

Lie Groups

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One can study Lie Groups from several points of view. The course is aimed to understand the structure of Lie Groups.

Definition 1.1. A smooth manifold M is a Hausdorff space which is locally Euclidean with a smooth atlas i.e. (i) given any $x \in M$, \exists a chart (U, ϕ) , $x \in U \subset M$ with $\phi : U \rightarrow \phi(U)$ open in \mathbb{R}^m .

(ii) We have collection $\{(U, \phi)\}$ of charts such that

$$\phi(U \cap V) \xrightarrow{\psi \circ \phi^{-1}} \psi(U \cap V)$$

is a diffeomorphism.

Suppose $f : M \rightarrow N$ is a continuous map between manifolds. We say that f is smooth if for $(U, \phi) \in \Pi(M)$, $(V, \psi) \in \Pi(N)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}$ is smooth.

TO DO : Construction of tangent bundle and vector bundle

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Definition 2.1. G is a Lie group if

1. G is a smooth manifold
2. G is also a group s.t

$$\begin{aligned}\mu : G \times G &\rightarrow G \\ (g, h) &\mapsto gh\end{aligned}$$

and

$$\begin{aligned}i : G &\rightarrow G \\ g &\mapsto g^{-1}\end{aligned}$$

are smooth maps.

Definition 2.2. A real (or complex) vector space V together with a bilinear map

$$[,] : V \times V \rightarrow V$$

is called a **Lie Algebra** if

1. $[X, Y] = -[Y, X]$ - skew symmetry
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ - Jacobi identity

Example. 1. $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, V any f.d vector space over \mathbb{R} or \mathbb{C} .

2. $(\mathbb{R}^\times, \cdot)$, $(\mathbb{C}^\times, \cdot)$

3. $S^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$

4. $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$

5. $\mathbb{R}^n / \mathbb{Z}^n \cong (\mathbb{R}^n / \mathbb{Z}^n) \cong (S^1)^n$

6. Suppose $\Gamma \subset V$ is a discrete subgroup. Then V/Γ is a Lie group.

7. N = unipotent upper triangular matrices, B = upper triangular matrices. As manifolds $N \cong \mathbb{R}^{\binom{n}{2}}$ and $B \cong (\mathbb{R}^\times)^n \times N$.

8. $\mathrm{SL}_n(\mathbb{R}) = \{X \in \mathrm{GL}_n(\mathbb{R}) \mid \det X = 1\}$, $\mathrm{SL}_n(\mathbb{C})$.

9. $O(n)$, $SO(n)$.

10. $U(n)$, $SU(n)$.

11. \mathbb{H}^\times, S^3 with quaternion multiplication.
12. $Sp(n) = \{X \in GL_n(\mathbb{R}) | X \text{ preserves quaternion structure as a subset of } \text{Aut}_{\mathbb{H}} \mathbb{H}^n\}$

Problem. $V/\Gamma \cong \mathbb{R}^k \times (S^1)^{n-k}$ for n -dimensional vector space V .

Theorem 2.1. Suppose G is a compact, connected, simple Lie group. Then G is locally isomorphic to

1. $SU(n), n \geq 2$ denoted by A_{n-1}
2. $SO(2n+1), n \geq 2$ denoted by B_n
3. $Sp(n), n \geq 1$ denoted by C_n
4. $SO(2n), n \geq 2$ denoted by D_n

or one of the following exceptional Lie group G_2, F_4, E_6, E_7, E_8 .

Problem. Prove that $SL_n(\mathbb{R})$ and $O(n)$ are smooth manifold, hence Lie groups.

Examples of Lie algebra -

- Example.**
1. $(V, [\cdot, \cdot] \equiv 0)$ is called trivial Lie algebra.
 2. $(\mathfrak{gl}_n(\mathbb{R}), [A, B] = AB - BA), \mathfrak{gl}_n(\mathbb{C})$
 3. $\mathfrak{sl}_n(\mathbb{R}) (\mathfrak{sl}_n(\mathbb{C}))$ is the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R}) (\mathfrak{gl}_n(\mathbb{C}))$ consisting of trace 0.
 4. \mathfrak{so}_n is Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ consisting of skew-symmetric matrices.

Definition 2.3. A vector field X on a Lie group G is called left invariant if $(L_g)_*(X_h) = X_{gh}$

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Recall $\mathbb{H} = \{a + bi + cj + dk : (a, b, c, d) \in \mathbb{R}^4, i^2 = -1, j^2 = -1, k^2 = -1, ij = k, jk = l, ki = j\}$ is the quaternion division algebra with the norm

$$||a + bi + cj + dk||^2 = a^2 + b^2 + c^2 + d^2$$

which satisfies $||q_1 \cdot q_2|| = ||q_1|| \cdot ||q_2||$

We can put this multiplication on $S^3 \cong SU(2)$ to get a compact Lie group. To get the

isomorphism $SU(2) \cong S^3$, we define a map

$$\begin{aligned}\phi : S^3 &\rightarrow SU(2) \\ (a, b, c, d) &\mapsto \begin{bmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{bmatrix}\end{aligned}$$

which is an algebra isomorphism.

Definition 3.1. The Lie algebra of G is the space of all left-invariant vector fields on G .

We have an isomorphism

$$\begin{aligned}\mathfrak{g} = \text{Lie}(G) &\rightarrow T_e G \\ X &\mapsto X_e\end{aligned}$$

Example. Let $G = \mathbb{R}^n$, with identity element $0 \in \mathbb{R}^n$ and left-invariant vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$. Then the Lie bracket is

$$[\cdot, \cdot] \equiv 0$$

Remark. In general for any abelian Lie group G , the Lie bracket is $[\cdot, \cdot] \equiv 0$.

Theorem 3.1. Let G be a connected Lie group. Then

1. $\text{Lie}(G) = \mathfrak{g}$ is isomorphic as a vector space to $T_e(G)$.
2. Left-invariant vector fields are smooth.
3. $\text{Lie}(G)$ is closed under Lie bracket.

Proof. 1. Let X be a left-invariant vector field on G . We need to show that Xf is smooth for each $f \in C^\infty(G)$.

$$\begin{aligned}(Xf)(g) &= X_g f \\ &= (d\lambda_g X_e) f \\ &= X_e(f \circ \lambda_g)\end{aligned}$$

To show that Xf is smooth, it suffices to show that $X_e(f \circ \lambda_g)$ is smooth. We realize $X_e(f \circ \lambda_g)$ as evaluation of a smooth function on a smooth function.

Let Y be a smooth vector field on G such that $Y_e = X_e$

$$Y_e(f \circ \lambda_g) = X_e(f \circ \lambda_g)$$

We look at λ_g as the composition of

$$\begin{aligned} G &\xrightarrow{i_g^2} G \times G \xrightarrow{\mu} G \\ x &\mapsto (g, x) \mapsto gx \end{aligned}$$

Regard Y as the vector field $(0, Y)$ on $G \times G$. Now

$$\begin{aligned} (0, Y)(f \circ \mu) \circ i_e^1(g) &= (0, Y)_{(g, e)}(f \circ \mu) \\ &= 0_g(f \circ \mu \circ i_g^1) + Y_e(f \circ \mu \circ i_g^2) \\ &= Y_e(f \circ \lambda_g) \end{aligned}$$

which proves the smoothness.

2. Let X, Y left-invariant vector fields on G . We must show that $[X, Y]$ is a left-invariant vector field.

$$\begin{aligned} d\lambda_g([X, Y]_e)f &= [X, Y]_g f \\ &= [X, Y]_e(f \circ \lambda_g) \\ &= X_e(Y(f \circ \lambda_g)) - Y_e(X(f \circ \lambda_g)) \\ &= X_e(d\lambda_g(Yf)) - Y_e(d\lambda_g(Yf)) \end{aligned}$$

□

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Lemma 4.1. Suppose $\psi : M \rightarrow N$ is a smooth map. Let X_1, X_2 be vector fields on M , Y_1, Y_2 be vector fields on N such that X_i is ψ -related to Y_i . Then $[X_1, X_2]$ is ψ -related to $[Y_1, Y_2]$.

Proof. Notice that

$$\begin{aligned} d\psi[X_1, X_2](f) &= [X_1, X_2](f \circ \psi) \\ &= X_1(X_2 f \circ \psi) - X_2(X_1 f \circ \psi) \\ &= X_1(d\psi X_2 f) - X_2(Y_1 f \circ \psi) \\ &= X_1(Y_2 f \circ \psi) - X_2(Y_1 f \circ \psi) \\ &= d\psi X_1(Y_2 f) - d\psi X_2(Y_1 f) \\ &= Y_1 Y_2 f \circ \psi - Y_2 Y_1 f \circ \psi \\ &= [Y_1, Y_2](f) \circ \psi \end{aligned}$$

□

This lemma proves that the set of left-invariant vector fields forms a Lie algebra.

Consider the Lie group $GL_n(\mathbb{R})$. We want to verify the Lie algebra structure on $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ with the isomorphism

$$\begin{aligned} Lie(GL_n(\mathbb{R})) &\rightarrow \mathfrak{gl}_n(\mathbb{R}) \\ X &\mapsto X_e \end{aligned}$$

Lemma 4.2.

$$\beta([X, Y]) = [\beta(X), \beta(Y)]$$

Proof. Evaluating the bracket on coordinate function x_{ij} .

$$[X, Y]_e(x_{ij}) = X_e(Y(x_{ij})) - Y_e(X(x_{ij})) \quad (1)$$

Now

$$\begin{aligned} Y(x_{ij})(g) &= d\lambda_g Y_e(x_{ij}) \\ &= Y_e(x_{ij} \circ \lambda_g) \\ &= \sum_k x_{ik}(g) Y_e(x_{kj}) \end{aligned}$$

Considering the above as function of g and substituting this in Eq. (1) we get

$$\begin{aligned} [X, Y]_e(x_{ij}) &= X_e Y_e(x_{ij}) - Y_e X_e(x_{ij}) \\ &= \sum_k \{X_e(x_{ik}) Y_e(x_{kj}) - Y_e(x_{ik}) X_e(x_{kj})\} \\ &= [X_e, Y_e](x_{ij}) \end{aligned}$$

□

Definition 4.1. A Lie subgroup H of a Lie group G is a submanifold $H \xrightarrow{\alpha} G$ where α is smooth and a group homomorphism.

We say that H is closed Lie subgroup if it is Lie subgroup such that $H \rightarrow \alpha(H)$ is a diffeomorphism.

Example. Consider the map $\mathbb{R} \rightarrow S^1 \times S^1$ given by

$$t \mapsto (e^{2\pi i t}, e^{2\pi i \sqrt{2} t})$$

The image is a Lie subgroup of $S^1 \times S^1$ but it is not a closed Lie subgroup. It is also known as “Skew-line” in the torus.

Definition 4.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $f : \mathfrak{g} \rightarrow \mathfrak{h}$ be a vector space homomorphism. Then we say that f is a Lie algebra homomorphism if

$$f([X, Y]) = [f(X), f(Y)]$$

Theorem 4.3. Suppose that $\psi : G \rightarrow H$ is a Lie group homomorphism. Let X be a left-invariant vector field on G . Extend $d\psi(X_e) = Y_e \in T_e H$ to a left-invariant vector field Y on H . Then X and Y are ψ -related. This implies $d\psi_e : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} TG & \xrightarrow{d\psi} & TH \\ \downarrow X & & \downarrow Y \\ G & \xrightarrow{\psi} & H \end{array}$$

We want to show that $Y \circ \psi = d\psi \circ X$. Now

$$\lambda_{\psi(g)} \circ \psi = \psi \circ \lambda_g$$

so

$$\begin{aligned} Y_{\psi(g)} &= d\lambda_{\psi(g)} Y_e \\ &= d\lambda_{\psi(g)} d\psi X_e \\ &= d(\lambda_{\psi(g)} \circ \psi)(X_e) \\ &= d(\psi \circ \lambda_g)(X_e) \\ &= d\psi d\lambda_g(X_e) \\ &= d\psi X_g \end{aligned}$$

□

Theorem 4.4. Let G, H be Lie groups with G connected. Let

$$\phi, \psi : G \rightarrow H$$

be homomorphism of Lie groups such that

$$d\phi = d\psi : T_e G \rightarrow T_e H$$

Then $\phi = \psi$.

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I missed the class.

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Theorem 6.1. If $\phi : H \rightarrow G$ is a homomorphism of Lie groups and if ω is a left-invariant differential form on G , then $\phi^*(\omega)$ is again a left-invariant form on H .

Suppose that $\phi : H \rightarrow G$ is a homomorphism of Lie groups. Let $\omega_1, \dots, \omega_d$ be a basis for $E_{\text{inv}}^1(G)$. Then

$$\mathcal{I}_\phi = \langle \{ \pi_1^* \phi^*(\omega_j) - \pi_2^*(\omega_j) \} : 1 \leq j \leq d \rangle$$

is a left-invariant differential ideal of $H \times G$.

Lemma 6.2. Suppose X_1, \dots, X_d is a basis of \mathfrak{g} dual to $\omega_1, \dots, \omega_d$. Suppose the Lie bracket is given by

$$[X_i, X_j] = \sum c_{ij}^k X_k$$

Then the C^∞ functions c_{ij}^k are constant. Further,

$$d\omega_i = -c_{kj}^i \omega_k \wedge \omega_j$$

Proof. Notice that

$$\begin{aligned} d\omega_k(X_i, X_j) &= -\omega_k([X_i, X_j]) \\ &= -c_{ij}^k \end{aligned}$$

which is a constant because a left-invariant 1-form evaluated on a left-invariant vector field is a constant. \square

Remark. The c_{ij}^k are called the structural constants of G with respect to the basis $\{X_i\}$ of \mathfrak{g} .

Proof. Theorem 4.4. Notice that $\mathcal{I}_\psi = \mathcal{I}_\phi$ since $d\phi = d\psi$ and these are invariant differential ideals; hence integral manifolds of \mathcal{I}_ϕ and \mathcal{I}_ψ passing through (e, e) are the same. Thus, $\phi = \psi$. \square

Lemma 6.3. Suppose G is any Hausdorff topological group which is connected. Suppose $e \in U \subset G$ is any open set. Then

$$G = \bigcup_{n \geq 1} U^n$$

where $U^n = \{x_1 \cdots x_n | x_i \in U\}$

Proof. Since $e \in U$ is open, $U^{-1} = \{x^{-1} | x \in U\}$ is also an open neighborhood of e . Let $V = U \cap U^{-1}$. Note that

$$H \doteq \bigcup_{n \geq 1} V^n$$

is a subgroup of G , and it is open. Since the cosets gH are also open it follows that $G = \cup_g H$ being connected must be H . □

Theorem 6.4. Let G be a Lie group and $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of \mathfrak{g} . Then there exists connected Lie subgroup H of G such that $T_e H = \mathfrak{h}$.

Proof. Consider the distribution \mathcal{D} defined as

$$\mathcal{D}_g = \{X_g | X \in \mathfrak{h}\}$$

on G . Suppose X_1, \dots, X_c is a basis of \mathfrak{h} . Then \mathcal{D} is generated by X_1, \dots, X_c and \mathcal{D} is involutive. □

Corollary. (a) There is a one-to-one correspondence between connected Lie subgroups of G and Lie subalgebras of \mathfrak{g} .

(b) Suppose $(H, i) \leftrightarrow \mathfrak{h} \subset \mathfrak{g}$. Then (H, i) is an embedded manifold if and only if H is closed.

Theorem 6.5. Suppose that $A \subset G$ is an abstract subgroup of G and if A has a manifold structure such that $(A, i) \rightarrow G$ is a submanifold. Then the manifold structure is unique, A is a Lie group and hence (A, i) is a Lie subgroup of G .

Theorem 6.6. (Adó) Suppose that \mathfrak{g} is a finite dimensional Lie algebra. Then \mathfrak{g} can be realized as a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

Given any connected Lie group G , it has a universal cover $\tilde{G} \xrightarrow{\pi} G$. Choose $\tilde{e} \in \pi^{-1}(e) \in \tilde{G}$ such that the following diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \longrightarrow & \tilde{G} \\ \pi \times \pi \downarrow & & \downarrow \pi \\ G \times G & \longrightarrow & G \end{array}$$

commutes.