Mean Curvature Flow

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Preface

Plan

1. Huisken's 1984 original paper.
2. Monotonicity formula and its application to type-I singularities.
3. Huisken Sinestrari paper on convexity estimates using Stampacchia trick.
4. Noncollapsing.

TO DO :

1. Highlight definition and new terms with **boldface** whenever applicable.
2. Look into capitalization of Mean/mean and make it consistent.

Why Mean curvature flow

Why specifically we want to study this flow and if we do what results can we achieve? To start with it is a very natural flow to consider on hypersurfaces in Euclidean space. It bends the higher curved parts with more speed than lower curved parts in order to uniformize the curvature across the hypersurface. Also, the parabolic nature of the equation directly gives short time existence and uniqueness; so we know given a hypersurface we have one way to evolve to possibly study its geometry. For its twin "Ricci flow" as Huisken calls it the motivation was uniformizing Riemannian manifolds with an eye towards Poincaré conjecture. This is of-course with benefit of hindsight after Perelman's seminal resolution using surgery methods.

Now what do we want to do with the very natural Mean curvature flow on hypersurfaces? A generic answer is to study the geometry of hypersurfaces and attempt a classification. This is severely restricted by the assumption of mean convexity ($H > 0$ everywhere) which makes maximum principle work in a number of cases. Huisken's result on the convergence of convex hypersurfaces into round sphere is the first step towards it, but it doesn't achieve much topologically. A uniformly convex hypersurface is diffeomorphic to unit sphere by Gauss map to begin with. For non-convex hypersurface singularities might develop which prohibit a direct analysis. To overcome this we blow-up the manifold near singularity and this limiting process gives an ancient solution. So we shift our attention to a classification of ancient solutions which is still a difficult problem. Angenent-Daskalopoulos-Sesum and Brendle-Choi have obtained results in this direction without self-similarity conditions

Another direction the Mean curvature flow is being explored is the Lagrangian Mean curvature flow in order to find special Lagrangians inside symplectic manifolds. In the case of Calabi-Yau manifolds the condition of being Lagrangian is preserved under Mean curvature flow.

Mean curvature flow can also be potentially used to find nice codimension one hypersurface inside Riemannian manifolds. While Brendle's proof of Lawson conjecture didn't directly involve the flow however it did use some techniques coming from his sharp estimate analysis of the inscribed radius in noncollapsing.

Organization

Acknowledgments

Contents

1. Introduction to Mean curvature flow

Mean curvature flow is the negative gradient flow of the volume functional on hypersurfaces.

1.1. Fundamentals of hypersurfaces

SET UP THE FOLLOWING -
1. SECOND FUNDAMENTAL FORM
2. MEAN CURVATURE
3. CONVEXITY RESULTS
4. MCF AS GRADIENT FLOW OF AREA
5. CODAZZI IDENTITY, SIMONS IDENTITY
6. UMBILLIC HYPERSURFACES ARE SPHERES

Let M^n be a smooth n-dimensional manifold with a smooth immersion $X : M^n \to \mathbb{R}^{n+1}$. If X is a diffeomorphism onto its image, we say X is an **embedding** and its image $\mathcal{M}^n = X(M^n)$ has the structure of a smooth n-dimensional submanifold of $\mathbb{R}^{n+1}$. We say that M^n is an **immersed hypersurface** and $\mathcal{M}$ is an **embedded hypersurface** respectively. Throughout this book we will denote the embedded manifold $X(M)$ by script $\mathcal{M}$ to differentiate between the domain and its image. Let $(U, \{x^i\})$ be a coordinate system on M^m, in Euclidean coordinates the pushforward of tangent vectors will be

$$dX(\partial_i) := \frac{\partial X}{\partial x^i} = \partial_i X$$

where $dX : TM^n \to T\mathbb{R}^{n+1}$ is the derivative of X. Since dX is an injection for each point in M^n, we can define an inner product on TM^n which in local coordinates is given by

$$g(\partial_i, \partial_j) = \langle \partial_i X, \partial_j X \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on Euclidean space. Further we can define the Levi-Civita connection on M^n from the Levi-Civita connection on $\mathbb{R}^{n+1}$. Let $X_p \in T_p\mathbb{R}^{n+1}$ be a vector and $Y : U \to T\mathbb{R}^{n+1}|_U$ be a local vector field in a neighborhood U containing p. The Levi-Civita connection of Y with respect to X on $\mathbb{R}^{n+1}$ is given by

$$D_{X_p} Y = X_p(Y^i)\partial_i$$

where $Y = (Y^1, \ldots, Y^{n+1})$ are the components of Y in the standard coordinates. Using the immersion condition, we define a connection on TM^n induced from D. Let $x \in M^n$

and $u \in T_pM^n, \tilde{v} \in TM^n|_U$ for some open set U containing x. Define a connection ∇ by

$$dX(\nabla_u \tilde{v}) = D_{dX(u)}(\tilde{V}) \tag{1.1.1}$$

where $\tilde{V}$ is an extension of $dX(\tilde{v})$ to an open set of $\mathbb{R}^{n+1}$ containing $X(U)$.

Lemma 1.1.1. The connection defined by Eq. (1.1.1) is well-defined and is the unique Levi-Civita connection on (M^n, g).

When X is an embedding, the restriction of the tangent bundle of $T\mathbb{R}^{n+1}|_{\mathcal{M}}$ can be decomposed as the direct sum

$$T\mathcal{M} \oplus N\mathcal{M}$$

where $N\mathcal{M}$ is the **normal bundle** which can be described as

$$N\mathcal{M} = \{(p, \nu) \in T\mathbb{R}^{n+1}|_{\mathcal{M}} : \langle u, \nu \rangle = 0 \text{ for all } u \in T_p\mathcal{M}\}.$$

For dimension reasons, the normal bundle at each point is one-dimensional. We fix a choice of unit normal ν_p for each $p \in \mathcal{M}$. This leads to **tangential projection** $\cdot^T : T\mathbb{R}^{n+1} \to T\mathcal{M}$ and **normal projection** $\cdot^\perp : T\mathbb{R}^{n+1} \to N\mathcal{M}$ maps of vectors in $T\mathbb{R}^{n+1}$ given by

$$u^T = u - \langle u, \nu \rangle \nu, \quad \text{and} \quad u^\perp = \langle u, v \rangle \nu$$

respectively. We can define the Levi-Civita connection on a embedded hypersurface $\mathcal{M}$ using the normal projection,

$$\nabla_u V = (D_u V)^T$$

where u is a vector and V is a local vector field. Notice that this is consistent with Eq. (1.1.1) since dX^{-1} is the tangential component. The next step is to calculate the Christoffel symbols of the connection ∇. For local coordinates $(U, \{x^i\})$ in M^n, the Christoffel symbols $\Gamma_{ij}^k : U \to \mathbb{R}$ are obtained by the formula

$$\nabla_{\partial_i X} \partial_j X = \partial_i(\partial_j X) + \Gamma_{ij}^k \partial_k X$$

which simplifies to

$$\Gamma_{ij}^k \partial_k X = \partial_i(\partial_j X) - (\partial_i(\partial_j X))^T = (\partial_i(\partial_j X))^\perp$$

1.2. Mean curvature flow

Definition 1.2.1. A one-parameter family of immersion $X : M^n \times I \to \mathbb{R}^{n+1}$ is said to evolve by **Mean Curvature Flow** (MCF) if

$$\frac{\partial}{\partial t} X(p,t) = \vec{H}(X(p,t)) = -H(X(p,t))\nu(X(p,t)) \quad \forall (p,t) \in M^n \times I. \tag{1.2.1}$$

Notice that the mean curvature vector $\vec{H} = -H\nu$ is independent of the direction of normal ν. The following lemma demonstrates the similarity of mean curvature flow with heat equation

Lemma 1.2.1. The mean curvature vector is equal to Laplace-Beltrami operator of the hypersurface

$$\vec{H} = -H\nu = \Delta_{\mathcal{M}} X.$$

Proof. □

1.2.1. Examples of the mean curvature flow

It is difficult to solve the Mean curvature flow PDE on an arbitrary hypersurface. The limited number of examples come from ansatz or special cases,

1. **Shrinking spheres**: Let $\mathbb{S}^n(r) \subset \mathbb{R}^{n+1}$ be sphere of dimension n with radius r. Since the mean curvature $H = \frac{n}{r}$ is constant across the sphere, me make the ansatz that the hypersurface remain spherical under mean curvature flow. Let $\mathcal{M}_t = \mathbb{S}^n(r(t))$ be the solution, then the PDE is reduced to an ODE given by

$$\frac{d}{dt} r(t) = -\frac{n}{r(t)} \tag{1.2.2}$$

whose solution is $r(t) = \sqrt{r_0^2 - 2nt}$ with $r(0) = r_0$. So the shrinking spheres $\mathbb{S}^n(\sqrt{r_0^2 - 2nt})$ are a solution to the mean curvature flow for $t \in [0, \frac{r_0^2}{2n})$.

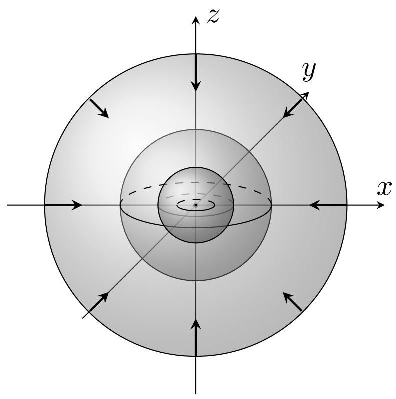

Figure 1.1.: Shrinking spheres of dimension 2

2. **Evolution of Graphs**: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The graph of f in $\mathbb{R}^{n+1}$,

$$\mathcal{M} = \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$$

is a smooth hypersurface. The mean curvature vector at $(x, f(x))$ for the hypersurface can be calculated to be,

$$\sqrt{1+|\nabla f|^2}\,\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right).$$

Ecker and Huisken proved in [EH89] that graphs evolving under the mean curvature flow remain graphs. So a family of graphs $\mathcal{M}_t = \{(x, f_t(x)) : x \in \mathbb{R}^{n+1}\}$ with the condition

$$\frac{\partial}{\partial t} f_t(x) = \sqrt{1+|\nabla f_t|^2}\,\mathrm{div}\left(\frac{\nabla f_t}{\sqrt{1+|\nabla f_t|^2}}\right)$$

is a solution of the mean curvature flow.

3. **Minimal surfaces**: Minimal surfaces are the critical points of the volume functional. A hypersurface $\mathcal{M}$ is said to be a **minimal hypersurface** if it satisfies $H(x) = 0$ for all $x \in \mathcal{M}$. Hence, minimal hypersurfaces are stationary solutions of the mean curvature flow.

4. **Products of solutions with Euclidean space** : Suppose $\mathcal{M}_t^n \subset \mathbb{R}^{n+1}$ is a solution of the mean curvature flow. It is easy to verify that mean curvature vector of the product $\mathcal{M}_t^n \times \mathbb{R}^m \subset \mathbb{R}^{n+1} \times \mathbb{R}^m$ is given by

$$\vec{H}(x, y) = (H(x)\nu(x), 0),$$

which implies that the time-parametrized product $\mathcal{N}_t = \mathcal{M}_t \times \mathbb{R}^{n+1}$ is a solution of the mean curvature flow as well.

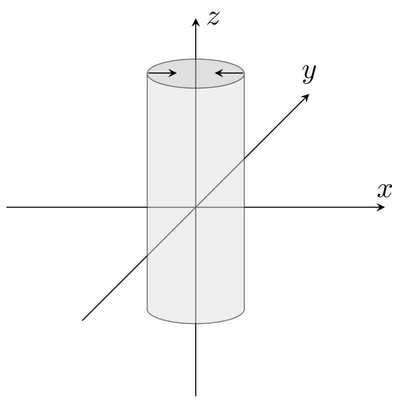

Figure 1.2.: Cylinder $S^1 \times \mathbb{R}$

1.2.2. Mean curvature flow as gradient of the area functional

Let $\mathcal{M}_0 \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ be a variation with $X(\cdot, 0) = \mathcal{M}_0$. Considering area as a function of time over the variation, we get

$$\frac{d}{dt}\text{Area}(\mathcal{M}_t) = \int_{\mathcal{M}_t} \langle \partial_t X, H\nu \rangle \tag{1.2.3}$$

Using this, the gradient of the area functional is

$$\nabla\text{Area} = H\nu$$

so the most efficient way to reduce the volume is to choose the variation so that

$$\partial_t X = -\nabla\text{Area} = -H\nu$$

which is the mean curvature flow. In particular, we get the following equation for evolution of area under mean curvature flow,

$$\frac{d}{dt}\text{Area}(\mathcal{M}_t) = -\int_{\mathcal{M}_t} H^2$$

which is the steepest descent of area in the space of hypersurface up to speed-parametrization.

1.2.3. Short-time existence

The following theorem about the uniqueness and existence of mean curvature flow is proved in [ACGL22].

Theorem 1.2.2. Short time existence : Let $X_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of a compact manifold without boundary. There exists an $\epsilon > 0$ and a smooth solution $X : M^n \times [0, \epsilon) \to \mathbb{R}^{n+1}$ to MCF, with $X(\cdot, 0) = X_0$. Moreover, the solution is unique.

1.3. Maximum principle

We can extend the maximum principle on Euclidean space to general Riemannian manifolds in the following fashion.

Lemma 1.3.1 (Scalar maximum principle)**.** Let $g(t) \in [0, T)$ be a 1-parameter family of Riemannian metrics on a closed manifold $\mathcal{M}^n$ and $\beta : \mathcal{M}^n \times [0, T) \to \mathbb{R}$ be a locally bounded function. Let $u : \mathcal{M}^n \times [0, T) \to \mathbb{R}$ be a C^2 function satisfying the following inequality

$$\frac{\partial}{\partial t} u(x, t) \geq \Delta_{g(t)} u + \beta u$$

If $u(x,0) \geq 0$ for all $x \in \mathcal{M}^n$, then $u(x,t) \geq 0$ for all $(x,t) \in \mathcal{M}^n \times [0,T)$.

Proof. □

The above maximum principle can be further extended to tensors. This was done by Hamilton in [Ham82] in the context of Ricci flow. Let $M = M_{ij} dx^i \otimes dx^j$ be a symmetric 2-tensor. We say M is non-negative if $v^T M v = M_{ij} v^i v^j \geq 0$ for all vectors v. Let $N_{ij} = p(M_{ij}, g_{ij})$ be a tensor formed by contracting products of M_{ij} with itself using the metric. Also suppose that whenever v is a null-eigenvector of M_{ij} (i.e. $M_{ij} v^j = 0$), we have $N_{ij} v^i v^j \geq 0$. Then the following maximum principle holds

Lemma 1.3.2 (Tensor maximum principle)**.** Let $g(t) \in [0,T)$ be a 1-parameter family of Riemannian metrics on a closed manifold $\mathcal{M}^n$. Let M_{ij} be a symmetric non-negative tensor evolving by the equation

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + N_{ij} \text{ for all } (x,t) \in \mathcal{M}^n \times [0,T)$$

where $N_{ij} = p(M_{ij}, g_{ij})$ satisfies the null-eigenvector condition above. If M is non-negative at $t = 0$, then it remains non-negative on $[0,T)$.

1.3.1. Applications to mean curvature flow

The tensor maximum principle can be used to prove that convexity is preserved under mean curvature flow.

Lemma 1.3.3. If

1.4. Evolution equations

To understand the properties of mean curvature flow it is essential to know the evolution of geometric quantities of the hypersurface. Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a smooth solution of mean curvature flow, so

$$\partial_t X(x,t) = \overrightarrow{H}(x,t) = -H(x,t)\nu(x,t)$$

Let $\{x^i\}$ be a local coordinate in M^n. Then the induced metric on the hypersurface is given by $g = X^*(\delta)$ where δ is the flat metric on $\mathbb{R}^{n+1}$. This gives

$$g_{ij} = \delta(X_*(\partial_i), X_*(\partial_j)) = \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle$$

Lemma 1.4.1. Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a solution of mean curvature flow. Then the evolution equation of metric, normal, second fundamental form, and mean curvature is given by

$$\partial_t g_{ij} = -2Hh_{ij} \tag{1.4.1}$$

$$\partial_t \nu = \nabla H \tag{1.4.2}$$

$$\partial_t h_{ij} = \Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij} \tag{1.4.3}$$

$$\partial_t H = \Delta H + |A|^2 H \tag{1.4.4}$$

Proof. 1. In local coordinates we have

$$\begin{aligned}
\partial_t g_{ij} &= \partial_t \langle \partial_i X, \partial_j X\rangle \\
&= \langle \partial_i(\partial_t X), \partial_j X\rangle + \langle \partial_i X, \partial_t(\partial_j X)\rangle \\
&= \langle \partial_t(-H\nu), \partial_j X\rangle + \langle \partial_i X, \partial_j(-H\nu)\rangle \\
&= -H\langle \partial_t \nu, \partial_j X\rangle - H\langle \partial_i X, \partial_t \nu\rangle \\
&= -2Hh_{ij}
\end{aligned}$$

2. Since $\langle \nu, \nu\rangle = 1$, we have $2\langle \partial_t \nu, \nu\rangle = 0$, so the vector $\partial_t \nu$ is in the tangent plane of the hypersurface. We can write it as a linear combination of tangent vectors $\{\partial_j X\}$ to get

$$\begin{aligned}
\partial_t \nu &= \langle \partial_t \nu, \partial_i X\rangle \partial_j X g^{ij} = -\langle \nu, \partial_i X\,(\partial_t X)\rangle \partial_j X g^{ij} \\
&= \langle \nu, \partial_i\,(H\nu)\rangle \partial_j X g^{ij} \\
&= \partial_i H \partial_j X g^{ij} + H\langle \nu, \partial_i \nu\rangle \partial_j X g^{ij} \\
&= \partial_i H \partial_j X g^{ij} = \nabla H
\end{aligned}$$

3. From the relations

$$\partial_i \partial_j X = \Gamma^k_{ij}\partial_k X - h_{ij}\nu \quad \text{and} \quad \partial_j \nu = h_{jl}g^{lm}\partial_m X$$

we get

$$\begin{aligned}
\partial_t h_{ij} &= -\partial_t \langle \partial_i \partial_j X, \nu\rangle \\
&= \langle \partial_i \partial_j (H\nu), \nu\rangle - \left\langle \partial_i \partial_j X, \partial_l H \partial_m X g^{lm}\right\rangle \\
&= \partial_i \partial_j H + H\left\langle \partial_i \left(h_{jm}g^{ml}\partial_l X\right), \nu\right\rangle - \left\langle \Gamma^k_{ij}\partial_k X - h_{ij}\nu, \partial_l H \partial_m X g^{lm}\right\rangle \\
&= \partial_i \partial_j H - \Gamma^k_{ij}\partial_k H + Hh_{jm}g^{ml}\left\langle \Gamma^p_{il}\partial_p X - h_{il}\nu, \nu\right\rangle \\
&= \nabla_i \nabla_j H - Hh_{il}g^{lm}h_{mj}.
\end{aligned}$$

4. Utilizing the previous evolution equation with product formula of derivatives,

$$\begin{aligned}\partial_t H &= \partial_t(g^{ij}h_{ij}) = (\partial_t g^{ij})h_{ij} + g^{ij}\partial_t h_{ij} \\ &= -g^{ik}(\partial_t g_{kl})g^{lj}h_{ij} + g^{ij}(\Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij}) \\ &= -g^{ik}(-2Hh_{kl})g^{lj}h_{ij} + \Delta(g^{ij}h_{ij}) - 2Hg^{ij}g^{lm}h_{il}h_{mj} + |A|^2 H \\ &= 2H|A|^2 + \Delta H - 2H|A|^2 + |A|^2 H \\ &= \Delta H + |A|^2 H.\end{aligned}$$

□

Corollary. If mean curvature is positive everywhere on the initial hypersurface, then it remains so throughout the flow.

Proof. We apply maximum principle to the evolution equation of H. □

Remark. This property of mean curvature holds even when the hypersurface is embedded in an arbitrary Riemannian manifold with positive Ricci curvature.

1.5. The Avoidance Principle

Let $X_i : M_i^n \times [0,T) \to \mathbb{R}^{n+1}, i = 1,2$ be properly immersed solutions to mean curvature flow such that at least one of M_1^n or M_2^n is compact. If the hypersurfaces are disjoint initially, i.e. $X_1(M^{n_1},0) \cap X_2(M_2^n,0) = \phi$, then they remain so. Define the distance function $d : M_1^n \times M_2^n \times [0,T) \to \mathbb{R}$ between the solutions by

$$d(x,y,t) = |X_2(y,t) - X_1(y,t)|.$$

as a function of time. From the assumption of compactness $d_0 := \inf_{(x,y)\in M_1^n \times M_2^n} d(x,y,0) > 0$

Theorem 1.5.1. If X_1 and X_2 are solutions to mean curvature flow on closed manifolds with $d_0 > 0$, then $d(x,y,t) > 0$ for all x,y,t. In particular, if $X_1(x,0) \neq X_2(y,0)$ for all $x \in M_1^n$ and $y \in M_2^n$, then $X_1(x,t) \neq X_2(y,t)$ for all $x \in M_1^n$, $y \in M_2^n$ and $t \in [0,T)$.

Proof. Assume on the contrary $d(x,y,t)$ is not everywhere strictly greater that d_0. Then there exists a t_0 such that $d(x_0,y_0,t_0) = d_0 - \delta$ □

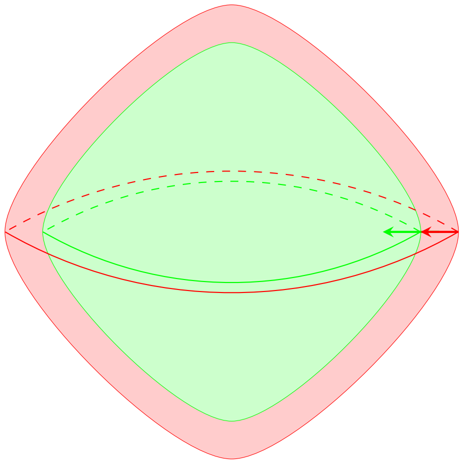

Figure 1.3.: Disjoint hypersurfaces flowing under MCF

Remark. From the proof we can conclude that the distance between the hypersurfaces is a non-decreasing function. Another way to see this is that d_t satisfies a heat-type parabolic equation

TO DO

on which maximum principle is applicable.

Proof. Fix $\epsilon > 0$ and suppose that $e^{\epsilon(1+t)}d(x, y, t)$ is not strictly greater than d_0. That is there exists a time $t_0 > 0$ such that $e^{\epsilon(1+t)}d(\cdot, \cdot, t)$ reaches d_0. That is,

$$e^{\epsilon(1+t)}d(\cdot, \cdot, t) > 0 \quad \text{for } t < t_0.$$

and

$$e^{\epsilon(1+t)}d(x_0, y_0, t_0) = d_0 \quad \text{for some } (x_0, y_0) \in M_1^n \times M_2^n.$$

Then

$$\partial_t(e^{\epsilon(1+t)}d)|_{(x_0,y_0,z_0)} \leq 0.$$

Let D denote the Euclidean directional derivative, by ∇_i the covariant derivative induced on M_i^n by X_i, and by ∇ the covariant derivative on $M_1^n \times M_2^n$ induced by ∇_1 and ∇_2, we find

$$\nabla^2.$$

$\square$

Remark. We can phrase the avoidance principle as : disjointness is an immortal property and jointness is an ancient property.

1.6. Long time existence

In this subsection we will prove that blowing up of second fundamental form is the only obstruction for continuing the flow. The proof goes by contradiction, relying on the Bernstein type estimates. This technique is very similar to the one Hamilton used for Ricci flow [Ham82].

Theorem 1.6.1. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a solution of the mean curvature flow with M^n compact. If X is maximal then $T < \infty$ and

$$\sup_{M^n \times [0,T)} |A| = \infty.$$

Proof. Assume on the contrary that $\sup_{M^n \times [0,T)} |A|^2 \leq C$. Our aim is to prove that the manifold $X(\cdot,t) = \mathcal{M}_t$ converges to a smooth limit $\mathcal{M}_T$ as $t \to T$ which will give a contradiction by applying short time existence of mean curvature flow. □

1.7. Huisken's theorem

Huisken's theorem proves the convergence of compact, uniformly convex hypersurface to sphere under Mean curvature flow in finite time.

Theorem 1.7.1. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$, $n \geq 2$ be a maximal solution of MCF such that M^n is compact and $X_0 = X(\cdot,0)$ is convex embedding. Then $X_t = X(\cdot,t)$ is a convex embedding for all $t > 0$ and X_t converges to a point $p \in \mathbb{R}^{n+1}$ as $t \to T$. Further the rescaled embeddings $\tilde{X}_t : M^n \to \mathbb{R}^{n+1}$ defined by

$$\tilde{X}_t(x) := \frac{X_t(x) - p}{\sqrt{2n(T-t)}}$$

converge uniformly in the smooth topology to a smooth embedding whose image coincides with the unit sphere S^n.

1.7.1. Pinching estimate

The quantity $\frac{|A|^2 - \frac{H^2}{n}}{H^2} = \frac{1}{n}\sum_{i<j}\left(\frac{\kappa_i}{H} - \frac{\kappa_j}{H}\right)^2$ is scaling invariant and measures the roundness of the hypersurface. If we compute the evolution equation we get

$$EQUATION \tag{1.7.1}$$

but the maximum principle is not directly applicable because of the positive last term. So we do not directly get a pointwise L^∞ bound we do the next best thing possible which is L^p bounds. After obtaining this there is a sophisticated iteration argument developed by Stampacchia which allows us to produce an L^∞ estimate. See here

Theorem 1.7.2. There exists constants δ and $C_0 < \infty$ depending only on $\mathcal{M}_0$ such that

$$|A|^2 - \frac{H^2}{n} \leq C_0 H^{2-\delta}$$

for $t \in (0, T]$.

Let $f_\sigma = \frac{|A|^2 - \frac{H^2}{n}}{H^{2-\sigma}} = \left(\frac{|A|^2 - \frac{H^2}{n}}{H^2}\right) H^\sigma$, then the evolution equation of f_σ is

Lemma 1.7.3. The evolution of f_σ is given by

$$\frac{\partial}{\partial t} f_\sigma =$$

1.8. Monotonicity Formula

Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be one-parameter family of immersions flowing by mean curvature. Let $\{x^i\}$ be local coordinates around a point $p \in M$. Then the metric and second fundamental form are given by

$$g_{ij} = \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle, \qquad h_{ij} = \left\langle \eta, \frac{\partial^2 X}{\partial x^i \partial x^j} \right\rangle.$$

If we scale the solution by a factor of λ, defined by $\tilde{X}(x,t) = \lambda X(x,t)$ we get the following metric and second fundamental form

$$\tilde{g_{ij}} = \left\langle \frac{\partial \tilde{X}}{\partial x^i}, \frac{\partial \tilde{X}}{\partial x^j} \right\rangle = \lambda^2 g_{ij}, \qquad \tilde{h_{ij}} = \left\langle \eta, \frac{\partial^2 \tilde{X}}{\partial x^i \partial x^j} \right\rangle = \lambda h_{ij}$$

so the scaled mean curvature is given by $\tilde{H} = \tilde{g}^{ij} \tilde{h}_{ij} = \frac{1}{\lambda} H$. This implies the scaled solutions satisfy the evolution equation

$$\frac{\partial \tilde{X}}{\partial t} = \lambda \frac{\partial X}{\partial t} = -\lambda^2 \tilde{H} \eta$$

or

$$\frac{\partial \tilde{X}}{\partial (\lambda^2 t)} = -\tilde{H} \eta$$

Hence if scale time by λ^2, then $\tilde{X}$ is also a solution of the mean curvature flow.

Mean curvature flow is invariant under parabolic scaling, i.e. if $X : M^n \times I \to \mathbb{R}^{n+1}$ is solution, then so is $X_\lambda(x,t) = \lambda X(x, \lambda^2 t)$. We construct a weighted area functional which is invariant under *parabolic* scaling along any solution to mean curvature flow which will be monotonous.

Let $\rho(x,t)$ be the backward heat kernel at (X_0, t_0), i.e.,

$$\rho(x,t) = \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} \cdot \exp\left(-\frac{|X(x,t) - X_0|^2}{4(t_0 - t)}\right), \qquad t < t_0$$

Theorem 1.8.1. If M_t is a solution of mean curvature flow for $t < t_0$, then we have the formula

$$\frac{d}{dt}\int_{M_t} \rho(x,t) d\mu_t = -\int_{M_t} \rho(x,t)\left(H - \frac{\langle X(x,t) - X_0, \eta\rangle}{2(t_0 - t)}\right)^2 d\mu_t$$

Proof. To simplify the formula assume that $(X_0, t_0) = (0,0)$. We know that $\frac{d}{dt}\mu_t = -H^2\mu_t$, so differentiating ρ with respect to time we get,

$$\begin{aligned}
\frac{d}{dt}\int_{M_t} \rho(x,t) d\mu_t &= \int_{M_t} \rho(x,t)(-H^2) d\mu_t + \int_{M_t} \frac{\partial}{\partial t}\rho(x,t) d\mu_t \\
&= -\int_{M_t} \rho(x,t) H^2 d\mu_t + \int_{M_t}\left(\frac{\langle X(x,t), H(x,t)\eta\rangle}{2(-t)}\rho(x,t)\right) d\mu_t \\
&\quad + \int_{M_t}\left(\frac{n}{2(4\pi)(-t)}(4\pi)\rho(x,t) - \frac{|X(x,t)|^2}{4(-t)^2}\rho(x,t)\right) d\mu_t \\
&= \int_{M_t} \rho\left(\frac{n}{2(-t)} + \frac{\langle X, H\eta\rangle}{2(-t)} - \frac{|X|^2}{4(-t)^2} - H^2\right) d\mu_t \qquad (1.8.1)
\end{aligned}$$

Now $\Delta X = -H\eta$, using this relation for second term and divergence theorem we get

$$\begin{aligned}
\int_{M_t} \rho \langle X, H\eta\rangle d\mu_t &= -\int_{M_t} \rho \langle X, \Delta X\rangle d\mu_t \\
&= -\sum_{k=1}^{n+1}\int_{M_t} \rho X_k \Delta X_k d\mu_t \\
&= \sum_{k=1}^{n+1}\int_{M_t} \langle \nabla(\rho X_k), \nabla X_k\rangle d\mu_t \\
&= \sum_{k=1}^{n+1}\int_{M_t} \left(\langle \nabla\rho, \nabla X_k\rangle X_k + \rho\langle \nabla X_k, \nabla X_k\rangle\right) d\mu_t \qquad (1.8.2)
\end{aligned}$$

Let $(U, \{x^i\})$ be some local coordinates on the hypersurface. In these coordinates we can write $\nabla\rho = g^{ij}\partial_i\rho\partial_j$, so $\langle\nabla\rho, \nabla X_k\rangle = \nabla\rho(X_k) = g^{ij}(\partial_i\rho)(\partial_j X_k)$ which implies

$$\begin{aligned}\sum_{k=1}^{n+1}\langle\nabla\rho, \nabla X_k\rangle X_k &= \sum_{k=1}^{n+1} g^{ij}(\partial_i\rho)(\partial_j X_k)X_k \\ &= g^{ij}(\partial_i\rho)\langle X, \partial_j X\rangle \\ &= g^{ij}\rho\left(\frac{-\langle X, \partial_i X\rangle}{2(-t)}\right)\langle X, \partial_j X\rangle \\ &= -\frac{\rho}{2(-t)}|X^T|^2 \end{aligned} \tag{1.8.3}$$

and

$$\sum_{k=1}^{n+1}\rho\langle\nabla X_k, \nabla X_k\rangle = \sum_{k=1}^{n+1}\rho g^{ij}(\partial_i X_k)(\partial_j X_k) = \rho g^{ij}\langle\partial_i X, \partial_j X\rangle = \rho g^{ij}g_{ij} = n\rho \tag{1.8.4}$$

Substituting Eq. (1.8.3) and Eq. (1.8.4) into Eq. (1.8.2) and multiplying by $\frac{1}{2(-t)}$, we get

$$\int_{M_t}\rho\frac{\langle X, H\eta\rangle}{2(-t)}d\mu_t = \int_{M_t}\rho\left(\frac{n}{2(-t)} - \frac{1}{4(-t)^2}|X^T|^2\right)d\mu_t$$

or

$$\int_{M_t}\frac{n\rho}{2(-t)}d\mu_t = \int_{M_t}\rho\left(\frac{\langle X, H\eta\rangle}{2(-t)} + \frac{1}{4(-t)^2}|X^T|^2\right)d\mu_t \tag{1.8.5}$$

where X^T denotes the tangential part of the vector X. Substituting Eq. (1.8.5) into Eq. (1.8.1)

$$\begin{aligned}\frac{d}{dt}\int_{M_t}\rho(x,t)d\mu_t &= \int_{M_t}\rho\left(\frac{\langle X, H\eta\rangle}{(-t)} - \frac{|X|^2}{4(-t)^2} - H^2 + \frac{1}{4(-t)^2}|X^T|^2\right)d\mu_t \\ &= -\int_{M_t}\rho\left(H - \frac{\langle X, \eta\rangle}{2(-t)}\right)^2 d\mu_t.\end{aligned}$$

□

1.8.1. Rescaled Monotonicity formula

From section (?) we know that the curvature blows up at the maximal time T and satisfies the inequality

$$\max_{p\in M}|A(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}$$

Definition 1.8.1. Let T be the maximal time of existence of a mean curvature flow. If there exists a constant $C > 1$ such that

$$\max_{p \in M} |A(p,t)| \leq \frac{C}{\sqrt{2(T-t)}}$$

we say the flow is developing at time T a *type I singularity.*

Conversely, if such a constant does not exist, that is

$$\limsup_{t \to T} \max_{p \in M} |A(p,t)| \sqrt{T-t} = \infty$$

we say that we have a *type II singularity.* We will restrict ourselves to type I singularity for the rest of this section.

1.9. Surfaces of positive mean curvature

From the maximum principle we know that if mean curvature of the initial hypersurface M_0 is positive then it will stay positive on M_t. For self-similar solutions, we know that the limiting hypersurface will satisfy the equation $H = \langle x, \nu \rangle$. We prove that sphere is the only compact hypersurface of positive mean curvature moving under self-similarity

Theorem 1.9.1. If M^n, $n \geq 2$, is compact with non-negative mean curvature H and satisfies the equation $H = -\langle X, \nu \rangle$, then M^n is a sphere of radius $\sqrt{n}$.

Proof. Suppose the hypersurface satisfies $H = -\langle X, \nu \rangle$. Let $e_1, \ldots, e_n$ be an orthonormal frame on M^n, then

$$\begin{aligned} \nabla_i H &= -\langle D_{e_i} X, \nu \rangle - \langle X, \nabla_{e_i} \nu \rangle \\ &= -\langle e_i, \nu \rangle - \langle X, \langle \nabla_{e_i} \nu, e_l \rangle e_l \rangle \\ &= \langle X, e_l \rangle h_{il} \end{aligned} \tag{1.9.1}$$

$$\nabla_i \nabla_j H = h_{ij} - H h_{il} h_{lj} + \langle x, e_l \rangle \nabla_l h_{ij}$$

□

2. Convexity estimates

As observed in the previous chapter, a convex hypersurface evolving under (rescaled) mean curvature flow converges smoothly to a round sphere. This convergence cannot be expected for a general mean-convex hypersurfaces as singularities might appear. Huisken and Sinestrari proved in [HS99a, HS99b] that mean convex hypersurface are asymptotically convex i.e. blowing the flow near singularity gives a convex ancient solution.

2.1. Elementary symmetric polynomials and cones

Mean curvature of a hypersurface at a point is the sum of principal curvatures which is a symmetric function. Similarly, Gauss curvature is the product of the principal curvatures. The study of elementary symmetric functions of principal curvatures will be crucial to analyze the convexity of singularities. We begin by recalling the definition of elementary symmetric polynomials.

Definition 2.1.1. For any $k = 1, \ldots, n$, the **k-th elementary symmetric polynomial** $S_k : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$S_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ with the convention $S_0 \equiv 1$.

Associated to each k we can also define the domain of positivity of first k elementary symmetric polynomials Γ_k given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \ldots, S_k(\lambda) > 0\}$$

It is easy to see that Γ_k are cones in the Euclidean space and satisfy $\Gamma_k \subset \Gamma_{k+1}$. In this formulation a hypersurface is mean-convex if the vector $(\kappa_1, \ldots, \kappa_n)$ is in Γ_1. The following proposition was proved in [HS99a] regarding the cones Γ_k.

Proposition 2.1.1. Let $A = \{x \in \mathbb{R}^n : x_1 > 0, \ldots, x_n > 0\}$ denote the positive cone. The sets Γ_k coincide with the connected component of the domain $\{\lambda \in \mathbb{R}^{n+1} : S_k(\lambda) > 0\}$ containing the positive cone A. Further, the cone Γ_n coincides with the positive cone A.

This establishes a hierarchy of convexity with last one being uniformly convex where the principal curvature vector $(\kappa_1, \dots, \kappa_n) \in \Gamma_n$ for all points in the hypersurface. The main result of the chapter is the following theorem.

Theorem 2.1.2. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth solution of the mean curvature flow with $n \geq 2$ such that $X(M^n, 0) = \mathcal{M}_0$ is compact and of positive mean curvature. Then, for any $\eta > 0$ there exists a constant $C_\eta > 0$ depending only on n, η and $\mathcal{M}_0$ such that

$$S_k \geq -\eta H^k - C_{\eta,k} \tag{2.1.1}$$

on $\mathcal{M}_t$ for any $t \in [0,T)$.

This means that the negative part of S_k cannot grow faster that H^k. We will only prove the theorem for $k = 2$ adapted from [HS99b]. A complete proof is done using induction in [HS99a].

2.2. Estimate of S_2

For any $\eta \in \mathbb{R}$ and $\sigma \in [0,2]$ let

$$g_{\sigma,\eta} = \left(\frac{|A|^2}{H^2} - (1+\eta)\right) H^\sigma = \frac{|A|^2 - (1+\eta)H^2}{H^{2-\sigma}} = \frac{-2S_2 - \eta H^2}{H^{2-\sigma}}.$$

Our aim is to derive a uniform bound of $g_{\sigma,\eta}$ which using Young's inequality will imply the desired estimate. The proof of Theorem 2.1.2 for $k = 2$ is divided into two parts. First part is obtaining an L^p estimate of $g_{\sigma,\eta}$ and the second part is utilizing Stampacchia lemma using Michael-Simon inequality in order to get an L^∞ bound. In order to prove the first part we derive the evolution equation of $g_{\sigma,\eta}$ using the product rule but before that we need the following lemmas.

Lemma 2.2.1. Following equality holds:

$$|\nabla A \cdot H - \nabla H \otimes A|^2 = |\nabla A|^2 H^2 + |A|^2 |\nabla H|^2 - \left\langle \nabla |A|^2, \nabla H \right\rangle H. \tag{2.2.1}$$

Proof. Computing the norm,

$$\begin{aligned} |\nabla A \cdot H - \nabla H \otimes A|^2 &= \langle \nabla A \cdot H - \nabla H \otimes A, \nabla A \cdot H - \nabla H \otimes A \rangle \\ &= |\nabla A|^2 H^2 + |\nabla H|^2 |A|^2 - 2H \langle \nabla A, \nabla H \otimes A \rangle \\ &= |\nabla A|^2 H^2 + |\nabla H|^2 |A|^2 - \left\langle \nabla |A|^2, \nabla H \right\rangle H. \end{aligned}$$

□

Lemma 2.2.2. The quantity $\frac{|A|^2}{H^2}$ satisfies the differential equation

$$\frac{\partial}{\partial t}\frac{|A|^2}{H^2} = \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2. \tag{2.2.2}$$

Proof. Computing the time derivative we get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{|A|^2}{H^2} &= \frac{1}{H^2}\frac{\partial|A|^2}{\partial t} - 2\frac{|A|^2}{H^3}\frac{\partial H}{\partial t}\\ &= \frac{1}{H^2}\left(\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4\right) - 2\frac{|A|^2}{H^3}\left(\Delta H + |A|^2H\right)\\ &= \frac{\Delta|A|^2}{H^2} - 2\frac{|\nabla A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3}.\end{aligned}$$

Recall the division formula for Laplacian,

$$\Delta\left(\frac{u}{v}\right) = \frac{\Delta u}{v} - u\frac{\Delta v}{v^2} - \frac{2}{v^2}\langle\nabla u, \nabla v\rangle + 2\frac{u}{v^3}|\nabla v|^2.$$

Calculating the Laplacian-Beltrami operator using this,

$$\begin{aligned}\Delta\frac{|A|^2}{H^2} &= \frac{\Delta|A|^2}{H^2} - |A|^2\frac{\Delta H^2}{H^4} - \frac{2}{H^4}\left\langle\nabla|A|^2, \nabla H^2\right\rangle + \frac{2|A|^2}{H^6}|\nabla H^2|^2\\ &= \frac{\Delta|A|^2}{H^2} - |A|^2\left(\frac{2H\Delta H + 2|\nabla H|^2}{H^4}\right) - \frac{2}{H^4}\left\langle\nabla|A|^2, 2H\nabla H\right\rangle + 8\frac{|A|^2}{H^6}|\nabla H|^2\\ &= \frac{\Delta|A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3} + 6|A|^2\frac{|\nabla H|^2}{H^4} - \frac{4}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle\end{aligned}$$

which substituted in the time derivative gives

$$\begin{aligned}\frac{\partial}{\partial t}\frac{|A|^2}{H^2} &= \Delta\frac{|A|^2}{H^2} - 6|A|^2\frac{|\nabla H|^2}{H^4} + \frac{4}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle - 2\frac{|\nabla A|^2}{H^2}\\ &= \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle\nabla H, \frac{\nabla|A|^2}{H^2} - \frac{2}{H^3}|A|^2\nabla H\right\rangle\\ &\qquad - \frac{2}{H^4}\left(|A|^2|\nabla H|^2 + |\nabla A|^2H^2 - H\left\langle\nabla|A|^2, \nabla H\right\rangle\right)\\ &= \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle\nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2.\end{aligned}$$

□

Using this we compute the time derivative of $g_{\sigma,\eta}$.

Lemma 2.2.3. The evolution equation of $g_{\sigma,\eta}$ is given by

$$\begin{aligned}\frac{\partial g_{\sigma,\eta}}{\partial t} =& \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma(1-\sigma)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &- \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta}. \end{aligned} \tag{2.2.3}$$

Proof. We can write $g_{\sigma,\eta} = \left(\frac{|A|^2}{H^2} - (1+\eta)\right) H^\sigma$ so

$$\begin{aligned}\frac{\partial g_{\sigma,\eta}}{\partial t} &= \left\{\Delta \frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \nabla \frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A \cdot H - \nabla H \otimes A|^2\right\} H^\sigma \\ &\quad + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\left(\Delta H^\sigma - \sigma(\sigma-1)H^{\sigma-2}|\nabla H|^2 + \sigma|A|^2 H^\sigma\right) \\ &= \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\left\langle \nabla H, \nabla \frac{|A|^2}{H^2}\right\rangle H^\sigma - \frac{\sigma(\sigma-1)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta} \\ &= \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\left(\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma}{H} g_{\sigma,\eta}|\nabla H|^2\right) - \frac{\sigma(\sigma-1)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta} \\ &= \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma(1-\sigma)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta}.\end{aligned}$$

□

Applying maximum principle on Lemma 2.2.2 get that $\frac{|A|^2}{H^2}$ is uniformly bounded so there exists a positive constant depending only on $\mathcal{M}_0$ such that

$$|A|^2 \leq \tilde{c_0} H^2 \quad \text{on} \quad \mathcal{M}_t,$$

for all time $t \in [0, T)$. This also implies $g_{\sigma,\eta} \leq c_0 H^\sigma$ but as H blows-up this isn't sufficient to prove the uniform bound. The following estimate of the good term in Eq. (2.2.3) will be required for the L^p estimate.

Lemma 2.2.4. [HS99b] If $(1+\eta)H^2 \leq |A|^2 \leq c_0 H^2$ for some $\eta, c_0 > 0$. Then

1. $-2Z \geq \eta H^2 |A|^2$
2. $|\nabla A \cdot H - \nabla H \otimes A|^2 \geq \frac{\eta^2}{4n(n-1)^2 c_0} H^2 |\nabla H|^2$

For the rest of proof we will restrict $\eta, \sigma \in (0,1)$ and c_i will denote a constant depending only on n, η and $\mathcal{M}_0$. For brevity we will write $g = g_{\sigma,\eta}$ as long as σ, η are fixed. Let

$g_+ = \max g(x,t), 0$ denote the positive part of g. Then $g_+^p \in C^1(\mathcal{M} \times [0,T))$ for $p > 1$ and

$$\partial_t g_+^p = p g_+^{p-1} \partial_t g, \qquad \nabla(g_+^p) = p g_+^{p-1} \nabla g.$$

Lemma 2.2.5. There exists constant c_2, c_3 such that

$$\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu \leq -\frac{p(p-1)}{2}\int_{\mathcal{M}} g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int_{\mathcal{M}} \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$- p\int_{\mathcal{M}} \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu + p\sigma \int_{\mathcal{M}} |A|^2 g_+^p d\mu \qquad (2.2.4)$$

for any $p \geq c_2$.

Proof. Differentiating with respect to time and using Lemma 2.2.3 for $p \geq 2$

$$\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu = \int \left(p g_+^{p-1}\partial_t g - H^2 g_+^p\right) d\mu$$
$$\leq \int p g_+^{p-1}\left(\Delta g + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g\rangle - \frac{2}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H h_{kl}|^2\right) d\mu$$
$$+ \sigma |A|^2 g \qquad (2.2.5)$$

Using integration by parts,

$$\int p g_+^{p-1}\Delta g d\mu = -p\int \left\langle \nabla g_+^{p-1}, \nabla g\right\rangle d\mu \qquad (2.2.6)$$
$$= -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu \qquad (2.2.7)$$

Also from Lemma 2.2.4 we deduce that if $c_1 \geq 4n(n-1)^2 c_0 \eta^{-2}$

$$\frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 \geq \frac{g_+^{p-1}}{c_1 H^{2-\sigma}}|\nabla H|^2$$
$$\geq \frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2 + \frac{1}{2c_0 c_1}\frac{g_+^p}{H^2}|\nabla H|^2 \qquad (2.2.8)$$

To handle the gradient term, let $p \geq \max\{2, 1+4c_0c_1\}$ to obtain

$$2(1-\sigma)p\frac{g_+^{p-1}}{H}\langle \nabla H, \nabla g\rangle \leq 2p\frac{g_+^{p-1}}{H}|\nabla H||\nabla g|$$
$$\leq \frac{p}{2c_0c_1}\frac{g_+^p}{H^2}|\nabla H|^2 + 2c_0c_1 p g_+^{p-2}|\nabla g|^2 \quad \text{[Peter-Paul inequality]}$$
$$\leq p\frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 - p\frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2$$
$$+ \frac{p(p-1)}{2}g_+^{p-2}|\nabla g|^2 \qquad \text{[Using Eq. (2.2.8)]}$$

Substituting this back in Eq. (2.2.5) and using integration by parts from Eq. (2.2.7),

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}} g_+^p d\mu \leq & -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu + p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu \\ & + \frac{p(p-1)}{2}\int g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\ & - 2p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu + p\sigma\int |A|^2 g_+^p d\mu\end{aligned}$$

which gives the desired inequality with $c_3 = \frac{1}{2c_1}$. □

To handle the bad positive term appearing in Eq. (2.2.4) we use the following lemma

Lemma 2.2.6. There exists a constant c_4 such that

$$\begin{aligned}\frac{1}{c_4}\int |A|^2 g_+^p d\mu \leq \left(p + \frac{p}{\beta}\right)\int g_+^{p-2}|\nabla g|^2 + (1+\beta p)\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\ + \int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu\end{aligned}$$

for any $\beta > 0, p > 2$.

Proof. The Laplacian-Beltrami operator satisfies,

$$\Delta(f^\sigma) = \sigma f^{\sigma-1}\Delta f + \sigma(\sigma-1)f^{\sigma-2}|\nabla f|^2$$

We have an expression for in Lemma 2.2.2 we know that

$$
\begin{aligned}
\Delta g &= \Delta\left(\frac{|A|^2}{H^2}\right)H^\sigma + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\Delta H^\sigma + 2\left\langle \nabla\frac{|A|^2}{H^2}, \nabla H^\sigma\right\rangle \\
&= \left(\frac{\Delta|A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3} + 6|A|^2\frac{|\nabla H|^2}{H^4} - \frac{4}{H^3}\left\langle \nabla|A|^2, \nabla H\right\rangle\right)H^\sigma \\
&\quad + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\left(\sigma H^{\sigma-1}\Delta H + \sigma(\sigma-1)H^{\sigma-2}|\nabla H|^2\right) \\
&\quad + 2\sigma H^{\sigma-1}\left\langle \frac{\nabla|A|^2}{H^2} - 2\frac{|A|^2}{H^3}\nabla H, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{|A|^2}{H^{3-\sigma}} - \sigma(1+\eta)H^{\sigma-1}\right)\Delta H + 6\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 - \frac{4}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle \\
&\quad + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2\sigma}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle - 4\sigma\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H + (6-4\sigma)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&\quad - \frac{2}{H^{4-\sigma}}H\left\langle \nabla|A|^2, \nabla H\right\rangle + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H + (6-4\sigma)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&\quad - \frac{2}{H^{4-\sigma}}\left(|\nabla A|^2H^2 + |A|^2|\nabla H|^2 - |\nabla A\cdot H - \nabla H\otimes A|^2\right) + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 \\
&\quad + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2 - 2|\nabla A|^2}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&\quad - 4(\sigma-1)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle \nabla|A|^2, \nabla H\right\rangle.
\end{aligned}
$$

Now similar to time derivative Lemma 2.2.3, we calculate inner product of ∇g with ∇H,

$$
\begin{aligned}
\langle \nabla g, \nabla H\rangle &= \left\langle \nabla\frac{|A|^2}{H^2}, \nabla H\right\rangle H^\sigma + \sigma\left(\frac{|A|^2}{H^2} - (1+\eta)\right)H^{\sigma-1}|\nabla H|^2 \\
&= \left\langle \frac{\nabla|A|^2}{H^2}, \nabla H\right\rangle H^\sigma - 2\frac{|A|^2}{H^{3-\sigma}}|\nabla H|^2 + \sigma\frac{g}{H}|\nabla H|^2,
\end{aligned}
$$

Using Simon's identity [?] and previous expression to eliminate the last mixed inner product term

$$
\begin{aligned}
\Delta g =& \frac{\Delta|A|^2 - 2|\nabla A|^2}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&- 4(\sigma-1)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 \\
&+ \frac{2(\sigma-1)}{H}\left(\langle \nabla g, \nabla H\rangle + 2\frac{|A|^2}{H^{3-\sigma}}|\nabla H|^2 - \sigma\frac{g}{H}|\nabla H|^2\right) \\
=& \frac{2\left\langle h_{ij}, \nabla_i\nabla_j H\right\rangle + 2Z}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&- \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H}\left\langle \nabla g, \nabla H\right\rangle \qquad (2.2.9)
\end{aligned}
$$

Recall Green's identity for compact manifold without boundary,

$$
\int_M u\Delta v = -\int_M \left\langle \nabla u, \nabla v\right\rangle.
$$

Multiplying Eq. (2.2.9) by $g_+^p H^{-\sigma}$ and using Green's identity the left-hand side evaluates to

$$
\begin{aligned}
A = \int g_+^p H^{-\sigma}\Delta g d\mu &= -\int \left\langle \nabla(g_+^p H^{-\sigma}), \nabla g\right\rangle d\mu \\
&= -p\int \frac{1}{H^\sigma} g_+^{p-1}|\nabla g|^2 d\mu + \sigma\int \frac{g_+^p}{H^{1+\sigma}}\left\langle \nabla g, \nabla H\right\rangle d\mu \qquad (2.2.10)
\end{aligned}
$$

while the right-hand side is

$$
\begin{aligned}
B =& 2\int \frac{\left\langle h_{ij}, \nabla_i\nabla_j H\right\rangle g_+^p}{H^2} d\mu + 2\int \frac{g_+^p Z}{H^2} d\mu + 2\int \frac{g_+^p}{H^4}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu \\
&+ (\sigma-2)\int \frac{g_+^{p+1}}{H^{1+\sigma}}\Delta H d\mu - 2(1+\eta)\int \frac{g_+^p}{H}\Delta H d\mu - \sigma(\sigma-1)\int \frac{g_+^{p+1}}{H^{2+\sigma}}|\nabla H|^2 d\mu \\
&+ 2(\sigma-1)\int \frac{g_+^{p+1}}{H^{1+\sigma}}\left\langle \nabla g, \nabla H\right\rangle d\mu \qquad (2.2.11)
\end{aligned}
$$

For the first term of Eq. (2.2.11) we can use divergence-type theorem for tensors to get,

$$
\begin{aligned}
2\int \frac{\left\langle h_{ij}, \nabla_i\nabla_j H\right\rangle g_+^p}{H^2} d\mu &= -2\int \left\langle \mathrm{tr}_{ik}\left(\nabla_k\left(\frac{g_+^p h_{ij}}{H^2}\right)\right), \nabla_j H\right\rangle d\mu \\
&= -2p\int \frac{g_+^{p-1}}{H^2}\left\langle \nabla^i g \otimes h_{ij}, \nabla_j H\right\rangle d\mu \\
&\quad + 4\int \frac{g_+^p}{H^3}\left\langle \nabla^i H \otimes h_{ij}, \nabla_j H\right\rangle d\mu - 2\int \frac{g_+^p}{H^2}\left\langle \nabla^i h_{ij}, \nabla_j H\right\rangle d\mu \\
&\qquad (2.2.12)
\end{aligned}
$$

Using Codazzi equation $\nabla^i h_{ij} = \nabla_j h_i^i$ for the last term,

$$2\int \frac{\langle h_{ij}, \nabla_i\nabla_j H\rangle\, g_+^p}{H^2} d\mu = -2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g \nabla_j H\rangle\, d\mu \\ + 4\int \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i H \nabla_j H\rangle\, d\mu - 2\int \frac{g_+^p}{H^2}|\nabla H|^2 d\mu \quad (2.2.13)$$

Applying Green's formula on ΔH terms in Eq. (2.2.11) and putting together Eq. (2.2.10), Eq. (2.2.11) and Eq. (2.2.13)

$$\begin{aligned} &- p\int \frac{1}{H^\sigma} g_+^{p-1}|\nabla g|^2 d\mu + \underbrace{\sigma\int \frac{g_+^p}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle\, d\mu}_{1} \\ &= -2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g\nabla_j H\rangle\, d\mu + 4\int \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i H\nabla_j H\rangle\, d\mu \underbrace{- 2\int \frac{g_+^p}{H^2}|\nabla H|^2 d\mu}_{2} \\ &+ 2\int \frac{g_+^p Z}{H^2} d\mu + 2\int \frac{g_+^p}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu \underbrace{- (\sigma-2)(p+1)\int \frac{g_+^p}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle\, d\mu}_{1} \\ &\underbrace{+ (\sigma-2)(1+\sigma)\int \frac{g_+^{p+1}}{H^{2+\sigma}}|\nabla H|^2 d\mu}_{3} + 2(1+\eta)p\int \frac{g_+^{p-1}}{H}\langle \nabla g, \nabla H\rangle\, d\mu \\ &\underbrace{- 2(1+\eta)\int \frac{g_+^p}{H^2}|\nabla H|^2 d\mu}_{2} \underbrace{- \sigma(\sigma-1)\int \frac{g_+^{p+1}}{H^{2+\sigma}}|\nabla H|^2 d\mu}_{3} \underbrace{+ 2(\sigma-1)\int \frac{g_+^{p+1}}{H^{1+\sigma}}\langle \nabla g, \nabla H\rangle\, d\mu}_{1} \end{aligned}$$

which gives

$$\begin{aligned} -2\int \frac{g_+^p Z}{H^2} d\mu = {} & p\int \frac{1}{H^\sigma} g_+^{p-1}|\nabla g|^2 d\mu - 2p\int \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g\nabla_j H\rangle\, d\mu \\ & + 4\int \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i H\nabla_j H\rangle\, d\mu + 2\int \frac{g_+^p}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu \\ & + p\int \left((2-\sigma)\frac{g_+^p}{H^{1+\sigma}} + 2(1+\eta)\frac{g_+^{p-1}}{H}\right)\langle \nabla g, \nabla H\rangle\, d\mu \\ & - 2\int \left(\frac{g_+^{p+1}}{H^{2+\sigma}} + (2+\eta)\frac{g_+^p}{H^2}\right)|\nabla H|^2 d\mu \end{aligned} \quad (2.2.14)$$

From Lemma 2.2.4 $-2Z \geq \eta H^2|A|^2$ and using utilizing $g \leq c_0 H^\sigma$ (and $|A| \leq c_0 H$) with Cauchy-Schwarz inequality in Eq. (2.2.14),

$$\eta \int g_+^p |A|^2 d\mu \leq c_0 p \int g_+^{p-2} |\nabla g|^2 d\mu + 4p(c_0+1) \int \frac{g_+^{p-1}}{H} |\nabla g||\nabla H| d\mu$$
$$+ 4c_0^2 \int \frac{g_+^{p-1}}{H^{2-\sigma}} |\nabla H|^2 d\mu + 2c_0 \int \frac{g_+^{p-1}}{H^{4-\sigma}} |\nabla A \cdot H - \nabla H \otimes A|^2 d\mu \tag{2.2.15}$$

Also, for any $\beta > 0$,

$$\begin{aligned} 2\frac{g_+^{p-1}}{H}|\nabla H||\nabla g| &\leq \frac{g_+^{p-2}}{\beta}|\nabla g|^2 + \beta \frac{g_+^p}{H^2}|\nabla H|^2 \\ &= \frac{g_+^{p-2}}{\beta}|\nabla g|^2 + c_0\beta \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 \end{aligned} \tag{2.2.16}$$

Combining Eq. (2.2.14), Eq. (2.2.15) and Eq. (2.2.16) proves the lemma. □

Proposition 2.2.7. For any $\eta \in (0,1)$ there exists constants c_5, c_6 such that the $L^p(\mathcal{M})$ norm of $(g_{\sigma,\eta})_+$ is non-decreasing function of t if the following holds

$$p \geq c_5, \qquad \sigma \leq (c_6 p)^{-\frac{1}{2}}.$$

Proof. Choose $\beta \sim p^{-\frac{1}{2}}$ and $\sigma \sim cp^{-\frac{1}{2}}$ in the previous lemma. □

Lemma 2.2.8 (Stampacchia lemma). Let $\psi : [k_0, \infty) \to \mathbb{R}$ be a non-negative, non-increasing function which satisfies

$$\psi(h) \leq \frac{C}{(h-k)^\alpha}\psi(k)^\beta \text{ for all } h > k > k_0 \tag{2.2.17}$$

for some constants $C > 0$, $\alpha > 0$ and $\beta > 1$. Then

$$\psi(k_0 + d) = 0, \tag{2.2.18}$$

where $d^\alpha = C\psi(k_0)^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}$.

We complete the proof of Theorem 2.1.2 using Stampacchia lemma which gives an L^∞ bound from the L^p bounds.

Proof. Let $k \geq k_0$, where

$$k_0 = \sup_{\sigma\in[0,1]} \sup_{\mathcal{M}_0} g_{\sigma,\eta}$$

Define $v = (g_{\sigma,\eta} - k)_+^{\frac{p}{2}}$ and $A(k,t) = \{x \in \mathcal{M}_t : v(x,t) > 0\}$. Differentiating v with respect to time we get for p large enough (similar to Lemma 2.2.5)

$$\frac{d}{dt}\int_{\mathcal{M}_t} v^2 d\mu + \int_{\mathcal{M}_t} |\nabla v|^2 d\mu \leq \sigma p \int_{\mathcal{M}_t} |A|^2 v^2 d\mu \leq c_0 \sigma p \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu \tag{2.2.19}$$

Also from the Michael-Simon result in [MS73], we have a Sobolev-type inequality given by

$$\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \leq C(n)\int_{\mathcal{M}_t} |\nabla v|^2 d\mu + C(n)\left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}}\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \tag{2.2.20}$$

where $q = \frac{n}{n-2}$ if $n > 2$ and an arbitrary number greater than 1 if $n = 2$. We can estimate the H^n factor in the integral on $A(k,t)$ using the previous proposition and the equality

$$\int_{\mathcal{M}_t} H^n g_{\sigma,\eta}^p d\mu = \int_{\mathcal{M}_t} g_{\sigma',\eta}^p d\mu$$

where $\sigma' = \sigma + \frac{n}{p}$. Let

$$p \geq \max\{c_5, 4n^2 c_6\} \quad \text{and} \quad \sigma \leq (4c_6 p^{-\frac{1}{2}})$$

so that

$$\sigma' = \sigma + \frac{n}{p} \leq \frac{1}{2\sqrt{c_6 p}} + \frac{1}{\sqrt{p}}\frac{n}{\sqrt{p}} \leq \frac{1}{\sqrt{c_6 p}}$$

which allows us to use Proposition 2.2.7,

$$\begin{aligned}
\left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}} &\leq \left(\int_{A(k,t)} H^n \left(\frac{g_{\sigma,\eta}^p}{k}\right) d\mu\right)^{\frac{2}{n}} \\
&= k^{-\frac{2p}{n}}\left(\int_{A(k,t)} g_{\sigma',\eta}^p d\mu\right)^{\frac{2}{n}} \\
&\leq k^{-\frac{2p}{n}}\left(\int_{\mathcal{M}_t} (g_{\sigma',\eta})_+^p d\mu\right)^{\frac{2}{n}} \\
&\leq k^{-\frac{2p}{n}}\left(\int_{\mathcal{M}_0} (g_{\sigma',\eta})_+^p d\mu\right)^{\frac{2}{n}} \\
&\leq \left(\frac{|\mathcal{M}_0| k_0}{k}\right)^{\frac{2p}{n}}
\end{aligned}$$

We can fix $k_1 > k_0$ such that for any $k \geq k_1$ the term $\int_{A(k,t)} H^n d\mu$ in Eq. (2.2.20) is less than $\frac{1}{2C(n)}$. For such k, using Eq. (2.2.19) with Eq. (2.2.20) to eliminate the gradient term,

$$\frac{d}{dt}\int_{\mathcal{M}_t} v^2 d\mu + \frac{1}{2C(n)}\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \leq c_0 \sigma p \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu. \tag{2.2.21}$$

Let $t_0 \in [0,T]$ be the time when $\sup_{t\in[0,T)} \int_{\mathcal{M}_t} v^2 d\mu$ is attained (we let $t_0 = T$ if it is not attained in the interior). Integrating Eq. (2.2.21) from 0 to t_0,

$$\int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{2C(n)} \int_0^{t_0} \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q}} dt \le c_0 \sigma p \int_0^{t_0} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \tag{2.2.22}$$

where we used the fact that $k > k_0 \ge \sup_{\mathcal{M}_0} g_{\sigma,\eta}$ so $\int_{\mathcal{M}_0} v^2 d\mu = 0$. Now integrating Eq. (2.2.21) from t_0 to T,

$$\int_{\mathcal{M}_T} v^2 d\mu - \int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{2C(n)} \int_{t_0}^{T} \left(\int_{\mathcal{M}_t} v^{2q} \right)^{\frac{1}{q}} dt \le c_0 \sigma p \int_{t_0}^{T} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt. \tag{2.2.23}$$

Throwing away $\int_{\mathcal{M}_T} v^2 d\mu$ term and adding Eq. (2.2.22) to half of Eq. (2.2.23),

$$\frac{1}{2} \int_{\mathcal{M}_{t_0}} v^2 d\mu + \frac{1}{4C(n)} \int_0^{T} \left(\int_{\mathcal{M}_t} v^{2q} \right)^{\frac{1}{q}} dt \le c_0 \sigma p \int_0^{T} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt$$

which is same as

$$\sup_{[0,T)} \int_{\mathcal{M}_t} v^2 d\mu + \int_0^{T} \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q}} dt \le 2\max\{1, 2C(n)\} c_0 \sigma p \int_0^{T} \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt. \tag{2.2.24}$$

Recall the interpolation inequality for L^p spaces for any $f \in L^q \cap L^r$,

$$||f||_{q_0} \le ||f||_q^{\alpha} ||f||_r^{1-\alpha}$$

where $\frac{1}{q_0} = \frac{\alpha}{q} + \frac{1-\alpha}{q}$ and $1 < q_0 < q$. Setting $r = 1, \alpha = \frac{1}{q_0}$ and $f = v^2$ we get

$$\left(\int_{\mathcal{M}_t} v^{2q_0} d\mu \right)^{\frac{1}{q_0}} \le \left(\int_{\mathcal{M}_t} v^{2q} d\mu \right)^{\frac{1}{q_0 q}} \left(\int_{\mathcal{M}_t} v^{2} d\mu \right)^{1-\frac{1}{q_0}}. \tag{2.2.25}$$

Integrating this in time and using Young's inequality,

$$\begin{aligned}
\left(\int_0^T \int_{A(k,t)} v^{2q_0} d\mu dt \right)^{\frac{1}{q_0}} &\le \left(\sup_{[0,T)} \int_{A(k,t)} v^2 d\mu \right)^{1-\frac{1}{q_0}} \left(\int_0^T \left(\int_{A(k,t)} v^{2q} d\mu \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q_0}} \\
&\le \frac{\sup_{[0,T)} \int_{A(k,t)} v^2 d\mu}{\frac{q_0}{q_0-1}} + \frac{\int_0^T \left(\int_{A(k,t)} v^{2q} d\mu \right)^{\frac{1}{q}} dt}{q_0} \\
&\le \sup_{[0,T)} \int_{A(k,t)} v^2 d\mu + \int_0^T \left(\int_{A(k,t)} v^{2q} d\mu \right)^{\frac{1}{q}} dt \\
&\le c_8 \sigma p \int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt
\end{aligned}$$

where $c_8 = 2\max\{1, 2C(n)\}c_0$. Set $\psi(k) = \int_0^T \int_{A(k,t)} d\mu dt$. We will obtain bounds on ψ which along with Stampacchia lemma will imply uniform bound of $g_{\sigma,\eta}$. Now Eq. (2.2.24) and Hölder inequality yields,

$$\int_0^T \int_{A(k,t)} v^2 d\mu dt \leq \left(\int_0^T \int_{A(k,t)} 1 d\mu dt\right)^{1-\frac{1}{q_0}} \left(\int_0^T \int_{A(k,t)} v^{2q_0} d\mu dt\right)^{\frac{1}{q_0}} \tag{2.2.26}$$

$$\leq c_8 \sigma p \psi(k)^{1-\frac{1}{q_0}} \int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \tag{2.2.27}$$

Let $r > 1$ which will be chosen later. Applying Hölder again on the right side with weights r and $\frac{r}{r-1}$,

$$\int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \leq \left(\int_0^T \int_{A(k,t)} d\mu dt\right)^{1-\frac{1}{r}} \left(\int_0^T \int_{A(k,t)} H^{2r} g^{pr}_{\sigma,\eta} d\mu dt\right)^{\frac{1}{r}}$$

$$= \psi(k)^{1-\frac{1}{r}} \left(\int_0^T \int_{A(k,t)} g^{pr}_{\sigma'',\eta} d\mu dt\right)^{\frac{1}{r}}$$

where $\sigma'' = \sigma + \frac{2}{p}$. For r large enough and p, σ^{-1} small enough from Proposition 2.2.7 there exists a constant $c_9 > 0$ independent of time such that

$$\int_0^T \int_{A(k,t)} H^2 g^p_{\sigma,\eta} d\mu dt \leq c_9^{\frac{1}{r}} \psi(k)^{1-\frac{1}{r}}. \tag{2.2.28}$$

Combining Eq. (2.2.27) and Eq. (2.2.28) for all $h > k \geq k_1$, we have

$$\begin{aligned}(h-k)^p \psi(h) &= \int_0^T \int_{A(h,t)} (h-k)^p d\mu dt \\ &\leq \int_0^T \int_{A(k,t)} v^2 d\mu dt \\ &\leq c_8 \sigma p c_9^{\frac{1}{r}} \psi(k)^{2-\frac{1}{r}-\frac{1}{q_0}}.\end{aligned}$$

Let $\gamma = 2 - \frac{1}{r} - \frac{1}{q_0}$ and $c_{10} = c_8 c_9^{\frac{1}{r}}$. Fix $r > \frac{q_0}{q_0-1}$ (so $\gamma > 1$) and p large enough, σ small enough while satisfying the hypothesis of Proposition 2.2.7 such that $\sigma p < 1$ then gives

$$\psi(h) \leq \frac{c_{10}}{(h-k)^p} \psi(k)^\gamma \tag{2.2.29}$$

Stampacchia lemma now implies $\psi(k) = 0$ for all $k \geq k_1 + d$ where $d^p = c_{10} 2^{\frac{\gamma p}{\gamma-1}+1} \psi(k_1)^{\gamma-1}$. Hence,

$$g_{\sigma,\eta} \leq k_1 + d \leq K := k_1 + c_{10} 2^{\frac{\gamma p}{\gamma-1}+1} (|\mathcal{M}_0| T)^{\gamma-1}$$

or

$$|A|^2 - (1+\eta) H^2 \leq K H^{2-\sigma}$$

so by Young's inequality there exists a constant C_η such that,

$$|A|^2 - H^2 \le \eta H^2 + KH^{2-\sigma} \le 2\eta H^2 + 2C_\eta.$$

Notice that $|A|^2 - H^2 = -\sum_{i\neq j} \kappa_i\kappa_j = -2S_2$ which implies the desired estimate. □

2.3. Asymptotic convexity

As mentioned in Section 1.8.1, we classify the singularities based on the blow-up rate of $|A|^2$. Recall from maximum principle on Lemma 2.2.2 there exists a c_0 such that $|A|^2 \le c_0 H^2$ and from algebra we get $H^2 \le n|A|^2$ so $|A|^2$ and H^2 have same rate of growth. We will focus on the growth of H^2.

The estimates obtained in the previous section will be very useful to obtain an asymptotic analysis of type II singularities. Following [HS99b] suppose a maximal solution $X : M \times [0, T) \to \mathbb{R}^{n+1}$ develops a type II singularity. Choose a sequence of points $\{(x_k, t_k)\}$ in spacetime as follows. For each integer $k \ge 1$, let $t_k \in [0, T - \frac{1}{k}]$, $x_k \in M$ such that

$$H^2(x_k, t_k)\left(T - \frac{1}{k} - t_k\right) = \sup_{(x,t)\in M\times\left[0,T-\frac{1}{k}\right]} H^2(x, t)\left(T - \frac{1}{k} - t\right) \tag{2.3.1}$$

Set $L_k = H(x_k, t_k)$, $\alpha_k = -L_k^2 t_k$ and $\omega_k = L_k^2(T - \frac{1}{k} - t_k)$.

Lemma 2.3.1. For singularities of type II, the following holds as $k \to \infty$,

$$t_k \to T, \quad L_k \to \infty, \quad \alpha_k \to -\infty, \text{ and } \omega_k \to \infty.$$

Proof. Fix $M > 0$. As the singularity is of type II, there exists a $t_M \in [0, T)$ and $x_M \in \mathcal{M}$ such that $H^2(x_M, t_M)(T - t_m) > 2M$. □

Now we will rescale the hypersurfaces to analyze the limiting behavior. For each $k \ge 1$, define a family of immersions by

$$X_k(x, t) = L_k(X(x, L_k^{-2}t + t_k) - X(x_k, t_k)) \text{ for } t \in [\alpha_k, \omega_k].$$

Let A_k and H_k denote the fundamental form of the rescaled immersions. Then by the definition of L_k and X_k we have

$$X_k(x_k, 0) = 0 \quad \text{and} \quad H_k(x_k, 0) = 1.$$

Further, observe that

$$H_k^2(x, t) = L_k^{-2} H^2(x, L_k^{-2}t + t_k) \le \frac{T - \frac{1}{k} - t_k}{T - \frac{1}{k} - t_k - L_k^{-2}t} = \frac{\omega_k}{\omega_k - t}.$$

From the previous lemma $\omega_k \to \infty$, so for any $\epsilon > 0$ and $\overline{\omega}$, there exists a k_0 such that

$$\max_{x \in M} H_k(x,t) \leq 1 + \epsilon$$

for any $k \geq k_0$ and $t \in [\alpha_{k_0}, \overline{\omega}]$. This curvature bound implies analogous bounds on second fundamental form as well as its covariant derivatives. Invoking Theorem A.2.1 there exists a subsequence of X_k converging uniformly on compact subsets of $\mathbb{R}^{n+1} \times \mathbb{R}$ to a limiting solution X_∞ of the mean curvature flow. This proves the asymptotic convexity of the flow in the following sense.

Theorem 2.3.2. Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth maximal solution of the mean curvature flow with $X(\cdot, 0) = \mathcal{M}_0$ compact and of positive mean curvature. Further assume that the flow develops a singularity of type II. Then there exists a sequence of rescaled flow $X_k(\cdot, t)$ converging smoothly on every compact set to a mean curvature flow $X_\infty(\cdot, t)$ which is defined for $t \in (-\infty, \infty)$. Also, the limit hypersurface X_∞ is convex (not necessarily uniformly convex) for each $t \in (-\infty, \infty)$ and satisfies $0 < H_\infty \leq 1$ everywhere with equality at least at one point.

3. Noncollapsing

Noncollapsing in mean curvature flow is a powerful result which gives a geometric idea about the structure of singularities. It can be used to rule out certain singularity profiles for mean convex mean curvature flow.

3.1. Inscribed curvature

Let $\mathcal{M} \in \mathbb{R}^{n+1}$ be a smooth hypersurface which is the boundary of an open Ω. For $x \in \mathcal{M}$, we want to find the radius of the largest inscribed sphere in $\mathcal{M}$ touching it at x. For any $y \in \mathcal{M}\backslash\{x\}$, the radius of the sphere passing through x and y and touching $\mathcal{M}$ at x is given by

$$r(x,y) = \frac{||x-y||^2}{2\langle x-y, N(x)\rangle} \tag{3.1.1}$$

where $N(x)$ is the outward unit normal vector of $\mathcal{M}$ at x. To get the **inradius** which would be radius of the largest sphere inscribed in $\mathcal{M}$, we take the infimum over all points $y \in \mathcal{M}\backslash\{x\}$ to get

$$r(x) = \inf_{y\in\mathcal{M}\backslash\{x\}} r(x,y) \tag{3.1.2}$$

Similarly the **inscribed curvature** $k : \mathcal{M} \to [0,\infty)$ is given by the reciprocal of the inradius,

$$k(x) = \frac{1}{r(x)} = \sup_{y\in\mathcal{M}\backslash\{x\}} \frac{2\langle x-y, N(x)\rangle}{||x-y||^2} \tag{3.1.3}$$

Definition 3.1.1. A mean convex hypersurface $\mathcal{M}$ is said to be α**-noncollapsed** if for every $x \in \mathcal{M}$ there exists an open ball B of radius $\frac{\alpha}{H(x)}$ entirely contained in Ω. In terms of inscribed curvature, this is same as the inequality

$$k(x) \geq \alpha H(x) \quad \text{for all} \quad x \in \mathcal{M}. \tag{3.1.4}$$

Theorem 3.1.1 (Noncollapsing)**.** Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth solution of the mean curvature with $X(\cdot,0) = \mathcal{M}_0$ compact and α-noncollapsed. Then $X(\cdot,t) = \mathcal{M}_t$ is α-noncollapsed for all $t \in [0,T)$.

We'll prove a stronger result which will imply noncollapsing.

Theorem 3.1.2. Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth solution of the mean curvature flow with $\mathcal{M}_0$ properly embedded. Then

$$\frac{\partial}{\partial t} k \leq \Delta k - 2 \sum_{\kappa_i < k} \frac{(\nabla_i k)^2}{k - \kappa_i} \tag{3.1.5}$$

where in inequality holds in the viscosity sense.

A. Convergence of Manifolds

For understanding the singularity of mean curvature flow, it is essential to understand the convergence of manifolds. We will develop a convergence criterion applicable for a remarkable number of results using Arzelà-Ascoli argument. We will follow the exposition from [CCG+07] and [ACGL22].

A.1. Cheeger-Gromov convergence

Definition A.1.1. Let $K \subset M$ be a compact set and let $\{g_k\}_{k\in\mathbb{N}}$, g_∞, and g be Riemannian metric on $\mathcal{M}$. For $p \in \{0\} \cup \mathbb{N}$, we say that g_k **converges in C^p to g_∞ uniformly in** K if for every $\epsilon > 0$ there exists $k_0 = k_0(\epsilon)$ such that for $k \geq k_0$,

$$\sup_{0\leq\alpha\leq p} \sup_{x\in K} |\nabla^\alpha(g_k - g_\infty)|_g < \epsilon$$

Given a manifold $\mathcal{M}$, a sequence of open sets $\{U_i\}$ is said to be an **exhaustion of** $\mathcal{M}$ if for every compact set $K \subset \mathcal{M}$ there exists an integer i_0 such that $K \in U_i$ for all $i \geq i_0$.

Definition A.1.2. Let $\mathcal{M}$ be a smooth manifold with exhaustion $\{U_k\}$ and Riemannian metrics $\{g_k\}$. We say that (U_k, g_k) converges in C^∞ to $(\mathcal{M}, g_\infty)$ if for any compact set $K \subset \mathcal{M}$ and any $p > 0$ there exists $k_0 = k_0(K, p)$ such that g_k converges in C^p to g_∞ uniformly in K for every $p \geq k_0$.

Definition A.1.3. A **pointed Riemannian manifold** is a 3-tuple (M, g, O) where $\mathcal{M}$ is a Riemannian manifold with metric g and $O \in \mathcal{M}$ is a choice of point usually called basepoint. If g is a complete metric we say that the 3-tuple is a **complete pointed Riemannian manifold**.

Definition A.1.4. A sequence $\{(\mathcal{M}_k^n, g_k, O_k)\}$ of complete pointed Riemannian manifolds **converges** to a complete Riemannian manifold $(M_\infty^n, g_\infty, O_\infty)$ if there exists

1. an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of $\mathcal{M}_\infty$ by open sets with $O_k \in U_K$ and

2. a sequence of diffeomorphisms $\Phi_k : U_k \to V_k = \Phi(U_k) \in \mathcal{M}_k$ with $\Phi(O_k) = O_\infty$

such that $(U_k, \Phi^*(g_k|_{V_k}))$ converges in C^∞ to $(\mathcal{M}_\infty, g_\infty)$ on compact sets in $\mathcal{M}_\infty$.

The above convergence is referred to as **Cheeger-Gromov convergence** in C^∞.

Definition A.1.5. A family of Riemannian manifolds is said to have **bounded geometry** if there exists positive constant C_p such that

$$|\nabla^p \mathrm{Rm}| \leq C_p$$

for all $p \in \{0\} \cup \mathbb{N}$.

Theorem A.1.1 (Compactness theorem)**.**

A.2. Applications to mean curvature flow

Theorem A.2.1. Let $X_k : M_k^n \times I_k \to \mathbb{R}^{n+1}$ be sequence of solutions of the mean curvature flow and $\{x_k\}$ be a sequence of points with $x_k \in M_k^n$ such that

1. $0 \in I_k$ and $X_k(x_k, 0) = 1$ for every k.

2.

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