Today I Tried

Devesh Rajpal

April 2024

(15/4/24) Today I tried to cast the self similar solutions of α-Gauss flow in the Monge ampere type using support function. The equation $\langle X(x), \nu(x)\rangle = cK^\alpha$ can be written as

$$h = c\left(\frac{\det(\overline{\nabla}^2 h + \overline{g}h)}{\det(\overline{g})}\right)^{-\alpha}$$

where $\overline{\nabla}^2 h$ is the 2-tensor defined using the standard connection on S^n. It is easy to calculate

$$\overline{\nabla}_i \overline{\nabla}_j h = \partial_i \partial_j h - (\overline{\nabla}_i \partial_j)h$$

(16/4/24) Today I tried spherical coordinates on the α-Gauss flow. In the parametrization $x = \cos\theta\cos\phi, y = \cos\theta\sin\phi, z = \cos\theta$, we have

$$\overline{g} = \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta \end{bmatrix}$$

and

$$K^{-1} = \frac{\det\left(\begin{bmatrix} h_{\theta\theta} & h_{\theta\phi} - \frac{\tan\theta}{2}h_\phi \\ h_{\theta\phi} - \frac{\tan\theta}{2}h_\phi & h_{\phi\phi} - \frac{\cos\theta}{2}h_\theta \end{bmatrix} + h\begin{bmatrix} 1 & 0 \\ 0 & \cos\theta \end{bmatrix}\right)}{\cos\theta}$$

which is quite ugly.

(17/4/24) Today I learned about the homogeneous degree 1 extension of the support function. Let $h : S^n \to \mathbb{R}$ be the support function of a strictly convex hypersurface. We extend this to $H : \mathbb{R}^{n+1} \to \mathbb{R}$ by defining,

$$H(x) = |x| h\left(\frac{x}{|x|}\right).$$

Note that H is just continuous but not necessarily differentiable at 0. It is easy to see that $DH(\lambda x) = DH(x)$. Let $x \in \mathbb{R}^{n+1}$ and v be a unit vector,

$$\begin{aligned} D_v H(x) &= h\left(\frac{x}{|x|}\right) D_v|x| + |x| D_v h\left(\frac{x}{|x|}\right) \\ &= \frac{v\cdot x}{|x|} h\left(\frac{x}{|x|}\right) + |x|\overline{\nabla}_{v^T} h\left(\frac{x}{|x|}\right) \end{aligned}$$

where $\overline{\nabla}_{v^T}$ is the covariant derivative on S^n in the direction $v^T \in T_x S^n$. Let $x \in S^n$ and substitute $v \in \{e_1, \ldots, e_{n+1}\}$ to get

$$DH(x) = xh(x) + \overline{\nabla} h(x)$$

which is the inverse of Gauss map! Thus, $G^{-1}(x) = DH(x)$, and also weirdly $D_x H(x) = DH(x)$ so the steepest accent is in the normal direction.

(18/4/24) Today I learned about a possible reducible symmetric group to try to construct self-similar solutions of the α-Gauss curvature flow. As considered previously the setup is with support functions. The sphere $h \equiv 1$ is an equilibrium point of the normalized α-Gauss flow. The construction of Γ symmetric solutions in Ben's paper is using spherical harmonics (eigenfunctions of the Laplacian) and some general version of the stable/unstable manifold theorem. The linearized version of normalized α-Gauss flow at $h \equiv 1$ is given by

$$\frac{\partial u}{\partial t} = \alpha(\Delta + n)u + u$$

so if $\Delta \psi = -\lambda \psi$, then $h \equiv 1$ is strictly unstable in the direction ψ (what does this really mean?). Another important fact is that entropy is a min for unstable direction, the Hessian of entropy at $h \equiv 1$ satisfies

$$D^2 Z_h(\psi, \psi) > 0.$$

The new idea is to use an affine boost along with the spherical harmonics to possible control the isoperimetric ratio. The considered example of a reducible group was generated by $x \mapsto -x$ and a 3-fold rotation symmetric group in yz plane along with reflection of the triangle (so the dihedral group D_3 in the yz plane). Consider a one-parameter family of affine transformations which stretches the x-direction,

$$T_\lambda = \begin{bmatrix} e^{2\lambda} & 0 & 0 \\ 0 & e^{-\lambda} & 0 \\ 0 & 0 & e^{-\lambda} \end{bmatrix}.$$

Now we can consider a one parameter family of flows produced by the unstable direction $\epsilon\psi + T_\lambda$ and the expectation is that since the entropy of the solutions with $\lambda = 0, \infty$ is ∞ (to check) one can possibly use a mountain pass theorem (in homotopies of λ variable) to create a critical point which will be a self-similar solution of the flow.

(19/4/24) Today I learned about the α-Gaussin entropy of a bounded convex hypersurface. In the book it is defined as

$$E_\alpha(\mathcal{M}) \doteqdot \begin{cases} \left(\frac{\mathrm{Vol}(\mathcal{M}^n)}{|B^{n+1}|}\right)^{\frac{n}{n+1}} \exp\left(\frac{1}{|S^n|}\int_{\mathcal{M}^n} K \log K d\mu\right) & \text{if } \alpha = 1 \\ \left(\frac{\mathrm{Vol}(\mathcal{M}^n)}{|B^{n+1}|}\right)^{\frac{n}{n+1}} \left(\frac{1}{|S^n|}\int_{\mathcal{M}^n} K^\alpha d\mu\right)^{\frac{1}{\alpha-1}} & \text{if } \alpha \neq 1 \end{cases}$$

It turns out that the α-Gaussian entropy is non-increasing under α-Gauss flow. The next task is to understand its property on normalized α-Gauss flow.

(22/4/24) Today I learned about the Brunn-Minkowski inequality. It states that for convex bodies $A, B \subset \mathbb{R}^n$ and $\lambda \in [0,1]$, we have

$$\mathrm{Vol}(\lambda A + (1-\lambda)B)^{\frac{1}{n}} \geq \lambda \mathrm{Vol}(A)^{\frac{1}{n}} + (1-\lambda)\mathrm{Vol}(B)^{\frac{1}{n}}$$

which is same as saying that $\mathrm{Vol}(\,\cdot\,)^{\frac{1}{n}}$ is a concave function on the set of convex bodies.

(23/4/24) Today I learned about the proof of monotonicity of α-Gaussian entropy under α-Gaussian flow using Brunn-Minkowski inequality, the proof is really slick.