1. Fundamentals of Probability

FRM Part 1: Quantitative Analysis

Devere Anthony Weaver

---

1 Sample Space, Event Space, and Events

A *sample space*, Ω, is a set containing all possible outcomes of an experiment. The set of outcomes depends on the problem being studied.

Events are subsets of a sample space. We denote an event with ω. Events can contain one ore more of the values in the sample space, or it may contain no elements, $\emptyset$.

The *event space* consists of **all combinations of outcomes** to which probabilities can be assigned. The event space is often denoted using $\mathcal{F}$. If an event space has a finite number of outcomes, we refer to it as a *discrete probability space.*

1.1 Probability

Probability is used to measure the likelihood of an event. Probabilities are always between 0 and 1.

The *frequentest interpretation* focuses on objective probability. It interprets probability as the frequency with which an event would occur if a set of independent experiments was run.

Probability can also be interpreted from a subjective point of view where probability reflects or incorporates an individual's beliefs about the likelihood of an event occurring. Since these are beliefs about likelihood, they may differ across individuals and don't have to agree with the objective probability of an event.

Probability is defined over event spaces and we tend to think about them as mathematical sets. These sets have three important fundamental operations for our purposes. Suppose we're given two sets A and B. Then,

- $\cap$ = the intersection operator
  - $A \cap B$ is the set of outcomes that appears in both A and B
- $\cup$ = the union operator
  - $A \cup B$ is the set of outcomes that appears in either A, or B, or both.
- A^c = the complement of the set A, it's the set of all outcomes that are not in A

Mutually exclusive events are events that can't occur together.

1.2 Fundamental Principles of Probability

The three Axioms of Probability (in plain English) are

1. Any event A in the event space $\mathcal{F}$ have $Pr(A) \geq 0$.
2. The probability of all events in Ω is one and thus $Pr(\Omega) = 1$.
3. If the events A_1 and A_2 are *mutually exclusive*, then $Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2)$.

These principles imply two useful properties of probability. The first being the the probability of an event or its complement must be 1.

$$Pr(A \cup A^c) = P(A) + P(A^c) = 1. \tag{1}$$

The second being the probability of the union of any two sets can be decomposed into

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B). \tag{2}$$

The elements in the intersection are in both sets, therefore subtracting this term avoids double counting.

1.3 Conditional Probability

We're often interested in the probability of an event happening only if another event happens first. This is known as *conditional probability* since we're computing a probability on the condition that another event occurs.

As an example, we might want to determine the probability that a large financial institution fails given that another large financial institution has also failed.

We can use conditional probability to incorporate additional information to update unconditional probabilities. The conditional probability of an event A occurring given that event B has occurred is given by

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}. \tag{3}$$

In this case, we treat event B as if it is now the event space and event Anis an event inside this new restricted space.

An important application of conditional probability is the Law of Total Probability which states that the total probability of an event can be reconstructed using conditional probabilities under the condition that the probability of the sets being conditioned is equal to 1.

$$Pr(A) = Pr(A|B)Pr(B) + Pr(A|B^c)Pr(B^c) \tag{4}$$

---

2 Independence

Two events are *independent* if the probability that one event occurs doesn't depend on whether the other event occurs.

When the two events A and B are independent,

$$Pr(A \cap B) = Pr(A)Pr(B). \tag{5}$$

This implies that for independent events, the conditional probability of the event is equal to the unconditional probability of the event. So, given two events A and B that are independent, we have

$$Pr(B|A) = \frac{Pr(B \cap A)}{Pr(A)} = \frac{Pr(B)Pr(A)}{Pr(A)} = Pr(B). \tag{6}$$

Similarly, $Pr(A|B) = Pr(A)$ if A and B are independent events.

One thing to note is that if A and B are mutually exclusive, then they can't also be independent. This is because the outcome of the first event will affect the outcome of the second event.

Independence can be generalized,

$$Pr(A_1 \cap A_2 \cap \cdots \cap A_n) = Pr(A_1) \times Pr(A_2) \times \cdots \times Pr(A_n). \tag{7}$$

2.1 Conditional Independence

Like probability, independence can be redefined to hold conditional on another event.

Two events A and B are conditionally independent if

$$Pr(A \cap B|C) = Pr(A|C)Pr(B|C) \tag{8}$$

The conditional probability above is the probability of an outcome that is in both A and B occurring given that an outcome in C occurs.

---

3 Bayes' Rule

Bayes' rule provides a method to construct conditional probabilities using other probability measures.

$$Pr(B|A) = \frac{Pr(A|B)Pr(B)}{Pr(A)}. \tag{9}$$

Apparently Bayes' rule has many financial risk management applications.

---

4 Further Reading

The following is a list of references for learning more in-depth about probability and offer more practice with the key concepts:

- *Statistics* by McClave and Sincich
- *Modern Mathematical Statistics with Applications* by Devore and Berk
- *Introduction to Mathematical Statistics with Applications* by Wackerly