

Today I Tried

Devesh Rajpal

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(15/4/24) Today I tried to cast the self similar solutions of α -Gauss flow in the Monge-Ampere type using support function. The equation $\langle X(x), \nu(x) \rangle = cK^\alpha$ can be written as

$$h = c \left(\frac{\det(\bar{\nabla}^2 h + \bar{g}h)}{\det(\bar{g})} \right)^{-\alpha}$$

where $\bar{\nabla}^2 h$ is the 2-tensor defined using the standard connection on S^n . It is easy to calculate

$$\bar{\nabla}_i \bar{\nabla}_j h = \partial_i \partial_j h - (\bar{\nabla}_i \partial_j) h$$

(16/4/24) Today I tried spherical coordinates on the α -Gauss flow. In the parametrization $x = \cos \theta \cos \phi, y = \cos \theta \sin \phi, z = \cos \theta$, we have

$$\bar{g} = \begin{bmatrix} 1 & 0 \\ 0 & \cos \theta \end{bmatrix}$$

and

$$K^{-1} = \frac{\det \left(\begin{bmatrix} h_{\theta\theta} & h_{\theta\phi} - \frac{\tan \theta}{2} h_\phi \\ h_{\theta\phi} - \frac{\tan \theta}{2} h_\phi & h_{\phi\phi} - \frac{\cos \theta}{2} h_\theta \end{bmatrix} + h \begin{bmatrix} 1 & 0 \\ 0 & \cos \theta \end{bmatrix} \right)}{\cos \theta}$$

which is quite ugly.

(17/4/24) Today I learned about the homogeneous degree 1 extension of the support function. Let $h : S^n \rightarrow \mathbb{R}$ be the support function of a strictly convex hypersurface. We extend this to $H : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by defining,

$$H(x) = |x| h \left(\frac{x}{|x|} \right).$$

Note that H is just continuous but not necessarily differentiable at 0. It is easy to see that $DH(\lambda x) = DH(x)$. Let $x \in \mathbb{R}^{n+1}$ and v be a unit vector,

$$\begin{aligned} D_v H(x) &= h \left(\frac{x}{|x|} \right) D_v |x| + |x| D_v h \left(\frac{x}{|x|} \right) \\ &= \frac{v \cdot x}{|x|} h \left(\frac{x}{|x|} \right) + |x| \bar{\nabla}_{v^T} h \left(\frac{x}{|x|} \right) \end{aligned}$$

where $\bar{\nabla}_{v^T}$ is the covariant derivative on S^n in the direction $v^T \in T_x S^n$. Let $x \in S^n$ and substitute $v \in \{e_1, \dots, e_{n+1}\}$ to get

$$DH(x) = xh(x) + \bar{\nabla}h(x)$$

which is the inverse of Gauss map! Thus, $G^{-1}(x) = DH(x)$, and also weirdly $D_x H(x) = DH(x)$ so the steepest ascent is in the normal direction.

(18/4/24) Today I learned about a possible reducible symmetric group to try to construct self-similar solutions of the α -Gauss curvature flow. As considered previously the setup is with support functions. The sphere $h \equiv 1$ is an equilibrium point of the normalized α -Gauss flow. The construction of Γ symmetric solutions in Ben's paper is using spherical harmonics (eigenfunctions of the Laplacian) and some general version of the stable/unstable manifold theorem. The linearized version of normalized α -Gauss flow at $h \equiv 1$ is given by

$$\frac{\partial u}{\partial t} = \alpha(\Delta + n)u + u$$

so if $\Delta\psi = -\lambda\psi$, then $h \equiv 1$ is strictly unstable in the direction ψ (what does this really mean?). Another important fact is that entropy is a min for unstable direction, the Hessian of entropy at $h \equiv 1$ satisfies

$$D^2 Z_h(\psi, \psi) > 0.$$

A new idea is to use an affine boost along with the spherical harmonics to possibly control the isoperimetric ratio. The considered example of a reducible group was generated by $x \mapsto -x$ and a 3-fold rotation symmetric group in yz plane along with reflection of the triangle (so the dihedral group D_3 in the yz plane). Consider a one-parameter family of affine transformations which stretches the x -direction,

$$T_\lambda = \begin{bmatrix} e^{2\lambda} & 0 & 0 \\ 0 & e^{-\lambda} & 0 \\ 0 & 0 & e^{-\lambda} \end{bmatrix}.$$

Now we can consider a one parameter family of flows produced by the unstable direction $\epsilon\psi + T_\lambda$ and the expectation is that since the entropy of the solutions with $\lambda = 0, \infty$ is ∞ (to check) one can possibly use a mountain pass theorem (in homotopies of λ variable) to create a critical point which will be a self-similar solution of the flow.

(19/4/24) Today I learned about the α -Gaussian entropy of a bounded convex hypersurface. In the book it is defined as

$$E_\alpha(\mathcal{M}) \doteq \begin{cases} \left(\frac{\text{Vol}(\mathcal{M}^n)}{|B^{n+1}|} \right)^{\frac{n}{n+1}} \exp \left(\frac{1}{|S^n|} \int_{\mathcal{M}^n} K \log K d\mu \right) & \text{if } \alpha = 1 \\ \left(\frac{\text{Vol}(\mathcal{M}^n)}{|B^{n+1}|} \right)^{\frac{n}{n+1}} \left(\frac{1}{|S^n|} \int_{\mathcal{M}^n} K^\alpha d\mu \right)^{\frac{1}{\alpha-1}} & \text{if } \alpha \neq 1 \end{cases}$$

It turns out that the α -Gaussian entropy is non-increasing under α -Gauss flow. The next task is to understand its property on normalized α -Gauss flow.

(22/4/24) Today I learned about the Brunn-Minkowski inequality. It states that for convex bodies $A, B \subset \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have

$$\text{Vol}(\lambda A + (1 - \lambda)B)^{\frac{1}{n}} \geq \lambda \text{Vol}(A)^{\frac{1}{n}} + (1 - \lambda) \text{Vol}(B)^{\frac{1}{n}}$$

which is same as saying that $\text{Vol}(\cdot)^{\frac{1}{n}}$ is a concave function on the set of convex bodies.

(23/4/24) Today I learned about the proof of monotonicity of α -Gaussian entropy under α -Gaussian flow using Brunn-Minkowski inequality.

(24/4/24) Today I tried to finish the monotonicity of entropy proof.

(29/4/24) Today I learned about Jacobi fields in the general context of calculus of variations. Let $I(u) = \int F(t, u(t), \dot{u}(t))$ which we want to minimize.

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(3/5/24) Today I learned about Mountain Pass Theorem(MPT).