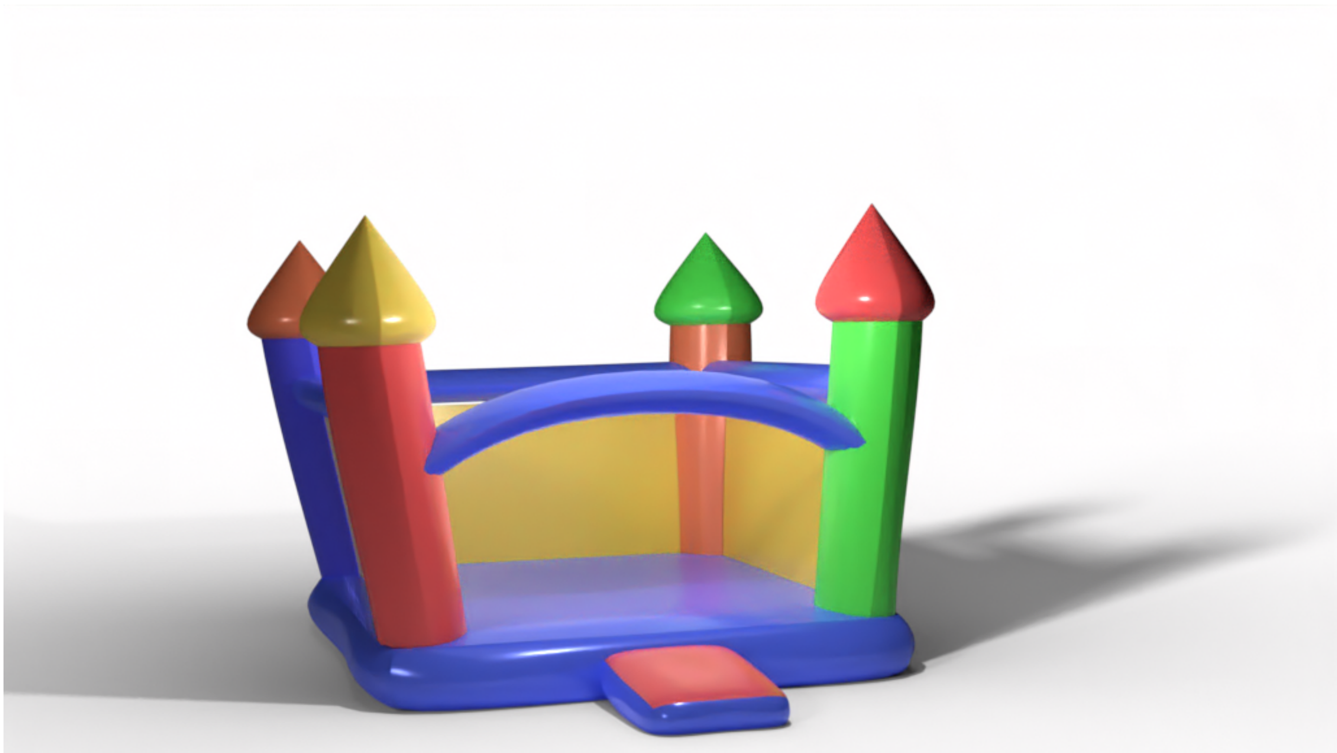

Figure 1: tmp

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1 INTRODUCTION

The consumption of geometric surface models by physics-based animation algorithms is fraught with difficulty. For volumetric objects, this process often involves identifying and discretizing the interior of the modelled object, typically either as a tetrahedral or hexahedral mesh.This procedure is both expensive and difficult, especially if the surface model is constructed from higher-order boundary representations, or if the volumetric discretization is required to be conforming or feature aligned. Removing explicit volumetric discretization from the physics-based-animation pipeline can avoid these difficulties and also provide a more unified modelling and simulation experience.

NURBS (Non-uniform Rational B Splines) are a popular higher-order modelling primitive which are used for computer-aided design (CAD), computational fabrication and computer animation. NURBS primitives were the first geometric representation used for physics-based animation, yet, despite over three decades of research, animation of NURBS objects remains a challenge.

Isogeometric Analysis (IGA) is a physics simulation methodology that uses the control variables of the NURBS model as the degrees-of-freedom (DOFS) of the simulation itself. Unfortunately IGA approaches for volumetric objects still require background volumetric structures, typically regular grids that make satisfaction of boundary conditions difficult (which makes collision resolution difficult) or more complicated cut-cell grids which introduce non-trivial root finding problems into the mix. Crucially, these simulation schemes typically assume models arise from engineering applications and meet tight geometric criteria such as that the mesh is manufacturable. These inputs are much cleaner than those produced by a typical modeller.

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We present the first truly meshless (no volumetric discretization generated) algorithm for direct, nonlinear elastodynamic simulation of NURBs models. Our nonlinear elastodynamic simulation scheme requires only a boundary description of the object (we do not strictly require a solid model) and approriate physical parameters. Because we explicitly use the NURBS boundary representation in the simulation its is straightforward to handle Dirichilet, Neumann boundary conditions and to apply contact resolution. Crucially because we broadly target animation not necessarily simulation for engineering or manufacturing we don't require that models satisfy the rigourous geometric requirements common for these applications

Our approach is an extension of the recently developed Virtual Element Method (VEM) for solving PDEs on domains tiled with arbitrary polygons. We establish a connection between VEM and the well-known shape matching simulation algorithm which enables us to derive equations of motion for an arbitrary NURBS model using Lagrangian Mechanics. Importantly, we show how to replace volumetric data structures for integration with ray casting approaches which enables our meshless approach to isogeometric elastodynamic physically-based animation of volumetric structures.

2 RELATED WORK

2.1 Shape Matching Related Papers

(1) *Shape Matching* [Müller et al. 2005]: Meshless simulation by fitting a polynomial describing the shapes deformation using only the nodal values.
(2) *Lattice Shape Matching* [Rivers and James 2007]: Voxelize model to construct a lattice of cubes. Use these lattice cubes to construct overlapping shape matching regions (just like clustered shape matching?). The original mesh is deformed using trilinear interpolation of lattice vertex positions.
(3) *Robust Real-Time Deformation of Incompressible Surface Meshes* [Diziol et al. 2011]: Shape matching on trimeshes with overlapping regions (clustered shape matching). Adds an additional volume preservation constraint. Position based dynamics approach to satisfying volume preservation.
(4) *Shape-Up: Shaping Discrete Geometry with Projections* [Bouaziz et al. 2012]: Shape constraints by least squares fitting (like in shape matching). They have some "proximity function" indicating distance to least-squares fit, then uses projection operators to minimize proximity function (pretty much just shape matching).
(5) *Shape Matching with Oriented Particles* [Müller and Chentanez 2011]: More general form for shape matching, permitting wider range of motion. Also they use shape matching projection operators for skinning, much like we do. For each skinning point, they specify weights with up to 4 particles (each with their own projection operator)
(6) *Fast Adaptive Shape Matching Deformations* [Steinemann et al. 2008]: Essentially same thing as Lattice Shape Matching, but instead they use an octree instead of a basic voxel grid for shape matching. It's not super significant to mention this paper, but it does make clear that much of the followup work after shape matching never didn't emphasize it's utility as a meshless boundary only method. They kept converting it to a mesh-based method!
(7) *A Geometric Deformation Model for Stable Cloth Simulation* [Stumpp et al. 2008] Shape matching for cloth simulation.

2.2 Other Meshless Methods in Graphics

(1) *Point Based Animation* [Müller et al. 2004]: Purely particle based, MLS to approximate derivatives. Appears to be among the earliest meshless methods in graphics based on continuum mechanics.
(2) *Position Based Dynamics* [Müller et al. 2007]: Operates directly on particle positions by forming set of constraints and solving for particle positions that satisfy these constraints. Meshless
(3) *Projective Dynamics* [Bouaziz et al. 2014]: Similar to position based dynamics but solves the constraints implicitly by minimizing energy potentials. Mesh-based. Global solver unlike PBD which satisfies constraints locally using Gauss-Seidel.

2.3 Virtual Element and other Element Methods

(1) *Mimetic Finite Differences* [Brezzi et al. 2005] [Lipnikov et al. 2014] (they double dipped!): Considered a close relative to VEM and framed as the predecessor to VEM (in VEM papers). I still haven't read on MFD yet. From PolyDDF: "extension of finite volume and finite difference techniques to polygons that first discretizes a prime operator (typically, the gradient or the divergence) via a boundary integral, and then derives other operators by mimicking continuous structural properties."
(2) *Basic principles of Virtual Element Method* [Veiga et al. 2012] Original VEM paper
(3) *The Hitchhiker's Guide to Virtual Element Method* [Beirão da Veiga et al. 2014]: More understandable versions of original VEM paper.
(4) *Discrete Differential Operators on Polygonal Meshes* [De Goes et al. 2020]: Extends VEM to do discrete differential geometry on arbitrary polygonal meshes.
(5) *FLexible Simulation of Deformable Models using Discontinuous Galerkin FEM* [Kaufmann et al. 2008]: Uses ordinary hexahedral elements, but uses a cut cell-based approach to support arbitrary polyhedra on the surface.
(6) *Generalizing the finite element method: Diffuse approximation and diffuse elements* [Nayroles et al. 1992] Predecessor to Element Free Galerkin. FEM interpolation replaced with a local Moving Least Square interpolation.
(7) *Element-free Galerkin methods* [Belytschko et al. 1994]: similar to DEM, but more accurate gradients (not sure of all the differences). In MLS methods, they solve least squares for each particle in the domain, weighting nearby particles with a Gaussian-like density function. In contrast to us, we only compute least squares fitting on the boundary, and then precompute some weighting for each particle to the projection operators on the boundaries.
(8) *Unified Simulation of Elastic Rods, Shells, and Solids* [Martin et al. 2010]: Propose Generalized moving least squares (GMLS) to resolve limitation of MLS shape functions that

Table 1: Algorithms for simulation of NURBS models. "Not Done" denotes features that are not demonstrated in publication but are theoretically achievable. "?" denotes that the feature was not demonstrated or discussed in literature. *Meshless* indicates whether the algorithm requires a volumetric mesh, *Boundary* indicates whether the algorithm uses boundary degrees-of-freedom directly in the simulation *Relaxed* indicates the algorithm supports relaxed modelling requirements and does not requiring fabrication quality inputs, *Code* describes whether or not there is a publicly available implementation of the method.

Algorithm	Meshless	Boundary	Relaxed	Code
X-CAD []	No (Cutcells)	No	No	No
Ours []	Yes	Yes	Yes	Planned

require many particles in the support of a point (that are not coplonar).

2.4 Physics based Skinning

(1) *Skinning Siggraph Course* [Jacobson et al. 2014]: The linear weighting of polynomials at the exterior is very similar to skinning.
(2) *Linear Subspace Design for Real-Time Shape Deformation* [Wang et al. 2015]: Linear deformation subspace that uses linear blend skinning and generalized barycentric coordinates. Similarly, we have "handles" but they are represented by entire NURBS patches, and our coordinates are the output of a polynomial, whereas barycentric coordinates are linear (I only skimmed this paper, not sure if this description is fair).
(3) *Complementary dynamics* [Zhang et al. 2020]: Physics based skinning, orthogonality constraint can be seen as similar to our stability term (conformity term, error term, whatever it's called :))
(4) *Physically-Based Character Skinning* [Deul and Bender 2013]: Linear blend skinning with multiple layers of skin simulated via oriented particle shape matching and position based dynamics to enforce distance constraints (avoiding unwanted intersections).

2.5 Isogeometric Analysis

(1) *Isogeometric Analysis Book* [Cottrell et al. 2009]: The book everyone references when they write Isogeometric analysis in their papers.
(2) *Dynamic NURBS* [Terzopoulos and Qin 1994]: Outline of how to simulate on NURBS with the control points as the degrees of freedom. Method used in our work.
(3) *XCAD* [Hafner et al. 2019]: Optimize CAD models. CAD embedded in hexahedral mesh, complex integration strategy. Uncut hexahedral elements simulated ordinarily, cut elements use XFEM that add additional DOF to account for new element shapes.
(4) *Development of a quadratic finite element formulation based on the XFEM and NURBS* [Haasemann et al. 2011]: XFEM to handle curve surface integration of NURBs patches. Complex subdividing of "X-Elements" to produce cut cells along NURBS surfaces.
(5) *A NURBS enhanced extended finite element approach for unfitted CAD analysis* [Legrain 2013]: Pretty much the same as the quadratic, but allows higher-order approximation and better handling of interface.
(6) *A NURBS-based interface-enriched generalized finite element method for problems with complex discontinuous gradient fields* [Safdari et al. 2015]: Similar to other XFEM NURBS approaches. Uses NURBS-based enrichment functions with cut cells. Additional DOFs added to handle discontinuities.
(7) *A NURBS-based generalized finite element scheme for 3D simulation of heterogeneous materials* [Safdari et al. 2016]: Similar to previous "NIGFEM" paper above, but now in 3D.
(8) *Swept Volume Parameterization for Isogeometric* [Aigner et al. 2009] To provide volumetric simulation of NURBS they introduce a new NURBS volume parameterization (B-Spline Volumes ... jesus christ)
(9) *A finite volume method on NURBS geometries and its application in isogeometric fluid–structure interaction* [Heinrich et al. 2012]: Combines NURBS paramaterization with finite volume method (requiring a mesh for the volume).
(10) *NURBS-Enhanced Finite Element Method (NEFEM)* [Sevilla et al. 2008]: Similar to the above example. They run an order FVM simulation and deal with the interface in a complicated manner (could this be considered XFEM?).

2.6 Quadrature

(1) *A new method for meshless integration in 2D and 3D Galerkin meshfree methods* [Khosravifard and Hematiyan 2010]: Strategy we use for integrating over CAD model volumes. Raycast along single dimension, find intersections, generate quadrature points in the intervals inside the object.
(2) Adaptive image-based intersection [Wang et al. 2012]: related to our meshless integration strategy in that we could use this to account for errors in the above approach (increase ray density where we estimate high error to be)
(3) Efficient and accurate numerical quadrature for immersed boundary methods [Kudela et al. 2015]: Finite Cell Method. "Immerses" a shape in a set of cells (mesh!) and computes quadrature over this. To handle curved surface they use an octree to subdivide to the desired level of accuracy.
(4) Higher-Order Finite Elements for Embedded Simulation [Longva et al. 2020]: Another Finite Cell method like the above, but with a new quadrature generation method (the ones with the circles in the triangles)
(5) *Highly accurate surface and volume integration on implicit domains by means of moment-fitting* [Müller et al. 2013] and

[Müller et al. 2017]: XCAD paper extends upon this method
still need to read these

2.7 Misc

(1) TRACKS: Toward Directable Thin Shells [Bergou et al. 2007]: Petrov-Galerkin test functions for weak-form constraints that handles artifacts due to pointwise constraints.
(2) FEM simulation of 3D deformable solids [Sifakis and Barbic 2012]
(3) Fusion 360 Gallery [Willis et al. 2020] : source of some models

REFERENCES

Martin Aigner, Christoph Heinrich, Bert Jüttler, Elisabeth Pilgerstorfer, Bernd Simeon, and Anh-Vu Vuong. 2009. Swept Volume Parameterization for Isogeometric Analysis. 19–44. https://doi.org/10.1007/978-3-642-03596-8_2

L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo. 2014. The Hitchhiker's Guide to the Virtual Element Method. *Mathematical Models and Methods in Applied Sciences* 24, 08 (2014), 1541–1573. https://doi.org/10.1142/S021820251440003X arXiv:https://doi.org/10.1142/S021820251440003X

T. Belytschko, Y. Y. Lu, and L. Gu. 1994. Element-free Galerkin methods. *Internat. J. Numer. Methods Engrg.* 37, 2 (1994), 229–256. https://doi.org/10.1002/nme.1620370205 arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/nme.1620370205

Miklós Bergou, Saurabh Mathur, Max Wardetzky, and Eitan Grinspun. 2007. TRACKS: Toward Directable Thin Shells. *ACM Trans. Graph.* 26, 3 (July 2007), 50–es. https://doi.org/10.1145/1276377.1276439

Sofien Bouaziz, Mario Deuss, Yuliy Schwartzburg, Thibaut Weise, and Mark Pauly. 2012. Shape-Up: Shaping Discrete Geometry with Projections. *Comput. Graph. Forum* 31, 5 (Aug. 2012), 1657–1667. https://doi.org/10.1111/j.1467-8659.2012.03171.x

Sofien Bouaziz, Sebastian Martin, Tiantian Liu, Ladislav Kavan, and Mark Pauly. 2014. Projective Dynamics: Fusing Constraint Projections for Fast Simulation. *ACM Trans. Graph.* 33, 4, Article 154 (July 2014), 11 pages. https://doi.org/10.1145/2601097.2601116

Franco Brezzi, Konstantin Lipnikov, and Valeria Simoncini. 2005. A family of mimetic finite difference methods on polygonal and polyhedral meshes. *Mathematical Models and Methods in Applied Sciences* 15 (04 2005). https://doi.org/10.1142/S0218202505000832

J. Cottrell, Thomas Hughes, and Yuri Bazilevs. 2009. *Isogeometric Analysis: Toward integration of CAD and FEA.* https://doi.org/10.1002/9780470749081.ch7

Fernando De Goes, Andrew Butts, and Mathieu Desbrun. 2020. Discrete Differential Operators on Polygonal Meshes. *ACM Trans. Graph.* 39, 4, Article 110 (July 2020), 14 pages. https://doi.org/10.1145/3386569.3392389

Crispin Deul and Jan Bender. 2013. Physically-Based Character Skinning. https://doi.org/10.2312/PE.vriphys.vriphys13.025-034

R. Diziol, J. Bender, and D. Bayer. 2011. Robust Real-Time Deformation of Incompressible Surface Meshes. In *Proceedings of the 2011 ACM SIGGRAPH/Eurographics Symposium on Computer Animation* (Vancouver, British Columbia, Canada) *(SCA '11)*. Association for Computing Machinery, New York, NY, USA, 237–246. https://doi.org/10.1145/2019406.2019438

G. Haasemann, M. Kästner, S. Prüger, and V. Ulbricht. 2011. Development of a quadratic finite element formulation based on the XFEM and NURBS. *Internat. J. Numer. Methods Engrg.* 86, 4-5 (2011), 598–617. https://doi.org/10.1002/nme.3120 arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/nme.3120

Christian Hafner, Christian Schumacher, Espen Knoop, Thomas Auzinger, Bernd Bickel, and Moritz Bächer. 2019. X-CAD: Optimizing CAD Models with Extended Finite Elements. *ACM Trans. Graph.* 38, 6, Article 157 (Nov. 2019), 15 pages. https://doi.org/10.1145/3355089.3356576

Ch. Heinrich, B. Simeon, and St. Boschert. 2012. A finite volume method on NURBS geometries and its application in isogeometric fluid–structure interaction. *Mathematics and Computers in Simulation* 82, 9 (2012), 1645 – 1666. https://doi.org/10.1016/j.matcom.2012.03.008

Alec Jacobson, Zhigang Deng, Ladislav Kavan, and J. P. Lewis. 2014. Skinning: Real-Time Shape Deformation (Full Text Not Available). In *ACM SIGGRAPH 2014 Courses* (Vancouver, Canada) *(SIGGRAPH '14)*. Association for Computing Machinery, New York, NY, USA, Article 24, 1 pages. https://doi.org/10.1145/2614028.2615427

Peter Kaufmann, Sebastian Martin, Mario Botsch, and Markus Gross. 2008. Flexible Simulation of Deformable Models Using Discontinuous Galerkin FEM. In *Proceedings of the 2008 ACM SIGGRAPH/Eurographics Symposium on Computer Animation* (Dublin, Ireland) *(SCA '08)*. Eurographics Association, Goslar, DEU, 105–115.

Amir Khosravifard and Mohammad Rahim Hematiyan. 2010. A new method for meshless integration in 2D and 3D Galerkin meshfree methods. *Engineering Analysis with Boundary Elements* 34, 1 (2010), 30 – 40. https://doi.org/10.1016/j.enganabound.2009.07.008

László Kudela, Nils Zander, Tino Bog, Stefan Kollmannsberger, and Ernst Rank. 2015. Efficient and accurate numerical quadrature for immersed boundary methods. *Advanced Modeling and Simulation in Engineering Sciences* 2 (06 2015). https://doi.org/10.1186/s40323-015-0031-y

Grégory Legrain. 2013. A NURBS enhanced extended finite element approach for unfitted CAD analysis. *Computational Mechanics* 52 (04 2013). https://doi.org/10.1007/s00466-013-0854-7

Konstantin Lipnikov, Gianmarco Manzini, and Mikhail Shashkov. 2014. Mimetic Finite Difference Method. *J. Comput. Phys.* 257 (Jan. 2014), 1163–1227. https://doi.org/10.1016/j.jcp.2013.07.031

Andreas Longva, Fabian Löschner, Tassilo Kugelstadt, José Antonio Fernández-Fernández, and Jan Bender. 2020. Higher-Order Finite Elements for Embedded Simulation. *ACM Trans. Graph.* 39, 6, Article 181 (Nov. 2020), 14 pages. https://doi.org/10.1145/3414685.3417853

Sebastian Martin, Peter Kaufmann, Mario Botsch, Eitan Grinspun, and Markus Gross. 2010. Unified Simulation of Elastic Rods, Shells, and Solids. *ACM Trans. Graph.* 29, 4, Article 39 (July 2010), 10 pages. https://doi.org/10.1145/1778765.1778776

Matthias Müller and Nuttapong Chentanez. 2011. Solid Simulation with Oriented Particles. In *ACM SIGGRAPH 2011 Papers* (Vancouver, British Columbia, Canada) *(SIGGRAPH '11)*. Association for Computing Machinery, New York, NY, USA, Article 92, 10 pages. https://doi.org/10.1145/1964921.1964987

Matthias Müller, Bruno Heidelberger, Matthias Teschner, and Markus Gross. 2005. Meshless Deformations Based on Shape Matching. *ACM Trans. Graph.* 24, 3 (July 2005), 471–478. https://doi.org/10.1145/1073204.1073216

M. Müller, R. Keiser, A. Nealen, M. Pauly, M. Gross, and M. Alexa. 2004. Point Based Animation of Elastic, Plastic and Melting Objects. In *Proceedings of the 2004 ACM SIGGRAPH/Eurographics Symposium on Computer Animation* (Grenoble, France) *(SCA '04)*. Eurographics Association, Goslar, DEU, 141–151. https://doi.org/10.1145/1028523.1028542

B. Müller, S. Krämer-Eis, F. Kummer, and M. Oberlack. 2017. A high-order discontinuous Galerkin method for compressible flows with immersed boundaries. *Internat. J. Numer. Methods Engrg.* 110, 1 (2017), 3–30. https://doi.org/10.1002/nme.5343 arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/nme.5343

B. Müller, F. Kummer, and M. Oberlack. 2013. Highly accurate surface and volume integration on implicit domains by means of moment-fitting. *Internat. J. Numer. Methods Engrg.* 96, 8 (2013), 512–528. https://doi.org/10.1002/nme.4569 arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/nme.4569

Matthias Müller, Bruno Heidelberger, Marcus Hennix, and John Ratcliff. 2007. Position based dynamics. *Journal of Visual Communication and Image Representation* 18, 2 (2007), 109 – 118. https://doi.org/10.1016/j.jvcir.2007.01.005

B. Nayroles, G. Touzot, and P. Villon. 1992. Generalizing the finite element method: Diffuse approximation and diffuse elements. *Computational Mechanics* 10 (1992), 307–318.

Alec R. Rivers and Doug L. James. 2007. FastLSM: Fast Lattice Shape Matching for Robust Real-Time Deformation. In *ACM SIGGRAPH 2007 Papers* (San Diego, California) *(SIGGRAPH '07)*. Association for Computing Machinery, New York, NY, USA, 82–es. https://doi.org/10.1145/1275808.1276480

Masoud Safdari, Ahmad R. Najafi, Nancy R. Sottos, and Philippe H. Geubelle. 2015. A NURBS-based interface-enriched generalized finite element method for problems with complex discontinuous gradient fields. *Internat. J. Numer. Methods Engrg.* 101, 12 (2015), 950–964. https://doi.org/10.1002/nme.4852 arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/nme.4852

Masoud Safdari, Ahmad R. Najafi, Nancy R. Sottos, and Philippe H. Geubelle. 2016. A NURBS-based generalized finite element scheme for 3D simulation of heterogeneous materials. *J. Comput. Phys.* 318 (2016), 373 – 390. https://doi.org/10.1016/j.jcp.2016.05.004

Ruben Sevilla, Sonia Mendez, and Antonio Huerta. 2008. Nurbs-enhanced finite element method (NEFEM). *Internat. J. Numer. Methods Engrg.* 76 (10 2008), 56–83. https://doi.org/10.1002/nme.2311

Eftychios Sifakis and Jernej Barbic. 2012. FEM Simulation of 3D Deformable Solids: A Practitioner's Guide to Theory, Discretization and Model Reduction. In *ACM SIGGRAPH 2012 Courses* (Los Angeles, California) *(SIGGRAPH '12)*. Association for Computing Machinery, New York, NY, USA, Article 20, 50 pages. https://doi.org/10.1145/2343483.2343501

Denis Steinemann, Miguel A. Otaduy, and Markus Gross. 2008. Fast Adaptive Shape Matching Deformations. In *Proceedings of the 2008 ACM SIGGRAPH/Eurographics Symposium on Computer Animation* (Dublin, Ireland) *(SCA '08)*. Eurographics Association, Goslar, DEU, 87–94.

Thomas Stumpp, Jonas Spillmann, Markus Becker, and Matthias Teschner. 2008. A Geometric Deformation Model for Stable Cloth Simulation. *VRIPHYS 2008 - 5th Workshop on Virtual Reality Interactions and Physical Simulations*, 39–46. https://doi.org/10.2312/PE/vriphys/vriphys08/039-046

Demetri Terzopoulos and Hong Qin. 1994. Dynamic NURBS with Geometric Constraints for Interactive Sculpting. *ACM Trans. Graph.* 13, 2 (April 1994), 103–136. https://doi.org/10.1145/176579.176580

L. Veiga, Franco Brezzi, Andrea Cangiani, G. Manzini, L. Marini, and Alessandro Russo. 2012. Basic principles of Virtual Element Methods. *Mathematical Models and Methods in Applied Sciences* 23 (11 2012). https://doi.org/10.1142/S0218202512500492

Bin Wang, François Faure, and Dinesh K. Pai. 2012. Adaptive image-based intersection volume. *ACM Trans. Graph.* 31, 4 (2012), 97:1–97:9. https://doi.org/10.1145/2185520.2185593

Yu Wang, Alec Jacobson, Jernej Barbič, and Ladislav Kavan. 2015. Linear Subspace Design for Real-Time Shape Deformation. *ACM Trans. Graph.* 34, 4, Article 57 (July 2015), 11 pages. https://doi.org/10.1145/2766952

Karl D. D. Willis, Yewen Pu, Jieliang Luo, Hang Chu, Tao Du, Joseph G. Lambourne, Armando Solar-Lezama, and Wojciech Matusik. 2020. Fusion 360 Gallery: A Dataset and Environment for Programmatic CAD Reconstruction. *arXiv preprint arXiv:2010.02392* (2020).

Jiayi Eris Zhang, Seungbae Bang, David I. W. Levin, and Alec Jacobson. 2020. Complementary dynamics. *ACM Transactions on Graphics* 39, 6 (Nov 2020), 1–11. https://doi.org/10.1145/3414685.3417819

Dump of some equations

$$\mathbf{M}(\mathbf{X}) = (X_0 - \bar{X}_0, X_1 - \bar{X}_1)$$

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_i^n x_i$$

$$\bar{\mathbf{X}} = \frac{1}{n}\sum_i^n X_i$$