The Newmark Method *

Michael Karow

January 18, 2016

Let $u : [t_1, t_e] \to \mathbb{R}^n$ be a solution of the equation

$$f(\ddot{u}(t), \dot{u}(t), u(t), t) = 0.$$

Furthermore, let $t_1 < t_2 < \ldots < t_{n-1} < t_n = t_e$ be a partition of the interval $[t_1, t_e]$. We want to generate approximations u_j, $\dot{u}_j$, $\ddot{u}_j$ for $u(t_j)$, $\dot{u}(t_j)$, $\ddot{u}(t_j)$ which satisfy

$$u_1 = u(t_1), \quad \dot{u}_1 = \dot{u}(t_1), \quad \ddot{u}_1 = \ddot{u}(t_1). \tag{1}$$

and

$$f(\ddot{u}_{j+1}, \dot{u}_{j+1}, u_{j+1}, t_{j+1}) = 0, \qquad j = 1, \ldots, n-1. \tag{2}$$

For the derivation of an algorithm we consider Taylor-expansions with remainder:

$$\begin{aligned} u(t_{j+1}) &= u(t_j) + \dot{u}(t_j)\, h_j + \frac{\ddot{u}(\tau_1)}{2}\, h_j^2 \qquad &\tau_1 \in [t_j, t_{j+1}], \qquad (3)\\ \dot{u}(t_{j+1}) &= \dot{u}(t_j) + \ddot{u}(\tau_2)\, h_j, &\tau_2 \in [t_j, t_{j+1}], \qquad (4) \end{aligned}$$

where

$$h_j = t_{j+1} - t_j$$

is the stepsize. In these formulae we replace $u(t_{j+1}), u(t_j), \dot{u}(t_j), \dot{u}(t_{j+1})$ by their approximations. The unknown quantities $\ddot{u}(\tau_1)$ und $\ddot{u}(\tau_2)$ are replaced by weighted means of $\ddot{u}_j$ and $\ddot{u}_{j+1}$. This yields the formulae

$$\begin{aligned} u_{j+1} &= u_j + \dot{u}_j\, h_j + ((\frac{1}{2} - \beta)\, \ddot{u}_j + \beta\, \ddot{u}_{j+1})\, h_j^2, \qquad (5)\\ \dot{u}_{j+1} &= \dot{u}_j + ((1-\gamma)\, \ddot{u}_j + \gamma\, \ddot{u}_{j+1})\, h_j. \qquad (6) \end{aligned}$$

Here, $\beta \in [0, \frac{1}{2}]$, $\gamma \in [0, 1]$ are fixed values. The Equations (1), (2), (5) und (6) define the Newmark method. For practical reasons we collect the j-dependent parts of the right hand sides of (5) and (6):

$$\begin{aligned} u_j^* &= u_j + \dot{u}_j\, h_j + (\frac{1}{2} - \beta)\, \ddot{u}_j\, h_j^2, \qquad (7)\\ \dot{u}_j^* &= \dot{u}_j + (1-\gamma)\, \ddot{u}_j\, h_j. \qquad (8) \end{aligned}$$

With these quantities the equation (5) and (6) can be written as

$$u_{j+1} = u_j^* + \beta\, \ddot{u}_{j+1}\, h_j^2, \qquad \dot{u}_{j+1} = \dot{u}_j^* + \gamma\, \ddot{u}_{j+1}\, h_j. \tag{9}$$

This combined with (2) yields

$$f(\, \ddot{u}_{j+1},\ \dot{u}_j^* + \gamma\ \ddot{u}_{j+1}\, h_j,\ u_j^* + \beta\, \ddot{u}_{j+1}\, h_j^2,\ t_{j+1}\,) = 0. \tag{10}$$

*Newmark, N. M. (1959) A method of computation for structural dynamics. Journal of Engineering Mechanics, ASCE, 85 (EM3) 67-94.

Now, the Newmark algorithm is the following:

- Let $u_1 = u(t_1), \quad \dot{u}_1 = \dot{u}(t_1), \quad \ddot{u}_1 = \ddot{u}(t_1)$.
- For $j = 1$ to $n-1$:
  (a) Compute u_j^* and $\dot{u}_j^*$ using (7) and (8).
  (b) Compute the solution $\ddot{u}_{j+1}$ of (10).
  (c) Compute u_{j+1} and $\dot{u}_{j+1}$ using (9).

The Newmark method was originally developed for functions of the form

$$f(\ddot{u}, \dot{u}, u, t) = M\ddot{u} + D\dot{u} + Su - p(t).$$

In this case step (b) of the algorithm is:

(b') Solve the equation

$$(M + \gamma\, h_j\, D + \beta\, h_j^2\, S)\ddot{u}_{j+1} = p(t_{j+1}) - D\,\dot{u}_j^* - S\,u_j^*.$$

Optimal parameters are: $\beta = 1/4$, $\gamma = 1/2$.
With these parameters the method is stable for any step size h. Furthermore, energy is preseved if $D = 0$, $p = 0$. See the next section.

The Newmark method and the balance of energy
We consider the differential equation

$$M\ddot{u}(t) + D\dot{u}(t) + Su(t) = p(t) \tag{11}$$

where $M, D, S \in \mathbb{R}^{n\times n}$ are symmetric, M and S are positive definite, and D is positive semidefinite. In mechanics M is called the mass matrix S is called the stiffness matrix and D is called the damping matrix. The vector $u(t)$ is the state of the mechanical system and $p(t) \in \mathbb{R}^n$ is the load vector which contains the external forces. The energy of the system is defined as

$$E(t) = \frac{1}{2}\left(\dot{u}(t)^\top M\dot{u}(t) + u(t)^\top Su(t)\right).$$

From (11) we have the following balance of energy:

$$\dot{E} = \dot{u}^\top p - \dot{u}^\top D\dot{u}. \tag{12}$$

Here $\dot{u}^\top p$ is the power of the external force p and $\dot{u}^\top D\dot{u}$ is the dissipated energy. In particular, if $D = 0$ (no damping) and $p = 0$ (no external forces) then E is constant.

We consider now the discretized balance of energy for the Newmark method with paramters $\beta = 1/4$ and $\gamma = 1/2$. Let $E_j = \frac{1}{2}(\dot{u}_j^\top M\dot{u}_j + u_j^\top Su_j)$. Then for $\beta = 1/4$, $\gamma = 1/2$,

$$E_{j+1} - E_j = (u_{j+1} - u_j)^\top \frac{p(t_j) + p(t_{j+1})}{2} - (u_{j+1} - u_j)^\top D\frac{\dot{u}_j + \dot{u}_{j+1}}{2} \tag{13}$$

and

$$\frac{E_{j+1} - E_j}{h_j} = \left(\frac{\dot{u}_{j+1} + \dot{u}_j}{2}\right)^\top \frac{p(t_j) + p(t_{j+1})}{2} - \left(\frac{\dot{u}_{j+1} + \dot{u}_j}{2}\right)^\top D\frac{\dot{u}_j + \dot{u}_{j+1}}{2}. \tag{14}$$

In particular, $E_{j+1} = E_j$ if $D = 0$ and $p(t_j) = p(t_{j+1}) = 0$.

Proof: on the next page.

Proof of equations (13) and (14): By definition of the Newmark method we have

$$
\begin{aligned}
u_{j+1} &= u_j + \dot{u}_j\, h_j + \frac{\ddot{u}_j + \ddot{u}_{j+1}}{4}\, h_j^2, \qquad \text{(since } \beta = 1/4) && (15)\\
\dot{u}_{j+1} &= \dot{u}_j + \frac{\ddot{u}_j + \ddot{u}_{j+1}}{2}\, h_j, \qquad \text{(since } \gamma = 1/2) && (16)\\
M\ddot{u}_j + Su_j &= p_j - D\dot{u}_j, && (17)\\
M\ddot{u}_{j+1} + Su_{j+1} &= p_{j+1} - D\dot{u}_{j+1}. && (18)
\end{aligned}
$$

By multiplying the sum of (17) and(18) from the left with $(u_{j+1} - u_j)^\top$ we obtain

$$L = (u_{j+1} - u_j)^\top (p_{j+1} + p_j) - (u_{j+1} - u_j)^\top D(\dot{u}_{j+1} + \dot{u}_j), \tag{19}$$

where L is defined as

$$L := (u_{j+1} - u_j)^\top M(\ddot{u}_{j+1} + \ddot{u}_j) + (u_{j+1} - u_j)^\top S(u_{j+1} + u_j). \tag{20}$$

Because of (19) equation (13) is a consequence of the identity

$$L = 2(E_{j+1} - E_j) = \dot{u}_{j+1}^\top M\dot{u}_{j+1} - \dot{u}_j^\top M\dot{u}_j + u_{j+1}^\top Su_{j+1} - u_j^\top Su_j, \tag{21}$$

which we show now. Since S is symmetric we have $u_{j+1}^\top Su_j = u_j^\top Su_{j+1}$. This yields

$$
\begin{aligned}
(u_{j+1} - u_j)^\top S(u_{j+1} + u_j) &= u_{j+1}^\top Su_{j+1} + u_{j+1}^\top Su_j - u_j^\top Su_{j+1} + u_j^\top Su_j\\
&= u_{j+1}^\top Su_{j+1} - u_j^\top Su_j.
\end{aligned}
$$

Thus,

$$L = (u_{j+1} - u_j)^\top M(\ddot{u}_{j+1} + \ddot{u}_j) + u_{j+1}^\top Su_{j+1} - u_j^\top Su_j.$$

It remains to verify that

$$(u_{j+1} - u_j)^\top M(\ddot{u}_{j+1} + \ddot{u}_j) = \dot{u}_{j+1}^\top M\dot{u}_{j+1} - \dot{u}_j^\top M\dot{u}_j. \tag{22}$$

However, from (15) and (16) it follows that

$$\ddot{u}_{j+1} + \ddot{u}_j = \frac{2}{h_j}(\dot{u}_{j+1} - \dot{u}_j), \qquad u_{j+1} - u_j = \frac{h_j}{2}(\dot{u}_{j+1} + \dot{u}_j). \tag{23}$$

This together with the symmetry of M yields (22). (14) is immediate from (13) and the second equation of (23). $\square$