Mean Curvature Flow

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Preface

Plan

1. Huisken's 1984 original paper
2. Monotonicity formula and its application to type-I singularities
3. Huisken Sinestrari paper on convexity estimates using Stampacchia trick
4. Noncollapsing

Why Mean curvature flow

Why specifically we want to study this flow and if we do what results can we achieve? To start with it is a very natural flow to consider on hypersurfaces in Euclidean space. It bends the higher curved parts with more speed than lower curved parts in order to uniformize the curvature across the hypersurface. Also, the parabolic nature of the equation directly gives short time existence and uniqueness; so we know given a hypersurface we have one way to evolve to possibly study its geometry. For its twin "Ricci flow" as Huisken calls it the motivation was uniformizing Riemannian manifolds with an eye towards Poincaré conjecture. This is of-course with benefit of hindsight after Perelman's seminal resolution using surgery methods.

Now what do we want to do with the very natural Mean curvature flow on hypersurfaces? A generic answer is to study the geometry of hypersurfaces and attempt a classification. This is severely restricted by the assumption of mean convexity ($H > 0$ everywhere) which makes maximum principle work in a number of cases. Huisken's result on the convergence of convex hypersurfaces into round sphere is the first step towards it, but it doesn't achieve much topologically. A uniformly convex hypersurface is diffeomorphic to unit sphere by Gauss map to begin with. For non-convex hypersurface singularities might develop which prohibit a direct analysis. To overcome this we blow-up the manifold near singularity and this limiting process gives an ancient solution. So we shift our attention to a classification of ancient solutions which is still a difficult problem. Angenent-Daskalopoulos-Sesum and Brendle-Choi have obtained results in this direction without self-similarity conditions

Another direction the Mean curvature flow is being explored is the Lagrangian Mean curvature flow in order to find special Lagrangians inside symplectic manifolds. In the case of Calabi-Yau manifolds the condition of being Lagrangian is preserved under Mean curvature flow.

Organization

Acknowledgments

Contents

1. Introduction to Mean curvature flow

1.1. Fundamentals of Hypersurfaces

SET UP THE FOLLOWING -
1. SECOND FUNDAMENTAL FORM
2. MEAN CURVATURE
3. CONVEXITY RESULTS
4. MCF AS GRADIENT FLOW OF AREA
5. CODAZZI IDENTITY, SIMONS IDENTITY
6. UMBILLIC HYPERSURFACES ARE SPHERES

Let M^n be a smooth n-dimensional manifold with a smooth immersion $X : M^n \to \mathbb{R}^{n+1}$. If X is a diffeomorphism onto its image, we say X is an embedding and its image $\mathcal{M}^n = X(M^n)$ has the structure of a smooth n-dimensional submanifold of $\mathbb{R}^{n+1}$. Let $(U, \{x^i\})$ be a coordinate system on M^m, in Euclidean coordinates the pushforward of tangent vectors will be

$$dX(\partial_i) := \frac{\partial X}{\partial x^i} = \partial_i X$$

where $dX : TM^n \to T\mathbb{R}^{n+1}$ is the derivative of X. Since dX is an injection for each point in M^n, we can define an inner product on $T(M^n)$ which in local coordinates is given by

$$g(\partial_i, \partial_j) = \langle \partial_i X, \partial_j X \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on Euclidean space. Further we can define the Levi-Civita connection on M^n from the Levi-Civita connection on $\mathbb{R}^{n+1}$. The Levi-Civita connection on $\mathbb{R}^{n+1}$ is given by

$$D_X Y = X(Y^i)\partial_i$$

(TO DO) Let $x \in M^n$ and $u \in T_p M^n, \tilde{v} \in T|_U M^n$ for some open set U containing x. Define a connection ∇ by

$$dX(\nabla_u \tilde{v}) = D_{dX(u)}(\tilde{V}) \tag{1.1.1}$$

where $\tilde{V}$ is an extension of $dX(\tilde{v})$ to an open set of $\mathbb{R}^{n+1}$ containing $X(U)$.

Theorem 1.1.1. The connection defined by Eq. (1.1.1) is well-defined and is the unique Levi-Civita connection on (M^n, g)

When X is an embedding, the restriction of the tangent bundle of $T\mathbb{R}^{n+1}|_{\mathcal{M}^n}$ can be decomposed as the direct sum

$$T\mathcal{M}^n \oplus N\mathcal{M}^n$$

where $N\mathcal{M}^n$ is the normal bundle which can be described as

$$N\mathcal{M}^n = \{(p,\nu) \in T\mathbb{R}^{n+1}|_{\mathcal{M}} : \langle u, \nu\rangle = 0 \text{ for all } u \in T_p\mathcal{M}\}$$

1.2. Evolution equations

Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a smooth 1-parameter family of immersions such that

$$\partial_t X(p,t) = \overrightarrow{H}(x,t) = -H(x,t)\nu(x,t)$$

Let $\{x^i\}$ be a local coordinate in M^n. Then the induced metric on the hypersurface is given by $g = X^*(\delta)$ where δ is the flat metric on $\mathbb{R}^{n+1}$. This gives

$$g_{ij} = \delta(X_*(\partial_i), X_*(\partial_j)) = \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle$$

Theorem 1.2.1. Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a solution of mean curvature flow. Then the evolution equation of metric, normal, second fundamental form, mean curvature is given by

$$\partial_t g_{ij} = -2Hh_{ij} \tag{1.2.1}$$

$$\partial_t \nu = \nabla H \tag{1.2.2}$$

$$\partial_t h_{ij} = \Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij} \tag{1.2.3}$$

$$\partial_t H = \Delta H + |A|^2 H \tag{1.2.4}$$

Proof. 1. In local coordinates we have

$$\begin{aligned}
\partial_t g_{ij} &= \frac{\partial}{\partial t}\left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle \\
&= \left\langle \frac{\partial^2 X}{\partial x^i \partial t}, \frac{\partial X}{\partial x^j} \right\rangle + \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j} \right\rangle \\
&= \left\langle \frac{\partial}{\partial t}(-H\nu), \frac{\partial X}{\partial x^j} \right\rangle + \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial}{\partial x^j}(-H\nu) \right\rangle \\
&= -H\left\langle \frac{\partial \nu}{\partial t}, \frac{\partial X}{\partial x^j} \right\rangle - H\left\langle \frac{\partial X}{\partial x^i}, \frac{\partial \nu}{\partial t} \right\rangle \\
&= -2Hh_{ij}
\end{aligned}$$

2. Since $\langle \nu, \nu \rangle = 1$, we have $2\langle \frac{\partial \nu}{\partial t}, \nu \rangle = 0$, so the vector $\frac{\partial \nu}{\partial t}$ is in the tangent plane of the hypersurface, hence we can write it as a linear combination of $\{\frac{\partial X}{\partial x^j}\}$ to get

$$\begin{aligned}\frac{\partial \nu}{\partial t} &= \left\langle \frac{\partial \nu}{\partial t}, \frac{\partial X}{\partial x^i} \right\rangle \frac{\partial X}{\partial X^j} g^{ij} = -\left\langle \nu, \frac{\partial}{\partial x^i}\left(\frac{\partial X}{\partial t}\right)\right\rangle \frac{\partial X}{\partial x^j} g^{ij} \\ &= \left\langle \nu, \frac{\partial}{\partial x^i}(H\nu)\right\rangle \frac{\partial X}{\partial x^j} g^{ij} \\ &= \frac{\partial H}{\partial x^i}\frac{\partial X}{\partial x^j} g^{ij} + H\left\langle \nu, \frac{\partial \nu}{\partial x^i}\right\rangle \frac{\partial X}{\partial x^j} g^{ij} \\ &= \frac{\partial H}{\partial x^i}\frac{\partial X}{\partial x^j} g^{ij} = \nabla H\end{aligned}$$

3. From the relation

$$\frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma^k_{ij}\frac{\partial X}{\partial x^k},$$

we get

$$\begin{aligned}\frac{\partial}{\partial t} h_{ij} &= -\frac{\partial}{\partial t}\left(\frac{\partial^2 X}{\partial x^i \partial x^j}, \nu\right) \\ &= \left(\frac{\partial^2}{\partial x^i \partial x^j}(H\nu), \nu\right) - .\end{aligned}$$

4. From the previous equation we get

$$\begin{aligned}\partial_t H = \partial_t(g^{ij}h_{ij}) &= \frac{\partial g^{ij}}{\partial t} h_{ij} + g^{ij}\frac{\partial h_{ij}}{\partial t} \\ &= -g^{ik}\frac{\partial g_{kl}}{\partial t} g^{lj} h_{ij} + g^{ij}(\Delta h_{ij} - 2H h_{il} g^{lm} h_{mj} + |A|^2 h_{ij}) \\ &= -g^{ik}(-2H h_{kl}) g^{lj} h_{ij} + \Delta(g^{ij}h_{ij}) - 2H g^{ij} g^{lm} h_{il} h_{mj} + |A|^2 H \\ &= 2H|A|^2 + \Delta H - 2H|A|^2 + |A|^2 H \\ &= \Delta H + |A|^2 H\end{aligned}$$

□

Corollary. If mean curvature is positive everywhere on the initial hypersurface, then it remains so throughout the flow.

Proof. We apply maximum principle to the evolution equation of H. □

Remark. This property of mean curvature holds even when the hypersurface is embedded in an arbitrary(OR WITH POSITIVE RICCI CURVATURE?) Riemannian manifold.

1.3. The Avoidance Principle

Let $X_i : M_i^n \times [0,T) \to \mathbb{R}^{n+1}, i = 1,2$ be properly immersed solutions to mean curvature flow such that at least one of M_1^n or M_2^n is compact. If the hypersurfaces are disjoint initially, i.e. $X_1(M^{n_1}, 0) \cap X_2(M_2^n, 0) = \phi$, then they remain so. Define the distance function $d : M_1^n \times M_2^n \times [0,T) \to \mathbb{R}$ between the solutions by

$$d(x,y,t) = |X_2(y,t) - X_1(y,t)|.$$

as a function of time. From the assumption of compactness $d_0 := \inf_{(x,y)\in M_1^n \times M_2^n} d(x,y,0) > 0$

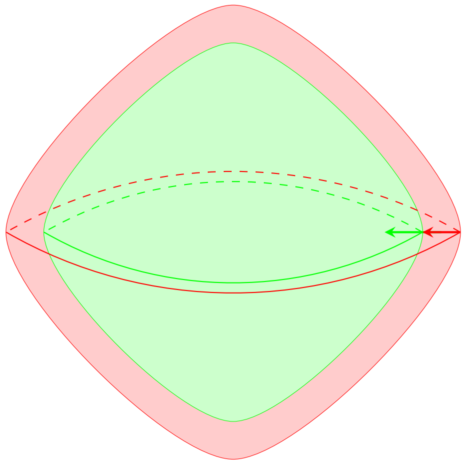

Figure 1.1.: Disjoint hypersurfaces flowing under MCF

Theorem 1.3.1. If X_1 and X_2 are solutions to mean curvature flow on closed manifolds with $d_0 > 0$, then $d(x,y,t) > 0$ for all x,y,t. In particular, if $X_1(x,0) \neq X_2(y,0)$ for all $x \in M_1^n$ and $y \in M_2^n$, then $X_1(x,t) \neq X_2(y,t)$ for all $x \in M_1^n$, $y \in M_2^n$ and $t \in [0,T)$.

Proof. Assume on the contrary $d(x,y,t)$ is not everywhere strictly greater that d_0. Then there exists a t_0 such that $d(x_0,y_0,t_0) = d_0 - \delta$ □

Remark. From the proof we can conclude that the distance between the hypersurfaces is a non-decreasing function. Another way to see this is that d_t satisfies a heat-type parabolic equation

TO DO

on which maximum principle is applicable.

Proof. Fix $\epsilon > 0$ and suppose that $e^{\epsilon(1+t)}d(x, y, t)$ is not strictly greater than d_0. That is there exists a time $t_0 > 0$ such that $e^{\epsilon(1+t)}d(\cdot, \cdot, t)$ reaches d_0. That is,

$$e^{\epsilon(1+t)}d(\cdot, \cdot, t) > 0 \quad \text{for } t < t_0.$$

and

$$e^{\epsilon(1+t)}d(x_0, y_0, t_0) = d_0 \quad \text{for some } (x_0, y_0) \in M_1^n \times M_2^n.$$

Then

$$\partial_t(e^{\epsilon(1+t)}d)|_{(x_0,y_0,z_0)} \leq 0.$$

Let D denote the Euclidean directional derivative, by ∇_i the covariant derivative induced on M_i^n by X_i, and by ∇ the covariant derivative on $M_1^n \times M_2^n$ induced by ∇_1 and ∇_2, we find

$$\nabla^2.$$

□

Remark. We can phrase the avoidance principle as : disjointness is an immortal property and jointness is an ancient property.

1.4. Huisken's theorem

Huisken's theorem proves the convergence of compact, uniformly convex hypersurface to sphere under Mean curvature flow in finite time.

Theorem 1.4.1. Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$, $n \geq 2$ be a maximal solution of MCF such that M^n is compact and $X_0 = X(\cdot, 0)$ is convex embedding. Then $X_t = X(\cdot, t)$ is a convex embedding for all $t > 0$ and X_t converges to a point $p \in \mathbb{R}^{n+1}$ as $t \to T$. Further the rescaled embeddings $\tilde{X}_t : M^n \to \mathbb{R}^{n+1}$ defined by

$$\tilde{X}_t(x) := \frac{X_t(x) - p}{\sqrt{2n(T - t)}}$$

converge uniformly in the smooth topology to a smooth embedding whose image coincides with the unit sphere S^n.

1.4.1. Long time existence

In this subsection we will prove that blow up second fundamental form is the only obstruction for continuing the flow.

1.4.2. Pinching estimate

The quantity $\frac{|A|^2 - \frac{H^2}{n}}{H^2} = \frac{1}{n}\sum_{i<j}\left(\frac{\kappa_i}{H} - \frac{\kappa_j}{H}\right)^2$ is scaling invariant and measures the roundness of the hypersurface. If we compute the evolution equation we get

$$EQUATION \tag{1.4.1}$$

but the maximum principle is not directly applicable because of the positive last term. So we do not directly get a pointwise L^∞ bound we do the next best thing possible which is L^p bounds. After obtaining this there is a sophisticated iteration argument developed by Stampacchia which allows us to produce an L^∞ estimate. See here

Theorem 1.4.2. There exists constants δ and $C_0 < \infty$ depending only on $\mathcal{M}_0$ such that

$$|A|^2 - \frac{H^2}{n} \leq C_0 H^{2-\delta}$$

for $t \in (0, T]$.

Let $f_\sigma = \frac{|A|^2 - \frac{H^2}{n}}{H^{2-\sigma}} = \left(\frac{|A|^2 - \frac{H^2}{n}}{H^2}\right) H^\sigma$, then the evolution equation of f_σ is

Lemma 1.4.3. The evolution of f_σ is given by

$$\frac{\partial}{\partial t} f_\sigma =$$

(Huisken' remark on this from the lecture series : the bad term can only be controlled using integral estimates 58 min mark)

(Sinestrari intuition about Stampacchia iteration : yt link)

1.4.3. Width-pinching

Let $\Omega \subset \mathbb{R}^{n+1}$ be a compact, convex body with boundary $\mathcal{M} = \partial\Omega$. Define the support function $\sigma : S^n \to \mathbb{R}$ as follows

$$\sigma(z) := \sup_{x \in \Omega} \langle x, z \rangle .$$

Using this we define the width function of Ω by

$$w(z) = \sigma(z) + \sigma(-z) \qquad \text{for all } z \in S^n$$

which as the name suggests measures the width of Ω in the direction z. Let w_+ and w_- denote the maximum and minimum width. It is easy to see that maximum width is the diameter of Ω, so

$$w_+ = \sup_{x,y \in \Omega} ||x - y||.$$

If $\mathcal{M}$ is smooth uniformly convex hypersurface, then we can write the support function in terms of Gauss map $G : \partial\Omega \to S^n$ as

$$\sigma(z) = \left\langle G^{-1}(z), z \right\rangle,$$

so

$$\sigma(G(z)) = \langle z, G(z) \rangle .$$

It is a lemma by Andrews that the pinching of principal curvatures implies the pinching of the widths, which will be later important to prove convergence of the hypersurface into a point.

Lemma 1.4.4. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$ be a uniformly convex body with compact, smooth boundary $\mathcal{M} = \partial\Omega$. Suppose the principal curvatures of $\mathcal{M}$ are pinched by a constant $C < \infty$, i.e. $\kappa_n(x) \leq C\kappa_1(x)$ for all $x \in \mathcal{M}$. Then the widths of $\mathcal{M}$ are pinched by the same constant, so

$$w_+ \leq Cw_-$$

Proof. We will first prove it for $n = 2$, which presents the main idea. Let $e \in S^2$ be unit vector in $\mathbb{R}^3$ and (TO DO LATER) $\square$

We know the solution of MCF of spheres, so it will be useful to relate the widths of the hypersurface to the spheres inscribed and circumscribed about it. Let

$$\rho_+ := \inf\{r : \Omega \subset B_r(x) \text{ for some } x \in \mathbb{R}^n\}$$

denote the circumradius and

$$\rho_- := \sup\{B_r(X) \subset \Omega \text{ for some } x \in \mathbb{R}^n\}$$

denote the inradius radius. There is also a relation between inscribed and circumscribed radius to widths of the hypersurface as given by the following lemma

Lemma 1.4.5. On any bounded convex body $\Omega \subset \mathbb{R}^{n+1}$,

$$\rho_+ \leq \sqrt{n+1}w_+ \quad \text{and} \quad \rho_- \geq \frac{1}{n+2}w_-.$$

Remark. These are not the best bounds however they are sufficient for our purpose.

Combining the two lemmas, we get

$$\rho_+ \leq \frac{n+2}{\sqrt{n+1}}C\rho_- \tag{1.4.2}$$

Let $\{\mathcal{M}_t\}_{t\in[0,T)}$ be family of hypersurfaces flowing under MCF with maximal time T and the initial pinching constant given by

$$C = \sup_{\mathcal{M}_0} \frac{\kappa_n}{\kappa_1}$$

If we can show that $\rho_- \to 0$ as $t \to T$, then inequality (1.4.2) would imply $\rho_+ \to 0$. This proves the convergence of hypersurfaces to a point.

Theorem 1.4.6. The inradius ρ_t of Ω_t tends to zero as $t \to T$. Thus, $\mathcal{M}_t$ converges to some point $p \in \mathbb{R}^{n+1}$ as $t \to T$.

1.5. Monotonicity Formula

Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be one-parameter family of immersions flowing by mean curvature. Let $\{x^i\}$ be local coordinates around a point $p \in M$. Then the metric and second fundamental form are given by

$$g_{ij} = \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle, \qquad h_{ij} = \left\langle \eta, \frac{\partial^2 X}{\partial x^i \partial x^j} \right\rangle$$

If we scale the solution by a factor of λ, defined by $\tilde{X}(x,t) = \lambda X(x,t)$ we get the following metric and second fundamental form

$$\tilde{g_{ij}} = \left\langle \frac{\partial \tilde{X}}{\partial x^i}, \frac{\partial \tilde{X}}{\partial x^j} \right\rangle = \lambda^2 g_{ij}$$

$$\tilde{h_{ij}} = \left\langle \eta, \frac{\partial^2 \tilde{X}}{\partial x^i \partial x^j} \right\rangle = \lambda h_{ij}$$

so the scaled mean curvature is given by $\tilde{H} = \tilde{g}^{ij}\tilde{h}_{ij} = \frac{1}{\lambda}H$. This implies the scaled solutions satisfy the evolution equation

$$\frac{\partial \tilde{X}}{\partial t} = \lambda \frac{\partial X}{\partial t} = -\lambda^2 \tilde{H}\eta$$

$$\frac{\partial \tilde{X}}{\partial (\lambda^2 t)} = -\tilde{H}\eta$$

Hence if scale time by λ^2, then $\tilde{X}$ is also a solution of the MCF.

Mean curvature flow is invariant under parabolic scaling, i.e. if $X : M^n \times I \to \mathbb{R}^{n+1}$ is solution, then so is $X_\lambda(x,t) = \lambda X(x, \lambda^2 t)$. We construct a weighted area functional which is invariant under *parabolic* scaling along any solution to mean curvature flow which will be monotonous.

Let $\rho(x,t)$ be the backward heat kernel at (X_0, t_0), i.e.,

$$\rho(x,t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}} \cdot \exp\left(-\frac{|X(x,t)-X_0|^2}{4(t_0-t)}\right), \qquad t < t_0$$

Theorem 1.5.1. If M_t is a solution of mean curvature flow for $t < t_0$, then we have the formula

$$\frac{d}{dt}\int_{M_t} \rho(x,t) d\mu_t = -\int_{M_t} \rho(x,t)\left(H - \frac{\langle X(x,t)-X_0, \eta\rangle}{2(t_0-t)}\right)^2 d\mu_t$$

Proof. To simplify the formula assume that $(X_0, t_0) = (0,0)$. We know that $\frac{d}{dt}\mu_t = -H^2\mu_t$, so differentiating ρ with respect to time we get,

$$\begin{aligned}
\frac{d}{dt}\int_{M_t} \rho(x,t) d\mu_t &= \int_{M_t} \rho(x,t)(-H^2) d\mu_t + \int_{M_t} \frac{\partial}{\partial t}\rho(x,t) d\mu_t \\
&= -\int_{M_t} \rho(x,t) H^2 d\mu_t + \int_{M_t} \left(\frac{\langle X(x,t), H(x,t)\eta\rangle}{2(-t)}\rho(x,t)\right) d\mu_t \\
&\quad + \int_{M_t}\left(\frac{n}{2(4\pi)(-t)}(4\pi)\rho(x,t) - \frac{|X(x,t)|^2}{4(-t)^2}\rho(x,t)\right) d\mu_t \\
&= \int_{M_t} \rho\left(\frac{n}{2(-t)} + \frac{\langle X, H\eta\rangle}{2(-t)} - \frac{|X|^2}{4(-t)^2} - H^2\right) d\mu_t
\end{aligned} \tag{1.5.1}$$

Now $\Delta X = -H\eta$, using this relation for second term and divergence theorem we get

$$\begin{aligned}
\int_{M_t} \rho \langle X, H\eta\rangle d\mu_t &= -\int_{M_t} \rho \langle X, \Delta X\rangle d\mu_t \\
&= -\sum_{k=1}^{n+1}\int_{M_t} \rho X_k \Delta X_k d\mu_t \\
&= \sum_{k=1}^{n+1}\int_{M_t} \langle \nabla(\rho X_k), \nabla X_k\rangle d\mu_t \\
&= \sum_{k=1}^{n+1}\int_{M_t} \left(\langle \nabla\rho, \nabla X_k\rangle X_k + \rho\langle \nabla X_k, \nabla X_k\rangle\right) d\mu_t
\end{aligned} \tag{1.5.2}$$

Let $(U, \{x^i\})$ be some local coordinates on the hypersurface. In these coordinates we can write $\nabla\rho = g^{ij}\partial_i\rho\partial_j$, so $\langle\nabla\rho, \nabla X_k\rangle = \nabla\rho(X_k) = g^{ij}(\partial_i\rho)(\partial_j X_k)$ which implies

$$\begin{aligned}\sum_{k=1}^{n+1}\langle\nabla\rho, \nabla X_k\rangle X_k &= \sum_{k=1}^{n+1} g^{ij}(\partial_i\rho)(\partial_j X_k)X_k \\ &= g^{ij}(\partial_i\rho)\langle X, \partial_j X\rangle \\ &= g^{ij}\rho\left(\frac{-\langle X, \partial_i X\rangle}{2(-t)}\right)\langle X, \partial_j X\rangle \\ &= -\frac{\rho}{2(-t)}|X^T|^2 \end{aligned} \tag{1.5.3}$$

and

$$\sum_{k=1}^{n+1}\rho\langle\nabla X_k, \nabla X_k\rangle = \sum_{k=1}^{n+1}\rho g^{ij}(\partial_i X_k)(\partial_j X_k) = \rho g^{ij}\langle\partial_i X, \partial_j X\rangle = \rho g^{ij}g_{ij} = n\rho \tag{1.5.4}$$

Substituting Eq. (1.5.3) and Eq. (1.5.4) into Eq. (1.5.2) and multiplying by $\frac{1}{2(-t)}$, we get

$$\int_{M_t}\rho\frac{\langle X, H\eta\rangle}{2(-t)}d\mu_t = \int_{M_t}\rho\left(\frac{n}{2(-t)} - \frac{1}{4(-t)^2}|X^T|^2\right)d\mu_t$$

or

$$\int_{M_t}\frac{n\rho}{2(-t)}d\mu_t = \int_{M_t}\rho\left(\frac{\langle X, H\eta\rangle}{2(-t)} + \frac{1}{4(-t)^2}|X^T|^2\right)d\mu_t \tag{1.5.5}$$

where X^T denotes the tangential part of the vector X. Substituting Eq. (1.5.5) into Eq. (1.5.1)

$$\begin{aligned}\frac{d}{dt}\int_{M_t}\rho(x,t)d\mu_t &= \int_{M_t}\rho\left(\frac{\langle X, H\eta\rangle}{(-t)} - \frac{|X|^2}{4(-t)^2} - H^2 + \frac{1}{4(-t)^2}|X^T|^2\right)d\mu_t \\ &= -\int_{M_t}\rho\left(H - \frac{\langle X, \eta\rangle}{2(-t)}\right)^2 d\mu_t\end{aligned}$$

□

1.5.1. Rescaled Monotonicity formula

From section (?) we know that the curvature blows up at the maximal time T and satisfies the inequality

$$\max_{p\in M}|A(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}$$

Definition 1.5.1. Let T be the maximal time of existence of a mean curvature flow. If there exists a constant $C > 1$ such that

$$\max_{p \in M} |A(p,t)| \leq \frac{C}{\sqrt{2(T-t)}}$$

we say the flow is developing at time T a *type I singularity*.

Conversely, if such a constant does not exist, that is

$$\limsup_{t \to T} \max_{p \in M} |A(p,t)| \sqrt{T-t} = \infty$$

we say that we have a *type II singularity*. We will restrict ourselves to type I singularity for the rest of this section.

1.6. Surfaces of positive mean curvature

From the maximum principle we know that if mean curvature of the initial hypersurface M_0 is positive then it will stay positive on M_t. For self-similar solutions, we know that the limiting hypersurface will satisfy the equation $H = \langle x, \nu \rangle$. We prove that sphere is the only compact hypersurface of positive mean curvature moving under self-similarity

Theorem 1.6.1. If M^n, $n \geq 2$, is compact with non-negative mean curvature H and satisfies the equation $H = -\langle X, \nu \rangle$, then M^n is a sphere of radius $\sqrt{n}$.

Proof. Suppose the hypersurface satisfies $H = -\langle X, \nu \rangle$. Let $e_1, \ldots, e_n$ be an orthonormal frame on M^n, then

$$\begin{aligned} \nabla_i H &= -\langle D_{e_i} X, \nu \rangle - \langle X, \nabla_{e_i} \nu \rangle \\ &= -\langle e_i, \nu \rangle - \langle X, \langle \nabla_{e_i} \nu, e_l \rangle e_l \rangle \\ &= \langle X, e_l \rangle h_{il} \end{aligned} \tag{1.6.1}$$

$$\nabla_i \nabla_j H = h_{ij} - H h_{il} h_{lj} + \langle x, e_l \rangle \nabla_l h_{ij}$$

□

1.7. Maximum principle

We can extend the maximum principle on Euclidean space to general Riemannian manifolds.

Lemma 1.7.1. Let

2. Convexity estimates

2.1. Estimate on the scalar curvature

It was established in section Section 1.7 that mean-convexity and uniform convexity is preserved under MCF. While 2-convexity isn't preserved under mean curvature flow; we can still derive an asymptotic result which allows to study the singularity.

Theorem 2.1.1. Let $\mathcal{M}_t, t \in [0, T)$ be solution of the mean curvature flow with $n \geq 2$ such that $\mathcal{M}_0$ is compact and of positive mean curvature. Then, for any $\eta > 0$ there exists a constant $C_\eta > 0$ depending only on $n, \eta, \mathcal{M}_0$ such that

$$\lambda_1 \geq -\eta H - C_\eta \tag{2.1.1}$$

on $\mathcal{M}_t$ for any $t \in [0, T)$.

Let $g_{\sigma,\eta} = \dfrac{|A|^2 - (1+\eta)H^2}{H^{2-\sigma}}$

Lemma 2.1.2. Following equality holds:

$$|\nabla A \cdot H - \nabla H \otimes A|^2 = |\nabla A|^2 H^2 + |A|^2 |\nabla H|^2 - \left\langle \nabla |A|^2, \nabla H \right\rangle H. \tag{2.1.2}$$

Proof. Computing the norm,

$$\begin{aligned} |\nabla A \cdot H - \nabla H \otimes A|^2 &= \langle \nabla A \cdot H - \nabla H \otimes A, \nabla A \cdot H - \nabla H \otimes A \rangle \\ &= |\nabla A|^2 H^2 + |\nabla H|^2 |A|^2 - 2H \langle \nabla A, \nabla H \otimes A \rangle \\ &= |\nabla A|^2 H^2 + |\nabla H|^2 |A|^2 - \left\langle \nabla |A|^2, \nabla H \right\rangle H. \end{aligned}$$

□

The proof is divided into two parts. First part is obtaining an L^p estimate of $g_{\sigma,\eta}$ and the second part is Stampacchia iteration using Michael-Simon inequality in order to get an L^∞ bound.

Lemma 2.1.3. The quantity $\frac{|A|^2}{H^2}$ satisfies the differential equation

$$\frac{\partial}{\partial t}\frac{|A|^2}{H^2} = \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A \cdot H - \nabla H \otimes A|^2. \tag{2.1.3}$$

Proof. Computing the time derivative we get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{|A|^2}{H^2} &= \frac{1}{H^2}\frac{\partial|A|^2}{\partial t} - 2\frac{|A|^2}{H^3}\frac{\partial H}{\partial t}\\ &= \frac{1}{H^2}\left(\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4\right) - 2\frac{|A|^2}{H^3}\left(\Delta H + |A|^2 H\right)\\ &= \frac{\Delta|A|^2}{H^2} - 2\frac{|\nabla A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3}\end{aligned}$$

Now calculating the Laplacian we get,

$$\begin{aligned}\Delta\frac{|A|^2}{H^2} &= \frac{\Delta|A|^2}{H^2} - |A|^2\frac{\Delta H^2}{H^4} - \frac{2}{H^4}\left\langle\nabla|A|^2, \nabla H^2\right\rangle + \frac{2|A|^2}{H^6}|\nabla H^2|^2\\ &= \frac{\Delta|A|^2}{H^2} - |A|^2\left(\frac{2H\Delta H + 2|\nabla H|^2}{H^4}\right) - \frac{2}{H^4}\left\langle\nabla|A|^2, 2H\nabla H\right\rangle + 8\frac{|A|^2}{H^6}|\nabla H|^2\\ &= \frac{\Delta|A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3} + 6|A|^2\frac{|\nabla H|^2}{H^4} - \frac{4}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle\end{aligned}$$

which combined gives

$$\begin{aligned}\frac{\partial}{\partial t}\frac{|A|^2}{H^2} &= \Delta\frac{|A|^2}{H^2} - 6|A|^2\frac{|\nabla H|^2}{H^4} + \frac{4}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle - 2\frac{|\nabla A|^2}{H^2}\\ &= \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle\nabla H, \frac{\nabla|A|^2}{H^2} - \frac{2}{H^3}|A|^2\nabla H\right\rangle\\ &\quad - \frac{2}{H^4}\left(|A|^2|\nabla H|^2 + |\nabla A|^2H^2 - H\left\langle\nabla|A|^2, \nabla H\right\rangle\right)\\ &= \Delta\frac{|A|^2}{H^2} + \frac{2}{H}\left\langle\nabla H, \nabla\frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A\cdot H - \nabla H\otimes A|^2.\end{aligned}$$

□

Applying maximum principle we get that $\frac{|A|^2}{H^2}$ is uniformly bounded so there exists a positive constant depending only on $\mathcal{M}_0$ such that

$$|A|^2 \leq c_0 H^2 \quad \text{on} \quad \mathcal{M}_t,$$

for all time $t \in [0, T)$.

Make it $g_{\sigma,\eta} \leq c_0 H^\sigma$

Recall Simon's identity from [Hui84]

$$\frac{1}{2}\Delta|A|^2 = \langle h_{ij}, \nabla_i\nabla_j H\rangle + |\nabla A|^2 + Z \tag{2.1.4}$$

TO DO : Write why this isn't enough to prove the required decay

Using this we compute the time derivative of $g_{\sigma,\eta}$

Lemma 2.1.4. The evolution equation of $g_{\sigma,\eta}$ is given by

$$\begin{aligned}\frac{\partial g_{\sigma,\eta}}{\partial t} =&\Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma(1-\sigma)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &- \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma |A|^2 g_{\sigma,\eta}. \end{aligned} \tag{2.1.5}$$

Proof. We can write $g_{\sigma,\eta} = \left(\frac{|A|^2}{H^2} - (1+\eta)\right) H^\sigma$ so

$$\begin{aligned}\frac{\partial g_{\sigma,\eta}}{\partial t} &= \left\{\Delta \frac{|A|^2}{H^2} + \frac{2}{H}\left\langle \nabla H, \nabla \frac{|A|^2}{H^2}\right\rangle - \frac{2}{H^4}|\nabla A \cdot H - \nabla H \otimes A|^2\right\} H^\sigma \\ &\quad + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\left(\Delta H^\sigma - \sigma(\sigma-1)H^{\sigma-2}|\nabla H|^2 + \sigma|A|^2 H^\sigma\right) \\ &= \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\left\langle \nabla H, \nabla \frac{|A|^2}{H^2}\right\rangle H^\sigma - \frac{\sigma(\sigma-1)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta} \\ &= \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\left(\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma}{H} g_{\sigma,\eta}|\nabla H|^2\right) - \frac{\sigma(\sigma-1)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta} \\ &= \Delta g_{\sigma,\eta} + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g_{\sigma,\eta}\rangle - \frac{\sigma(1-\sigma)}{H^2} g_{\sigma,\eta}|\nabla H|^2 \\ &\quad - \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \sigma|A|^2 g_{\sigma,\eta}. \end{aligned}$$

□

Let $g_+ = \max g(x,t), 0$ denote the positive part of g. Then $g_+^p \in C^1(\mathcal{M} \times [0,T))$ for $p > 1$

Remark. How Sinestrari does it in the lecture : $\forall \eta > 0, \exists C_\eta > 0$ s.t.

$$\lambda_1 \geq -\eta H - C_\eta$$

for all $t < T$ on $\mathcal{M}_t$

Lemma 2.1.5. If $(1+\eta)H^2 \leq |A|^2 \leq c_0 H^2$ for some $\eta, c_0 > 0$. Then

$$|\nabla A \cdot H - \nabla H \otimes A|^2 \geq \frac{\eta^2}{4n(n-1)^2 c_0} H^2 |\nabla H|^2$$

Proof. We break the tensor as follows

$$|\nabla A \cdot H - \nabla H \otimes A|^2 = |\nabla A \cdot H - \frac{1}{2}(\nabla H \otimes A + A \otimes \nabla H) - \frac{1}{2}(\nabla H \otimes A-)$$

DO IT IN ORIGINAL NOTATION MAYBE. □

Lemma 2.1.6. There exists constant c_2, c_3 such that

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathcal{M}} g_+^p d\mu \leq & -\frac{p(p-1)}{2}\int_{\mathcal{M}} g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int_{\mathcal{M}} \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\ & - p\int_{\mathcal{M}} \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 d\mu + p\sigma \int_{\mathcal{M}} |A|^2 g_+^p d\mu\end{aligned}$$

Proof. Differentiating with respect to time and using **??** for $p \geq 2$

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathcal{M}} g_+^p d\mu &= \int \left(p g_+^{p-1}\partial_t g - H^2 g_+^p\right) d\mu \\ &\leq \int p g_+^{p-1}\left(\Delta g + 2\frac{(1-\sigma)}{H}\langle \nabla H, \nabla g\rangle - \frac{2}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H h_{kl}|^2\right) d\mu \\ &\quad + \sigma |A|^2 g \end{aligned} \tag{2.1.6}$$

Using integration by parts,

$$\begin{aligned}\int p g_+^{p-1}\Delta g d\mu &= -p\int \left\langle \nabla g_+^{p-1}, \nabla g\right\rangle d\mu & (2.1.7)\\ &= -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu & (2.1.8)\end{aligned}$$

Also from proposition **??** we deduce that if $c_1 \geq 4n(n-1)^2 c_0 \eta^{-2}$

$$\begin{aligned}\frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 &\geq \frac{g_+^{p-1}}{c_1 H^{2-\sigma}}|\nabla H|^2 \\ &\geq \frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2 + \frac{1}{2c_0 c_1}\frac{g_+^p}{H^2}|\nabla H|^2 \end{aligned} \tag{2.1.9}$$

To handle the gradient term, let $p \geq \max\{2, 1 + 4c_0 c_1\}$ to obtain

$$\begin{aligned}2(1-\sigma)p\frac{g_+^{p-1}}{H}\langle \nabla H, \nabla g\rangle &\leq 2p\frac{g_+^{p-1}}{H}|\nabla H||\nabla g| \\ &\leq \frac{p}{2c_0 c_1}\frac{g_+^p}{H^2}|\nabla H|^2 + 2c_0 c_1 p g_+^{p-2}|\nabla g|^2 \quad \text{[Peter-Paul inequality]} \\ &\leq p\frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 - p\frac{g_+^{p-1}}{2c_1 H^{2-\sigma}}|\nabla H|^2 \\ &\quad + \frac{p(p-1)}{2}g_+^{p-2}|\nabla g|^2 \qquad \text{[Using Eq. (2.1.9)]}\end{aligned}$$

Substituting this back in Eq. (2.1.6) and using integration by parts from Eq. (2.1.8),

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathcal{M}} g_+^p d\mu \leq & -p(p-1)\int g_+^{p-2}|\nabla g|^2 d\mu + p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu \\ & + \frac{p(p-1)}{2}\int g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{c_3}\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\ & - 2p\int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu + p\sigma\int |A|^2 g_+^p d\mu\end{aligned}$$

which gives the desired inequality with $c_3 = \frac{1}{2c_1}$. □

To handle the bad positive term appearing in the last we use the following lemma

Lemma 2.1.7. There exists a constant c_4 such that

$$\begin{aligned}\frac{1}{c_4}\int |A|^2 g_+^p d\mu \leq \left(p+\frac{p}{\beta}\right)\int g_+^{p-2}|\nabla g|^2 + (1+\beta p)\int \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\ + \int \frac{g_+^{p-1}}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 d\mu\end{aligned}$$

for any $\beta > 0, p > 2$.

Proof. From the calculation of Laplacian in Lemma 2.1.3 we know that

$$
\begin{aligned}
\Delta g &= \Delta\left(\frac{|A|^2}{H^2}\right)H^\sigma + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\Delta H^\sigma + 2\left\langle \nabla\frac{|A|^2}{H^2}, \nabla H^\sigma\right\rangle \\
&= \left(\frac{\Delta|A|^2}{H^2} - 2|A|^2\frac{\Delta H}{H^3} + 6|A|^2\frac{|\nabla H|^2}{H^4} - \frac{4}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle\right)H^\sigma \\
&\quad + \left(\frac{|A|^2}{H^2} - (1+\eta)\right)\left(\sigma H^{\sigma-1}\Delta H + \sigma(\sigma-1)H^{\sigma-2}|\nabla H|^2\right) \\
&\quad + 2\sigma H^{\sigma-1}\left\langle\frac{\nabla|A|^2}{H^2} - 2\frac{|A|^2}{H^3}\nabla H, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{|A|^2}{H^{3-\sigma}} - \sigma(1+\eta)H^{\sigma-1}\right)\Delta H + 6\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 - \frac{4}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle \\
&\quad + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2\sigma}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle - 4\sigma\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H + (6-4\sigma)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&\quad - \frac{2}{H^{4-\sigma}}H\left\langle\nabla|A|^2, \nabla H\right\rangle + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2}{H^{2-\sigma}} + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H + (6-4\sigma)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 \\
&\quad - \frac{2}{H^{4-\sigma}}\left(|\nabla A|^2H^2 + |A|^2|\nabla H|^2 - |\nabla A\cdot H - \nabla H\otimes A|^2\right) + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 \\
&\quad + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle \\
&= \frac{\Delta|A|^2 - 2|\nabla A|^2}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A\cdot H - \nabla H\otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&\quad - 4(\sigma-1)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\left\langle\nabla|A|^2, \nabla H\right\rangle.
\end{aligned}
$$

Now similar to time derivative we calculate the gradient of g with H,

$$
\begin{aligned}
\langle\nabla g, \nabla H\rangle &= \left\langle\nabla\frac{|A|^2}{H^2}, \nabla H\right\rangle H^\sigma + \sigma\left(\frac{|A|^2}{H^2} - (1+\eta)\right)H^{\sigma-1}|\nabla H|^2 \\
&= \left\langle\frac{\nabla|A|^2}{H^2}, \nabla H\right\rangle H^\sigma - 2\frac{|A|^2}{H^{3-\sigma}}|\nabla H|^2 + \sigma\frac{g}{H}|\nabla H|^2,
\end{aligned}
$$

Also recall Simon's identity, and using this to eliminate the mixed inner product term

$$\begin{aligned}
\Delta g =& \frac{\Delta|A|^2 - 2|\nabla A|^2}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&- 4(\sigma-1)\frac{|A|^2}{H^{4-\sigma}}|\nabla H|^2 + \sigma(\sigma-1)\frac{g}{H^2}|\nabla H|^2 \\
&+ \frac{2(\sigma-1)}{H^{3-\sigma}}\left(\langle \nabla g, \nabla H\rangle + 2\frac{|A|^2}{H^{3-\sigma}}|\nabla H|^2 - \sigma\frac{g}{H}|\nabla H|^2\right) \\
=& \frac{2\langle h_{ij}, \nabla_i\nabla_j H\rangle + 2Z}{H^{2-\sigma}} + \frac{2}{H^{4-\sigma}}|\nabla A \cdot H - \nabla H \otimes A|^2 + \left((\sigma-2)\frac{g}{H} - 2(1+\eta)H^{\sigma-1}\right)\Delta H \\
&- \sigma(\sigma-1)\frac{g}{H}|\nabla H|^2 + \frac{2(\sigma-1)}{H^{3-\sigma}}\langle \nabla g, \nabla H\rangle
\end{aligned}$$

Multiplying this equation by $g_+^p H^{-\sigma}$

$$-\int \frac{2Z}{H^2} g_+^p d\mu =$$

□

3. Noncollapsing

Noncollapsing in mean curvature flow is a powerful result which gives a geometric idea about the structure of singularities.

A. Convergence of Manifolds

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