3. Common Univariate Random Variables

FRM Part 1: Quantitative Analysis

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1 Parametric Distributions

Distributions can be divided into to broad categories:

- *parametric* - can be described by a mathematical function and they are easier to work with generally, but force use to make assumptions that might not be supported by real-world data
- *nonparametric* - basically just a collection of data; they can fit data without assumptions but have difficulty generalizing (e.g. it's just a collection of data)

NOTE: No derivations/proofs of the PDFs and CDFs exist in this document, I just didn't feel like typing all that out. If you want to see the derivations and proofs, check out my corresponding written version of these notes...but I'm almost certainly the only person looking at these so...idk.

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2 Uniform Distribution

The PDF for the uniform distribution is given by

$$Unif(b_1, b_2) := f(x) = \begin{cases} c & \forall\ b_1 < x < b_2 \text{ such that } b_1 < b_2 \\ 0 & \forall\ b_1 > x > b_2. \end{cases}$$

Recall the probability of any outcome must be one. Using this information, we can compute c (again, I didn't type up the steps, but it's straightforward to compute.

The set up...

$$\int_{-\infty}^{\infty} Unif(b_1, b_2)\, dx = \int_{b_1}^{b_2} c\, dx = cx|_{b_1}^{b_2} = 1.$$

Do the algebra, set it equal to 1 and solve for c to get

$$c = \frac{1}{b_2 - b_1}.$$

Thus, we can use the above equation to solve for any constant c given any b_1, b_2.

The formulas for computing the mean and the variance for the uniform distribution can also be easily computed by integration.

$$\mu = \int_{b_1}^{b_2} cx\, dx = \frac{b_2 + b_1}{2}.$$

Likewise, the variance is

$$\sigma^2 = \int_{b_1}^{b_2} c(x - \mu)^2\, dx = \frac{1}{12}(b_2 - b_1)^2.$$

The *standard uniform distribution* is a uniform distribution in which $b_1 = 0$ and $b_2 = 1$.

The uniform distribution is a very important building block for computer models in finance since this distribution is typically the foundation for pseudo-random number generators in most computers.

To calculate the CDF for the uniform distribution, we again integrate,

$$P(X \leq a) = \int_{b_1}^{a} c\, dz = cz|_{b_1}^{a} = \frac{a - b_1}{b_2 - b_1}.$$

So the, the CDF is

$$F(x) = \begin{cases} 0 & x < b_1 \\ \frac{x-b_1}{b_2-b_1} & b_1 \leq x \leq b_2 \\ 1 & x \geq b_2. \end{cases}$$

When $a = b_1$, we're at the minimum and the CDF $= 0$. Likewise, when $a = b_2$, we're at the maximum and the CDF $= 1$.

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3 Bernoulli Distribution

Recall, a Bernoulli random variable takes on the value of 1 or 0. It is common to label the outcome 1 as a ”success” and 0 as a ”failure”.

Don't be confused however, in risk management and finance, typically 1 is used to denote the unusual or undesirable state, despite the name ”success”.

Define p as the probability the random variable $X = 1$. Then the PMF is given by

$$p(x) = \begin{cases} 1 - p & x = 0 \\ p & x = 1. \end{cases}$$

The mean is given by

$$\mu = p.$$

The variance is given by

$$\sigma^2 = p(1 - p).$$

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4 Binomial Distribution

A binomial random variable can be thought of as a collection of Bernoulli random variables. It measures the total number of successes from n independent Bernoulli random variables, where each has a probability of success equal to p. So, we can use binomial distributions to model counts of independent events.

A binomial distribution has two parameters:

1. $n :=$ the number of independent experiments
2. $p :=$ the probability that each experiment is successful

The binomial PDF is given by,

$$P[X = x] = \binom{n}{x} p^x (1 - p)^{n-x}.$$

This gives us the probability of any x number of “things”. Recall,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The mean of the binomial random variable is

$$\mu = np.$$

The variance is given by

$$\sigma^2 = np(1 - p).$$

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5 Poisson Distribution

For a Poisson random variable X, the PMF is given by

$$P[X = n] = \frac{\lambda^n}{n!} e^{-\lambda}.$$

λ is a constant value and more importantly the mean is given by

$$\mu = \sigma^2 = \lambda.$$

The Poisson distribution is used to model the occurrence of events over time. For example the number of bond defaults over time or the number of crashes in equity markets, etc.

So n is the number of events that occur in an interval and λ is the expected number of events in the interval (the rate parameter).

If (i) the rate at which events occur over time is constant and (ii) the probability of any one event occurring is independent of all other events, then the events follow a *Poisson process*. We model Poisson processes by

$$P[X = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

where t is equal to the amount of elapsed time for the interval. So, the number of expected events before time t is equal to λt.

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6 Normal Distribution

It's a classic.

Of course the normal distribution is one of the most popular distributions in finance and risk management.

Some neat properties include:

- symmetrical with the mean and median equal at the highest point of the PDF
- skew is always 0 (due to it being symmetrical, see point above)
- kurtosis if always 3 with any excess kurtosis equal to 0

So for a continuous random variable X, the PDF for a normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Note: To hell with this proof and derivation.

Another important property of the normal distribution is that any linear combination of independent normal random variables is also normal.

Just know this is a fact and check the handwritten notes to see why this is the case.

Recall, a normal distribution that has $\mu = 0$ and $\sigma = 1$ is called *standard normal*

$$\phi = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

There is no explicit solution for the normal distribution's CDF nor its inverse. We have to use numerical approximation algorithms to approximate a solution and at this date, all statistical software packages already implement them.

The following are standardized distances from the mean:

- $1\sigma = 68.2\%$ of mass
- $2\sigma = 95.4\%$ of mass
- $3\sigma = 99.7\%$

6.1 Approximating Discrete Random Variables

Recall that a normal distribution can approximate a binomial random variable if both np and $n(1-p)$ are sufficiently large. This typically occurs when they are greater than 10.

The Poisson can also be approximated by a normal random variable3 when λ is large. The parameters for a normal distribution are both λ and this occurs when $\lambda \geq 1000$.

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7 Lognormal Distribution

Recall, log returns are normally distributed, but we can use some other distribution to model standard returns directly. The lognormal distribution has the following PDF,

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\ln x - \mu}{2})^2}.$$

One thing to note is that if a variable has a lognormal distribution, then the log of that variable has a normal distribution.

While we can use this distribution to model standard returns, it's often just easier to use the log returns and model those with the normal distribution since the normal distribution has better properties.

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8 Monte Carlo Simulations

This isn't the section that covers Monte Carlo simulations on the FRM exam; however, I've deemed this is a good place to introduce the concept.

We can approximate solutions of problems with no exact solution by creating a Monte Carlo simulation (MCS).

An MCS consists a number of trials where for each trial, we feed random inputs into a system of equations. By collecting the outputs from the system, we can estimate the statistical properties of the output variables.

Month Carlo simulations are covered in a later section of the FRM and of course more in-depth in other resources.

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9 Chi-Squared Distribution, χ^2

If we have k standard normal variables $Z_1, Z_2, \ldots Z_k$, then the sum of their squares, S, has a chi-squared distribution:

$$S = \sum_{i=1}^{k} Z_i^2 \sim \chi_k^2$$

where k is the degrees of freedom. The chi-squared distribution is symmetrical and nonnegative with $\mu = k$ and $\sigma^2 = 2k$.

As k gets large, the distribution becomes more symmetrical and as k approaches infinity, the distribution approaches a normal distribution.

The PDF is given by,

$$f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}.$$

The gamma function is given by,

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx.$$

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10 Student's t Distribution

If Z is a standard normal distribution and U is chi-squared with k degrees of freedom, which is independent of Z, then,

$$X = \frac{Z}{\sqrt{\frac{U}{k}}}$$

follows a t distribution with k degrees of freedom.

The PDF is given by something suppppeerrr ugly, and we're not even going to try to commit it to memory. What we should commit to memory is some of the properties about the shape of the distribution and how it changes with k.

The t distribution has a mean of 0 and for small values of k, it will have a shape like the normal distribution but with excess kurtosis (fatter tails). As k increases, the kurtosis will decrease and as k goes to infinity, the distribution converges to the standard normal.

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11 F Distribution

If U_1 and U_2 are iid and chi-squared with k_1 and k_2, then

$$X = \frac{\frac{U_1}{k_1}}{\frac{U_2}{k_2}} \sim F(k_1, k_2).$$

X follows an F distribution with parameters k_1 and k_2.

Like the Student's t distribution, the PDf for the F distribution is hella complicated and not worth trying to commit to memory. What is important is knowing the properties of its shape, among other things.

The mean is given by,

$$\mu = \frac{k_2}{k_2 - 2},\ k_2 > 2.$$

The variance is given by

$$\sigma^2 = \frac{2k_2^2(k_1 + k_2 - 2)}{k_1(k_2 - 2)^2(k_2 - 4)},\ k_2 > 4.$$

As the k parameters increase, the mean and mode converge to 1. As they go to infinity, the distribution converges to the normal distribution.

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12 Beta Distribution

This distribution is very important for Bayesian analysis methods. The beta distribution is given by the following PDF,

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1},\ 0 \le x \le 1.$$

The mean is given by

$$\mu = \frac{a}{a+b}.$$

The variance is given by

$$\sigma^2 = \frac{ab}{(a+b)^2(a+b+1)}.$$

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13 Mixture Distributions

A *mixture distribution* is a distribution that results form a weighted average distribution of density functions,

$$f(x) = \sum_{i=1}^{n} w_i f_i(x).$$

Each $f_i(x)$ is a component distribution and the w_i are the mixing proportion or the weights.

In a typical mixture distribution, the components are parametric and the weights are nonparametric (they're based on empirical data).

By adding more component distributions, we can approximate data with increased precision but this means they're less useful in general cases.

These type of models are very useful in risk management. Securities with a skewed mixed distribution or with excess kurtosis are often considered riskier than those without since extreme events can occur more frequently.