Differential Geometry

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Levi-Civita connection

Theorem 1.1. Let (M, g) be a Riemannian manifold. There exists a unique connection ∇ on M which satisfies the following for any vector fields $X, Y, Z \in \Gamma(TM)$

1. Compatibility with metric :

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

2. Torsion-free :

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

Such a connection is called the **Levi-Civita** connection on M.

Proof. Assume such a connection exists, then

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{1}$$

$$Yg(Z, X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \tag{2}$$

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{3}$$

Adding Eq. (1) and Eq. (2) and subtracting Eq. (3) from the sum

$$Xg(Y, Z) + Yg(Z, Z) - Zg(X, Y) = g(Y, [X, Z]) + g(X, [Y, Z]) + 2g(\nabla_X Y, Z) - g(Z, [X, Y])$$

Or

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,Z) - Zg(X,Y) + g(Z,[X,Y]) - g(Y,[X,Z]) - g(X,[Y,Z]) \tag{4}$$

This gives us a formula for determining the connection ∇ which proves the existence and uniqueness. Further it can be proved that properties 1. and 2. are satisfied with this formula. $\square$

Let $\gamma : (\mathbb{R}, \delta) \to (M, g)$ be a smooth curve on M. Let $X \in \Gamma(TM)$ be a vector field. Define $X_\gamma(t) = X(\gamma(t))$ to be its restriction on γ. Then

$$\frac{D}{dt}X(t) \doteqdot \nabla_{\gamma'(t)} X|_{\gamma(t)}$$

which is called covariant differentiation along the curve γ.

Definition 1.1. A curve $\gamma : (a, b) \to M$ is called a **geodesic** if

$$\nabla_{\gamma'(t)}\gamma'(t) = 0 \qquad \forall t \in (a, b)$$

2 Oct 18, 2022

Parallel Transport along a curve

Let (M, g) be a smooth Riemannian manifold. Let $\gamma : \mathbb{R} \to M$ be a smooth curve and $X \in \Gamma(TM)$ is a vector field. Restricting X to γ we get a vector field along γ which we define by $X(t) \doteqdot X|_{\gamma(t)}$. Let

$$\frac{D}{dt}X(t) \doteqdot (\nabla_{\gamma'(t)} X)|_{\gamma(t)}$$

Definition 2.1. A vector field X along γ is called **parallel** along γ if $\frac{D}{dt}X(t) = 0$. If $t_0 < t_1$, we say that $X(t_1)$ is obtained from $X(t_0)$ by parallel transport along γ.

In a local chart $\{x^i\}$, let γ be given by $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$ and $X|_{\gamma(t)} = X^i \frac{\partial}{\partial x^i}\big|_{\gamma(t)}$ or

$$X(t) = X^i(t) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}$$

The equation for parallel transport in local coordinates becomes

$$\frac{D}{dt}X(t) = 0 \tag{5}$$

$$\left(\nabla_{\gamma_*(\frac{\partial}{\partial t})}X\right)\Big|_{\gamma(t)} = 0 \tag{6}$$

$$\nabla_{\gamma'(t)}\left(X^i\frac{\partial}{\partial x^i}\right)\Big|_{\gamma(t)} = 0 \tag{7}$$

$$\frac{dX^i}{dt}\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} + X^i\nabla_{\gamma'(t)}\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} = 0 \tag{8}$$

$$\frac{dX^i}{dt}\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} + X^i\frac{d\gamma^k(t)}{dt}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i} = 0 \tag{9}$$

$$\frac{dX^i}{dt}\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} + X^i\frac{d\gamma^k(t)}{dt}\Gamma^l_{ki}\frac{\partial}{\partial x^l}\Big|_{\gamma(t)} = 0 \tag{10}$$

$$\left(\frac{dX^l}{dt} + X^i\frac{d\gamma^k(t)}{dt}\Gamma^l_{ki}\right)\frac{\partial}{\partial x^l}\Big|_{\gamma(t)} = 0 \tag{11}$$

which is a system of linear equations hence there exits a unique solution for all the time it is defined.

If $g = (g_{ij})$ in the chart $\{x^i\}$, then from the compatibility of ∇, we get

$$\begin{aligned}\frac{\partial}{\partial x^i}g_{jk} &= g\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) + g\left(\frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right)\\ &= \Gamma^l_{ij}g_{kl} + \Gamma^l_{ik}g_{lj}\end{aligned}$$

Using other cyclic relations, we get

$$\Gamma^k_{ij} = g^{kl}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

Geometric interpretation of torsion

For any connection ∇, the torsion tensor is defined by

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

Let $\gamma(s,t)$ be variation of a curve. Then

$$\left[\frac{D}{ds}, \frac{D}{dt}\right] = 0 \Longleftrightarrow \tau \equiv 0$$

3 Oct 20, 2022

Geodesics and the Exponential map

Let (M, g) be a smooth Riemannian manifold and $\gamma : [0, \tau] \to M$ be a smooth curve on M. Then

$$\frac{d}{dt} g\left(\gamma'(t), \gamma'(t)\right) = 2g\left(\frac{D}{dt}(\gamma'(t)), \gamma'(t)\right)$$

If γ is a geodesic, then parallel transport along γ is denoted by $P_{\gamma(t_0)\to\gamma(t_1)}X(t)$ for $X \in \Gamma(TM)$. Now we want to find an expression of P in local coordinates. Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector at $\gamma(0)$. We want to solve the differential equation

$$\frac{D}{dt}X(t) = 0$$

along γ where $\frac{D}{dt} = \nabla_{\gamma'(t)}$. This was done in Eq. (11) which is a linear system of equation and the solution exists for while we are in a coordinate patch. To get a general solution, we cover the image of γ by finitely many coordinate patches and repeat the process (since the image is compact, we can cover it by finitely many patches).

Now we use this to define the exponential map at a point.

Proposition 3.1 (Do Carmo Proposition 2.7)**.** Given $p \in M$, there exists a neighborhood V of $p \in M$, a number $\epsilon > 0$ and a C^∞ mapping $\gamma : (-2, 2) \times U \to M$, $U = \{(q, w) \in TM | \, q \in V, w \in T_qM, |w| < \epsilon\}$ such that $\gamma(t, p, w), t \in (-2, 2)$ is the unique geodesic of M which at the instant $t = 0$, passes through q with velocity w, for which $q \in V$ and for every $w \in T_qM$, with $|w| < \epsilon$.

Definition 3.1. Let $p \in M$ and V, ϵ, U as defined in the previous proposition, then the **exponential map** is defined by

$$\exp(p, v) = \gamma\left(1, p, \frac{v}{|v|}\right), \qquad \text{for all } (p, v) \in U$$

So exponential map sends a vector v to a point which is obtained by going out the length $|v|$ in the direction of v by the unique geodesic. Using this we can write exponential map as

$$\exp_p : B_\epsilon(0) \subset T_pM \to M$$

by $\exp_p(v) = \exp(p, v)$ where B_ϵ the open ball with ceneter at origin 0 of T_pM and of radius ϵ.

Definition 3.2. The **injectivity radius** at point p of the manifold (M, g) is defined to be the supremum of ϵ's for which the exponential map $\exp_p : B_\epsilon(0) \to M$ is a diffeomorphism. It is denoted by $\mathrm{inj}_M(p)$.

Further, define injectivity radius of M to be the infimum of $\mathrm{inj}_M(p)$ among all the points,

$$\mathrm{inj}M \doteqdot \inf_{p\in M} \mathrm{inj}_M(p)$$

Remark. There exists a complete Riemannian manifold with $\mathrm{inj}(M) = 0$. For sphere of radius R we know that $\mathrm{inj}(p) = \pi R$. Attach a sphere of radius $\frac{1}{2n}$ at each point $(n, 0)$on the plane $\mathbb{R}^2$ by cutting out a small open set at south pole . Then we get a (geodesic/metric) complete manifold where the infimum of the injectivity radius at the north pole of the attached spheres is 0.