The eigenvalue method for undamped vibrations

Michael Karow*

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Problem: Find all solutions of the homogeneous DAE (differential algebraic equation)

$$\underbrace{\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}}_{M_e} \begin{bmatrix} \ddot{w} \\ \ddot{\mu} \end{bmatrix} + \underbrace{\begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix}}_{S_e} \begin{bmatrix} w \\ \mu \end{bmatrix} = 0. \tag{1}$$

<u>Assumptions on matrices:</u> $M \in \mathbb{R}^{N\times N}$ symmetric positive definite, $S \in \mathbb{R}^{N\times N}$ symmetric positive semidefinite, $S_e \in \mathbb{R}^{(N+K)\times(N+K)}$ invertible (implication: the columns of $C \in \mathbb{R}^{N\times K}$ are linearly independent).

<u>Notation:</u> $A = S_e^{-1} M_e$. <u>Ansatz:</u> First, we determine the solutions of (1) of the form

$$\begin{bmatrix} w(t) \\ \mu(t) \end{bmatrix} = \sigma(t) \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix}, \qquad \sigma(t) \in \mathbb{R}, \quad \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix} \in \mathbb{R}^{N+K} \setminus \{0\} \text{ fixed}, \quad \sigma \not\equiv 0. \qquad (*)$$

Proposition 1. $(*)$ is a solution of (1) if and only if

$$\ddot{\sigma} + \frac{1}{\lambda}\sigma = 0, \quad \text{where} \quad A \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix} = \lambda \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix}, \quad \lambda > 0.$$

Corollary: Let $\omega := 1/\sqrt{\lambda} > 0$. Then $\ddot{\sigma} + \omega^2 \sigma = 0$. Thus, all solutions σ are of the form

$$\sigma(t) = \alpha \cos(\omega t) + \frac{\beta}{\omega} \sin(\omega t) = \rho \cos(\omega t - \phi), \quad \text{where} \quad \alpha, \beta, \phi \in \mathbb{R} \text{ arbitrary}, \rho \geq 0.$$

We have $\alpha = \sigma(0)$, $\beta = \dot{\sigma}(0)$, $\rho = \sqrt{\alpha^2 + (\beta/\omega)^2}$, ϕ such that $\alpha = \alpha \cos(\phi)$, $\beta/\omega = \rho \sin(\phi)$. ($\rho$ and ϕ are the polar coordinates of the vector $[\alpha \quad \beta/\omega]^\top \in \mathbb{R}^2$.) Of course, the functions σ can also be written in the form $\sigma(t) = \rho \sin(\omega t - \tilde{\phi})$, $\tilde{\phi} = \phi - \pi/2$ (sinusoidal functions).

Interpretation in beam project: The solutions $(*)$ are standing waves.

Proof of Proposition 1. Insertion of $(*)$ in (1) yields

$$M_e \ddot{f} \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix} + S_e f \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix} = 0.$$

Reordering terms we get

$$\underbrace{S_e^{-1} M_e}_{A} \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix} = \underbrace{-\frac{f}{\ddot{f}}}_{=:\lambda} \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix} \qquad (**).$$

We show that $\lambda > 0$:

$$\begin{aligned} (**) \quad &\Rightarrow \quad \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix} = \lambda \begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix} \begin{bmatrix} w_0 \\ \mu_0 \end{bmatrix} \\ &\Rightarrow \quad M w_0 = \lambda S w_0 + C \mu_0, \quad C^\top w_0 = 0. \qquad (***) \end{aligned}$$

*karow@math.tu-berlin.de

If $w_0 = 0$ then $C\mu_0 = 0$, and then $\mu_0 = 0$, since the columns of C C are linearly independent. This is the trival case which had been excuded in $(*)$. Hence, $w_0 \neq 0$. Multiplication of the first equation in $(* * *)$ with $w_0^\top$ yields (since M is positive definite),

$$\underbrace{w_0^\top M w_0}_{>0} = \lambda\,(\underbrace{w_0^\top S w_0}_{\geq 0} + \underbrace{w_0^\top C}_{(C^\top w_0)^\top = 0} \quad \mu_0) \quad \Rightarrow \quad \lambda > 0 \qquad \square$$

Proposition 2 (proof omitted).

(i) There exist $N-K$ linearly independent eigenvectors $\begin{bmatrix} w_k \\ \mu_k \end{bmatrix}$, such that

(a) $A \begin{bmatrix} w_k \\ \mu_k \end{bmatrix} = \lambda_k \begin{bmatrix} w_k \\ \mu_k \end{bmatrix}, \quad \lambda_k > 0,$

(b) $w_j^\top M w_k = 0$ für $j \neq k$. (for $\lambda_j \neq \lambda_k$ statement (a) implies (b)).

(ii) All solutions ot the DAE (1) are of the form

$$\begin{bmatrix} w(t) \\ \mu(t) \end{bmatrix} = \sum_{k=1}^{N-K} \left(\alpha_k \cos(\omega_k t) + \frac{\beta_k}{\omega_k} \sin(\omega_k t) \right) \begin{bmatrix} w_k \\ \mu_k \end{bmatrix}, \qquad \omega_k = 1/\sqrt{\lambda_k}, \quad \alpha_k, \beta_k \in \mathbb{R}.$$

We have

$$\alpha_k = \frac{w_k^\top M w(0)}{w_k^\top M w_k}, \qquad \beta_k = \frac{w_k^\top M \dot{w}(0)}{w_k^\top M w_k} \qquad (* * **)$$

(consequence of (b), see remark below). The vectors $w(0)$, $\dot{w}(0) \in \mathbb{R}^N$ (initial values) can be chosen arbitrarily.

Remarks.

- The summands (ii) are called vibration modes.
- The matrix A has the (additional) eigenvalue 0 with algebraic multiplicity $2K$.
- Eigenvalues can be calculated using the MATLAB command `eig`. Eigenvectors to the eigenvalue 0 have to be ignored.
- Derivation of (b) for $\lambda_j \neq \lambda_k$: from (a) it follows that

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \mu_k \end{bmatrix} = \lambda_k \begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \mu_k \end{bmatrix}.$$

Multiplikation with $\begin{bmatrix} w_j \\ \mu_j \end{bmatrix}^\top$ yields

$$w_j^\top M w_k = \begin{bmatrix} w_j \\ \mu_j \end{bmatrix}^\top \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \mu_k \end{bmatrix} = \lambda_k \begin{bmatrix} w_j \\ \mu_j \end{bmatrix}^\top \begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \mu_k \end{bmatrix} \qquad (+)$$

The same derivation with exchanged indices yields

$$w_k^\top M w_j = \begin{bmatrix} w_k \\ \mu_k \end{bmatrix}^\top \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_j \\ \mu_j \end{bmatrix} = \lambda_j \begin{bmatrix} w_k \\ \mu_k \end{bmatrix}^\top \begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix} \begin{bmatrix} w_j \\ \mu_j \end{bmatrix} \qquad (++)$$

From the symmetry of the matrices we infer

$$w_j^\top M w_k = w_k^\top M w_j, \qquad \begin{bmatrix} w_j \\ \mu_j \end{bmatrix}^\top \begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \mu_k \end{bmatrix} = \begin{bmatrix} w_k \\ \mu_k \end{bmatrix}^\top \begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix} \begin{bmatrix} w_j \\ \mu_j \end{bmatrix}.$$

By subtracting (+) and (++) we get

$$0 = (\lambda_k - \lambda_j) \begin{bmatrix} w_j \\ \mu_j \end{bmatrix}^\top \begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \mu_k \end{bmatrix}, \text{ thus, } 0 = \begin{bmatrix} w_j \\ \mu_j \end{bmatrix}^\top \begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \mu_k \end{bmatrix}, \text{ thus, from (+), } w_j^\top M w_k = 0$$

- Derivation of $(* * **)$ from (b):
We have

$$\dot{w}(0) = \sum_{k=1}^{N-K} \left(-\alpha_k \omega_k \sin(\omega_k 0) + \beta_k \cos(\omega_k 0)\right) w_k = \sum_{k=1}^{N-K} \beta_k w_k.$$

Multiplikation with $w_j^\top M$ yields (because of (b)), $w_j^\top M \dot{w}(0) = \beta_j w_j^\top M w_j$, $j = 1, \dots N-K$. This implies the formula for β (replace j with k). The derivation for α is analogous.