Assignment 4

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Problem 1

(a) Write R function.

```
beta_binom <- function(Y, N, theta0, type, a = 1, b = 1,
                       title = "Prior and Posterior Distribution")
{
  # beta_binom() - returns a list of posterior mean and variance values where
  # the prior distribution follows Beta and the posterior is Binomial

  # Arguments:
  # Y = number of success for Binomial
  # N = number of trials for Binomial
  # a, b are size and rate parameters for prior Beta distribution
  # theta0 = hypothesized value for unknown parameter
  # type = type of hypothesis test to be conducted (one-sided or two-sided)
  A <- Y + a
  B <- N - Y + b

  if (type == "left" | type == "right") { # one sided
    # compute priors
    prior.h0 <- pbeta(theta0, a, b)
    prior.h1 <- 1 - prior.h0
    prior.odds <- prior.h1 / prior.h0

    # compute posterior probability for null, alternative, and odds
    posterior.h0 <- pbeta(theta0, A, B)
    posterior.h1 <- 1 - posterior.h0
    posterior.odds <- posterior.h1 / posterior.h0

    # Compute Bayes Factor for one-sided test
    bayes.factor <- posterior.odds / prior.odds
  } else if (type == "two sided") { # two sided
    # choose machine epsilon as epsilon
    #epsilon = .Machine$double.eps
    epsilon = 0.000000003

    # compute priors
```

```
    prior.h0<-pbeta((theta0+epsilon),a,b)-pbeta((theta0-epsilon),a,b)
    prior.h1 <- 1 - prior.h0
    prior.odds <- prior.h1 / prior.h0

    # compute posteriors
    posterior.h0<-pbeta((theta0+epsilon),A, B) - pbeta((theta0-epsilon),A,B)
    posterior.h1 <- 1 - posterior.h0
    posterior.odds <- posterior.h1 / posterior.h0

    # compute two-sides Bayes Factor
    bayes.factor <- posterior.odds / prior.odds
  }
  mylist <- list(prior.h0, posterior.h0, bayes.factor)
  names(mylist) <- c("Prior Null", "Posterior Null", "Bayes Factor")
  return(mylist)
}
```

(b) The values of a and b that give prior mean 0.7 and prior standard deviation 0.2 are 2.98 and 1.28 respectively.

(c) Conduct Bayesian analysis using $N = 30$ and $Y = 20$. Once with uninformative prior and again with the informative prior.

```
# uninformative prior
beta_binom(Y=20, N=30, theta0=0.5, type = "right", a = 1, b = 1)
```

```
## $`Prior Null`
## [1] 0.5
##
## $`Posterior Null`
## [1] 0.03537777
##
## $`Bayes Factor`
## [1] 27.26634
```

```
# here we would reject H0 in favor of H1
```

```
# informative prior
beta_binom(Y=20, N=30, theta0=0.5, type="right", a = 2.98, b = 1.28)
```

```
## $`Prior Null`
## [1] 0.1766437
##
## $`Posterior Null`
## [1] 0.02041971
##
## $`Bayes Factor`
## [1] 10.29202
```

```
# here we would reject H0 in favor of H1
```

(d) Conduct a two-tailed Bayesian analysis.

```r
# uninformative prior
beta_binom(Y=20, N=30, theta0=0.5, type = "two sided", a = 1, b = 1)
```

```
## $`Prior Null`
## [1] 6e-09
##
## $`Posterior Null`
## [1] 5.204578e-09
##
## $`Bayes Factor`
## [1] 1.152831
```

```r
# here we should fail to reject H0
```

```r
# informative prior
beta_binom(Y=20, N=30, theta0=0.5, type = "two sided", a = 2.98, b = 1.28)
```

```
## $`Prior Null`
## [1] 5.94622e-09
##
## $`Posterior Null`
## [1] 3.400073e-09
##
## $`Bayes Factor`
## [1] 1.748851
```

TODO: Add conclusions and add descriptive names to the outcomes.

Problem 2

(a) The posterior distribution for λ is given by,

$$f(\lambda|y) \propto Gamma(a + y, b + 1).$$

(b) The posterior mean is given by,

$$E(\lambda|y) = \frac{a + y}{b + 1}.$$

The posterior variance is given by,

$$Var(\lambda|y) = \frac{a}{b^2}.$$

(c) R Function

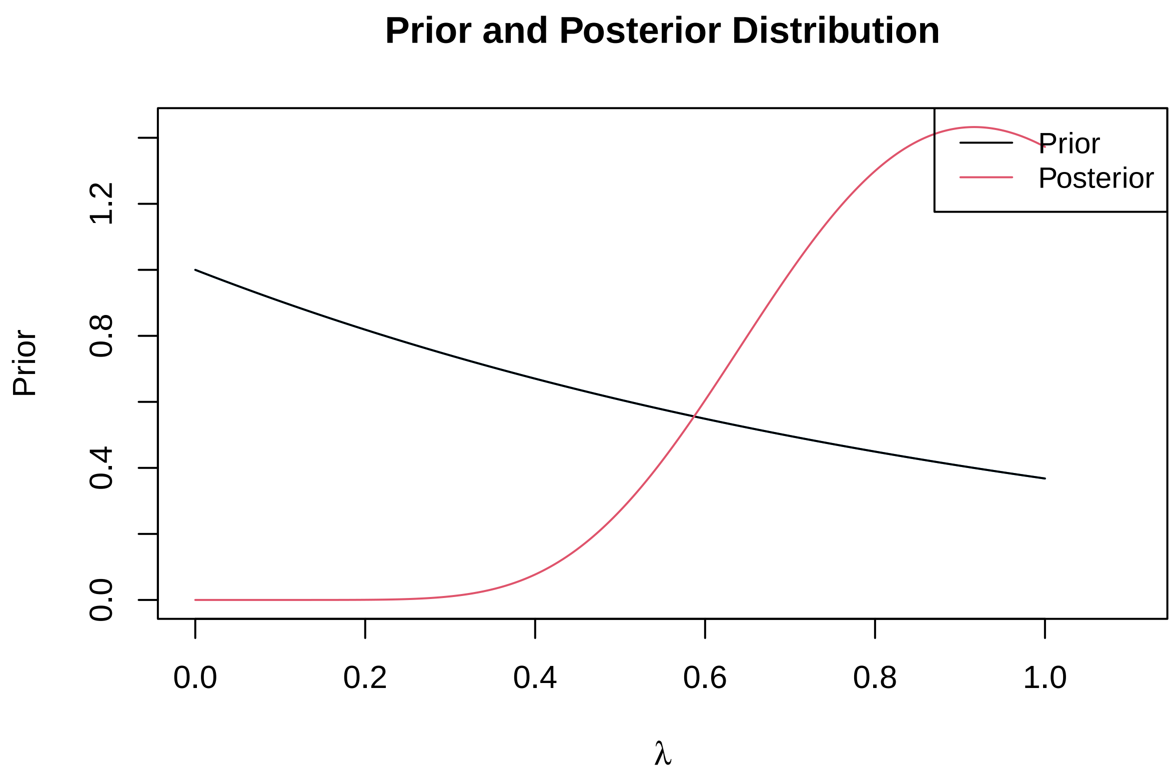
```
pois_gamma <- function(Y, a = 1, b = 1, title = "Prior and Posterior Distribution")
{
  # pois_gamma() - returns a list of posterior mean and variance values where
  # the prior distribution follows Beta and the posterior is Binomial

  # Arguments:
  # Y = observations of random variable
  # a  = shape parameter for Gamma distribution
  # b = rate parameter for Gamma distribution

  lambda <- seq(0,1,length=10000)    # values of theta following beta distribution

  # only plotting one prior
  prior = dgamma(lambda, a, b)
  posterior = dgamma(lambda, Y + a, Y + b)

  # plot the actual priors
  label = c("Prior", "Posterior")
  plot.new()
  plot(lambda,prior,
       type="l",
       col = 4,
       xlab=expression(lambda),
       ylab="Prior",
       ylim = c(0, max(posterior)),
       xlim = c(0, 1.1)
  )
  lines(lambda,prior,col=1)
  lines(lambda, posterior, col = 2)
  title(main = title)
  legend("topright",label,col=c(1,2), lty=1, cex=.9)

  # compute posterior mean and posterior standard deviation
  posterior_mean <- (Y + a) / (b + 1)
  posterior_var <- a / b^2
  posterior_sd <- sqrt(posterior_var)
```

```
  # return posterior mean and variance
  mylist <- list("posterior mean" = posterior_mean, "posterior standard deviation" = posterior_sd)
  return(mylist)
}
```

(d) The values of a and b that give prior mean 5 and prior standard deviation 2.5 are $a = 15.81139$ and $b = 3.162278$.

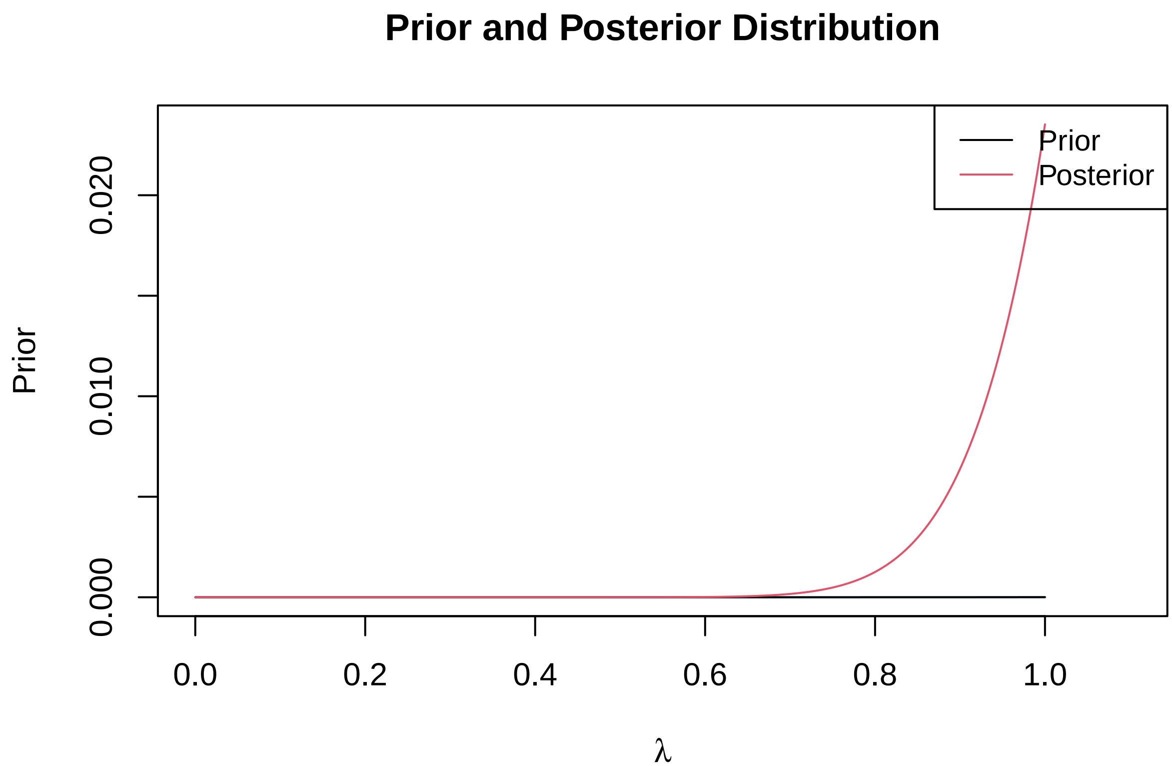
(e) Conduct Bayesian analysis using $Y = 11$, once with uninformative and again with informative.

```
# uninformative prior
pois_gamma(Y = 11)
```




```
## $`posterior mean`
## [1] 6
##
## $`posterior standard deviation`
## [1] 1
```

```
# informative prior
pois_gamma(Y = 11, a = 15.81139, b = 3.152278)
```




```
## $`posterior mean`
## [1] 6.457032
##
## $`posterior standard deviation`
## [1] 1.261422
```

(f) TODO: Summarize results

(g) Conduct upper-tailed hypothesis test.

```r
#==============================================================================
# H0: lambda <= 10
# H1: lambda > 10
#==============================================================================
# Data
Y <- 11

# Define one prior
a  <- c(0.5, 1, 10)
b  <- c(0.5, 1, 2)
lab <- c("a=b=0.5", "a=b=1", "a=10, b=2")

# posterior parameters
A <- a+Y
B <- b+1
```

```r
# parameter
lambda <-10

# compute priors
prior.h0 <- pgamma(lambda, shape = a, rate = b)
prior.h1 <- 1 - prior.h0
prior.odds <- prior.h1 / prior.h0

# compute posteriors
posterior.h0 <- pgamma(lambda,shape=A,rate=B)
posterior.h1 <- 1-posterior.h0
posterior.odds <- posterior.h1 / posterior.h0

# compute Bayes Factor
bayes.factor <- posterior.odds / prior.odds

# output results
output <- cbind(H0.prior=prior.h0,H0.post=posterior.h0,BF10=bayes.factor)
output
```

```
##       H0.prior   H0.post       BF10
## [1,] 0.9984346 0.8505984 112.027467
## [2,] 0.9999546 0.9786132 481.349238
## [3,] 0.9950046 0.9647154   7.285175
```

(h) Conduct two-tailed test.

```r
#==============================================================================
# H0: lambda = 10
# H1: lambda =/= 10
#==============================================================================
# Data
Y <- 11

# priors
a <- c(0.5, 1, 10)
b <- c(0.5, 1, 2)
lab <- c("a=b=0.5", "a=b=1", "a=10, b=2")

# posteriors
A <- a+Y
B <- b+1

# parameters and perturbation
lambda <- 10
epsilon <- 0.000001

# compute priors
prior.h0 <- pgamma(lambda+epsilon,shape=a,rate=b) - pgamma(lambda-epsilon,shape=a,rate=b)
prior.h1 <- 1-prior.h0
prior.odds <- prior.h1 / prior.h0
```

```r
# compute posteriors
posterior.h0 <- pgamma(lambda+epsilon,shape=A,rate=B) - pgamma(lambda-epsilon,shape=A,rate=B)
posteriror.h1 <- 1-posterior.h0
posterior.odds <- posteriror.h1 / posterior.h0

# compute bayes factor
bayes.factor <- posterior.odds / prior.odds

# output results
output <- cbind(H0.prior=prior.h0,H0.post=posterior.h0,BF10=bayes.factor)
output
```

```
##          H0.prior      H0.post        BF10
## [1,] 1.700073e-09 1.722409e-07 0.009870320
## [2,] 9.079981e-11 4.230041e-08 0.002146547
## [3,] 1.163261e-08 8.046690e-08 0.144563942
```

TODO: State conclusions and decisions.