Today I Tried

Devesh Rajpal

April 2024

(15/4/24) Today I tried to cast the self similar solutions of α-Gauss flow in the Monge-Ampere type using support function. The equation $\langle X(x), \nu(x)\rangle = cK^{\alpha}$ can be written as

$$h = c\left(\frac{\det(\overline{\nabla}^2 h + \overline{g}h)}{\det(\overline{g})}\right)^{-\alpha}$$

where $\overline{\nabla}^2 h$ is the 2-tensor defined using the standard connection on S^n. It is easy to calculate

$$\overline{\nabla}_i \overline{\nabla}_j h = \partial_i \partial_j h - (\overline{\nabla}_i \partial_j) h$$

(16/4/24) Today I tried spherical coordinates on the α-Gauss flow. In the parametrization $x = \cos\theta\cos\phi, y = \cos\theta\sin\phi, z = \cos\theta$, we have

$$\overline{g} = \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta \end{bmatrix}$$

and

$$K^{-1} = \frac{\det\left(\begin{bmatrix} h_{\theta\theta} & h_{\theta\phi} - \frac{\tan\theta}{2}h_{\phi} \\ h_{\theta\phi} - \frac{\tan\theta}{2}h_{\phi} & h_{\phi\phi} - \frac{\cos\theta}{2}h_{\theta} \end{bmatrix} + h\begin{bmatrix} 1 & 0 \\ 0 & \cos\theta \end{bmatrix}\right)}{\cos\theta}$$

which is quite ugly.

(17/4/24) Today I learned about the homogeneous degree 1 extension of the support function. Let $h : S^n \to \mathbb{R}$ be the support function of a strictly convex hypersurface. We extend this to $H : \mathbb{R}^{n+1} \to \mathbb{R}$ by defining,

$$H(x) = |x| h\left(\frac{x}{|x|}\right).$$

Note that H is just continuous but not necessarily differentiable at 0. It is easy to see that $DH(\lambda x) = DH(x)$. Let $x \in \mathbb{R}^{n+1}$ and v be a unit vector,

$$\begin{aligned} D_v H(x) &= h\left(\frac{x}{|x|}\right) D_v|x| + |x| D_v h\left(\frac{x}{|x|}\right) \\ &= \frac{v \cdot x}{|x|} h\left(\frac{x}{|x|}\right) + |x| \overline{\nabla}_{v^T} h\left(\frac{x}{|x|}\right) \end{aligned}$$

where $\overline{\nabla}_{v^T}$ is the covariant derivative on S^n in the direction $v^T \in T_x S^n$. Let $x \in S^n$ and substitute $v \in \{e_1, \ldots, e_{n+1}\}$ to get

$$DH(x) = xh(x) + \overline{\nabla}h(x)$$

which is the inverse of Gauss map! Thus, $G^{-1}(x) = DH(x)$, and also weirdly $D_x H(x) = DH(x)$ so the steepest accent is in the normal direction.

(18/4/24) Today I learned about a possible reducible symmetric group to try to construct self-similar solutions of the α-Gauss curvature flow. As considered previously the setup is with support functions. The sphere $h \equiv 1$ is an equilibrium point of the normalized α-Gauss flow. The construction of Γ symmetric solutions in Ben's paper is using spherical harmonics (eigenfunctions of the Laplacian) and some general version of the stable/unstable manifold theorem. The linearized version of normalized α-Gauss flow at $h \equiv 1$ is given by

$$\frac{\partial u}{\partial t} = \alpha(\Delta + n)u + u$$

so if $\Delta\psi = -\lambda\psi$, then $h \equiv 1$ is strictly unstable in the direction ψ (what does this really mean?). Another important fact is that entropy is a min for unstable direction, the Hessian of entropy at $h \equiv 1$ satisfies

$$D^2 Z_h(\psi, \psi) > 0.$$

A new idea is to use an affine boost along with the spherical harmonics to possible control the isoperimetric ratio. The considered example of a reducible group was generated by $x \mapsto -x$ and a 3-fold rotation symmetric group in yz plane along with reflection of the triangle (so the dihedral group D_3 in the yz plane). Consider a one-parameter family of affine transformations which stretches the x-direction,

$$T_\lambda = \begin{bmatrix} e^{2\lambda} & 0 & 0 \\ 0 & e^{-\lambda} & 0 \\ 0 & 0 & e^{-\lambda} \end{bmatrix}.$$

Now we can consider a one parameter family of flows produced by the unstable direction $\epsilon\psi + T_\lambda$ and the expectation is that since the entropy of the solutions with $\lambda = 0, \infty$ is ∞ (to check) one can possibly use a mountain pass theorem (in homotopies of λ variable) to create a critical point which will be a self-similar solution of the flow.

(19/4/24) Today I learned about the α-Gaussin entropy of a bounded convex hypersurface. In the book it is defined as

$$E_\alpha(\mathcal{M}) \doteqdot \begin{cases} \left(\frac{\mathrm{Vol}(\mathcal{M}^n)}{|B^{n+1}|}\right)^{\frac{n}{n+1}} \exp\left(\frac{1}{|S^n|}\int_{\mathcal{M}^n} K \log K d\mu\right) & \text{if } \alpha = 1 \\ \left(\frac{\mathrm{Vol}(\mathcal{M}^n)}{|B^{n+1}|}\right)^{\frac{n}{n+1}} \left(\frac{1}{|S^n|}\int_{\mathcal{M}^n} K^\alpha d\mu\right)^{\frac{1}{\alpha-1}} & \text{if } \alpha \neq 1 \end{cases}$$

It turns out that the α-Gaussian entropy is non-increasing under α-Gauss flow. The next task is to understand its property on normalized α-Gauss flow.

(22/4/24) Today I learned about the Brunn-Minkowski inequality. It states that for convex bodies $A, B \subset \mathbb{R}^n$ and $\lambda \in [0,1]$, we have

$$\mathrm{Vol}(\lambda A + (1-\lambda)B)^{\frac{1}{n}} \geq \lambda \mathrm{Vol}(A)^{\frac{1}{n}} + (1-\lambda)\mathrm{Vol}(B)^{\frac{1}{n}}$$

which is same as saying that $\mathrm{Vol}(\,\cdot\,)^{\frac{1}{n}}$ is a concave function on the set of convex bodies.

(23/4/24) Today I learned about the proof of monotonicity of α-Gaussian entropy under α-Gaussian flow using Brunn-Minkowski inequality.

(24/4/24) Today I tried to finish the monotonicity of entropy proof.

(29/4/24) Today I learned about Jacobi fields in the general context of calculus of variations. Let $I(u) = \int F(t, u(t), \dot{u}(t))$ which we want to minimize.

May 2024

(3/5/24) Today I learned about Mountain Pass Theorem(MPT).

(7/5/24) Today I am writing equations for nomalized α-Gauss curvature flow for once and all. Let $X(t)$ be a solution of the α-GCF, so that

$$\partial_t X = -K^\alpha \nu.$$

The evolution of volume under α-GCF is given by

$$\frac{d}{dt} V(X(t)) = -\int_{S^n} K^{\alpha-1} d\overline{\mu}.$$

We normalize the the equation for the scaled hypersurface to have constant volume $|B^{n+1}|$, by defining

$$\tilde{X}(\tau(t)) = \left(\frac{|B^{n+1}|}{V(X(t))}\right)^{\frac{1}{n+1}} X(t)$$

where τ is some parametrization of time to be defined later to make equations cleaner. Note that by scaling, the new Gauss curvature is $\tilde{K} = \left(\frac{|B^{n+1}|}{V(X(t))}\right)^{-\frac{n}{n+1}} K$. Differentiating with respect to τ,

$$\begin{aligned}
\frac{\partial \tilde{X}}{\partial \tau} &= \frac{\partial \tilde{X}}{\partial t}\frac{dt}{d\tau} \\
&= \left[-\left(\frac{|B^{n+1}|}{V(X(t))}\right)^{\frac{1}{n+1}} K^\alpha \nu + \frac{\tilde{X}}{(n+1)V}\int_{S^n} K^{\alpha-1} d\overline{\mu}\right]\frac{dt}{d\tau} \\
&= \left[-\left(\frac{|B^{n+1}|}{V(X(t))}\right)^{\frac{1+\alpha n}{1+n}} \tilde{K}^\alpha \nu + \frac{\tilde{X}}{(n+1)V}\left(\frac{|B^{n+1}|}{V(X(t))}\right)^{\frac{(\alpha-1)n}{n+1}} \int_{S^n} \tilde{K}^{\alpha-1}\right]\frac{dt}{d\tau} \\
&= \left[-\tilde{K}^\alpha \nu + \frac{\tilde{X}}{(n+1)|B^{n+1}|}\int_{S^n} \tilde{K}^{\alpha-1} d\overline{\mu}\right]\left(\frac{|B^{n+1}|}{V(X(t))}\right)^{-\frac{(1+\alpha n)}{1+n}} \frac{dt}{d\tau}.
\end{aligned}$$

Choose $\tau(t)$ such that the term outside square bracket is 1 so that the new evolution equation is

$$\begin{aligned}\frac{\partial \tilde{X}}{\partial \tau} &= -\tilde{K}^{\alpha}\nu + \frac{\tilde{X}}{(n+1)|B^{n+1}|}\int_{S^n}\tilde{K}^{\alpha-1}d\overline{\mu} \\ &= -\tilde{K}^{\alpha}\nu + \frac{\tilde{X}}{|S^n|}\int_{S^n}\tilde{K}^{\alpha-1}d\overline{\mu} = -\tilde{K}^{\alpha}\nu + \tilde{X}\fint_{S^n}\tilde{K}^{\alpha-1}d\overline{\mu}\end{aligned}$$

Let $\tilde{h}(z) = \sup\{p\cdot z : p\in\tilde{X}\} = \left(\frac{|B^{n+1}|}{V(X(t))}\right)^{\frac{1}{n+1}} h(z)$ be the support function of the rescaled hypersurface. Its evolution equation is

$$\begin{aligned}\frac{\partial \tilde{h}}{\partial \tau} &= -\tilde{K}^{\alpha} + \frac{\tilde{h}}{|S^n|}\int_{S^n}\tilde{K}^{\alpha-1}d\overline{\mu} \\ &= -\det_{\overline{g}}(\overline{\nabla}^2\tilde{h}+\overline{g}\tilde{h})^{-\alpha} + \frac{\tilde{h}}{|S^n|}\int_{S^n}\det_{\overline{g}}(\overline{\nabla}^2\tilde{h}+\overline{g}\tilde{h})^{1-\alpha}d\overline{\mu}\end{aligned}$$

where we have used the fact that Gauss curvature can be determined from the support function by the formula

$$K = \det_{\overline{g}}(\overline{\nabla}^2\tilde{h}+\overline{g}\tilde{h})^{-1}.$$

(9/5/24) Today I learned about some rigidity results of ancient solutions of α-GCF. If $X_t = \phi(t)X_0$ is a family of hypersurfaces moving homothetically, then under α-GCF it satisfies

$$K^{\alpha}[X_0] = -\lambda\langle X_0,\nu_0\rangle \quad \text{with} \quad \phi'(t)\phi^{n\alpha}(t) = \lambda.$$

If $\lambda > 0$ it is called **expanding solution** and if $\lambda < 0$ it is called **contracting/shrinking solution**.

In paper *Complete noncompact self-similar solutions of Gauss curvature flows I. Positive powers*, Urbas proves that all complete non-compact convex hypersurfaces which move homothetically under α-GCF for some $\alpha\in(0,\infty)$ satisfy the previous equation and further for contracting solutions ($\lambda<0$), X_0 is a hyperplane through the origin.

(10/5/24) Today I learned about Hersch's theorem on first eigenvalue of the Laplacian on a sphere (S^2, g). It states that given a Riemmanian metric g on sphere, the first non-zero eigenvalue of negative Laplacian $-\Delta_g(= -g^{ij}\nabla_i\nabla_j)$ satisfies

$$\lambda_1 \le \frac{8\pi}{\text{Area}(S^2,g)}.$$

The full proof is present in Schoen, Yau's *Lectures on Differential Geometry*. It centrally use the fact that there is only one conformal class of metrics in S^2 (uniformization theorem I guess) and then finds a nice conformal metric of g where the center of mass is at the origin.

(14/5/24) Today I learned about stable manifold theorem. The proof is based on Banach contraction theorem like in the case of Picard-Lindelof theorem. In Picard-Lindelof the idea was to define an integral operator which is a contraction and the fixed point of the operator solves the ODE. The idea here is very similar but it takes more effort to the set up the appropriate Banach space and define the operator and then prove that it is a contraction.