Lie Groups

Devesh Rajpal

Contents

1 4th January 23

One can study Lie Groups from several points of view. The course is aimed to understand the structure of Lie Groups.

Definition 1.1. A smooth manifold M is a Hausdorff space which is locally Euclidean with a smooth atlas i.e. (i) given any $x \in M$, $\exists$ a chart (U, ϕ), $x \in U \subset M$ with $\phi : U \to \phi(U)$ open in $\mathbb{R}^m$.
   (ii) We have collection $\{(U, \phi)\}$ of charts such that

$$\phi(U \cap V) \xrightarrow{\psi \circ \phi^{-1}} \psi(U \cap V)$$

is a diffeomorphism.

   Suppose $f : M \to N$ is a continuous map between manifolds. We say that f is smooth if for $(U, \phi) \in \mathrm{II}(M)$, $(V, \psi) \in \mathrm{II}(N)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}$ is smooth.
   TO DO : Construction of tangent bundle and vector bundle

2 9th Jan 2023

Definition 2.1. G is a Lie group if

1. G is a smooth manifold

2. G is also a group s.t

$$\mu : G \times G \to G$$
$$(g, h) \mapsto gh$$

and

$$i : G \to G$$
$$g \mapsto g^{-1}$$

are smooth maps.

Definition 2.2. A real (or complex) vector space V together with a bilinear map

$$[,] : V \times V \to V$$

is called a **Lie Algebra** if

1. $[X, Y] = -[Y, X]$ - skew symmetry

2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ - Jacobi identity

Example.   1. $(\mathbb{R}, +), (\mathbb{C}, +), V$ any f.d vector space over $\mathbb{R}$ or $\mathbb{C}$.

2. $(\mathbb{R}^\times, \cdot), (\mathbb{C}^\times, \cdot)$

3. $S^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$

4. $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$

5. $\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}^n/\mathbb{Z}^n) \cong (S^1)^n$

6. Suppose $\Gamma \subset V$ is a discrete subgroup. Then V/Γ is a Lie group.

7. $N =$ unipotent upper triangular matrices, $B =$ upper triangular matrices. As manifolds $N \cong \mathbb{R}^{\binom{n}{2}}$ and $B \cong (\mathbb{R}^\times)^n \times N$.

8. $\mathrm{SL}_n(\mathbb{R}) = \{X \in \mathrm{GL}_n(\mathbb{R}) \mid \det X = 1\}$, $\mathrm{SL}_n(\mathbb{C})$.

9. $O(n)$, $SO(n)$.

10. $U(n)$, $SU(n)$.

11. $\mathbb{H}^\times$, S^3 with quaternion multiplication.

12. $Sp(n) = \{X \in \mathrm{GL}_n(\mathbb{R}) \mid X \text{ preserves quaternion structure as a subset of } \mathrm{Aut}_{\mathbb{H}} \mathbb{H}^n\}$

Problem. $V/\Gamma \cong \mathbb{R}^k \times (S^1)^{n-k}$ for n-dimensional vector space V.

Theorem 2.1. Suppose G is a compact, connected, simple Lie group. Then G is locally isomorphic to

1. $SU(n), n \geq 2$ denoted by A_{n-1}

2. $SO(2n + 1), n \geq 2$ denoted by B_n

3. $Sp(n), n \geq 1$ denoted by C_n

4. $SO(2n), n \geq 2$ denoted by D_n

or one of the following exceptional Lie group G_2, F_4, E_6, E_7, E_8.

Problem. Prove that $\mathrm{SL}_n(\mathbb{R})$ and $O(n)$ are smooth manifold, hence Lie groups.

Examples of Lie algebra -

Example.   1. $(V, [\cdot, \cdot] \equiv 0)$ is called trivial Lie algebra.

2. $(\mathfrak{gl}_n(\mathbb{R}), [A, B] = AB - BA)$, $\mathfrak{gl}_n(\mathbb{C})$

3. $\mathfrak{sl}_n(\mathbb{R})$ $(\mathfrak{sl}_n(\mathbb{C}))$ is the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ $(\mathfrak{gl}_n(\mathbb{C}))$ consisting of trace 0.

4. $\mathfrak{so}_n$ is Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ consisting of skew-symmetric matrices.

Definition 2.3. A vector field X on a Lie group G is called left invariant if $(L_g)_*(X_h) = X_{gh}$

3 11th Jan 2023

Recall $\mathbb{H} = \{a + bi + cj + dk : (a,b,c,d) \in \mathbb{R}^4, i^2 = -1, j^2 = -1, k^2 = -1, ij = k, jk = l, ki = j\}$ is the quaternion division algebra with the norm

$$||a + bi + cj + dk||^2 = a^2 + b^2 + c^2 + d^2$$

which satisfies $||q_1 \cdot q_2|| = ||q_1|| \cdot ||q_2||$

We can put this multiplication on $S^3 \cong SU(2)$ to get a compact Lie group. To get the isomorphism $SU(2) \cong S^3$, we define a map

$$\phi : S^3 \to SU(2)$$

$$(a,b,c,d) \mapsto \begin{bmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{bmatrix}$$

which is an algebra isomorphism.

> **Definition 3.1.** The Lie algebra of G is the space of all left-invariant vector fields on G.

We have an isomorphism

$$\mathfrak{g} = \mathrm{Lie}(G) \to T_e G$$
$$X \mapsto X_e$$

> **Example.** Let $G = \mathbb{R}^n$, with identity element $0 \in \mathbb{R}^n$ and left-invariant vector fields $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$. Then the Lie bracket is
>
> $$[\cdot, \cdot] \equiv 0$$

> **Remark.** In general for any abelian Lie group G, the Lie bracket is $[\cdot, \cdot] \equiv 0$.

> **Theorem 3.1.** Let G be a connected Lie group. Then
>
> 1. $\mathrm{Lie}(G) = \mathfrak{g}$ is isomorphic as a vector space to $T_e(G)$.
>
> 2. Left-invariant vector fields are smooth.
>
> 3. $\mathrm{Lie}(G)$ is closed under Lie bracket.
>
> **Proof.** 1. Let X be a left-invariant vector field on G. We need to show that Xf is smooth for each $f \in C^\infty(G)$.
>
> $$\begin{aligned} (Xf)(g) &= X_g f \\ &= (d\lambda_g X_e) f \\ &= X_e(f \circ \lambda_g) \end{aligned}$$
>
> To show that Xf is smooth, it suffices to show that $X_e(f \circ \lambda_g)$ is smooth. We realize $X_e(f \circ \lambda_g)$ as evaluation of a smooth function on a smooth function.

Let Y be a smooth vector field on G such that $Y_e = X_e$

$$Y_e(f \circ \lambda_g) = X_e(f \circ \lambda_g)$$

We look at λ_g as the composition of

$$G \xrightarrow{i_g^2} G \times G \xrightarrow{\mu} G$$
$$x \mapsto (g, x) \mapsto gx$$

Regard Y as the vector field $(0, Y)$ on $G \times G$. Now

$$
\begin{aligned}
(0, Y)(f \circ \mu) \circ i_e^1(g) &= (0, Y)_{(g,e)}(f \circ \mu) \\
&= 0_g(f \circ \mu \circ i_g^1) + Y_e(f \circ \mu \circ i_g^2) \\
&= Y_e(f \circ \lambda_g)
\end{aligned}
$$

which proves the smoothness.

2. Let X, Y left-invariant vector fields on G. We must show that $[X, Y]$ is a left-invariant vector field.

$$
\begin{aligned}
d\lambda_g([X, Y]_e)f &= [X, Y]_g f \\
&= [X, Y]_e(f \circ \lambda_g) \\
&= X_e(Y(f \circ \lambda_g)) - Y_e(X(f \circ \lambda_g)) \\
&= X_e(d\lambda_g(Yf)) - Y_e(d\lambda_g(Yf))
\end{aligned}
$$

$\square$

4 18 Jan 2023

Lemma 4.1. Suppose $\psi : M \to N$ is a smooth map. Let X_1, X_2 be vector fields on M, Y_1, Y_2 be vector fields on N such that X_i is ψ-related to Y_i. Then $[X_1, X_2]$ is ψ-related to $[Y_1, Y_2]$.

Proof. Notice that

$$
\begin{aligned}
d\psi[X_1, X_2](f) &= [X_1, X_2](f \circ \psi) \\
&= X_1(X_2 f \circ \psi) - X_2(X_1 f \circ \psi) \\
&= X_1(d\psi X_2 f) - X_2(Y_1 f \circ \psi) \\
&= X_1(Y_2 f \circ \psi) - X_2(Y_1 f \circ \psi) \\
&= d\psi X_1(Y_2 f) - d\psi X_2(Y_1 f) \\
&= Y_1 Y_2 f \circ \psi - Y_2 Y_1 f \circ \psi \\
&= [Y_1, Y_2](f) \circ \psi
\end{aligned}
$$

$\square$

This lemma proves that the set of left-invariant vector fields forms a Lie algebra.

Consider the Lie group $\mathrm{GL}_n(\mathbb{R})$. We want to verify the Lie algebra structure on $\mathfrak{gl}_n(\mathbb{R}) =$

$M_n(\mathbb{R})$ with the isomorphism

$$Lie(\mathrm{GL}_n(\mathbb{R})) \to \mathfrak{gl}_n(\mathbb{R})$$
$$X \xmapsto{\beta} X_e$$

Lemma 4.2.
$$\beta([X,Y]) = [\beta(X), \beta(Y)]$$

Proof. Evaluating the bracket on coordinate function x_{ij}.

$$[X,Y]_e(x_{ij}) = X_e(Y(x_{ij})) - Y_e(X(x_{ij})) \tag{1}$$

Now

$$\begin{aligned}
Y(x_{ij})(g) &= d\lambda_g Y_e(x_{ij}) \\
&= Y_e(x_{ij} \circ \lambda_g) \\
&= \sum_k x_{ik}(g) Y_e(x_{kj})
\end{aligned}$$

Considering the above as function of g and substituting this in Eq. (1) we get

$$\begin{aligned}
[X,Y]_e(x_{ij}) &= X_e Y_e(x_{ij}) - Y_e X_e(x_{ij}) \\
&= \sum_k \{X_e(x_{ik})Y_e(x_{kj}) - Y_e(x_{ik})X_e(x_{kj})\} \\
&= [X_e, Y_e](x_{ij})
\end{aligned}$$

$\square$

Definition 4.1. A **Lie subgroup** H of a Lie group G is a submanifold $H \xrightarrow{\alpha} G$ where α is smooth and a group homomorphism.

We say that H is closed Lie subgroup if it is Lie subgroup such that $H \to \alpha(H)$ is a diffeomorphism.

Example. Consider the map $\mathbb{R} \to S^1 \times S^1$ given by

$$t \mapsto (e^{2\pi it}, e^{2\pi i \sqrt{2} t})$$

The image is a Lie subgroup of $S^1 \times S^1$ but it is not a closed Lie subgroup. It is also known as "Skew-line" in the torus.

Definition 4.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $f : \mathfrak{g} \to \mathfrak{h}$ be a vector space homomorphism. Then we say that f is a Lie algebra homomorphism if

$$f([X,Y]) = [f(X), f(Y)]$$

Theorem 4.3. Suppose that $\psi : G \to H$ is a Lie group homomorphism. Let X be a left-invariant vector field on G. Extend $d\psi(X_e) = Y_e \in T_e H$ to a left-invariant vector field Y on H. Then X and Y are ψ-related. This implies $d\psi_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
TG & \xrightarrow{\;\;d\psi\;\;} & TH \\[2pt]
X \Big\uparrow\Big\downarrow & & \Big\uparrow\Big\downarrow Y \\[2pt]
G & \xrightarrow{\;\;\psi\;\;} & H
\end{array}
$$

We want to show that $Y \circ \psi = d\psi \circ Y$. Now

$$
\lambda_{\psi(g)} \circ \psi = \psi \circ \lambda_g
$$

so

$$
\begin{aligned}
Y_{\psi(g)} &= d\lambda_{\psi(g)} Y_e \\
&= d\lambda_{\psi(g)} d\psi X_e \\
&= d(\lambda_{\psi(g)} \circ \psi)(X_e) \\
&= d(\psi \circ \lambda_g)(X_e) \\
&= d\psi \, d\lambda_g(X_e) \\
&= d\psi X_g
\end{aligned}
$$

$\square$

Theorem 4.4. Let G, H be Lie groups with G connected. Let

$$
\phi, \psi : G \to H
$$

be homomorphism of Lie groups such that

$$
d\phi = d\psi : T_e G \to T_e H
$$

Then $\phi = \psi$.

5 23 Jan 2023

I missed the class. Regardless here are some definitions from Warner covered on this day.

Definition 5.1. Let M be a smooth d-dimensional manifold. For any integer $1 \le c \le d$, a **c-dimensional distribution** $\mathscr{D}$ on manifold is a choice of c-dimensional subspace $\mathscr{D}_p \subset T_p M$. $\mathscr{D}$ is smooth if for each $p \in M$ there is an open neighborhood U of p and there are c smooth vector fields $X_1, \ldots, X_c$ on U which span $\mathscr{D}_m$ for each $p \in U$.
  We say $\mathscr{D}$ is **involutive** if $[X, Y] \in \mathscr{D}$ whenever $X, Y \in \mathscr{D}$.

Definition 5.2. A submanifold (N, ϕ) of M is an integral manifold of a distribution $\mathscr{D}$ if

$$d\phi(N_p) = \mathscr{D}_{\phi(p)}$$

Suppose there exists an integral manifold N for a distribution $\mathscr{D}$, then for the points on N the distribution $\mathscr{D}$ is necessarily involutive. Frobenius theorem states that it is sufficient condition for a distribution to be integral.

Theorem 5.1. (Frobenius) Let $\mathscr{D}$ be a c-dimensional involutive smooth distribution on M. Then there exists an integral manifold of $\mathscr{D}$ passing through each point of M.

Differential Ideals

Let $E^*(M) = \bigoplus_{i=0}^{\infty} E^i(M)$ denote the graded algebra of smooth differential forms over manifold M.

Definition 5.3. Let $\mathscr{D}$ be a smooth p-dimensional distribution on M. A q-form ω is said to **annihilate** $\mathscr{D}$ if for each $x \in M$

$$\omega_x(v_1, \ldots, v_q) = 0 \quad \text{whenever } v_1, \ldots, v_q \in \mathscr{D}_x$$

A form $\omega \in E^*(M)$ is said to annihilate $\mathscr{D}$ if each of the homogenous components of ω annihilate $\mathscr{D}$. Define

$$\mathscr{I}(\mathscr{D}) \doteq \{\omega \in E^*(M) : \omega \text{ annihilates } \mathscr{D}\}$$

Definition 5.4. An ideal $\mathscr{I} \in E^*(M)$ is called a **differential ideal** if it is closed under exterior differentiation; i.e.

$$d(\mathscr{I}) \subset \mathscr{I}.$$

Theorem 5.2. A smooth distribution $\mathscr{D}$ on M is involutive if and only if the ideal $\mathscr{I}(\mathscr{D})$ is a differential ideal.

6 25 Jan 2023

Theorem 6.1. If $\phi : H \to G$ is a homomorphism of Lie groups and if ω is a left-invariant differential form on G, then $\phi^*(\omega)$ is again a left-invariant form on H.

Suppose that $\phi : H \to G$ is a homomorphism of Lie groups. Let $\omega_1, \ldots, \omega_d$ be a basis for $E^1_{\text{inv}}(G)$. Then

$$\mathcal{I}_\phi = \langle \{\pi_1^* \phi^*(\omega_j) - \pi_2^*(\omega_j)\} : 1 \leq j \leq d \rangle$$

is a left-invariant differential ideal of $H \times G$.

Lemma 6.2. Suppose $X_1, \ldots, X_d$ is a basis of $\mathfrak{g}$ dual to $\omega_1, \ldots, \omega_d$. Suppose the Lie bracket

is given by

$$[X_i, X_j] = \sum c_{ij}^k X_k$$

Then the C^∞ functions c_{ij}^k are constant. Further,

$$d\omega_i = -c_{kj}^i \omega_k \wedge \omega_j$$

Proof. Notice that

$$d\omega_k(X_i, X_j) = -\omega_k([X_i, X_j])$$
$$= -c_{ij}^k$$

which is a constant because a left-invariant 1-form evaluated on a left-invariant vector field is a constant. $\square$

Remark. The c_{ij}^k are called the structural constants of G with respect to the basis $\{X_i\}$ of $\mathfrak{g}$.

Proof. Theorem 4.4. Notice that $\mathcal{I}_\psi = \mathcal{I}_\phi$ since $d\phi = d\psi$ and these are invariant differential ideals; hence integral manifolds of $\mathcal{I}_\phi$ and $\mathcal{I}_\psi$ passing through (e, e) are the same. Thus, $\phi = \psi$. $\square$

Lemma 6.3. Suppose G is any Hausdorff topological group which is connected. Suppose $e \in U \subset G$ is any open set. Then

$$G = \bigcup_{n \geq 1} U^n$$

where $U^n = \{x_1 \cdots x_n | x_i \in U\}$

Proof. Since $e \in U$ is open, $U^{-1} = \{x^{-1} | x \in U\}$ is also an open neighborhood of e. Let $V = U \cap U^{-1}$. Note that

$$H \doteq \bigcup_{n \geq 1} V^n$$

is a subgroup of G, and it is open. Since the cosets gH are also open it follows that $G = \cup_g H$ being connected must be H. $\square$

Theorem 6.4. Let G be a Lie group and $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists connected Lie subgroup H of G such that $T_e H = \mathfrak{h}$.

Proof. Consider the distribution $\mathscr{D}$ defined as

$$\mathscr{D}_g = \{X_g | X \in \mathfrak{h}\}$$

on G. Suppose $X_1, \ldots, X_c$ is a basis of $\mathfrak{h}$. Then $\mathscr{D}$ is generated by $X_1, \ldots, X_c$ and $\mathscr{D}$ is involutive. $\square$

Corollary. (a) There is a one-to-one correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.

(b) Suppose $(H, i) \leftrightarrow \mathfrak{h} \subset \mathfrak{g}$. Then (H, i) is an embedded manifold if and only if H is closed.

Theorem 6.5. Suppose that $A \subset G$ is an abstract subgroup of G and if A has a manifold structure such that $(A, i) \to G$ is a submanifold. Then the manifold structure is unique, A is a Lie group and hence (A, i) is a Lie subgroup of G.

Theorem 6.6. (Adó) Suppose that $\mathfrak{g}$ is a finite dimensional Lie algebra. Then $\mathfrak{g}$ can be realized as a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

Given any connected Lie group G, it has a universal cover $\tilde{G} \xrightarrow{\pi} G$. Choose $\tilde{e} \in \pi^{-1}(e) \in \tilde{G}$ such that the following diagram

$$
\begin{array}{ccc}
\tilde{G} \times \tilde{G} & \longrightarrow & \tilde{G} \\
{\scriptstyle \pi \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\
G \times G & \longrightarrow & G
\end{array}
$$

commutes.

7   30 Jan 2023

Lemma 7.1. Suppose that G is a connected Lie group. Then $\pi_1(G)$ is abelian.

Proof. Suppose $\sigma, \tau : I \to G$ be two loops. Define $\sigma \cdot \tau$ by

$$(\sigma \cdot \tau)(s) = \sigma(s) \cdot \tau(s)$$

Then we have

$$\sigma * \tau \cong \sigma \cdot \tau$$

where $*$ denote the product in the fundamental group $\pi_1(G)$ (given by concatenation) and $\cong$ denotes equivalent in homotopy. Also,

$$\sigma \cdot \tau \cdot \sigma^{-1} \cong \tau$$

which implies $\sigma\tau \cong \tau \cdot \sigma$ $\qquad \square$

Theorem 7.2. Suppose that G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ with G simply connected. Let $\tilde{\phi} : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then there exists a Lie group homomorphism

$$\phi : G \to H$$

such that $d\phi_e : T_e(G) = \mathfrak{g} \to \mathfrak{h} = T_e H$ is equal to $\tilde{\phi}$.

Proof. Let $\{\omega_i\}$ be a basis for invariant differential forms in $E^1(H)$. Let $\mathscr{I}$ be the ideal generated by $\{\pi_1^* \tilde{\phi}^*(\omega_j) - \pi_2^*(\omega_j) | 1 \leq j \leq d\}$. Then $\mathscr{I}$ is an invariant differential ideal of $G \times H$, so it comes from vanishing of an integrable submanifold of $G \times H$ passing through (e, e).

Then M is a Lie subgroup of $G \times H$ and $M \xrightarrow{p} G$ obtained by restriction of π_1 is a group homomorphism and also a local diffeomorphism. So $p : M \to G$ is a covering projection but G is simply connected so p is a diffeomorphism

$$G \xrightarrow{p^{-1}} M \hookrightarrow G \times H \to H.$$

$\square$

Corollary. 1. Suppose $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras and G and H are simply connected. Then $G \cong H$ as Lie groups.

2. There exists a one-to-one correspondence between (finite dimensional) Lie algebras and simply connected Lie groups.

3. The differential structure of a Lie group is determined by its Lie algebra.

If G is a topological group which is locally Euclidean, does it support a Lie group structure? The answer is yes but the proof is quite difficult.

Exponential map

Let X be a left-invariance vector field on G. We have a Lie algebra homomorphism

$$\text{Lie}(\mathbb{R}) \cong \mathbb{R} \to \mathfrak{g}$$

$$c\frac{d}{dt} \to cX$$

This yields a Lie group homomorphism

$$\mathbb{R} \xrightarrow{\exp_X} G$$

$$x \mapsto \exp_X(x)$$

then $d\exp_X(c\frac{d}{dt}) = cX$. The map

$$\mathfrak{g} \xrightarrow{\exp} G$$

$$X \mapsto \exp_X(1)$$

is called the **exponential map**.

Theorem 7.3. Let $X \in \text{Lie}(G)$. Then

1. $\exp(tX) = \exp_X(t)$

2. $\exp(t_1 X_1 + t_2 X) = \exp(t_1 X) \cdot \exp(t_2 X)$
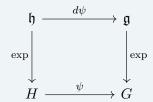
3. $\exp(-tX) = (\exp(tX))^{-1}$

4. $\exp : \mathfrak{g} \to G$ is smooth and $d\exp : T_0\mathfrak{g} \to T_eG = \mathfrak{g}$ is the identity map

5. $\lambda_g \circ \exp_X : \mathbb{R} \to G$ is the unique integral curve of X which is based at g.

6. The left-invariant vector fields are complete, i.e. their integral curves exist for all time.

7. The one-parameter group of diffeomorphism $\psi_{X,t}$ for $t \in \mathbb{R}$ is given by

$$\psi_{X,t} = \rho_{\exp_X(t)}$$

where ρ_g denote right-multiplication by g.

Theorem 7.4. Suppose $\psi : H \to G$ is a Lie group homomorphism. Then

$$
\begin{CD}
\mathfrak{h} @>d\psi>> \mathfrak{g} \\
@V\exp VV @VV\exp V \\
H @>\psi>> G
\end{CD}
$$

commutes.

[DO THIS COMMUTATIVE DIAGRAM.]

8 1 Feb 2023

Theorem 8.1. Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra where $\mathrm{Lie}(G)$. Let $A \subset G$ an abstract subgroup such that there exists a neighbourhood $0 \in V \subset \mathfrak{g}$ such that

$$\exp(V \cap \mathfrak{h}) = U \cap H$$

for some neighborhood $e \in U \subset G$. Then H has a unique manifold structure such that $(H, i) \hookrightarrow G$ is an embedded submanifold of G and H is closed in subset topology.

Remark. Lines with irrational slope in torus doesn't satisfy the hypothesis.

Matrix exponentiation

Recall that $\mathfrak{gl}(n, \mathbb{R})$ denotes the Lie algebra of $n \times n$ matrices over $\mathbb{R}$ and similarly for $\mathfrak{gl}(n, \mathbb{C})$.

Definition 8.1. Define a map

$$\mathfrak{gl}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$$

$$A \mapsto e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

It can be proved that the series is convergent with sup norm and further we have a lemma

Lemma 8.2. If $AB = BA$ then

$$e^{A+B} = e^A e^B$$

which can be used to prove that $e^A \in \mathrm{GL}(n, \mathbb{C})$, so the definition makes sense.

Fix A and consider the function

$$\mathbb{R} \ni t \mapsto e^{tA} \in \mathrm{GL}(n, \mathbb{C})$$

then its derivative is

$$\frac{d}{dt}\bigg|_{t=0} e^{tA} = A$$

because we can differentiate term by term in uniform convergence. This confirms Theorem 7.3 4th part.

The left-invariant vector field given by $A \in \mathfrak{gl}(n, \mathbb{C})$ is just multiplication by A on the right. Thus, $t \mapsto e^{tA}$ is the integral curve associated to the vector field $A \in \mathfrak{gl}(n, C)$ based at I. Hence, this is the exponential map in the cases of $\mathrm{GL}(n, \mathbb{C})$.

Theorem 8.3. The exponential map $\exp : \mathfrak{g} \to G$ is smooth.

Proof. Let $X \in \mathfrak{g}$ and consider the map

$$V : G \times \mathfrak{g} \to TG \times \mathfrak{g}$$
$$(g, X) \mapsto (X_g, 0)$$

then V is smooth. Also, V is left-invariant on $G \times \mathfrak{g}$. Consider the integral curve γ based at (g, X) of V. Then
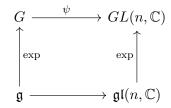
$$\gamma_V(t) = (g \exp_X(t), X)$$

because of left invariance so

$$\gamma_V(1) = (g \exp(X), X)$$

$$G \times \mathfrak{g} \xrightarrow{\gamma_V(1)} G \times \mathfrak{g} \xrightarrow{\pi} G$$
$$(e, X) \mapsto \gamma_V(1) \to \exp(X)$$

$\square$

9 6 Feb 2023

Note that exponential map commutes with Lie group homomorphisms. Using Ado's theorem we get that for any Lie group

$$
\begin{array}{ccc}
G & \xrightarrow{\ \psi\ } & GL(n, \mathbb{C}) \\
\uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\
\mathfrak{g} & \longrightarrow & \mathfrak{gl}(n, \mathbb{C})
\end{array}
$$

Consider the Lie group $\mathrm{SL}(n,\mathbb{C}) = \{X \in \mathrm{GL}(n,\mathbb{C}) | \det(X) = 1\}$, for any $A \in \mathfrak{gl}(n,\mathbb{C})$ upper triangular with diagonal entries $\lambda_1, \ldots, \lambda_n$ then

$$\det(e^A) = e^{\lambda_1 + \ldots + \lambda_n} = e^{\mathrm{tr}(A)}$$

Now $\mathfrak{sl}(n,\mathbb{C}) = \{A \in \mathfrak{gl}(n,\mathbb{C}) | \mathrm{tr}(A) = 0\}$, then $\mathfrak{sl}(n,\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ and exponential maps $\mathfrak{sl}(n,\mathbb{C})$ to the Lie subgroup $\mathrm{SL}(n,\mathbb{C})$. As $\mathrm{SL}(n,\mathbb{C})$ is a closed subgroup of $\mathrm{GL}(n,\mathbb{C})$ and dimension $2(n^2 - 1)$. Using Theorem 8.1 on an appropriate neighborhood we can complete the proof.

Lie subgroup	Lie subalgebra $\mathfrak{gl}(n,\mathbb{C})$
$U(n) \longleftrightarrow$	$u(n) = $ skew-Hermitian matrices
$\mathrm{SU}(n) \longleftrightarrow$	$su(n) = $ skew-Hermitian + trace $= 0$

Prove the above given correspondence using this lemma (TO DO).

Lemma 9.1. Suppose that $P \in \mathrm{GL}(n,\mathbb{C})$ and $A \in \mathfrak{gl}(n,\mathbb{C})$, then

$$Pe^A P^{-1} = e^{PAP^{-1}}.$$

Theorem 9.2 (Baker-Campbell-Hausdorff formula). Let $\mathfrak{g}$ be a Lie algebra corresponding to a connected Lie group G. Then in a neighborhood U of the identity, the multiplication $U \times U \to G$ is completely determined by Lie algebra structure of $\mathfrak{g}$. There is a formula for $Z = Z(X,Y)$, $X,Y \in V \subset \mathfrak{g}$, where $e^X \cdot e^Y = e^Z$

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \ldots$$

Consider

$$e^{tX} \cdot e^{tY} = \left(\sum \frac{t^k X^k}{k!} \right) \left(\sum \frac{t^l Y^l}{l!} \right)$$

$$= \sum_{m \geq 0} \left(\sum_{k+l=m} \frac{X^k Y^l}{k!l!} \right) t^m$$

Suppose $Z = tZ_1 + t^2 Z_2 + t^3 Z_3 \ldots$, then

$$e^Z = 1 + (tZ_1 + t^2 Z_2 + \ldots) + \frac{(tZ_1 + t^2 Z_2 +)}{2!} + \ldots$$

$$= 1 + t(Z_1) + t^2 \left(Z_2 + \frac{Z_1^2}{2!} \right)$$

So we get $Z_1 = X + Y$,

$$\frac{X^2}{2!} + XY + \frac{Y^2}{2!} = Z_2 + \frac{Z_1^2}{2!}$$

$$= Z_2 + \frac{1}{2} \left(X^2 + XY + YX + Y^2 \right)$$

so $Z_2 = XY - \frac{1}{2}(XY + YX) = \frac{1}{2}(XY - YX) = \frac{1}{2}[X,Y]$

Theorem 9.3. Suppose that $\psi : R \to G$ is a continuous homomorphism. The ψ is smooth.

Proof. It is enough to show that ψ is smooth at 0. Let U be a star-like neighborhood of $0 \in \mathfrak{g}$ such that $\exp |_U : U \to G$ is a diffeomorphism onto $\exp(U)$. Let $U' = \{\frac{X}{2}|X \in U\}$. Choose $Y \in U'$ and let $\psi(t_0) = \exp(Y)$. Choose $t_0 > 0$ such that

$$\psi([-t_0, t_0]) \subset \exp(U')$$

Let $n \geq 2$, and suppose that $X \in U'$ such that $\exp(X) = \psi(\frac{t_0}{n})$. Claim $nX = Y$ $\qquad\square$

10 6 Feb

11 8 Feb

12 13 Feb

Definition 12.1. Let $\mathfrak{a} \in \mathfrak{g}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$. We say that $\mathfrak{a}$ is an **ideal** in $\mathfrak{g}$ if $[X, Y] \in \mathfrak{a}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$.

Theorem 12.1. Suppose $A \subset g$ is a connected Lie subgroup of a connected Lie group G. Then A is normal in G if and only if $\mathfrak{a} = \operatorname{Lie}(A)$ is an ideal in $\mathfrak{g}$.

Proof. Suppose that $\mathfrak{a} \subset \mathfrak{g}$ is an ideal. Let $g \in G$, $h \in A$. We must show that $ghg^{-1} \in A$, to do this it is enough to show this for g in a neighborhood of e and h in a neighborhood of e in A. So we may write $g = \exp X, h = \exp Y$

$$\begin{aligned}
ghg^{-1} &= \exp \circ \operatorname{Ad}_g(Y) \\
&= \exp \operatorname{Ad}_{\exp(X)}(Y) \\
&= (\exp(\exp(id_X))) \\
&= \exp\left(I + \operatorname{ad}_X + \frac{\operatorname{ad}_X^2}{2!} + \dots\right)(Y) \\
&= \exp\left(Y + [X, Y] + \frac{[X, [X, Y]]}{2!} + \dots\right) \in A
\end{aligned}$$

Now assume A is normal in G. Let $X \in \mathfrak{g}$, $Y \in \mathfrak{a}$. Write $g_t = \exp tX$. We know that

$$\begin{aligned}
A \ni g_t(\exp(sY))g_t^{-1} &= \exp(\operatorname{Ad}_{g_t}(sY)) \\
&= \exp(s \operatorname{Ad}_{g_t}) \\
&= \exp(s \exp \operatorname{ad}_{tX}(Y))
\end{aligned}$$

This implies $\exp \operatorname{ad}_{tX}(Y) \in \mathfrak{a}$ so $Y + t[X, Y] + \frac{t^2}{2!}[X, [X, Y]] + \dots$ and using $\left.\frac{d}{dt}\right|_{t=0} \exp \operatorname{ad}_{tX}(Y) = [X, Y] \in \mathfrak{a}$. $\qquad\square$

Definition 12.2. The center of a Lie algebra $\mathfrak{g}$ is the vector space $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \,|\, [X, Y] = 0 \,\forall\, Y \in \mathfrak{g}\}$.

Remark. Note that $\mathfrak{z}$ is trivial Lie subalgebra of $\mathfrak{g}$.

Theorem 12.2. Let $Z = Z(G)$ be the center of G. Then $Z(G) = \ker(\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}))$.

Proof. If $\mathfrak{g} \in Z(G)$, then $i_g : G \to G = \mathrm{id}_G$ where i_g is the conjugation map. Taking the differential, this implies $A_g : \mathfrak{g} \to \mathfrak{g}$ is identity, hence $g \in \ker(\mathrm{Ad})$.

Suppose that $g \in \ker(\mathrm{Ad})$, so $\mathrm{Ad}_g(X) = X$. Let $X \in \mathfrak{g}$ then

$$\exp tX = \exp(t \, \mathrm{Ad}_g(X))$$
$$= g \exp(tX) g^{-1}$$

so g commutes with elements $\exp(tX)$ in a neighborhood of e but that is enough since elements of the form $\exp tX$ for any $t \in \mathbb{R}, X \in \mathfrak{g}$ generate G. Therefore, $g \in Z(G)$. $\qquad\square$

Proposition 12.3. If $X, Y \in \mathfrak{g}$ are such that $[X, Y] = 0$. Then

$$\exp(X + Y) = \exp(X) \exp(Y).$$

Proof. Let $\mathfrak{a} = \mathbb{R}X + \mathbb{R}Y$. Then $\mathfrak{a}$ is abelian subalgebra of $\mathfrak{g}$. Then the corresponding Lie subgroup A is abelian. Define $\alpha : \mathbb{R} \to G$ such that

$$\alpha(t) = \exp(tX) \exp(tY) \in A$$

It follows that $\alpha(s + t) = \alpha(s)\alpha(t)$ since A is abelian. Now $\alpha(t) = \exp(tZ)$ for some $Z \in \mathfrak{g}$ where $Z = \dfrac{d}{dt}\bigg|_{t=0} \alpha(t)$.

$$\frac{d}{dt}\alpha(t) = \frac{d}{dt}\bigg|_{t=0} \exp(tX) + \frac{d}{dt}\bigg|_{t=0} \exp(tY)$$
$$= X_e + Y_e$$

So $Z_e = X_e + Y_e$ and $\exp(tZ) = \exp(tX) \exp(tY)$ for all $t \in \mathbb{R}$. $\qquad\square$

13 15 Feb

Motivation. We will try to look into automorphism group of Lie group now and the expectation is that it is a Lie group itself.

Let $\psi : V \otimes V \to V$ be a linear map. Consider the sets

$$A_\psi(V) = \{\alpha \in \mathrm{GL}(V) | (\alpha u, \alpha v) = \alpha((u, v))\},$$

i.e. the diagram commutes

$$
\begin{array}{ccc}
V \otimes V & \xrightarrow{\ \psi\ } & V \\
\downarrow{\scriptstyle \alpha \otimes \alpha} & & \downarrow{\scriptstyle \alpha} \\
V \otimes V & \xrightarrow{\ \psi\ } & V
\end{array}
$$

and

$$Dev_\psi(V) = \{f \in \mathrm{End}(V) \mid f(\psi(u,v)) = \psi(f(u),v) + \psi(u,f(v))\}$$

Proposition 13.1. 1. $A_\psi(V)$ is a closed subgroup of $\mathrm{GL}(V)$.

2. $Dev_\psi(V)$ is a Lie subalgebra of $\mathfrak{g}(V)$.

Proof. TO DO $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 13.2. Lie algebra of $A_\psi(V)$ equals $Dev_\psi(V)$.

Proof. Let $\mathfrak{a} = Lie(A_\psi(V)) \subset \mathfrak{g}(V) = \mathrm{End}(V)$. We must show that $\mathfrak{a} = Dev_\psi(V)$. Suppose that $f \in \mathfrak{a}$, then $\exp(tf) \in A_\psi(V)$ for all t. We need to show that

$$f \circ \psi = \psi \circ (f \otimes 1 + 1 \otimes f)$$

To do this, let $u,v \in V$, then

$$
\begin{aligned}
\exp tf(u,v) &= (\exp tf(u), \exp tf(v)) \\
&= (u,v) + (tf(u),v) + (u,tf(v)) + \text{higher powers of } t
\end{aligned}
$$

so

$$f(u,v) = \frac{d}{dt}\Big|_{t=0} \exp tf(u,v) = (f(u),v) + (u,f(v))$$

so $f \in Dev_\psi(V)$.

Let $f \in Dev_\psi(V)$, we must show that

$$
\begin{aligned}
\exp(tf)(u,v) &= (\exp(tf)u, \exp(tf)v) \\
i.e \qquad \exp(tf) \circ \psi &= \psi \circ (\exp(tf) \otimes \exp(tf)) \qquad \forall u,v \in V \text{ and } \forall t \in \mathbb{R}
\end{aligned}
$$

As $f \in Dev_\psi(V)$, we have

$$
\begin{aligned}
f \circ \psi &= \psi \circ (f \otimes 1 + 1 \otimes f) \\
f^2 \circ \psi &= f \circ f \circ \psi \\
&= f \circ \psi \circ (f \otimes 1 + 1 \otimes f) \\
&= \psi \circ (f \otimes 1 + 1 \otimes f)^2
\end{aligned}
$$

By induction,

$$f^n \circ \psi = \psi \circ (f \otimes 1 + 1 \otimes f)$$

17

and $f \otimes 1, 1 \otimes f : V \otimes V \to V \otimes V$ commutes. It follows that

$$\exp(tf) \circ \psi = \sum \left(\frac{t^k f^k}{k!} \circ \psi \right)$$

$$= \sum \frac{t^k}{k!} \psi \circ (f \otimes 1 + 1 \otimes f)^k$$

$$= \psi \circ \sum \frac{t^k}{k!} (f \otimes 1 + 1 \otimes f)^k$$

$$= \psi \circ \exp(tf \otimes 1 + 1 \otimes tf)$$
$$= \psi \circ (tf \otimes 1) \circ \exp(1 \otimes tf)$$
$$= \psi \circ \exp(tf \otimes tf)$$
$$= \psi(\exp(tf) \otimes \exp(tf))$$

$\square$

Let $V = \mathfrak{g} = \mathrm{Lie}(G)$ and $\psi = [\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ be the Lie bracket. Then

$$A_\psi(V) = \mathrm{Aut}_{Lie}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$$

and

$$\mathrm{Der}_\psi(V) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$$

by the theorem. Note that $G \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathfrak{g})$ factors through $G \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ and $\mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathfrak{g})$.

Let V be a finite dimensional vector space. Consider a bilinear form

$$B : V \times V \to F,$$

equipped with a linear map

$$V \otimes V \to F$$

An element $g \in \mathrm{GL}(V)$ is B-invariant if

$$(u, v) = (gu, gv) \qquad \forall u, v \in V$$

An element $f \in \mathrm{End}(V)$ is B-invariant if

$$(fu, v) + (u, fv) = 0$$

Then $O_B(V) = \{g \in \mathrm{GL}(V) | g \text{ is } B\text{-invariant}\}$ is a closed Lie subgroup of $\mathrm{GL}(V)$ with Lie algebra B-invariant linear map endomorphisms of V.

Example. Take $V = \mathbb{R}^n$ and B is the standard inner product. Then $O_B(V) = O(n)$.

14 1 March

Missed

15 6 March

Missed

16 8 March

Missed

17 13 March

Fundamental group of Lie groups

Reference - Hall (?)

Complexification

Let V be a real vector space. Then the complexification is the vector space $V \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}$. If V is a Lie algebra, then $V_{\mathbb{C}}$ is a Lie algebra where the bracket operates on $V_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of that on V. It is given by

$$[X + iY, X' + iY'] = [X, X'] - [Y, Y'] + i\left([X, Y'] + [X', Y]\right)$$

for all $X, Y, X', Y' \in V$. Suppose that V is a real Lie algebra and W is a complex Lie algebra. Suppose $f : V \to W$ is a Lie algebra homomorphism where W is regarded as a $\mathbb{R}$-Lie algebra. Then f extends to a unique complex Lie algebra homomorphism

$$f_{\mathbb{C}} : V \otimes \mathbb{C} \to W$$

Suppose that $W = V + iV$ as $\mathbb{C}$ vector space and where $V \cap iV = 0$ (internal direct sum). Then we say that V is a real form of W.

Suppose W is a complex Lie algebra and V is a real Lie subalgebra contained in W which is a real form of W. Then

$$V_{\mathbb{C}} \equiv W$$

as $\mathbb{C}$-Lie algebra.

Q. Given a Lie algebra, when is it the Lie algebra of a compact Lie group?

A. Something about Killing form and non-degeneracy of complexified Lie algebra and semisimple Lie algebra.

18 20 March

Suppose that $\psi : H \to G$ is a Lie algebra homomorphism into a connected $\mathbb{C}$-Lie group G. Then $d\psi : \mathfrak{h} \to \mathfrak{g}$ extends to a complex Lie algebra homomorphism

$$\mathfrak{h}_{\mathbb{C}} \xrightarrow{d\psi \otimes \mathbb{C}} \mathfrak{g}.$$

> **Definition 18.1.** We say that $\psi : H \to G$ is a complexification of H if for any complex Lie group L and any real Lie group homomorphism $f : H \to L$, there exists a unique complex Lie group homomorphism $\phi : G \to L$ such that
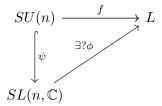> $$f = \phi \circ \psi.$$

Also,

Definition 18.2. A homomorphism of Lie groups $\psi : G \to L$ is a complex Lie group homomorphism if G, L are complex and $d\psi : \mathfrak{g} \to \mathfrak{l}$ is a complex Lie algebra homomorphism.
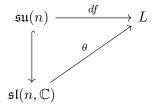
Idea : Like we can complexify a real Lie algebra, we would like to have a concept of complexification of a Lie group but we may not be able to do so for all real Lie groups.

Given a complex Lie group G, any (H, ψ) whose complexification is G will be called a real form. For a given complex Lie group there can be more than one real form. E.g consider $SU(n) \subset \mathrm{SL}(n, \mathbb{C})$ and it can be proven that $SU(n)$ is a real form of $\mathrm{SL}(n, \mathbb{C})$ (also called compact form since $SU(n)$ is compact) by dimension analysis.

Consider the diagram

$$
\begin{array}{ccc}
SU(n) & \xrightarrow{\ \ f\ \ } & L \\
\ \downarrow{\psi} & \nearrow{\scriptstyle \exists?\phi} & \\
SL(n, \mathbb{C}) & &
\end{array}
$$

where L is a complex Lie group. At the Lie algebra level

$$
\begin{array}{ccc}
\mathfrak{su}(n) & \xrightarrow{\ \ df\ \ } & L \\
\ \downarrow & \nearrow{\scriptstyle \theta} & \\
\mathfrak{sl}(n, \mathbb{C}) & &
\end{array}
$$

where $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) + i\mathfrak{su}(n)$, $\exists\, \mathrm{SL}(n, \mathbb{C}) \xrightarrow{\phi} L$ such that $d\phi = \theta$ since $\mathrm{SL}(n, \mathbb{C})$ is simply connected. Then ϕ restricts to f since $d\phi|_{\mathfrak{su}(n)} = df$.

Theorem 18.1. Let K be a compact connected Lie group. Then there exists a complex Lie group $K_{\mathbb{C}}$ and a Lie group homomorphism $f : K \to K_{\mathbb{C}}$ such that

1. $f_* : \pi_1(K) \to \pi_1(K_{\mathbb{C}})$ is an isomorphism.

2. $\mathrm{Lie}(K_{\mathbb{C}}) = \mathrm{Lie}(K) \otimes \mathbb{C}$.

3. K_C is the complexification of K.

Theorem 18.2. Suppose that G is a complex linear connected semisimple Lie group. Then any maximal compact Lie subgroup $K \subset G$ is a real form of G.

19 22 March

Let β be a symmetric bilinear form on V, where V is a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Let Q be the associated quadratic form

$$
Q : V \to \mathbb{R} \qquad \text{or} \qquad Q : V \to \mathbb{C}
$$

$$
Q(\lambda v) = \lambda^2 v
$$

We have $Q(V) = \beta(v, v)$, and $\beta(u, v) = \frac{Q(u+v) - Q(u) - Qv}{2}$. Suppose that (V, β), (V, β') are quadratic spaces. Then we say that $(V, \beta), (V, \beta')$ are equivalent if there exists $T : V \to V$ such that

$$\beta'(u, v) = \beta(Tu, TV) \quad \forall u, v \in V$$

Suppose that $v_1, \ldots, v_n$ is a basis for V. Then the matrix of β is $B = (\beta(v_i, v_j))$.

Let B, B' be the matrices of β, β'. Then $(V, \beta), (V, \beta')$ are equivalent if there exists $T \in M_n(F)$ such that

$$B = {}^t T B T$$

where ^{t} denotes transpose. Now if $x = (x_1 \ldots x_n)^t$, $y = (y_1, \ldots, y_n)^t$ are vectors in $F^n \equiv V$, then

$$x^t B y = \beta(x, y)$$

and

$$\begin{aligned}
\beta'(x, y) &= \beta(Tx, Ty) \\
&= x^t T^t B T y \\
&= x^t B' y
\end{aligned}$$

which proves the statement. Suppose $E_1 \subset E$, (E, β) is a quadratic space. Then $(E_1, \beta|_{E_1})$ is a quadratic space.

$$E_1^\perp = \{x \in E : \beta(x, y) = 0 \, \forall y \in E_1\}$$

Lemma 19.1. Suppose that $E_1 \subset E$ and $(E, \beta|_{E_1})$ is non-degenerate. Then

$$E = E_1 \oplus E_1^\perp = E_1 \perp E_1^\perp$$

If (E, β) is non-degenerate, then $(E_1^\perp, \beta_{E_1^\perp})$ is also non-degenerate.

Proof. TO DO. □

Example - Consider the quad space (H, β) where $H = \mathbb{R}^2$ and $Q((x, y)) = x^2 - y^2$. Then $(H, \beta) \cong (H, \beta')$ where $Q'((x, y)) = xy$. One can calculate that

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$B' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so the forms are non-degenerate and similar using transformation $T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Suppose (V, β) is non-singular, if $\beta|_E \equiv 0$, then $\dim E \leq \frac{1}{2} \dim V$. Further

Lemma 19.2. If (V, β) is non-singular then

$$V = V_1 \oplus \cdots \oplus V_n$$

where each V_i is 1-dimensional and $(V_i, \beta|_{V_i})$ is non-degenerate, $V_i \perp V_j$ if $i \neq j$, i.e. there exists a basis of V with respect to the matrix B of β is diagonal.

Proof. The proof is by induction on dimension. First suppose that $v \in V$ is non-zero then choose $V_1 = Fv$ then

$$V = V_1 \oplus V_1^{\perp}$$

and $(V_1^{\perp}, \beta|_{V_1^{\perp}})$ is non-degenerate. Apply induction to $(V_1^{\perp}, \beta|_{V_1^{\perp}})$.

Suppose $\beta(v, v) = 0$. Choose by non-degeneracy of β a vector $V \in V$ such that $\beta(u, v) = 0$. Notice that $\beta(u + v, u + v) = 2\beta(u, v) \neq 0$ which lands us in earlier case. $\square$

Now suppose that (V, β) is arbitrary. Let $V_0 = \mathrm{rad}(\beta) = \{x \in V : \beta(x, y) = 0 \forall y \in V\}$ Consider the quotient $(\frac{V}{V_0}, \overline{\beta})$ with

$$\overline{\beta}(u + V_0, v + V_0) = \beta(u, v)$$

and $\mathrm{rad}(\beta) = 0$, so $(\frac{V}{V_0}, \overline{\beta})$ is non-degenerate. Main theorem

Theorem 19.3. Over $\mathbb{R}$ any non-degenerate β is equivalent to the bilinear form with basis
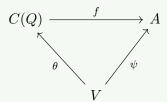
$$\begin{bmatrix} I_k & \\ & -I_l \end{bmatrix}$$

with $k + l = n$. Moreover, k, l are uniquely determined by β.

Definition 19.1. Let (V, Q) be a quadratic space. The **Clifford algebra** $C(Q)$ associated to it is an algebra over F with a homomorphism $\theta : V \to C(Q)$ such that

1. $\theta(x)^2 = Q(x)$

2. $C(Q)$ is universal with respect to 1st property, i.e. if $\psi : V \to A$ is any vector space homomorphism to an F-algebra such that

$$\psi(x)^2 = Q(x)$$

then there exists a unique algebra homomorphism f such that

$$
\begin{array}{ccc}
C(Q) & \xrightarrow{\ f\ } & A \\
& & \\
{\scriptstyle \theta} \nwarrow & & \nearrow {\scriptstyle \psi} \\
& V &
\end{array}
$$

commutes.

We can construct the Clifford algebra by

$$C(Q) = \frac{T(V)}{\langle x \otimes x - \psi(x) \rangle}$$

where $T(V)$ is the tensor algebra of V.

Example. 1. $V = \mathbb{R}$, $Q(x) = -x^2$ then

$$T(V) = \mathbb{R} \oplus \mathbb{R}e_1 \oplus \mathbb{R}(e_1 \otimes e_1) \oplus \ldots$$

and $C(Q) = \mathbb{R} \oplus \mathbb{R}e_1$, $e_1^2 = -1$ so $C(Q) \cong \mathbb{C}$.
  2. $V = \mathbb{R}$, $Q'(x) = x^2$, then

$$C(Q') = \mathbb{R} \oplus \mathbb{R}e_1$$

with $e_1^2 = 1$ so it is the polynomial ring $\frac{\mathbb{R}[x]}{(x^2-1)}$.