Lie Groups

Devesh Rajpal

Contents

1 4th January 23

One can study Lie Groups from several points of view. The course is aimed to understand the structure of Lie Groups.

Definition 1.1. A smooth manifold M is a Hausdorff space which is locally Euclidean with a smooth atlas i.e. (i) given any $x \in M$, $\exists$ a chart (U, ϕ), $x \in U \subset M$ with $\phi : U \to \phi(U)$ open in $\mathbb{R}^m$.

(ii) We have collection $\{(U, \phi)\}$ of charts such that

$$\phi(U \cap V) \xrightarrow{\psi \circ \phi^{-1}} \psi(U \cap V)$$

is a diffeomorphism.

Suppose $f : M \to N$ is a continuous map between manifolds. We say that f is smooth if for $(U, \phi) \in \amalg(M)$, $(V, \psi) \in \amalg(N)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}$ is smooth.

TO DO : Construction of tangent bundle and vector bundle

2 9th Jan 2023

Definition 2.1. G is a Lie group if

1. G is a smooth manifold
2. G is also a group s.t

$$\begin{aligned} \mu : G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

and

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth maps.

Definition 2.2. A real (or complex) vector space V together with a bilinear map

$$[,] : V \times V \to V$$

is called a **Lie Algebra** if

1. $[X, Y] = -[Y, X]$ - skew symmetry
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ - Jacobi identity

Example.

1. $(\mathbb{R}, +), (\mathbb{C}, +)$, V any f.d vector space over $\mathbb{R}$ or $\mathbb{C}$.
2. $(\mathbb{R}^\times, \cdot), (\mathbb{C}^\times, \cdot)$
3. $S^1 = \{z \in \mathbb{C}^\times || z| = 1\}$
4. $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$
5. $\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}^n/\mathbb{Z}^n) \cong (S^1)^n$
6. Suppose $\Gamma \subset V$ is a discrete subgroup. Then V/Γ is a Lie group.
7. N = unipotent upper triangular matrices, B = upper triangular matrices. As manifolds $N \cong \mathbb{R}^{\binom{n}{2}}$ and $B \cong (\mathbb{R}^\times)^n \times N$.
8. $\mathrm{SL}_n(\mathbb{R}) = \{X \in \mathrm{GL}_n(\mathbb{R}) | \det X = 1\}$, $\mathrm{SL}_n(\mathbb{C})$.
9. $O(n)$, $SO(n)$.

10. $U(n)$, $SU(n)$.

11. $\mathbb{H}^\times$, S^3 with quaternion multiplication.

12. $Sp(n) = \{X \in \mathrm{GL}_n(\mathbb{R}) | X$ preserves quaternion structure as a subset of $\mathrm{Aut}_{\mathbb{H}} \mathbb{H}^n\}$

Problem. $V/\Gamma \cong \mathbb{R}^k \times (S^1)^{n-k}$ for n-dimensional vector space V.

Theorem 2.1. Suppose G is a compact, connected, simple Lie group. Then G is locally isomorphic to

1. $SU(n), n \geq 2$ denoted by A_{n-1}
2. $SO(2n+1), n \geq 2$ denoted by B_n
3. $Sp(n), n \geq 1$ denoted by C_n
4. $SO(2n), n \geq 2$ denoted by D_n

or one of the following exceptional Lie group G_2, F_4, E_6, E_7, E_8.

Problem. Prove that $\mathrm{SL}_n(\mathbb{R})$ and $O(n)$ are smooth manifold, hence Lie groups.

Examples of Lie algebra -

Example. 1. $(V, [\cdot,\cdot] \equiv 0)$ is called trivial Lie algebra.

2. $(\mathfrak{gl}_n(\mathbb{R}), [A, B] = AB - BA)$, $\mathfrak{gl}_n(\mathbb{C})$
3. $\mathfrak{sl}_n(\mathbb{R})$ ($\mathfrak{sl}_n(\mathbb{C})$) is the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ ($\mathfrak{gl}_n(\mathbb{C})$) consisting of trace 0.
4. $\mathfrak{so}_n$ is Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ consisting of skew-symmetric matrices.

Definition 2.3. A vector field X on a Lie group G is called left invariant if $(L_g)_*(X_h) = X_{gh}$

3 11th Jan 2023

Recall $\mathbb{H} = \{a + bi + cj + dk : (a, b, c, d) \in \mathbb{R}^4,\ i^2 = -1, j^2 = -1, k^2 = -1, ij = k, jk = l, ki = j\}$ is the quaternion division algebra with the norm

$$||a + bi + cj + dk||^2 = a^2 + b^2 + c^2 + d^2$$

which satisfies $||q_1 \cdot q_2|| = ||q_1|| \cdot ||q_2||$

We can put this multiplication on $S^3 \cong SU(2)$ to get a compact Lie group. To get the isomorphism $SU(2) \cong S^3$, we define a map

$$\phi : S^3 \to SU(2)$$
$$(a, b, c, d) \mapsto \begin{bmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{bmatrix}$$

which is an algebra isomorphism.

Definition 3.1. The Lie algebra of G is the space of all left-invariant vector fields on G.

We have an isomorphism

$$\begin{aligned} \mathfrak{g} = \mathrm{Lie}(G) &\to T_eG \\ X &\mapsto X_e \end{aligned}$$

Example. Let $G = \mathbb{R}^n$, with identity element $0 \in \mathbb{R}^n$ and left-invariant vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$. Then the Lie bracket is

$$[\cdot, \cdot] \equiv 0$$

Remark. In general for any abelian Lie group G, the Lie bracket is $[\cdot, \cdot] \equiv 0$.

Theorem 3.1. Let G be a connected Lie group. Then

1. $\mathrm{Lie}(G) = \mathfrak{g}$ is isomorphic as a vector space to $T_e(G)$.
2. Left-invariant vector fields are smooth.
3. $\mathrm{Lie}(G)$ is closed under Lie bracket.

Proof. 1. Let X be a left-invariant vector field on G. We need to show that Xf is smooth for each $f \in C^\infty(G)$.

$$\begin{aligned} (Xf)(g) &= X_g f \\ &= (d\lambda_g X_e) f \\ &= X_e(f \circ \lambda_g) \end{aligned}$$

To show that Xf is smooth, it suffices to show that $X_e(f \circ \lambda_g)$ is smooth. We realize $X_e(f \circ \lambda_g)$ as evaluation of a smooth function on a smooth function.

Let Y be a smooth vector field on G such that $Y_e = X_e$

$$Y_e(f \circ \lambda_g) = X_e(f \circ \lambda_g)$$

We look at λ_g as the composition of

$$\begin{aligned} G \xrightarrow{i_g^2} G \times G &\xrightarrow{\mu} G \\ x \mapsto (g, x) &\mapsto gx \end{aligned}$$

Regard Y as the vector field $(0, Y)$ on $G \times G$. Now

$$\begin{aligned}(0,Y)(f\circ\mu)\circ i_e^1(g) &= (0,Y)_{(g,e)}(f\circ\mu)\\ &= 0_g(f\circ\mu\circ i_g^1) + Y_e(f\circ\mu\circ i_g^2)\\ &= Y_e(f\circ\lambda_g)\end{aligned}$$

which proves the smoothness.

2. Let X, Y left-invariant vector fields on G. We must show that $[X, Y]$ is a left-invariant vector field.

$$\begin{aligned}d\lambda_g([X,Y]_e)f &= [X,Y]_g f\\ &= [X,Y]_e(f\circ\lambda_g)\\ &= X_e(Y(f\circ\lambda_g)) - Y_e(X(f\circ\lambda_g))\\ &= X_e(d\lambda_g(Yf)) - Y_e(d\lambda_g(Yf))\end{aligned}$$

□

4 18 Jan 2023

Lemma 4.1. Suppose $\psi : M \to N$ is a smooth map. Let X_1, X_2 be vector fields on M, Y_1, Y_2 be vector fields on N such that X_i is ψ-related to Y_i. Then $[X_1, X_2]$ is ψ-related to $[Y_1, Y_2]$.

Proof. Notice that

$$\begin{aligned}d\psi[X_1,X_2](f) &= [X_1,X_2](f\circ\psi)\\ &= X_1(X_2f\circ\psi) - X_2(X_1f\circ\psi)\\ &= X_1(d\psi X_2f) - X_2(Y_1f\circ\psi)\\ &= X_1(Y_2f\circ\psi) - X_2(Y_1f\circ\psi)\\ &= d\psi X_1(Y_2f) - d\psi X_2(Y_1f)\\ &= Y_1Y_2f\circ\psi - Y_2Y_1f\circ\psi\\ &= [Y_1,Y_2](f)\circ\psi\end{aligned}$$

□

This lemma proves that the set of left-invariant vector fields forms a Lie algebra.

Consider the Lie group $\mathrm{GL}_n(\mathbb{R})$. We want to verify the Lie algebra structure on $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ with the isomorphism

$$\begin{aligned}Lie(\mathrm{GL}_n(\mathbb{R})) &\to \mathfrak{gl}_n(\mathbb{R})\\ X &\overset{\beta}{\mapsto} X_e\end{aligned}$$

Lemma 4.2.

$$\beta([X,Y]) = [\beta(X), \beta(Y)]$$

Proof. Evaluating the bracket on coordinate function x_{ij}.

$$[X,Y]_e(x_{ij}) = X_e(Y(x_{ij})) - Y_e(X(x_{ij})) \tag{1}$$

Now

$$\begin{aligned} Y(x_{ij})(g) &= d\lambda_g Y_e(x_{ij}) \\ &= Y_e(x_{ij} \circ \lambda_g) \\ &= \sum_k x_{ik}(g) Y_e(x_{kj}) \end{aligned}$$

Considering the above as function of g and substituting this in Eq. (1) we get

$$\begin{aligned} [X,Y]_e(x_{ij}) &= X_e Y_e(x_{ij}) - Y_e X_e(x_{ij}) \\ &= \sum_k \{X_e(x_{ik})Y_e(x_{kj}) - Y_e(x_{ik})X_e(x_{kj})\} \\ &= [X_e, Y_e](x_{ij}) \end{aligned}$$

□

Definition 4.1. A **Lie subgroup** H of a Lie group G is a submanifold $H \xrightarrow{\alpha} G$ where α is smooth and a group homomorphism.

We say that H is closed Lie subgroup if it is Lie subgroup such that $H \to \alpha(H)$ is a diffeomorphism.

Example. Consider the map $\mathbb{R} \to S^1 \times S^1$ given by

$$t \mapsto (e^{2\pi i t}, e^{2\pi i \sqrt{2} t})$$

The image is a Lie subgroup of $S^1 \times S^1$ but it is not a closed Lie subgroup. It is also known as "Skew-line" in the torus.

Definition 4.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $f : \mathfrak{g} \to \mathfrak{h}$ be a vector space homomorphism. Then we say that f is a Lie algebra homomorphism if

$$f([X,Y]) = [f(X), f(Y)]$$

Theorem 4.3. Suppose that $\psi : G \to H$ is a Lie group homomorphism. Let X be a left-invariant vector field on G. Extend $d\psi(X_e) = Y_e \in T_eH$ to a left-invariant vector field Y on H. Then X and Y are ψ-related. This implies $d\psi_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd} TG \arrow[r, "d\psi"] \arrow[d] & TH \arrow[d] \\ G \arrow[r, "\psi"] \arrow[u, bend left, "X"] & H \arrow[u, bend right, "Y"'] \end{tikzcd}$$

We want to show that $Y \circ \psi = d\psi \circ Y$. Now

$$\lambda_{\psi(g)} \circ \psi = \psi \circ \lambda_g$$

so

$$\begin{aligned} Y_{\psi(g)} &= d\lambda_{\psi(g)} Y_e \\ &= d\lambda_{\psi(g)} d\psi X_e \\ &= d(\lambda_{\psi(g)} \circ \psi)(X_e) \\ &= d(\psi \circ \lambda_g)(X_e) \\ &= d\psi d\lambda_g(X_e) \\ &= d\psi X_g \end{aligned}$$

□

Theorem 4.4. Let G, H be Lie groups with G connected. Let

$$\phi, \psi : G \to H$$

be homomorphism of Lie groups such that

$$d\phi = d\psi : T_eG \to T_eH$$

Then $\phi = \psi$.

5 23 Jan 2023

I missed the class. Regardless here are some definitions from Warner covered on this day.

Definition 5.1. Let M be a smooth d-dimensional manifold. For any integer $1 \leq c \leq d$, a **c-dimensional distribution** $\mathscr{D}$ on manifold is a choice of c-dimensional subspace $\mathscr{D}_p \subset T_pM$. $\mathscr{D}$ is smooth if for each $p \in M$ there is an open neighborhood U of p and there are c smooth vector fields $X_1, \ldots, X_c$ on U which span $\mathscr{D}_m$ for each $p \in U$.

We say $\mathscr{D}$ is **involutive** if $[X, Y] \in \mathscr{D}$ whenever $X, Y \in \mathscr{D}$.

Definition 5.2. A submanifold (N, ϕ) of M is an integral manifold of a distribution $\mathscr{D}$ if

$$d\phi(N_p) = \mathscr{D}_{\phi(p)}$$

Suppose there exists an integral manifold N for a distribution $\mathscr{D}$, then for the points on N the distribution $\mathscr{D}$ is necessarily involutive. Frobenius theorem states that it is sufficient condition for a distribution to be integral.

Theorem 5.1. **(Frobenius)** Let $\mathscr{D}$ be a c-dimensional involutive smooth distribution on M. Then there exists an integral manifold of $\mathscr{D}$ passing through each point of M.

Differential Ideals

Let $E^*(M) = \bigoplus_{i=0}^{\infty} E^i(M)$ denote the graded algebra of smooth differential forms over manifold M.

Definition 5.3. Let $\mathscr{D}$ be a smooth p-dimensional distribution on M. A q-form ω is said to **annihilate** $\mathscr{D}$ if for each $x \in M$

$$\omega_x(v_1, \ldots, v_q) = 0 \quad \text{whenever } v_1, \ldots, v_q \in \mathscr{D}_x$$

A form $\omega \in E^*(M)$ is said to annihilate $\mathscr{D}$ if each of the homogenous components of ω annihilate $\mathscr{D}$. Define

$$\mathscr{I}(\mathscr{D}) \doteqdot \{\omega \in E^*(M) : \omega \text{ annihilates } \mathscr{D}\}$$

Definition 5.4. An ideal $\mathscr{I} \in E^*(M)$ is called a **differential ideal** if it is closed under exterior differentiation; i.e.

$$d(\mathscr{I}) \subset \mathscr{I}.$$

Theorem 5.2. A smooth distribution $\mathscr{D}$ on M is involutive if and only if the ideal $\mathscr{I}(\mathscr{D})$ is a differential ideal.

6 25 Jan 2023

Theorem 6.1. If $\phi : H \to G$ is a homomorphism of Lie groups and if ω is a left-invariant differential form on G, then $\phi^*(\omega)$ is again a left-invariant form on H.

Suppose that $\phi : H \to G$ is a homomorphism of Lie groups. Let $\omega_1, \ldots, \omega_d$ be a basis for $E^1_{\text{inv}}(G)$. Then

$$\mathcal{I}_\phi = \langle \{\pi_1^* \phi^*(\omega_j) - \pi_2^*(\omega_j)\} : 1 \le j \le d \rangle$$

is a left-invariant differential ideal of $H \times G$.

Lemma 6.2. Suppose $X_1, \ldots, X_d$ is a basis of $\mathfrak{g}$ dual to $\omega_1, \ldots, \omega_d$. Suppose the Lie bracket is given by

$$[X_i, X_j] = \sum c_{ij}^k X_k$$

Then the C^∞ functions c_{ij}^k are constant. Further,

$$d\omega_i = -c_{kj}^i \omega_k \wedge \omega_j$$

Proof. Notice that

$$\begin{aligned} d\omega_k(X_i, X_j) &= -\omega_k([X_i, X_j]) \\ &= -c_{ij}^k \end{aligned}$$

which is a constant because a left-invariant 1-form evaluated on a left-invariant vector field is a constant. □

Remark. The c_{ij}^k are called the structural constants of G with respect to the basis $\{X_i\}$ of $\mathfrak{g}$.

Proof. Theorem 4.4. Notice that $\mathcal{I}_\psi = \mathcal{I}_\phi$ since $d\phi = d\psi$ and these are invariant differential ideals; hence integral manifolds of $\mathcal{I}_\phi$ and $\mathcal{I}_\psi$ passing through (e, e) are the same. Thus, $\phi = \psi$. □

Lemma 6.3. Suppose G is any Hausdorff topological group which is connected. Suppose $e \in U \subset G$ is any open set. Then

$$G = \bigcup_{n \geq 1} U^n$$

where $U^n = \{x_1 \cdots x_n | x_i \in U\}$

Proof. Since $e \in U$ is open, $U^{-1} = \{x^{-1} | x \in U\}$ is also an open neighborhood of e. Let $V = U \cap U^{-1}$. Note that

$$H \doteqdot \bigcup_{n \geq 1} V^n$$

is a subgroup of G, and it is open. Since the cosets gH are also open it follows that $G = \cup_g H$ being connected must be H.

□

Theorem 6.4. Let G be a Lie group and $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists connected Lie subgroup H of G such that $T_e H = \mathfrak{h}$.

Proof. Consider the distribution $\mathscr{D}$ defined as

$$\mathscr{D}_g = \{X_g | X \in \mathfrak{h}\}$$

on G. Suppose $X_1, \ldots, X_c$ is a basis of $\mathfrak{h}$. Then $\mathscr{D}$ is generated by $X_1, \ldots, X_c$ and $\mathscr{D}$ is involutive. □

Corollary. (a) There is a one-to-one correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.

(b) Suppose $(H, i) \leftrightarrow \mathfrak{h} \subset \mathfrak{g}$. Then (H, i) is an embedded manifold if and only if H is closed.

Theorem 6.5. Suppose that $A \subset G$ is an abstract subgroup of G and if A has a manifold structure such that $(A, i) \to G$ is a submanifold. Then the manifold structure is unique, A is a Lie group and hence (A, i) is a Lie subgroup of G.

Theorem 6.6. **(Adó)** Suppose that $\mathfrak{g}$ is a finite dimensional Lie algebra. Then $\mathfrak{g}$ can be realized as a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

Given any connected Lie group G, it has a universal cover $\tilde{G} \xrightarrow{\pi} G$. Choose $\tilde{e} \in \pi^{-1}(e) \in \tilde{G}$ such that the following diagram

$$\begin{tikzcd}
\tilde{G} \times \tilde{G} \arrow[r] \arrow[d, "\pi\times\pi"'] & \tilde{G} \arrow[d, "\pi"] \\
G \times G \arrow[r] & G
\end{tikzcd}$$

commutes.

7 30 Jan 2023

Lemma 7.1. Suppose that G is a connected Lie group. Then $\pi_1(G)$ is abelian.

Proof. Suppose $\sigma, \tau : I \to G$ be two loops. Define $\sigma \cdot \tau$ by

$$(\sigma \cdot \tau)(s) = \sigma(s) \cdot \tau(s)$$

Then we have

$$\sigma * \tau \cong \sigma \cdot \tau$$

where $*$ denote the product in the fundamental group $\pi_1(G)$ (given by concatenation) and $\cong$ denotes equivalent in homotopy. Also,

$$\sigma \cdot \tau \cdot \sigma^{-1} \cong \tau$$

which implies $\sigma\tau \cong \tau \cdot \sigma$ □

Theorem 7.2. Suppose that G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ with G simply connected. Let $\tilde{\phi} : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then there exists a Lie group homomorphism

$$\phi : G \to H$$

such that $d\phi_e : T_e(G) = \mathfrak{g} \to \mathfrak{h} = T_e H$ is equal to $\tilde{\phi}$.

Proof. Let $\{\omega_i\}$ be a basis for invariant differential forms in $E^1(H)$. Let $\mathscr{I}$ be the ideal generated by $\{\pi_1^*\tilde{\phi}^*(\omega_j) - \pi_2^*(\omega_j) | 1 \leq j \leq d\}$. Then $\mathscr{I}$ is an invariant differential ideal of $G \times H$, so it comes from vanishing of an integrable submanifold of $G \times H$ passing through (e, e).

Then M is a Lie subgroup of $G \times H$ and $M \xrightarrow{p} G$ obtained by restriction of π_1 is a group homomorphism and also a local diffeomorphism. So $p : M \to G$ is a covering projection but G is simply connected so p is a diffeomorphism

$$G \xrightarrow{p^{-1}} M \hookrightarrow G \times H \to H.$$

□

Corollary.
1. Suppose $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras and G and H are simply connected. Then $G \cong H$ as Lie groups.
2. There exists a one-to-one correspondence between (finite dimensional) Lie algebras and simply connected Lie groups.
3. The differential structure of a Lie group is determined by its Lie algebra.

If G is a topological group which is locally Euclidean, does it support a Lie group structure? The answer is yes but the proof is quite difficult.

Exponential map

Let X be a left-invariance vector field on G. We have a Lie algebra homomorphism

$$\begin{aligned} \mathrm{Lie}(\mathbb{R}) \cong \mathbb{R} &\to \mathfrak{g} \\ c\frac{d}{dt} &\to cX \end{aligned}$$

This yields a Lie group homomorphism

$$\begin{aligned} \mathbb{R} &\xrightarrow{\exp_X} G \\ x &\mapsto \exp_X(x) \end{aligned}$$

then $d\exp_X(c\frac{d}{dt}) = cX$. The map

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\exp} G \\ X &\mapsto \exp_X(1) \end{aligned}$$

is called the **exponential map**.

Theorem 7.3. Let $X \in \mathrm{Lie}(G)$. Then

1. $\exp(tX) = \exp_X(t)$
2. $\exp(t_1X_1 + t_2X) = \exp(t_1X) \cdot \exp(t_2X)$
3. $\exp(-tX) = (\exp(tX))^{-1}$
4. $\exp : \mathfrak{g} \to G$ is smooth and $d\exp : T_0\mathfrak{g} \to T_eG = \mathfrak{g}$ is the identity map

5. $\lambda_g \circ \exp_X : \mathbb{R} \to G$ is the unique integral curve of X which is based at g.
6. The left-invariant vector fields are complete, i.e. their integral curves exist for all time.
7. The one-parameter group of diffeomorphism $\psi_{X,t}$ for $t \in \mathbb{R}$ is given by

$$\psi_{X,t} = \rho_{exp_X(t)}$$

where ρ_g denote right-multiplication by g.

Theorem 7.4. Suppose $\psi : H \to G$ is a Lie group homomorphism. Then

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "d\psi"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ H \arrow[r, "\psi"] & G \end{tikzcd}$$

commutes.

[DO THIS COMMUTATIVE DIAGRAM.]

8 1 Feb 2023

Theorem 8.1. Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra where Lie(G). Let $A \subset G$ an abstract subgroup such that there exists a neighbourhood $0 \in V \subset \mathfrak{g}$ such that

$$\exp(V \cap \mathfrak{h}) = U \cap H$$

for some neighborhood $e \in U \subset G$. Then H has a unique manifold structure such that $(H, i) \hookrightarrow G$ is an embedded submanifold of G and H is closed in subset topology.

Remark. Lines with irrational slope in torus doesn't satisfy the hypothesis.

Matrix exponentiation

Recall that $\mathfrak{gl}(n, \mathbb{R})$ denotes the Lie algebra of $n \times n$ matrices over $\mathbb{R}$ and similarly for $\mathfrak{gl}(n, \mathbb{C})$.

Definition 8.1. Define a map

$$\mathfrak{gl}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$$
$$A \mapsto e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

It can be proved that the series is convergent with sup norm and further we have a lemma

Lemma 8.2. If $AB = BA$ then

$$e^{A+B} = e^A e^B$$

which can be used to prove that $e^A \in \mathrm{GL}(n, \mathbb{C})$ so the definition makes sense.

Fix A and consider the function

$$\mathbb{R} \ni t \mapsto e^{tA} \in \mathrm{GL}(n, \mathbb{C})$$

then its derivative is

$$\left.\frac{d}{dt}\right|_{t=0} e^{tA} = A$$

because we can differentiate term by term in uniform convergence. This confirms Theorem 7.3 4th part.

The left-invariant vector field given by $A \in \mathfrak{gl}(n, \mathbb{C})$ is just multiplication by A on the right. Thus, $t \mapsto e^{tA}$ is the integral curve associated to the vector field $A \in \mathfrak{gl}(n, C)$ based at I. Hence, this is the exponential map in the cases of $\mathrm{GL}(n, \mathbb{C})$.

Theorem 8.3. The exponential map $\exp : \mathfrak{g} \to G$ is smooth.

Proof. Let $X \in \mathfrak{g}$ and consider the map

$$\begin{aligned} V : G \times \mathfrak{g} &\to TG \times \mathfrak{g} \\ (g, X) &\mapsto (X_g, 0) \end{aligned}$$

then V is smooth. Also, V is left-invariant on $G \times \mathfrak{g}$. Consider the integral curve γ based at (g, X) of V. Then

$$\gamma_V(t) = (g \exp_X(t), X)$$

because of left invariance so

$$\gamma_V(1) = (g \exp(X), X)$$

$$\begin{aligned} G \times \mathfrak{g} &\xrightarrow{\gamma_V(1)} G \times \mathfrak{g} \xrightarrow{\pi} G \\ (e, X) &\mapsto \gamma_V(1) \to \exp(X) \end{aligned}$$

$\square$

9 6 Feb 2023

Note that exponential map commutes with Lie group homomorphisms. Using Ado's theorem we get that for any Lie group

$$\begin{tikzcd} G \arrow[r, "\psi"] & GL(n, \mathbb{C}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r] & \mathfrak{gl}(n, \mathbb{C}) \arrow[u, "\exp"'] \end{tikzcd}$$

Consider the Lie group $\mathrm{SL}(n,\mathbb{C}) = \{X \in \mathrm{GL}(n,\mathbb{C}) | \det(X) = 1\}$, for any $A \in \mathfrak{gl}(n,\mathbb{C})$ upper triangular with diagonal entries $\lambda_1, \dots, \lambda_n$ then

$$\det(e^A) = e^{\lambda_1 + \dots + \lambda_n} = e^{\mathrm{tr}(A)}$$

Now $\mathfrak{sl}(n,\mathbb{C}) = \{A \in \mathfrak{gl}(n,\mathbb{C}) | \mathrm{tr}(A) = 0\}$, then $\mathfrak{sl}(n,\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ and exponential maps $\mathfrak{sl}(n,\mathbb{C})$ to the Lie subgroup $\mathrm{SL}(n,\mathbb{C})$. As $\mathrm{SL}(n,\mathbb{C})$ is a closed subgroup of $\mathrm{GL}(n,\mathbb{C})$ and dimension $2(n^2-1)$. Using Theorem 8.1 on an appropriate neighborhood we can complete the proof.

Lie subgroup	Lie subalgebra $\mathfrak{gl}(n,\mathbb{C})$
$U(n) \longleftrightarrow$	$u(n) =$ skew-Hermitian matrices
$\mathrm{SU}(n) \longleftrightarrow$	$su(n) =$ skew-Hermitian + trace $= 0$

Prove the above given correspondence using this lemma (TO DO).

Lemma 9.1. Suppose that $P \in \mathrm{GL}(n,\mathbb{C})$ and $A \in \mathfrak{gl}(n,\mathbb{C})$, then

$$Pe^A P^{-1} = e^{PAP^{-1}}.$$

Theorem 9.2 (Baker-Campbell-Hausdorff formula)**.** Let $\mathfrak{g}$ be a Lie algebra corresponding to a connected Lie group G. Then in a neighborhood U of the identity, the multiplication $U \times U \to G$ is completely determined by Lie algebra structure of $\mathfrak{g}$. There is a formula for $Z = Z(X,Y)$, $X, Y \in V \subset \mathfrak{g}$, where $e^X \cdot e^Y = e^Z$

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \dots$$

Consider

$$\begin{aligned} e^{tX} \cdot e^{tY} &= \left(\sum \frac{t^k X^k}{k!}\right)\left(\sum \frac{t^l Y^l}{l!}\right) \\ &= \sum_{m \geq 0} \left(\sum_{k+l=m} \frac{X^k Y^l}{k!l!}\right) t^m \end{aligned}$$

Suppose $Z = tZ_1 + t^2 Z_2 + t^3 Z_3 \dots$, then

$$\begin{aligned} e^Z &= 1 + (tZ_1 + t^2 Z_2 + \dots) + \frac{(tZ_1 + t^2 Z_2 +)}{2!} + \dots \\ &= 1 + t(Z_1) + t^2 \left(Z_2 + \frac{Z_1^2}{2!}\right) \end{aligned}$$

So we get $Z_1 = X + Y$,

$$\begin{aligned} \frac{X^2}{2!} + XY + \frac{Y^2}{2!} &= Z_2 + \frac{Z_1^2}{2!} \\ &= Z_2 + \frac{1}{2}\left(X^2 + XY + YX + Y^2\right) \end{aligned}$$

so $Z_2 = XY - \frac{1}{2}(XY + YX) = \frac{1}{2}(XY - YX) = \frac{1}{2}[X,Y]$

Theorem 9.3. Suppose that $\psi : R \to G$ is a continuous homomorphism. The ψ is smooth.

Proof. It is enough to show that ψ is smooth at 0. Let U be a star-like neighborhood of $0 \in \mathfrak{g}$ such that $\exp|_U : U \to G$ is a diffeomorphism onto $\exp(U)$. Let $U' = \{\frac{X}{2} | X \in U\}$. Choose $Y \in U'$ and let $\psi(t_0) = \exp(Y)$. Choose $t_0 > 0$ such that

$$\psi([-t_0, t_0]) \subset \exp(U')$$

Let $n \geq 2$, and suppose that $X \in U'$ such that $\exp(X) = \psi(\frac{t_0}{n})$. Claim $nX = Y$ ☐

10 6 Feb

11 8 Feb

12 13 Feb

Definition 12.1. Let $\mathfrak{a} \in \mathfrak{g}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$. We say that $\mathfrak{a}$ is an **ideal** in $\mathfrak{g}$ if $[X, Y] \in \mathfrak{a}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$.

Theorem 12.1. Suppose $A \subset g$ is a connected Lie subgroup of a connected Lie group G. Then A is normal in G if and only if $\mathfrak{a} = \mathrm{Lie}(A)$ is an ideal in $\mathfrak{g}$.

Proof. Suppose that $\mathfrak{a} \subset \mathfrak{g}$ is an ideal. Let $g \in G$, $h \in A$. We must show that $ghg^{-1} \in A$, to do this it is enough to show this for g in a neighborhood of e and h in a neighborhood of e in A. So we may write $g = \exp X, h = \exp Y$

$$\begin{aligned}
ghg^{-1} &= \exp \circ \mathrm{Ad}_g(Y) \\
&= \exp \mathrm{Ad}_{\exp(X)}(Y) \\
&= (\exp(\exp(id_X))) \\
&= \exp\left(I + \mathrm{ad}_X + \frac{\mathrm{ad}_X^2}{2!} + \dots\right)(Y) \\
&= \exp\left(Y + [X, Y] + \frac{[X, [X, Y]]}{2!} + \dots\right) \in A
\end{aligned}$$

Now assume A is normal in G. Let $X \in \mathfrak{g}$, $Y \in \mathfrak{a}$. Write $g_t = \exp tX$. We know that

$$\begin{aligned}
A \ni g_t(\exp(sY))g_t^{-1} &= \exp(\mathrm{Ad}_{g_t}(sY)) \\
&= \exp(s\,\mathrm{Ad}_{g_t}) \\
&= \exp(s \exp \mathrm{ad}_{tX}(Y))
\end{aligned}$$

This implies $\exp \mathrm{ad}_{tX}(Y) \in \mathfrak{a}$ so $Y + t[X, Y] + \frac{t^2}{2!}[X, [X, Y]] + \dots$ and using $\frac{d}{dt}\Big|_{t=0} \exp \mathrm{ad}_{tX}(Y) = [X, Y] \in \mathfrak{a}$ ☐

Definition 12.2. The center of a Lie algebra $\mathfrak{g}$ is the vector space $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \,|\, [X, Y] = 0 \,\forall\, Y \in \mathfrak{g}\}$.

Remark. Note that $\mathfrak{z}$ is trivial Lie subalgebra of $\mathfrak{g}$.

Theorem 12.2. Let $Z = Z(G)$ be the center of G. Then $Z(G) = \ker(\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}))$.

Proof. If $\mathfrak{g} \in Z(G)$, then $i_g : G \to G = \mathrm{id}_G$ where i_g is the conjugation map. Taking the differential, this implies $A_g : \mathfrak{g} \to \mathfrak{g}$ is identity, hence $g \in \ker(\mathrm{Ad})$.

Suppose that $g \in \ker(\mathrm{Ad})$, so $\mathrm{Ad}_g(X) = X$. Let $X \in \mathfrak{g}$ then

$$\begin{aligned}\exp tX &= \exp(t\,\mathrm{Ad}_g(X)) \\ &= g\exp(tX)g^{-1}\end{aligned}$$

so g commutes with elements $\exp(tX)$ in a neighborhood of e but that is enough since elements of the form $\exp tX$ for any $t \in \mathbb{R}, X \in \mathfrak{g}$ generate G. Therefore $g \in Z(G)$. □

Proposition 12.3. If $X, Y \in \mathfrak{g}$ are such that $[X, Y] = 0$. Then

$$\exp(X + Y) = \exp(X)\exp(Y).$$

Proof. Let $\mathfrak{a} = \mathbb{R}X + \mathbb{R}Y$. Then $\mathfrak{a}$ is abelian subalgebra of $\mathfrak{g}$. Then the corresponding Lie subgroup A is abelian. Define $\alpha : \mathbb{R} \to G$ such that

$$\alpha(t) = \exp(tX)\exp(tY) \in A$$

It follows that $\alpha(s + t) = \alpha(s)\alpha(t)$ since A is abelian. Now $\alpha(t) = \exp(tZ)$ for some $Z \in \mathfrak{g}$ where $Z = \frac{d}{dt}|_{t=0}\alpha(t)$.

$$\begin{aligned}\frac{d}{dt}\alpha(t) &= \frac{d}{dt}|_{t=0}\exp(tX) + \frac{d}{dt}|_{t=0}\exp(tY) \\ &= X_e + Y_e\end{aligned}$$

So $Z_e = X_e + Y_e$ and $\exp(tZ) = \exp(tX)\exp(tY)$ for all $t \in \mathbb{R}$. □