5. Modern Portfolio Theory (MPT) and the Capital Asset Pricing Model (CAPM)

FRM Part 1: Foundations of Risk Management

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1 Risk vs. Return for Investors

An important point to note is that the trade-off between risk and reward is **not** between risk and actual (realized) return. The trade-off is between risk and *expected return*.

Recall, expected return is a weighted average of the possible returns, where the weight applied to a particular returns equals the probability of that return occurring. We can estimated the possible returns from historical data or it can be assessed objectively.

1.1 Quantifying Risk

A convenient measure to quantify risk when choosing an investment is to use the *standard deviation* of the return over one year.

$$\sigma = \sqrt{E(R^2) - [E(R)]^2}. \tag{1}$$

So, the mean gives the expected return of an individual asset and the standard deviation gives a measurement of risk for a single asset, but of course we can combine multiple assets into a portfolio.

Then, how do we measure the expected return and risk for a portfolio of assets?

First, consider the specific case of two investments with returns R_1 and R_2. To measure the return, we'll weight each investment by the proportion they take up in the total portfolio, w_1, w_2.

Then, to compute the expected value for this portfolio of two investments, we can use the following,

$$\mu_p = w_1 R_1 + w_2 R_2. \tag{2}$$

Next, to compute the standard deviation of this portfolio, we can use

$$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2} \tag{3}$$

where ρ is the Pearson's correlation coefficient between the two investments.

To compute these values for a portfolio with multiple investments, just extend the above equations.

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2 Modern Portfolio Theory (MPT)

Remember, a *good* financial model is one that helps explain the major explanatory variables from a noisy background. There will be many simplifying assumptions we'll need to use, but they don't necessarily invalidate the effectiveness of the models.

MPT asserts how investors should construct a portfolio based on certain assumptions about investor behavior and the properties of the capital market.

In this theory, Harry Markowitz informs us that a *rational investor* is one who is risk adverse and seeks to maximize utility. A rational investor selects their investments using:

- *expected profit (expected return)* - the average rate of return
- *risk* - how much returns vary around this average rate of return

In other words, an investor needs to evaluate potential investment allocations based upon the associated means and variances. The investor selects an asset based on its contribution to the **overall** mean and variance.

- we need to think of the risk of a single investment in terms of its interactions with other assets in the portfolio
- i.e. we need to consider the *covariability* of the assets return with respect to the return of the overall portfolio

From this, we can also infer that investing in assets that fluctuate in different directions can help offset idiosyncratic risk.

The following are the assumptions made for the MPT:

1. Capital markets are perfect.
    - no taxes nor transaction costs
    - all trades have costless access to all available information
    - there is perfect competition, no one has an edge
2. Returns are normally distributed.

2.1 The Efficient Frontier

The *efficient frontier* is a line that contains all portfolios of assets such that there are **no** other portfolios with a higher expected return rate at the same level of risk.

- i.e. the points on this line are the optimal portfolios at each level of risk when measuring in terms of standard deviation

The only way to achieve a higher expected return along the efficient frontier is to increase the risk level.

Along this line, there is a point called the *market portfolio* that includes all the risky assets in the economy, each entering the portfolio in proportion equal to its relative market value.

- the market portfolio is also sometimes called the *tangency portfolio*
- while it is debatable, an index such as the S&P500 can be used as a proxy for the market portfolio since we can't feasibly measure the entire set of all available assets

The market portfolio removes idiosyncratic risk via diversification.

Some issues with MPT include:

- The assumption of normality is an issue since empirical evidence doesn't support this assumption.
- Investors ignore skewness in selecting assets by only focusing on mean and variance of returns
- Implementation error may occur when estimating the model parameters with the use of historical data.

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3 The Capital Asset Pricing Model (CAPM)

The *capital asset pricing model (CAPM)* is a model that shows the relationship between the risk and expected return of a risky asset.

$$E(R) = R_F + \beta[E(R_M) - R_F] \tag{4}$$

where

- $E(R)$ is the expected rate of return for a given asset
- R_F is the risk-free rate of return
- β is the risk contribution of a given stock to the overall portfolio's risk
- $E(R_M)$ is the expected return of the market portfolio

Sharpe and Lintner showed the risk of an individual asset can be decomposed into:

- *diversifiable risk (idiosyncratic)* - risk that can neutralized through diversification
  - this risk is proxied by an asset's β
- *systematic risk (market risk)* - risk that can't be eliminated through diversification

CAPM builds-on an extends MPT. The following are some important assumptions for the model:

1. Access to information is freely available and instantly absorbed by market participants.
2. All market participants have the same expectation.
3. All market participants make their investment decisions on mean and variance returns.
4. No transaction costs, taxes, or fees exist.
5. Allocations can be made in an investment of any partial amount.
6. Market participants can borrow and lend at a common risk-free interest rate.
7. Any individual investor's allocation decision can't change market prices.

Another key assertion of CAPM is that investors weigh their personal portfolio as a combination of

- a risk-free asset (e.g. treasury bill, savings account, etc.)
  - in practice the most common estimate for risk-free rate of return is a 3-month U.S. treasury rate
- a risky asset (e.g. stocks)

The *market risk premium* is the premium that investors demand for taking on the risk of the market portfolio as opposed to the risk-free asset.

- i.e. it's the compensation for taking on increased risk

The market premium is given by

$$MarketPremium = E(R_M) - R_F, \tag{5}$$

where

- $E(R_M)$ is the expected return of the market portfolio
- R_F is the risk-free return

According to CAPM, if the market is in equilibrium, the price of an asset will reflect the relative contribution of its risk to the total risk of the market portfolio. We use Beta, β, as a factor in the model that captures this systematic risk contribution of a given asset.

$$\beta = \frac{Cov(R_I, R_M)}{\sigma_M^2} = \frac{\sigma_I}{\sigma_M}\rho_{I,M}, \tag{6}$$

where

- R_I = the returns on an asses I
- R_M = the returns on the market portfolio
- σ_M^2 = the total risk of the market portfolio
- σ_I = the standard deviation of returns on asset I
- σ_M = the standard deviation of returns on the market portfolio
- $\rho_{I,M}$ = the correlation coefficient between I and the market portfolio

So, again, β represents the portion of an individual asset's risk that can't be neutralized by diversification and for which the investor must be compensated for.

- a increased β implies an increased expected return
- a lower β implies an decreased expected return

β is important because we can use it to compute an asset's expected return (see equation 4 above). It is also important for managers who are attempting to create shareholder value. They can use it to compute:

- the hurdle rate
- RAROC
- EVA (expected value added)

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4 The Capital Market Line (CM) and the Security Market Line (SML)

The *capital market line* shows all the portfolios an investor can create once we allow for a risk-free asset, it is a mix of risky and risk-free assets.

We can estimate the return of a portfolio using the capital market line is given by

$$E(R_P) = R_F + \frac{E(R_M) - R_F}{\sigma_M}\sigma_P \tag{7}$$

It depicts the linear relationship between the expected rate of return on any asset and its systematic risk, β (image not included on this document).

- any portfolio on this line is β efficient
- any portfolio below this line gives inferior performance
- any portfolio above this line gives superior returns (i.e. a "winner")

If $\beta > 1$, we call this an "aggressive" portfolio.

If $0 < \beta < 1$ we call this a "defensive" portfolio.

If $\beta < 0$ then this means the asset moves counter to the market.

The *security market line* gives the relationship between the expected return for an *individual asset* and it's give, **not** the entire portfolio.

We can use a linear fitting algorithm, such as OLS, to compute β.

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5 Performance Measures

The following measures in this section are traditional measures based on CAPM with the idea that to get higher than average returns, we must assume a higher than average risk.

5.1 Sharpe Performance Index (SPI)

The invest performance index, SPI, is given by,

$$SPI = \frac{E(R_I) - R_F}{\sigma_I} \tag{8}$$

where

- I is an arbitrary asset.

Recall, the capital market line equation given by 7. In this equation, the slope is given by

$$\frac{E(R_M) - R_F}{\sigma_M}.$$

An SPI greater than this slope of the capital market line indicates superior performance. Similarly, an SPI below this slope indicates inferior performance.

5.2 Treynor Performance Index (TPI)

The TPI is given by,

$$TPI = \frac{E(R_I) - R_F}{\beta_I}. \tag{9}$$

Like SPI, TPI measures risk premium per unit of risk but it uses the β of an asset or portfolio instead of the standard deviation.

A TPI greater than the market premium indicates a positive α and thus superior performance. A TPI less than the market premium indicates a negative α and thus inferior performance.

5.2.1 Alpha, α

A superior return by a portfolio is referred to as α.

$$\alpha = R_P - R_F - \beta(R_M - R_F). \tag{10}$$

Alpha represents the extra return made from superior portfolio management. This is zero-sum, an investor can only make a positive alpha at the expense of other investors who are making a negative alpha.

5.3 Jensen's Performance Index (JPI)

By running a time-series regression of the portfolio excess rate of return on the market portfolio's excess rate of return, one can estimate the beta of portfolio I,

$$(R_{It} - R_{Ft}) = \hat{\alpha_I} + \hat{\beta_I}(R_{Mt} - R_{Ft}) + e_{It}. \tag{11}$$

If $\hat{\alpha_I}$ is significantly different from zero and is positive, then the performance of asset I is considered superior.

5.4 Sortino Ratio (SR)

SR is a modification of SPI. Like SPI, it still measures the risk-adjusted return of an asset or portfolio, but the focus with this measure is on downside risk.

$$SR = \frac{R_P - T}{\sqrt{\frac{1}{N}\sum_{t=1}^{N} min(0, R_{Pt} - T)^2}} \tag{12}$$

where

- the denominator is the downside deviation
- T is the target or required rate of return for the investment strategy or the *minimum accepted rate*

5.5 Information Ratio

The information ratio compares or adjusts performance relative to a benchmark. The information ratio is given by

$$IR = \frac{Averageactivereturn}{Trackingerror}. \tag{13}$$

The *tracking error* is the measure of the difference between a portfolio's returns and those of a benchmark it was meant to mimic or beat. It is the standard deviation of the *active returns* over some time period. The active returns is given by,

$$R_P - R_B. \tag{14}$$