Lie Groups

Devesh Rajpal

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One can study Lie Groups from several points of view. The course is aimed to understand the structure of Lie Groups.

Definition 1.1. A smooth manifold M is a Hausdorff space which is locally Euclidean with a smooth atlas i.e. (i) given any $x \in M$, $\exists$ a chart (U, ϕ), $x \in U \subset M$ with $\phi : U \to \phi(U)$ open in $\mathbb{R}^m$.

(ii) We have collection $\{(U, \phi)\}$ of charts such that

$$\phi(U \cap V) \xrightarrow{\psi \circ \phi^{-1}} \psi(U \cap V)$$

is a diffeomorphism.

Suppose $f : M \to N$ is a continuous map between manifolds. We say that f is smooth if for $(U, \phi) \in \amalg(M)$, $(V, \psi) \in \amalg(N)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}$ is smooth.

TO DO : Construction of tangent bundle and vector bundle

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Definition 2.1. G is a Lie group if

1. G is a smooth manifold
2. G is also a group s.t

$$\begin{aligned} \mu : G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

and

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth maps.

Definition 2.2. A real (or complex) vector space V together with a bilinear map

$$[,] : V \times V \to V$$

is called a **Lie Algebra** if

1. $[X, Y] = -[Y, X]$ - skew symmetry
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ - Jacobi identity

Example.
1. $(\mathbb{R}, +), (\mathbb{C}, +)$, V any f.d vector space over $\mathbb{R}$ or $\mathbb{C}$.
2. $(\mathbb{R}^\times, \cdot), (\mathbb{C}^\times, \cdot)$
3. $S^1 = \{z \in \mathbb{C}^\times | |z| = 1\}$
4. $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$
5. $\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}^n/\mathbb{Z}^n) \cong (S^1)^n$
6. Suppose $\Gamma \subset V$ is a discrete subgroup. Then V/Γ is a Lie group.
7. N = unipotent upper triangular matrices, B = upper triangular matrices. As manifolds $N \cong \mathbb{R}^{\binom{n}{2}}$ and $B \cong (\mathbb{R}^\times)^n \times N$.
8. $\mathrm{SL}_n(\mathbb{R}) = \{X \in \mathrm{GL}_n(\mathbb{R}) | \det X = 1\}$, $\mathrm{SL}_n(\mathbb{C})$.
9. $O(n)$, $SO(n)$.
10. $U(n)$, $SU(n)$.

11. $\mathbb{H}^\times$, S^3 with quaternion multiplication.
12. $Sp(n) = \{X \in \mathrm{GL}_n(\mathbb{R}) | X \text{ preserves quaternion structure as a subset of } \mathrm{Aut}_{\mathbb{H}} \mathbb{H}^n\}$

Problem. $V/\Gamma \cong \mathbb{R}^k \times (S^1)^{n-k}$ for n-dimensional vector space V.

Theorem 2.1. Suppose G is a compact, connected, simple Lie group. Then G is locally isomorphic to

1. $SU(n), n \geq 2$ denoted by A_{n-1}
2. $SO(2n+1), n \geq 2$ denoted by B_n
3. $Sp(n), n \geq 1$ denoted by C_n
4. $SO(2n), n \geq 2$ denoted by D_n

or one of the following exceptional Lie group G_2, F_4, E_6, E_7, E_8.

Problem. Prove that $\mathrm{SL}_n(\mathbb{R})$ and $O(n)$ are smooth manifold, hence Lie groups.

Examples of Lie algebra -

Example.
1. $(V, [\cdot,\cdot] \equiv 0)$ is called trivial Lie algebra.
2. $(\mathfrak{gl}_n(\mathbb{R}), [A, B] = AB - BA)$, $\mathfrak{gl}_n(\mathbb{C})$
3. $\mathfrak{sl}_n(\mathbb{R})$ ($\mathfrak{sl}_n(\mathbb{C})$) is the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ ($\mathfrak{gl}_n(\mathbb{C})$) consisting of trace 0.
4. $\mathfrak{so}_n$ is Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ consisting of skew-symmetric matrices.

Definition 2.3. A vector field X on a Lie group G is called left invariant if $(L_g)_*(X_h) = X_{gh}$

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Recall $\mathbb{H} = \{a+bi+cj+dk : (a,b,c,d) \in \mathbb{R}^4,\ i^2 = -1, j^2 = -1, k^2 = -1, ij = k, jk = l, ki = j\}$ is the quaternion division algebra with the norm

$$||a + bi + cj + dk||^2 = a^2 + b^2 + c^2 + d^2$$

which satisfies $||q_1 \cdot q_2|| = ||q_1|| \cdot ||q_2||$

We can put this multiplication on $S^3 \cong SU(2)$ to get a compact Lie group. To get the isomorphism $SU(2) \cong S^3$, we define a map

$$\begin{aligned} \phi : S^3 &\to SU(2) \\ (a,b,c,d) &\mapsto \begin{bmatrix} a+bi & c+di \\ -(c-di) & a-bi \end{bmatrix} \end{aligned}$$

which is an algebra isomorphism.

Definition 3.1. The Lie algebra of G is the space of all left-invariant vector fields on G.

We have an isomorphism

$$\begin{aligned} \mathfrak{g} = \text{Lie}(G) &\to T_e G \\ X &\mapsto X_e \end{aligned}$$

Example. Let $G = \mathbb{R}^n$, with identity element $0 \in \mathbb{R}^n$ and left-invariant vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$. Then the Lie bracket is

$$[\cdot,\cdot] \equiv 0$$

Remark. In general for any abelian Lie group G, the Lie bracket is $[\cdot,\cdot] \equiv 0$.

Theorem 3.1. Let G be a connected Lie group. Then

1. $\text{Lie}(G) = \mathfrak{g}$ is isomorphic as a vector space to $T_e(G)$.
2. Left-invariant vector fields are smooth.
3. $\text{Lie}(G)$ is closed under Lie bracket.

Proof. 1. Let X be a left-invariant vector field on G. We need to show that Xf is smooth for each $f \in C^\infty(G)$.

$$\begin{aligned} (Xf)(g) &= X_g f \\ &= (d\lambda_g X_e) f \\ &= X_e(f \circ \lambda_g) \end{aligned}$$

To show that Xf is smooth, it suffices to show that $X_e(f \circ \lambda_g)$ is smooth. We realize $X_e(f \circ \lambda_g)$ as evaluation of a smooth function on a smooth function.

Let Y be a smooth vector field on G such that $Y_e = X_e$

$$Y_e(f \circ \lambda_g) = X_e(f \circ \lambda_g)$$

We look at λ_g as the composition of

$$\begin{aligned} G \xrightarrow{i_g^2} G \times G \xrightarrow{\mu} G \\ x \mapsto (g, x) \mapsto gx \end{aligned}$$

Regard Y as the vector field $(0, Y)$ on $G \times G$. Now

$$\begin{aligned} (0, Y)(f \circ \mu) \circ i_e^1(g) &= (0, Y)_{(g,e)}(f \circ \mu) \\ &= 0_g(f \circ \mu \circ i_g^1) + Y_e(f \circ \mu \circ i_g^2) \\ &= Y_e(f \circ \lambda_g) \end{aligned}$$

which proves the smoothness.

2. Let X, Y left-invariant vector fields on G. We must show that $[X, Y]$ is a left-invariant vector field.

$$\begin{aligned} d\lambda_g([X, Y]_e)f &= [X, Y]_g f \\ &= [X, Y]_e(f \circ \lambda_g) \\ &= X_e(Y(f \circ \lambda_g)) - Y_e(X(f \circ \lambda_g)) \\ &= X_e(d\lambda_g(Yf)) - Y_e(d\lambda_g(Yf)) \end{aligned}$$

□

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Lemma 4.1. Suppose $\psi : M \to N$ is a smooth map. Let X_1, X_2 be vector fields on M, Y_1, Y_2 be vector fields on N such that X_i is ψ-related to Y_i. Then $[X_1, X_2]$ is ψ-related to $[Y_1, Y_2]$.

Proof. Notice that

$$\begin{aligned} d\psi[X_1, X_2](f) &= [X_1, X_2](f \circ \psi) \\ &= X_1(X_2 f \circ \psi) - X_2(X_1 f \circ \psi) \\ &= X_1(d\psi X_2 f) - X_2(Y_1 f \circ \psi) \\ &= X_1(Y_2 f \circ \psi) - X_2(Y_1 f \circ \psi) \\ &= d\psi X_1(Y_2 f) - d\psi X_2(Y_1 f) \\ &= Y_1 Y_2 f \circ \psi - Y_2 Y_1 f \circ \psi \\ &= [Y_1, Y_2](f) \circ \psi \end{aligned}$$

□

This lemma proves that the set of left-invariant vector fields forms a Lie algebra.

Consider the Lie group $\mathrm{GL}_n(\mathbb{R})$. We want to verify the Lie algebra structure on $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ with the isomorphism

$$\begin{aligned} Lie(\mathrm{GL}_n(\mathbb{R})) &\to \mathfrak{gl}_n(\mathbb{R}) \\ X &\overset{\beta}{\mapsto} X_e \end{aligned}$$

Lemma 4.2.

$$\beta([X,Y]) = [\beta(X), \beta(Y)]$$

Proof. Evaluating the bracket on coordinate function x_{ij}.

$$[X,Y]_e(x_{ij}) = X_e(Y(x_{ij})) - Y_e(X(x_{ij})) \tag{1}$$

Now

$$\begin{aligned} Y(x_{ij})(g) &= d\lambda_g Y_e(x_{ij}) \\ &= Y_e(x_{ij} \circ \lambda_g) \\ &= \sum_k x_{ik}(g) Y_e(x_{kj}) \end{aligned}$$

Considering the above as function of g and substituting this in Eq. (1) we get

$$\begin{aligned} [X,Y]_e(x_{ij}) &= X_e Y_e(x_{ij}) - Y_e X_e(x_{ij}) \\ &= \sum_k \{X_e(x_{ik})Y_e(x_{kj}) - Y_e(x_{ik})X_e(x_{kj})\} \\ &= [X_e, Y_e](x_{ij}) \end{aligned}$$

□

Definition 4.1. A **Lie subgroup** H of a Lie group G is a submanifold $H \xrightarrow{\alpha} G$ where α is smooth and a group homomorphism.

We say that H is closed Lie subgroup if it is Lie subgroup such that $H \to \alpha(H)$ is a diffeomorphism.

Example. Consider the map $\mathbb{R} \to S^1 \times S^1$ given by

$$t \mapsto (e^{2\pi i t}, e^{2\pi i \sqrt{2} t})$$

The image is a Lie subgroup of $S^1 \times S^1$ but it is not a closed Lie subgroup. It is also known as "Skew-line" in the torus.

Definition 4.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $f : \mathfrak{g} \to \mathfrak{h}$ be a vector space homomorphism. Then we say that f is a Lie algebra homomorphism if

$$f([X, Y]) = [f(X), f(Y)]$$

Theorem 4.3. Suppose that $\psi : G \to H$ is a Lie group homomorphism. Let X be a left-invariant vector field on G. Extend $d\psi(X_e) = Y_e \in T_eH$ to a left-invariant vector field Y on H. Then X and Y are ψ-related. This implies $d\psi_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
TG \arrow[r, "d\psi"] \arrow[d] & TH \arrow[d] \\
G \arrow[r, "\psi"] \arrow[u, bend left, "X"] & H \arrow[u, bend right, "Y"']
\end{tikzcd}$$

We want to show that $Y \circ \psi = d\psi \circ Y$. Now

$$\lambda_{\psi(g)} \circ \psi = \psi \circ \lambda_g$$

so

$$\begin{aligned}
Y_{\psi(g)} &= d\lambda_{\psi(g)} Y_e \\
&= d\lambda_{\psi(g)} d\psi X_e \\
&= d(\lambda_{\psi(g)} \circ \psi)(X_e) \\
&= d(\psi \circ \lambda_g)(X_e) \\
&= d\psi d\lambda_g (X_e) \\
&= d\psi X_g
\end{aligned}$$

□

Theorem 4.4. Let G, H be Lie groups with G connected. Let

$$\phi, \psi : G \to H$$

be homomorphism of Lie groups such that

$$d\phi = d\psi : T_eG \to T_eH$$

Then $\phi = \psi$.

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I missed the class. Regardless here are some definitions from Warner covered on this day.

Definition 5.1. Let M be a smooth d-dimensional manifold. For any integer $1 \leq c \leq d$, a **c-dimensional distribution** $\mathscr{D}$ on manifold is a choice of c-dimensional subspace $\mathscr{D}_p \subset T_pM$. $\mathscr{D}$ is smooth if for each $p \in M$ there is an open neighborhood U of p and there are c smooth vector fields $X_1, \dots, X_c$ on U which span $\mathscr{D}_m$ for each $p \in U$.

We say $\mathscr{D}$ is **involutive** if $[X, Y] \in \mathscr{D}$ whenever $X, Y \in \mathscr{D}$.

Definition 5.2. A submanifold (N, ϕ) of M is an integral manifold of a distribution $\mathscr{D}$ if

$$d\phi(N_p) = \mathscr{D}_{\phi(p)}$$

Suppose there exists an integral manifold N for a distribution $\mathscr{D}$, then for the points on N the distribution $\mathscr{D}$ is necessarily involutive. Frobenius theorem states that it is sufficient condition for a distribution to be integral.

Theorem 5.1. **(Frobenius)** Let $\mathscr{D}$ be a c-dimensional involutive smooth distribution on M. Then there exists an integral manifold of $\mathscr{D}$ passing through each point of M.

Differential Ideals

Let $E^*(M) = \bigoplus_{i=0}^{\infty} E^i(M)$ denote the graded algebra of smooth differential forms over manifold M.

Definition 5.3. Let $\mathscr{D}$ be a smooth p-dimensional distribution on M. A q-form ω is said to **annihilate** $\mathscr{D}$ if for each $x \in M$

$$\omega_x(v_1, \dots, v_q) = 0 \quad \text{whenever } v_1, \dots, v_q \in \mathscr{D}_x$$

A form $\omega \in E^*(M)$ is said to annihilate $\mathscr{D}$ if each of the homogenous components of ω annihilate $\mathscr{D}$. Define

$$\mathscr{I}(\mathscr{D}) \doteqdot \{\omega \in E^*(M) : \omega \text{ annihilates } \mathscr{D}\}$$

Definition 5.4. An ideal $\mathscr{I} \in E^*(M)$ is called a **differential ideal** if it is closed under exterior differentiation; i.e.

$$d(\mathscr{I}) \subset \mathscr{I}.$$

Theorem 5.2. A smooth distribution $\mathscr{D}$ on M is involutive if and only if the ideal $\mathscr{I}(\mathscr{D})$ is a differential ideal.

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Theorem 6.1. If $\phi : H \to G$ is a homomorphism of Lie groups and if ω is a left-invariant differential form on G, then $\phi^*(\omega)$ is again a left-invariant form on H.

Suppose that $\phi : H \to G$ is a homomorphism of Lie groups. Let $\omega_1, \ldots, \omega_d$ be a basis for $E^1_{\text{inv}}(G)$. Then

$$\mathcal{I}_\phi = \langle \{\pi_1^* \phi^*(\omega_j) - \pi_2^*(\omega_j)\} : 1 \leq j \leq d \rangle$$

is a left-invariant differential ideal of $H \times G$.

Lemma 6.2. Suppose $X_1, \ldots, X_d$ is a basis of $\mathfrak{g}$ dual to $\omega_1, \ldots, \omega_d$. Suppose the Lie bracket is given by

$$[X_i, X_j] = \sum c_{ij}^k X_k$$

Then the C^∞ functions c_{ij}^k are constant. Further,

$$d\omega_i = -c_{kj}^i \omega_k \wedge \omega_j$$

Proof. Notice that

$$\begin{aligned} d\omega_k(X_i, X_j) &= -\omega_k([X_i, X_j]) \\ &= -c_{ij}^k \end{aligned}$$

which is a constant because a left-invariant 1-form evaluated on a left-invariant vector field is a constant. □

Remark. The c_{ij}^k are called the structural constants of G with respect to the basis $\{X_i\}$ of $\mathfrak{g}$.

Proof. Theorem 4.4. Notice that $\mathcal{I}_\psi = \mathcal{I}_\phi$ since $d\phi = d\psi$ and these are invariant differential ideals; hence integral manifolds of $\mathcal{I}_\phi$ and $\mathcal{I}_\psi$ passing through (e, e) are the same. Thus, $\phi = \psi$. □

Lemma 6.3. Suppose G is any Hausdorff topological group which is connected. Suppose

$e \in U \subset G$ is any open set. Then
$$G = \bigcup_{n \geq 1} U^n$$
where $U^n = \{x_1 \cdots x_n | x_i \in U\}$

Proof. Since $e \in U$ is open, $U^{-1} = \{x^{-1} | x \in U\}$ is also an open neighborhood of e. Let $V = U \cap U^{-1}$. Note that
$$H \doteqdot \bigcup_{n \geq 1} V^n$$
is a subgroup of G, and it is open. Since the cosets gH are also open it follows that $G = \cup_g H$ being connected must be H.

□

Theorem 6.4. Let G be a Lie group and $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists connected Lie subgroup H of G such that $T_e H = \mathfrak{h}$.

Proof. Consider the distribution $\mathscr{D}$ defined as
$$\mathscr{D}_g = \{X_g | X \in \mathfrak{h}\}$$
on G. Suppose $X_1, \ldots, X_c$ is a basis of $\mathfrak{h}$. Then $\mathscr{D}$ is generated by $X_1, \ldots, X_c$ and $\mathscr{D}$ is involutive. □

Corollary. (a) There is a one-to-one correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.

(b) Suppose $(H, i) \leftrightarrow \mathfrak{h} \subset \mathfrak{g}$. Then (H, i) is an embedded manifold if and only if H is closed.

Theorem 6.5. Suppose that $A \subset G$ is an abstract subgroup of G and if A has a manifold structure such that $(A, i) \to G$ is a submanifold. Then the manifold structure is unique, A is a Lie group and hence (A, i) is a Lie subgroup of G.

Theorem 6.6. (Adó) Suppose that $\mathfrak{g}$ is a finite dimensional Lie algebra. Then $\mathfrak{g}$ can be realized as a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

Given any connected Lie group G, it has a universal cover $\tilde{G} \xrightarrow{\pi} G$. Choose $\tilde{e} \in \pi^{-1}(e) \in \tilde{G}$ such that the following diagram

$$\begin{tikzcd}
\tilde{G} \times \tilde{G} \arrow[r] \arrow[d, "\pi \times \pi"'] & \tilde{G} \arrow[d, "\pi"] \\
G \times G \arrow[r] & G
\end{tikzcd}$$

commutes.