① Bivariate Normal PDF:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)}{2(1-\rho^2)}\right\}$$

Marginal

$$\Downarrow$$

$$f_{X(x)} = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{-\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)}{2(1-\rho^2)}\right\} dy$$

$$\Rightarrow \frac{\exp\left\{\frac{-\left(\frac{x-\mu_x}{\sigma_x}\right)^2}{2(1-\rho^2)}\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right\} dy$$

$$\Rightarrow \frac{\exp\left\{\frac{-\left(\frac{x-\mu_x}{\sigma_x}\right)^2}{2(1-\rho^2)}\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\}$$

$$\Rightarrow \frac{\exp\left\{\frac{-\left(\frac{x-\mu_x}{\sigma_x}\right)^2}{2(1-\rho)^2}\right\}\exp\left\{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times$$

$$\int_{-\infty}^{\infty}\exp\left\{\frac{y-\mu_y}{\sigma_y}-\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\right\}^2$$

Let e=exp{}, running out of room, then

$$\Rightarrow \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_x}{\sigma_x}\right)^2(1-\rho^2)}}{2\pi\sigma_x\sigma_1\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_y}{\sigma_y}-\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\right)^2} dy$$

Next let $q=\left(\frac{y-\mu_x}{\sigma_x}-\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\right)\cdot\frac{1}{\sqrt{1-\rho^2}}$, then,

$$\Rightarrow dq=\frac{1}{\sqrt{1-\rho}\,\sigma_y}dy.$$

Simplifying, we get,

$$\Rightarrow \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}q^2}dq = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{2\pi\sigma_x}\cdot\sqrt{2\pi}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)} \equiv \text{Normal Distribution.}$$

$\therefore$ We've shown the marginal distribution for a bivariate normal is a normal distribution

B

7b) Find conditional Distribution of $f_{X|Y}(x)$.

Let X and Y be random variables. The conditional distributions are defined

$$\Rightarrow f_{X|Y}(x) = \frac{f(x,y)}{f(x)}, \quad f_{Y|X}(x) = \frac{f(x,y)}{f(y)},$$

where $f(x,y)$ is the joint density function and $f(x), f(y)$ are marginal density functions.

Let $f(x,y)$ be the bivariate normal distribution PDF, then without loss of generality

$$\Rightarrow f_{X|Y} = \frac{f(x,y)}{f(y)} = \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)}{2(1-\rho)^2}\right\}}{\frac{1}{\sqrt{2\pi}\,\sigma_y} \cdot e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_2}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - (1-\rho^2)\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)\right\}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}} \exp\left\{\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right) - \rho\left(\frac{y-\mu_2}{\sigma_2}\right)\right)^2\right\}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}} \exp\left\{\frac{1}{2\sigma_x^2(1-\rho^2)}\left(x - \left(\mu_1 + \sigma_x\rho\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^2\right\}$$

$$\equiv N\left(\mu_x + \rho\sigma_x\left(\frac{y-\mu_y}{\sigma_y}\right), \sigma_x^2(1-\rho^2)\right).$$

$\therefore$ The conditional distribution is normally distributed.

2) Setup

① Y = "final exam"

② $\mu_Y = 73$, $\sigma_Y = 12$

③ X = "average of 4"

④ $\mu_X = 70$, $\sigma_X = 15$

⑤ $\rho = .71$

AIM: Find given values,

(a) $E(Y|X) = \mu_{Y|X}$

$$\Rightarrow \mu_{Y|X} = \mu_X + \rho\sigma_X\left(\frac{X-\mu_Y}{\sigma_Y}\right)$$

$$= 70 + (.71)(15)\left(\frac{X-73}{12}\right)$$

$$= \boxed{70 + 10.65\left(\frac{X-73}{12}\right)}$$

(b) $V(Y|X) = \sigma^2_{Y|X}$

$$\Rightarrow \sigma^2_{Y|X} = \sigma^2_X(1-\rho^2)$$

$$= \sigma^2_X(0.4959)$$

$$= 225(0.4959)$$

$$= \boxed{111.5775}$$

(c) $P(Y > 90 \mid X = 80)$

$$\Rightarrow \mu_{Y|X=80} = 70 + 10.65\left(\frac{80-73}{12}\right) \approx 76.2125.$$

Thus,

$$\Rightarrow P\left(Z > \frac{90 - 76.2125}{10.56}\right)$$

$$\Rightarrow P(Z > 1.3056) \approx \boxed{0.09691}.$$

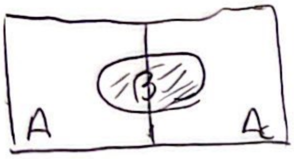
(3) Setup

Events:

- A = "disease"
- A^c = "no disease"
- B = "positive"
- B^c = "negative"

Prob:

- $P(A) = 0.01$
- $P(A^c) = 0.99$
- $P(B|A) = 0.97$
- $P(B|A^c) = 0.06$



(a) find $P(B)$

$$\Rightarrow P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$
$$= (.97)(.01) + (.06)(.99) \approx \boxed{0.0691}.$$

(b) $P(A|B)$

$$\Rightarrow P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)} = \frac{(.01)(.97)}{(.01)(.97) + (.99)(.06)}$$

$$= \frac{0.0097}{0.0691} \approx \boxed{0.14041}.$$

(c)
$$P(A^c|B^c) = \frac{P(A^c)[1 - P(B|A^c)]}{P(A)[1 - P(B|A)] + P(A^c)[1 - P(B|A^c)]}$$

$$= \frac{(.99)(1 - .06)}{(.01)(1 - .97) + (.99)(1 - .06)} \approx \boxed{.999}.$$

4) Setup

Events:

- A_1 = Species 1
- A_2 = Species 2
- A_3 = Species 3
- B = logged

Prob

- $P(B|A_1) = .1$
- $P(B|A_2) = .15$
- $P(B|A_3) = .5$

$$* \; P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) \approx \boxed{0.187}$$

$$\text{(a) } P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(B)} = \frac{(.45)(.10)}{(.45)(.1)+(.38)(.15)+(.65)(.17)} \approx \boxed{0.2406}.$$

$$\text{(b) } P(A_2|B) = \frac{P(A_2)P(B|A_2)}{P(B)} = \frac{(.38)(.15)}{0.187} \approx \boxed{0.3048}.$$

$$\text{(c) } P(A_3|B) = \frac{P(A_3)P(B|A_3)}{P(B)} = \frac{(.17)(.5)}{0.187} \approx \boxed{0.4545}.$$