Lie Groups

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One can study Lie Groups from several points of view. The course is aimed to understand the structure of Lie Groups.

Definition 1.1. A smooth manifold M is a Hausdorff space which is locally Euclidean with a smooth atlas i.e. (i) given any $x \in M$, $\exists$ a chart (U, ϕ), $x \in U \subset M$ with $\phi : U \to \phi(U)$ open in $\mathbb{R}^m$.

(ii) We have collection $\{(U, \phi)\}$ of charts such that

$$\phi(U \cap V) \xrightarrow{\psi \circ \phi^{-1}} \psi(U \cap V)$$

is a diffeomorphism.

Suppose $f : M \to N$ is a continuous map between manifolds. We say that f is smooth if for $(U, \phi) \in \amalg(M)$, $(V, \psi) \in \amalg(N)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}$ is smooth.

TO DO : Construction of tangent bundle and vector bundle

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Definition 2.1. G is a Lie group if

1. G is a smooth manifold
2. G is also a group s.t

$$\begin{aligned}\mu : G \times G &\to G \\ (g,h) &\mapsto gh\end{aligned}$$

and

$$\begin{aligned}i : G &\to G \\ g &\mapsto g^{-1}\end{aligned}$$

are smooth maps.

Definition 2.2. A real (or complex) vector space V together with a bilinear map

$$[,] : V \times V \to V$$

is called a **Lie Algebra** if

1. $[X, Y] = -[Y, X]$ - skew symmetry
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ - Jacobi identity

Example.
1. $(\mathbb{R}, +), (\mathbb{C}, +)$, V any f.d vector space over $\mathbb{R}$ or $\mathbb{C}$.
2. $(\mathbb{R}^\times, \cdot), (\mathbb{C}^\times, \cdot)$
3. $S^1 = \{z \in \mathbb{C}^\times | |z| = 1\}$
4. $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$
5. $\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}^n/\mathbb{Z}^n) \cong (S^1)^n$
6. Suppose $\Gamma \subset V$ is a discrete subgroup. Then V/Γ is a Lie group.
7. N = unipotent upper triangular matrices, B = upper triangular matrices. As manifolds $N \cong \mathbb{R}^{\binom{n}{2}}$ and $B \cong (\mathbb{R}^\times)^n \times N$.
8. $\mathrm{SL}_n(\mathbb{R}) = \{X \in \mathrm{GL}_n(\mathbb{R}) | \det X = 1\}$, $\mathrm{SL}_n(\mathbb{C})$.
9. $O(n)$, $SO(n)$.
10. $U(n)$, $SU(n)$.
11. $\mathbb{H}^\times$, S^3 with quaternion multiplication.

12. $Sp(n) = \{X \in \mathrm{GL}_n(\mathbb{R}) | X$ preserves quaternion structure as a subset of $\mathrm{Aut}_{\mathbb{H}} \mathbb{H}^n\}$

Problem. $V/\Gamma \cong \mathbb{R}^k \times (S^1)^{n-k}$ for n-dimensional vector space V.

Theorem 2.1. Suppose G is a compact, connected, simple Lie group. Then G is locally isomorphic to

1. $SU(n), n \geq 2$ denoted by A_{n-1}
2. $SO(2n+1), n \geq 2$ denoted by B_n
3. $Sp(n), n \geq 1$ denoted by C_n
4. $SO(2n), n \geq 2$ denoted by D_n

or one of the following exceptional Lie group G_2, F_4, E_6, E_7, E_8.

Problem. Prove that $\mathrm{SL}_n(\mathbb{R})$ and $O(n)$ are smooth manifold, hence Lie groups.

Examples of Lie algebra -

Example.
1. $(V, [\cdot,\cdot] \equiv 0)$ is called trivial Lie algebra.
2. $(\mathfrak{gl}_n(\mathbb{R}), [A,B] = AB - BA)$, $\mathfrak{gl}_n(\mathbb{C})$
3. $\mathfrak{sl}_n(\mathbb{R})$ $(\mathfrak{sl}_n(\mathbb{C}))$ is the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ $(\mathfrak{gl}_n(\mathbb{C}))$ consisting of trace 0.
4. $\mathfrak{so}_n$ is Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ consisting of skew-symmetric matrices.

Definition 2.3. A vector field X on a Lie group G is called left invariant if $(L_g)_*(X_h) = X_{gh}$

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Recall $\mathbb{H} = \{a + bi + cj + dk : (a,b,c,d) \in \mathbb{R}^4, i^2 = -1, j^2 = -1, k^2 = -1, ij = k, jk = l, ki = j\}$ is the quaternion division algebra with the norm

$$||a + bi + cj + dk||^2 = a^2 + b^2 + c^2 + d^2$$

which satisfies $||q_1 \cdot q_2|| = ||q_1|| \cdot ||q_2||$

We can put this multiplication on $S^3 \cong SU(2)$ to get a compact Lie group. To get the isomorphism $SU(2) \cong S^3$, we define a map

$$\begin{aligned} \phi : S^3 &\to SU(2) \\ (a,b,c,d) &\mapsto \begin{bmatrix} a+bi & c+di \\ -(c-di) & a-bi \end{bmatrix} \end{aligned}$$

which is an algebra isomorphism.

Definition 3.1. The Lie algebra of G is the space of all left-invariant vector fields on G.

We have an isomorphism

$$\begin{aligned} \mathfrak{g} = \mathrm{Lie}(G) &\to T_eG \\ X &\mapsto X_e \end{aligned}$$

Example. Let $G = \mathbb{R}^n$, with identity element $0 \in \mathbb{R}^n$ and left-invariant vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$. Then the Lie bracket is

$$[\cdot,\cdot] \equiv 0$$

Remark. In general for any abelian Lie group G, the Lie bracket is $[\cdot,\cdot] \equiv 0$.

Theorem 3.1. Let G be a connected Lie group. Then

1. $\mathrm{Lie}(G) = \mathfrak{g}$ is isomorphic as a vector space to $T_e(G)$.
2. Left-invariant vector fields are smooth.
3. $\mathrm{Lie}(G)$ is closed under Lie bracket.

Proof. 1. Let X be a left-invariant vector field on G. We need to show that Xf is smooth for each $f \in C^\infty(G)$.

$$\begin{aligned} (Xf)(g) &= X_g f \\ &= (d\lambda_g X_e) f \\ &= X_e(f \circ \lambda_g) \end{aligned}$$

To show that Xf is smooth, it suffices to show that $X_e(f \circ \lambda_g)$ is smooth. We realize $X_e(f \circ \lambda_g)$ as evaluation of a smooth function on a smooth function.

Let Y be a smooth vector field on G such that $Y_e = X_e$

$$Y_e(f \circ \lambda_g) = X_e(f \circ \lambda_g)$$

We look at λ_g as the composition of

$$\begin{aligned} G \xrightarrow{i_g^2} G \times G \xrightarrow{\mu} G \\ x \mapsto (g, x) \mapsto gx \end{aligned}$$

Regard Y as the vector field $(0, Y)$ on $G \times G$. Now

$$\begin{aligned} (0, Y)(f \circ \mu) \circ i_e^1(g) &= (0, Y)_{(g,e)}(f \circ \mu) \\ &= 0_g(f \circ \mu \circ i_g^1) + Y_e(f \circ \mu \circ i_g^2) \\ &= Y_e(f \circ \lambda_g) \end{aligned}$$

which proves the smoothness.

2. Let X, Y left-invariant vector fields on G. We must show that $[X, Y]$ is a left-invariant vector field.

$$\begin{aligned} d\lambda_g([X, Y]_e)f &= [X, Y]_g f \\ &= [X, Y]_e(f \circ \lambda_g) \\ &= X_e(Y(f \circ \lambda_g)) - Y_e(X(f \circ \lambda_g)) \\ &= X_e(d\lambda_g(Y f)) - Y_e(d\lambda_g(Y f)) \end{aligned}$$

□

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Lemma 4.1. Suppose $\psi : M \to N$ is a smooth map. Let X_1, X_2 be vector fields on M, Y_1, Y_2 be vector fields on N such that X_i is ψ-related to Y_i. Then $[X_1, X_2]$ is ψ-related to $[Y_1, Y_2]$.

Proof. Notice that

$$\begin{aligned} d\psi[X_1, X_2](f) &= [X_1, X_2](f \circ \psi) \\ &= X_1(X_2 f \circ \psi) - X_2(X_1 f \circ \psi) \\ &= X_1(d\psi X_2 f) - X_2(Y_1 f \circ \psi) \\ &= X_1(Y_2 f \circ \psi) - X_2(Y_1 f \circ \psi) \\ &= d\psi X_1(Y_2 f) - d\psi X_2(Y_1 f) \\ &= Y_1 Y_2 f \circ \psi - Y_2 Y_1 f \circ \psi \\ &= [Y_1, Y_2](f) \circ \psi \end{aligned}$$

□

This lemma proves that the set of left-invariant vector fields forms a Lie algebra.

Consider the Lie group $\mathrm{GL}_n(\mathbb{R})$. We want to verify the Lie algebra structure on $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ with the isomorphism

$$\begin{aligned} Lie(\mathrm{GL}_n(\mathbb{R})) &\to \mathfrak{gl}_n(\mathbb{R}) \\ X &\overset{\beta}{\mapsto} X_e \end{aligned}$$

Lemma 4.2.

$$\beta([X,Y]) = [\beta(X), \beta(Y)]$$

Proof. Evaluating the bracket on coordinate function x_{ij}.

$$[X,Y]_e(x_{ij}) = X_e(Y(x_{ij})) - Y_e(X(x_{ij})) \tag{1}$$

Now

$$\begin{aligned} Y(x_{ij})(g) &= d\lambda_g Y_e(x_{ij}) \\ &= Y_e(x_{ij} \circ \lambda_g) \\ &= \sum_k x_{ik}(g) Y_e(x_{kj}) \end{aligned}$$

Considering the above as function of g and substituting this in Eq. (1) we get

$$\begin{aligned} [X,Y]_e(x_{ij}) &= X_e Y_e(x_{ij}) - Y_e X_e(x_{ij}) \\ &= \sum_k \{X_e(x_{ik})Y_e(x_{kj}) - Y_e(x_{ik})X_e(x_{kj})\} \\ &= [X_e, Y_e](x_{ij}) \end{aligned}$$

□

Definition 4.1. A **Lie subgroup** H of a Lie group G is a submanifold $H \overset{\alpha}{\to} G$ where α is smooth and a group homomorphism.

We say that H is closed Lie subgroup if it is Lie subgroup such that $H \to \alpha(H)$ is a diffeomorphism.

Example. Consider the map $\mathbb{R} \to S^1 \times S^1$ given by

$$t \mapsto (e^{2\pi i t}, e^{2\pi i \sqrt{2} t})$$

The image is a Lie subgroup of $S^1 \times S^1$ but it is not a closed Lie subgroup. It is also known as "Skew-line" in the torus.

Definition 4.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $f : \mathfrak{g} \to \mathfrak{h}$ be a vector space homomorphism. Then we say that f is a Lie algebra homomorphism if

$$f([X,Y]) = [f(X), f(Y)]$$

Theorem 4.3. Suppose that $\psi : G \to H$ is a Lie group homomorphism. Let X be a left-invariant vector field on G. Extend $d\psi(X_e) = Y_e \in T_eH$ to a left-invariant vector field Y on H. Then X and Y are ψ-related. This implies $d\psi_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd} TG \arrow[r, "d\psi"] \arrow[d] & TH \arrow[d] \\ G \arrow[r, "\psi"] \arrow[u, bend left, "X"] & H \arrow[u, bend right, "Y"'] \end{tikzcd}$$

We want to show that $Y \circ \psi = d\psi \circ Y$. Now

$$\lambda_{\psi(g)} \circ \psi = \psi \circ \lambda_g$$

so

$$\begin{aligned} Y_{\psi(g)} &= d\lambda_{\psi(g)} Y_e \\ &= d\lambda_{\psi(g)} d\psi X_e \\ &= d(\lambda_{\psi(g)} \circ \psi)(X_e) \\ &= d(\psi \circ \lambda_g)(X_e) \\ &= d\psi d\lambda_g (X_e) \\ &= d\psi X_g \end{aligned}$$

□

Theorem 4.4. Let G, H be Lie groups with G connected. Let

$$\phi, \psi : G \to H$$

be homomorphism of Lie groups such that

$$d\phi = d\psi : T_eG \to T_eH$$

Then $\phi = \psi$.