Assignment #1

1. Definition: X and Y are independent if $f_{X|y}(x) = f_X(x)$ for all x and y.

   It can be shown that

   (a) $f_{X|y}(x) = f_X(x)$ for all x and y iff $f_{Y|x}(y) = f_Y(y)$ for all x and y.

   (b) X and Y are independent iff $f(x, y) = f_X(x)f_Y(y)$ for all x and y.

2. The bi-variate normal distribution has the joint PDF as

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{ -\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)}{2\left(1-\rho^2\right)} \right\}$$

   (a) Find the marginal distribution of $f_X(x)$ and $f_Y(y)$

   (b) Find the conditional distribution of $f_{X|y}(x)$

3. A fair coin is tossed four times, and the random variable X is the number of heads in the first three tosses and the random variable Y is the number of heads in the last three tosses.

   (a) What is the joint probability mass function of X and Y ?

   (b) What are the marginal probability mass functions of X and Y ?

   (c) Are the random variables X and Y independent?

   (d) What are the expectations and variances of the random variables X and Y ?

   (e) If there is one head in the last three tosses, what is the conditional probability mass function of X? What are the conditional expectation and variance of X?

4. Suppose that two continuous random variables X and Y have a joint probability density function

$$f(x, y) = A(e^{x+y} + e^{2x-y})$$

   for $1 \leq x \leq 2$ and $0 \leq y \leq 3$, and $f(x, y) = 0$ elsewhere.

   (a) What is the value of A?

   (b) What is $P(1.5 \leq X \leq 2, 1 \leq Y \leq 2)$?

   (c) Construct the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

   (d) Are the random variables X and Y independent?

   (e) If $Y = 0$, what is the conditional probability density function of X?

5. For a Calculus I class, the final exam score Y and the average of the four earlier tests X are bi-variate normal with mean $\mu_1 = 73$, standard deviation $\sigma_1 = 12$ and mean $\mu_2 = 70$, standard deviation $\sigma_2 = 15$. The correlation is $\rho = 0.71$. Find

   (a) $E(Y|X) = \mu_{Y|X}$

   (b) $V(Y|X) = \sigma^2_{Y|X}$

   (c) $P(Y > 90|X = 80)$ that is, the probability that the nal exam score exceeds 90 given that the average of the four earlier tests is 80