

# Lie Groups

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## 1 4th January 23

One can study Lie Groups from several points of view. The course is aimed to understand the structure of Lie Groups.

**Definition 1.1.** A smooth manifold  $M$  is a Hausdorff space which is locally Euclidean with a smooth atlas i.e. (i) given any  $x \in M$ ,  $\exists$  a chart  $(U, \phi)$ ,  $x \in U \subset M$  with  $\phi : U \rightarrow \phi(U)$  open in  $\mathbb{R}^m$ .

(ii) We have collection  $\{(U, \phi)\}$  of charts such that

$$\phi(U \cap V) \xrightarrow{\psi \circ \phi^{-1}} \psi(U \cap V)$$

is a diffeomorphism.

Suppose  $f : M \rightarrow N$  is a continuous map between manifolds. We say that  $f$  is smooth if for  $(U, \phi) \in \Pi(M)$ ,  $(V, \psi) \in \Pi(N)$  such that  $f(U) \subset V$  and  $\psi \circ f \circ \phi^{-1}$  is smooth.

TO DO : Construction of tangent bundle and vector bundle

## 2 9th Jan 2023

**Definition 2.1.**  $G$  is a Lie group if

1.  $G$  is a smooth manifold
2.  $G$  is also a group s.t

$$\begin{aligned}\mu : G \times G &\rightarrow G \\ (g, h) &\mapsto gh\end{aligned}$$

and

$$\begin{aligned}i : G &\rightarrow G \\ g &\mapsto g^{-1}\end{aligned}$$

are smooth maps.

**Definition 2.2.** A real (or complex) vector space  $V$  together with a bilinear map

$$[,] : V \times V \rightarrow V$$

is called a **Lie Algebra** if

1.  $[X, Y] = -[Y, X]$  - skew symmetry
2.  $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$  - Jacobi identity

**Example.** 1.  $(\mathbb{R}, +)$ ,  $(\mathbb{C}, +)$ ,  $V$  any f.d vector space over  $\mathbb{R}$  or  $\mathbb{C}$ .

2.  $(\mathbb{R}^\times, \cdot)$ ,  $(\mathbb{C}^\times, \cdot)$

3.  $S^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$

4.  $\mathrm{GL}_n(\mathbb{R})$ ,  $\mathrm{GL}_n(\mathbb{C})$

5.  $\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}^n/\mathbb{Z}^n) \cong (S^1)^n$

6. Suppose  $\Gamma \subset V$  is a discrete subgroup. Then  $V/\Gamma$  is a Lie group.

7.  $N$  = unipotent upper triangular matrices,  $B$  = upper triangular matrices. As manifolds  $N \cong \mathbb{R}^{\binom{n}{2}}$  and  $B \cong (\mathbb{R}^\times)^n \times N$ .

8.  $\mathrm{SL}_n(\mathbb{R}) = \{X \in \mathrm{GL}_n(\mathbb{R}) \mid \det X = 1\}$ ,  $\mathrm{SL}_n(\mathbb{C})$ .

9.  $O(n)$ ,  $SO(n)$ .

10.  $U(n)$ ,  $SU(n)$ .

11.  $\mathbb{H}^\times$ ,  $S^3$  with quaternion multiplication.

12.  $Sp(n) = \{X \in GL_n(\mathbb{R}) | X \text{ preserves quaternion structure as a subset of } \text{Aut}_{\mathbb{H}} \mathbb{H}^n\}$

**Problem.**  $V/\Gamma \cong \mathbb{R}^k \times (S^1)^{n-k}$  for  $n$ -dimensional vector space  $V$ .

**Theorem 2.1.** Suppose  $G$  is a compact, connected, simple Lie group. Then  $G$  is locally isomorphic to

1.  $SU(n), n \geq 2$  denoted by  $A_{n-1}$
2.  $SO(2n+1), n \geq 2$  denoted by  $B_n$
3.  $Sp(n), n \geq 1$  denoted by  $C_n$
4.  $SO(2n), n \geq 2$  denoted by  $D_n$

or one of the following exceptional Lie group  $G_2, F_4, E_6, E_7, E_8$ .

**Problem.** Prove that  $SL_n(\mathbb{R})$  and  $O(n)$  are smooth manifold, hence Lie groups.

Examples of Lie algebra -

**Example.** 1.  $(V, [\cdot, \cdot] \equiv 0)$  is called trivial Lie algebra.

2.  $(\mathfrak{gl}_n(\mathbb{R}), [A, B] = AB - BA), \mathfrak{gl}_n(\mathbb{C})$
3.  $\mathfrak{sl}_n(\mathbb{R}) (\mathfrak{sl}_n(\mathbb{C}))$  is the Lie subalgebra of  $\mathfrak{gl}_n(\mathbb{R}) (\mathfrak{gl}_n(\mathbb{C}))$  consisting of trace 0.
4.  $\mathfrak{so}_n$  is Lie subalgebra of  $\mathfrak{gl}_n(\mathbb{R})$  consisting of skew-symmetric matrices.

**Definition 2.3.** A vector field  $X$  on a Lie group  $G$  is called left invariant if  $(L_g)_*(X_h) = X_{gh}$

### 3 11th Jan 2023

Recall  $\mathbb{H} = \{a + bi + cj + dk : (a, b, c, d) \in \mathbb{R}^4, i^2 = -1, j^2 = -1, k^2 = -1, ij = k, jk = i, ki = j\}$  is the quaternion division algebra with the norm

$$||a + bi + cj + dk||^2 = a^2 + b^2 + c^2 + d^2$$

which satisfies  $||q_1 \cdot q_2|| = ||q_1|| \cdot ||q_2||$

We can put this multiplication on  $S^3 \cong SU(2)$  to get a compact Lie group. To get the isomorphism  $SU(2) \cong S^3$ , we define a map

$$\begin{aligned} \phi : S^3 &\rightarrow SU(2) \\ (a, b, c, d) &\mapsto \begin{bmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{bmatrix} \end{aligned}$$

which is an algebra isomorphism.

**Definition 3.1.** The Lie algebra of  $G$  is the space of all left-invariant vector fields on  $G$ .

We have an isomorphism

$$\begin{aligned}\mathfrak{g} = \text{Lie}(G) &\rightarrow T_e G \\ X &\mapsto X_e\end{aligned}$$

**Example.** Let  $G = \mathbb{R}^n$ , with identity element  $0 \in \mathbb{R}^n$  and left-invariant vector fields  $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ . Then the Lie bracket is

$$[\cdot, \cdot] \equiv 0$$

**Remark.** In general for any abelian Lie group  $G$ , the Lie bracket is  $[\cdot, \cdot] \equiv 0$ .

**Theorem 3.1.** Let  $G$  be a connected Lie group. Then

1.  $\text{Lie}(G) = \mathfrak{g}$  is isomorphic as a vector space to  $T_e(G)$ .
2. Left-invariant vector fields are smooth.
3.  $\text{Lie}(G)$  is closed under Lie bracket.

**Proof.** 1. Let  $X$  be a left-invariant vector field on  $G$ . We need to show that  $Xf$  is smooth for each  $f \in C^\infty(G)$ .

$$\begin{aligned}(Xf)(g) &= X_g f \\ &= (d\lambda_g X_e)f \\ &= X_e(f \circ \lambda_g)\end{aligned}$$

To show that  $Xf$  is smooth, it suffices to show that  $X_e(f \circ \lambda_g)$  is smooth. We realize  $X_e(f \circ \lambda_g)$  as evaluation of a smooth function on a smooth function.

Let  $Y$  be a smooth vector field on  $G$  such that  $Y_e = X_e$

$$Y_e(f \circ \lambda_g) = X_e(f \circ \lambda_g)$$

We look at  $\lambda_g$  as the composition of

$$\begin{aligned}G &\xrightarrow{i_g^2} G \times G \xrightarrow{\mu} G \\ x &\mapsto (g, x) \mapsto gx\end{aligned}$$

Regard  $Y$  as the vector field  $(0, Y)$  on  $G \times G$ . Now

$$\begin{aligned}(0, Y)(f \circ \mu) \circ i_e^1(g) &= (0, Y)_{(g, e)}(f \circ \mu) \\ &= 0_g(f \circ \mu \circ i_g^1) + Y_e(f \circ \mu \circ i_g^2) \\ &= Y_e(f \circ \lambda_g)\end{aligned}$$

which proves the smoothness.

2. Let  $X, Y$  left-invariant vector fields on  $G$ . We must show that  $[X, Y]$  is a left-invariant vector field.

$$\begin{aligned} d\lambda_g([X, Y]_e)f &= [X, Y]_gf \\ &= [X, Y]_e(f \circ \lambda_g) \\ &= X_e(Y(f \circ \lambda_g)) - Y_e(X(f \circ \lambda_g)) \\ &= X_e(d\lambda_g(Yf)) - Y_e(d\lambda_g(Xf)) \end{aligned}$$

□

## 4 18 Jan 2023

**Lemma 4.1.** Suppose  $\psi : M \rightarrow N$  is a smooth map. Let  $X_1, X_2$  be vector fields on  $M$ ,  $Y_1, Y_2$  be vector fields on  $N$  such that  $X_i$  is  $\psi$ -related to  $Y_i$ . Then  $[X_1, X_2]$  is  $\psi$ -related to  $[Y_1, Y_2]$ .

**Proof.** Notice that

$$\begin{aligned} d\psi[X_1, X_2](f) &= [X_1, X_2](f \circ \psi) \\ &= X_1(X_2f \circ \psi) - X_2(X_1f \circ \psi) \\ &= X_1(d\psi X_2f) - X_2(Y_1f \circ \psi) \\ &= X_1(Y_2f \circ \psi) - X_2(Y_1f \circ \psi) \\ &= d\psi X_1(Y_2f) - d\psi X_2(Y_1f) \\ &= Y_1Y_2f \circ \psi - Y_2Y_1f \circ \psi \\ &= [Y_1, Y_2](f) \circ \psi \end{aligned}$$

□

This lemma proves that the set of left-invariant vector fields forms a Lie algebra.

Consider the Lie group  $GL_n(\mathbb{R})$ . We want to verify the Lie algebra structure on  $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$  with the isomorphism

$$\begin{aligned} Lie(GL_n(\mathbb{R})) &\rightarrow \mathfrak{gl}_n(\mathbb{R}) \\ X &\xrightarrow{\beta} X_e \end{aligned}$$

**Lemma 4.2.**

$$\beta([X, Y]) = [\beta(X), \beta(Y)]$$

**Proof.** Evaluating the bracket on coordinate function  $x_{ij}$ .

$$[X, Y]_e(x_{ij}) = X_e(Y(x_{ij})) - Y_e(X(x_{ij})) \quad (1)$$

Now

$$\begin{aligned} Y(x_{ij})(g) &= d\lambda_g Y_e(x_{ij}) \\ &= Y_e(x_{ij} \circ \lambda_g) \\ &= \sum_k x_{ik}(g) Y_e(x_{kj}) \end{aligned}$$

Considering the above as function of  $g$  and substituting this in Eq. (1) we get

$$\begin{aligned} [X, Y]_e(x_{ij}) &= X_e Y_e(x_{ij}) - Y_e X_e(x_{ij}) \\ &= \sum_k \{X_e(x_{ik}) Y_e(x_{kj}) - Y_e(x_{ik}) X_e(x_{kj})\} \\ &= [X_e, Y_e](x_{ij}) \end{aligned}$$

□

**Definition 4.1.** A **Lie subgroup**  $H$  of a Lie group  $G$  is a submanifold  $H \xrightarrow{\alpha} G$  where  $\alpha$  is smooth and a group homomorphism.

We say that  $H$  is closed Lie subgroup if it is Lie subgroup such that  $H \rightarrow \alpha(H)$  is a diffeomorphism.

**Example.** Consider the map  $\mathbb{R} \rightarrow S^1 \times S^1$  given by

$$t \mapsto (e^{2\pi i t}, e^{2\pi i \sqrt{2} t})$$

The image is a Lie subgroup of  $S^1 \times S^1$  but it is not a closed Lie subgroup. It is also known as “Skew-line” in the torus.

**Definition 4.2.** Let  $\mathfrak{g}, \mathfrak{h}$  be Lie algebras and  $f : \mathfrak{g} \rightarrow \mathfrak{h}$  be a vector space homomorphism. Then we say that  $f$  is a Lie algebra homomorphism if

$$f([X, Y]) = [f(X), f(Y)]$$

**Theorem 4.3.** Suppose that  $\psi : G \rightarrow H$  is a Lie group homomorphism. Let  $X$  be a left-invariant vector field on  $G$ . Extend  $d\psi(X_e) = Y_e \in T_e H$  to a left-invariant vector field  $Y$  on  $H$ . Then  $X$  and  $Y$  are  $\psi$ -related. This implies  $d\psi_e : \mathfrak{g} \rightarrow \mathfrak{h}$  is a Lie algebra homomorphism.

**Proof.** Consider the commutative diagram

$$\begin{array}{ccc} TG & \xrightarrow{d\psi} & TH \\ \uparrow X & & \uparrow Y \\ G & \xrightarrow{\psi} & H \end{array}$$

We want to show that  $Y \circ \psi = d\psi \circ Y$ . Now

$$\lambda_{\psi(g)} \circ \psi = \psi \circ \lambda_g$$

so

$$\begin{aligned} Y_{\psi(g)} &= d\lambda_{\psi(g)}Y_e \\ &= d\lambda_{\psi(g)}d\psi X_e \\ &= d(\lambda_{\psi(g)} \circ \psi)(X_e) \\ &= d(\psi \circ \lambda_g)(X_e) \\ &= d\psi d\lambda_g(X_e) \\ &= d\psi X_g \end{aligned}$$

□

**Theorem 4.4.** Let  $G, H$  be Lie groups with  $G$  connected. Let

$$\phi, \psi : G \rightarrow H$$

be homomorphism of Lie groups such that

$$d\phi = d\psi : T_e G \rightarrow T_e H$$

Then  $\phi = \psi$ .

## 5 23 Jan 2023

I missed the class. Regardless here are some definitions from Warner covered on this day.

**Definition 5.1.** Let  $M$  be a smooth  $d$ -dimensional manifold. For any integer  $1 \leq c \leq d$ , a  **$c$ -dimensional distribution**  $\mathcal{D}$  on manifold is a choice of  $c$ -dimensional subspace  $\mathcal{D}_p \subset T_p M$ .  $\mathcal{D}$  is smooth if for each  $p \in M$  there is an open neighborhood  $U$  of  $p$  and there are  $c$  smooth vector fields  $X_1, \dots, X_c$  on  $U$  which span  $\mathcal{D}_m$  for each  $p \in U$ .

We say  $\mathcal{D}$  is **involutive** if  $[X, Y] \in \mathcal{D}$  whenever  $X, Y \in \mathcal{D}$ .

**Definition 5.2.** A submanifold  $(N, \phi)$  of  $M$  is an integral manifold of a distribution  $\mathcal{D}$  if

$$d\phi(N_p) = \mathcal{D}_{\phi(p)}$$

Suppose there exists an integral manifold  $N$  for a distribution  $\mathcal{D}$ , then for the points on  $N$  the distribution  $\mathcal{D}$  is necessarily involutive. Frobenius theorem states that it is sufficient condition for a distribution to be integral.

**Theorem 5.1. (Frobenius)** Let  $\mathcal{D}$  be a  $c$ -dimensional involutive smooth distribution on  $M$ . Then there exists an integral manifold of  $\mathcal{D}$  passing through each point of  $M$ .

## Differential Ideals

Let  $E^*(M) = \bigoplus_{i=0}^{\infty} E^i(M)$  denote the graded algebra of smooth differential forms over manifold  $M$ .

**Definition 5.3.** Let  $\mathcal{D}$  be a smooth  $p$ -dimensional distribution on  $M$ . A  $q$ -form  $\omega$  is said to **annihilate**  $\mathcal{D}$  if for each  $x \in M$

$$\omega_x(v_1, \dots, v_q) = 0 \quad \text{whenever } v_1, \dots, v_q \in \mathcal{D}_x$$

A form  $\omega \in E^*(M)$  is said to annihilate  $\mathcal{D}$  if each of the homogenous components of  $\omega$  annihilate  $\mathcal{D}$ . Define

$$\mathcal{I}(\mathcal{D}) \doteq \{\omega \in E^*(M) : \omega \text{ annihilates } \mathcal{D}\}$$

**Definition 5.4.** An ideal  $\mathcal{I} \in E^*(M)$  is called a **differential ideal** if it is closed under exterior differentiation; i.e.

$$d(\mathcal{I}) \subset \mathcal{I}.$$

**Theorem 5.2.** A smooth distribution  $\mathcal{D}$  on  $M$  is involutive if and only if the ideal  $\mathcal{I}(\mathcal{D})$  is a differential ideal.

## 6 25 Jan 2023

**Theorem 6.1.** If  $\phi : H \rightarrow G$  is a homomorphism of Lie groups and if  $\omega$  is a left-invariant differential form on  $G$ , then  $\phi^*(\omega)$  is again a left-invariant form on  $H$ .

Suppose that  $\phi : H \rightarrow G$  is a homomorphism of Lie groups. Let  $\omega_1, \dots, \omega_d$  be a basis for  $E_{\text{inv}}^1(G)$ . Then

$$\mathcal{I}_\phi = \langle \{\pi_1^* \phi^*(\omega_j) - \pi_2^*(\omega_j)\} : 1 \leq j \leq d \rangle$$

is a left-invariant differential ideal of  $H \times G$ .

**Lemma 6.2.** Suppose  $X_1, \dots, X_d$  is a basis of  $\mathfrak{g}$  dual to  $\omega_1, \dots, \omega_d$ . Suppose the Lie bracket is given by

$$[X_i, X_j] = \sum c_{ij}^k X_k$$

Then the  $C^\infty$  functions  $c_{ij}^k$  are constant. Further,

$$d\omega_i = -c_{kj}^i \omega_k \wedge \omega_j$$

**Proof.** Notice that

$$\begin{aligned} d\omega_k(X_i, X_j) &= -\omega_k([X_i, X_j]) \\ &= -c_{ij}^k \end{aligned}$$

which is a constant because a left-invariant 1-form evaluated on a left-invariant vector field



is a constant.  $\square$

**Remark.** The  $c_{ij}^k$  are called the structural constants of  $G$  with respect to the basis  $\{X_i\}$  of  $\mathfrak{g}$ .

**Proof.** Theorem 4.4. Notice that  $\mathcal{I}_\psi = \mathcal{I}_\phi$  since  $d\phi = d\psi$  and these are invariant differential ideals; hence integral manifolds of  $\mathcal{I}_\phi$  and  $\mathcal{I}_\psi$  passing through  $(e, e)$  are the same. Thus,  $\phi = \psi$ .  $\square$

**Lemma 6.3.** Suppose  $G$  is any Hausdorff topological group which is connected. Suppose  $e \in U \subset G$  is any open set. Then

$$G = \bigcup_{n \geq 1} U^n$$

where  $U^n = \{x_1 \cdots x_n | x_i \in U\}$

**Proof.** Since  $e \in U$  is open,  $U^{-1} = \{x^{-1} | x \in U\}$  is also an open neighborhood of  $e$ . Let  $V = U \cap U^{-1}$ . Note that

$$H \doteq \bigcup_{n \geq 1} V^n$$

is a subgroup of  $G$ , and it is open. Since the cosets  $gH$  are also open it follows that  $G = \bigcup_g gH$  being connected must be  $H$ .  $\square$

**Theorem 6.4.** Let  $G$  be a Lie group and  $\mathfrak{h} \subset \mathfrak{g}$  be a Lie subalgebra of  $\mathfrak{g}$ . Then there exists connected Lie subgroup  $H$  of  $G$  such that  $T_e H = \mathfrak{h}$ .

**Proof.** Consider the distribution  $\mathcal{D}$  defined as

$$\mathcal{D}_g = \{X_g | X \in \mathfrak{h}\}$$

on  $G$ . Suppose  $X_1, \dots, X_c$  is a basis of  $\mathfrak{h}$ . Then  $\mathcal{D}$  is generated by  $X_1, \dots, X_c$  and  $\mathcal{D}$  is involutive.  $\square$

**Corollary.** (a) There is a one-to-one correspondence between connected Lie subgroups of  $G$  and Lie subalgebras of  $\mathfrak{g}$ .

(b) Suppose  $(H, i) \leftrightarrow \mathfrak{h} \subset \mathfrak{g}$ . Then  $(H, i)$  is an embedded manifold if and only if  $H$  is closed.

**Theorem 6.5.** Suppose that  $A \subset G$  is an abstract subgroup of  $G$  and if  $A$  has a manifold structure such that  $(A, i) \rightarrow G$  is a submanifold. Then the manifold structure is unique,  $A$  is a Lie group and hence  $(A, i)$  is a Lie subgroup of  $G$ .

**Theorem 6.6. (Adó)** Suppose that  $\mathfrak{g}$  is a finite dimensional Lie algebra. Then  $\mathfrak{g}$  can be

realized as a subalgebra of  $\mathfrak{gl}(n, \mathbb{R})$ .

Given any connected Lie group  $G$ , it has a universal cover  $\tilde{G} \xrightarrow{\pi} G$ . Choose  $\tilde{e} \in \pi^{-1}(e) \in \tilde{G}$  such that the following diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \longrightarrow & \tilde{G} \\ \pi \times \pi \downarrow & & \downarrow \pi \\ G \times G & \longrightarrow & G \end{array}$$

commutes.

**7 30 Jan 2023**

**Lemma 7.1.** Suppose that  $G$  is a connected Lie group. Then  $\pi_1(G)$  is abelian.

**Proof.** Suppose  $\sigma, \tau : I \rightarrow G$  be two loops. Define  $\sigma \cdot \tau$  by

$$(\sigma \cdot \tau)(s) = \sigma(s) \cdot \tau(s)$$

Then we have

$$\sigma * \tau \cong \sigma \cdot \tau$$

where  $*$  denote the product in the fundamental group  $\pi_1(G)$  (given by concatenation) and  $\cong$  denotes equivalent in homotopy. Also,

$$\sigma \cdot \tau \cdot \sigma^{-1} \cong \tau$$

which implies  $\sigma\tau \cong \tau \cdot \sigma$  □

**Theorem 7.2.** Suppose that  $G$  and  $H$  are Lie groups with Lie algebras  $\mathfrak{g}$  and  $\mathfrak{h}$  with  $G$  simply connected. Let  $\tilde{\phi} : \mathfrak{g} \rightarrow \mathfrak{h}$  be a Lie algebra homomorphism. Then there exists a Lie group homomorphism

$$\phi : G \rightarrow H$$

such that  $d\phi_e : T_e(G) = \mathfrak{g} \rightarrow \mathfrak{h} = T_eH$  is equal to  $\tilde{\phi}$ .

**Proof.** Let  $\{\omega_i\}$  be a basis for invariant differential forms in  $E^1(H)$ . Let  $\mathcal{I}$  be the ideal generated by  $\{\pi_1^* \tilde{\phi}^*(\omega_j) - \pi_2^*(\omega_j) | 1 \leq j \leq d\}$ . Then  $\mathcal{I}$  is an invariant differential ideal of  $G \times H$ , so it comes from vanishing of an integrable submanifold of  $G \times H$  passing through  $(e, e)$ .

Then  $M$  is a Lie subgroup of  $G \times H$  and  $M \xrightarrow{p} G$  obtained by restriction of  $\pi_1$  is a group homomorphism and also a local diffeomorphism. So  $p : M \rightarrow G$  is a covering projection but  $G$  is simply connected so  $p$  is a diffeomorphism

$$G \xrightarrow{p^{-1}} M \hookrightarrow G \times H \rightarrow H.$$

□

**Corollary.** 1. Suppose  $\mathfrak{g} \cong \mathfrak{h}$  as Lie algebras and  $G$  and  $H$  are simply connected. Then  $G \cong H$  as Lie groups.

2. There exists a one-to-one correspondence between (finite dimensional) Lie algebras and simply connected Lie groups.

3. The differential structure of a Lie group is determined by its Lie algebra.

If  $G$  is a topological group which is locally Euclidean, does it support a Lie group structure? The answer is yes but the proof is quite difficult.

## Exponential map

Let  $X$  be a left-invariant vector field on  $G$ . We have a Lie algebra homomorphism

$$\begin{aligned} \text{Lie}(\mathbb{R}) &\cong \mathbb{R} \rightarrow \mathfrak{g} \\ c \frac{d}{dt} &\rightarrow cX \end{aligned}$$

This yields a Lie group homomorphism

$$\begin{aligned} \mathbb{R} &\xrightarrow{\exp_X} G \\ x &\mapsto \exp_X(x) \end{aligned}$$

then  $d\exp_X(c \frac{d}{dt}) = cX$ . The map

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\exp} G \\ X &\mapsto \exp_X(1) \end{aligned}$$

is called the **exponential map**.

**Theorem 7.3.** Let  $X \in \text{Lie}(G)$ . Then

1.  $\exp(tX) = \exp_X(t)$
2.  $\exp(t_1X + t_2X) = \exp(t_1X) \cdot \exp(t_2X)$
3.  $\exp(-tX) = (\exp(tX))^{-1}$
4.  $\exp : \mathfrak{g} \rightarrow G$  is smooth and  $d\exp : T_0\mathfrak{g} \rightarrow T_eG = \mathfrak{g}$  is the identity map
5.  $\lambda_g \circ \exp_X : \mathbb{R} \rightarrow G$  is the unique integral curve of  $X$  which is based at  $g$ .
6. The left-invariant vector fields are complete, i.e. their integral curves exist for all time.
7. The one-parameter group of diffeomorphism  $\psi_{X,t}$  for  $t \in \mathbb{R}$  is given by

$$\psi_{X,t} = \rho_{\exp_X(t)}$$

where  $\rho_g$  denote right-multiplication by  $g$ .

**Theorem 7.4.** Suppose  $\psi : H \rightarrow G$  is a Lie group homomorphism. Then

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{d\psi} & \mathfrak{g} \\ \exp \downarrow & & \downarrow \exp \\ H & \xrightarrow{\psi} & G \end{array}$$

commutes.

[DO THIS COMMUTATIVE DIAGRAM.]

**8 1 Feb 2023**

**Theorem 8.1.** Suppose that  $\mathfrak{h} \subset \mathfrak{g}$  is a Lie subalgebra where  $\text{Lie}(G)$ . Let  $A \subset G$  an abstract subgroup such that there exists a neighbourhood  $0 \in V \subset \mathfrak{g}$  such that

$$\exp(V \cap \mathfrak{h}) = U \cap H$$

for some neighborhood  $e \in U \subset G$ . Then  $H$  has a unique manifold structure such that  $(H, i) \hookrightarrow G$  is an embedded submanifold of  $G$  and  $H$  is closed in subset topology.

**Remark.** Lines with irrational slope in torus doesn't satisfy the hypothesis.

## Matrix exponentiation

Recall that  $\mathfrak{gl}(n, \mathbb{R})$  denotes the Lie algebra of  $n \times n$  matrices over  $\mathbb{R}$  and similarly for  $\mathfrak{gl}(n, \mathbb{C})$ .

**Definition 8.1.** Define a map

$$\begin{aligned} \mathfrak{gl}(n, \mathbb{C}) &\rightarrow \text{GL}(n, \mathbb{C}) \\ A &\mapsto e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \end{aligned}$$

It can be proved that the series is convergent with sup norm and further we have a lemma

**Lemma 8.2.** If  $AB = BA$  then

$$e^{A+B} = e^A e^B$$

which can be used to prove that  $e^A \in \text{GL}(n, \mathbb{C})$  so the definition makes sense.

Fix  $A$  and consider the function

$$\mathbb{R} \ni t \mapsto e^{tA} \in \text{GL}(n, \mathbb{C})$$

then its derivative is

$$\left. \frac{d}{dt} \right|_{t=0} e^{tA} = A$$

because we can differentiate term by term in uniform convergence. This confirms Theorem 7.3 4th part.

The left-invariant vector field given by  $A \in \mathfrak{gl}(n, \mathbb{C})$  is just multiplication by  $A$  on the right. Thus,  $t \mapsto e^{tA}$  is the integral curve associated to the vector field  $A \in \mathfrak{gl}(n, \mathbb{C})$  based at  $I$ . Hence, this is the exponential map in the cases of  $GL(n, \mathbb{C})$ .

**Theorem 8.3.** The exponential map  $\exp : \mathfrak{g} \rightarrow G$  is smooth.

**Proof.** Let  $X \in \mathfrak{g}$  and consider the map

$$\begin{aligned} V : G \times \mathfrak{g} &\rightarrow TG \times \mathfrak{g} \\ (g, X) &\mapsto (Xg, 0) \end{aligned}$$

then  $V$  is smooth. Also,  $V$  is left-invariant on  $G \times \mathfrak{g}$ . Consider the integral curve  $\gamma$  based at  $(g, X)$  of  $V$ . Then

$$\gamma_V(t) = (g \exp_X(t), X)$$

because of left invariance so

$$\gamma_V(1) = (g \exp(X), X)$$

$$\begin{aligned} G \times \mathfrak{g} &\xrightarrow{\gamma_V(1)} G \times \mathfrak{g} \xrightarrow{\pi} G \\ (e, X) &\mapsto \gamma_V(1) \rightarrow \exp(X) \end{aligned}$$

□

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Note that exponential map commutes with Lie group homomorphisms. Using Ado's theorem we get that for any Lie group

$$\begin{array}{ccc} G & \xrightarrow{\psi} & GL(n, \mathbb{C}) \\ \uparrow \exp & & \uparrow \exp \\ \mathfrak{g} & \longrightarrow & \mathfrak{gl}(n, \mathbb{C}) \end{array}$$

Consider the Lie group  $SL(n, \mathbb{C}) = \{X \in GL(n, \mathbb{C}) \mid \det(X) = 1\}$ , for any  $A \in \mathfrak{gl}(n, \mathbb{C})$  upper triangular with diagonal entries  $\lambda_1, \dots, \lambda_n$  then

$$\det(e^A) = e^{\lambda_1 + \dots + \lambda_n} = e^{\text{tr}(A)}$$

Now  $\mathfrak{sl}(n, \mathbb{C}) = \{A \in \mathfrak{gl}(n, \mathbb{C}) \mid \text{tr}(A) = 0\}$ , then  $\mathfrak{sl}(n, \mathbb{C})$  is a Lie subalgebra of  $\mathfrak{gl}(n, \mathbb{C})$  and exponential maps  $\mathfrak{sl}(n, \mathbb{C})$  to the Lie subgroup  $SL(n, \mathbb{C})$ . As  $SL(n, \mathbb{C})$  is a closed subgroup of  $GL(n, \mathbb{C})$  and dimension  $2(n^2 - 1)$ . Using Theorem 8.1 on an appropriate neighborhood we can complete the proof.

Lie subgroup	Lie subalgebra $\mathfrak{gl}(n, \mathbb{C})$
$U(n) \longleftrightarrow$	$u(n) = \text{skew-Hermitian matrices}$
$SU(n) \longleftrightarrow$	$su(n) = \text{skew-Hermitian} + \text{trace} = 0$

Prove the above given correspondence using this lemma (TO DO).

**Lemma 9.1.** Suppose that  $P \in GL(n, \mathbb{C})$  and  $A \in \mathfrak{gl}(n, \mathbb{C})$ , then

$$Pe^AP^{-1} = e^{PAP^{-1}}.$$

**Theorem 9.2 (Baker-Campbell-Hausdorff formula).** Let  $\mathfrak{g}$  be a Lie algebra corresponding to a connected Lie group  $G$ . Then in a neighborhood  $U$  of the identity, the multiplication  $U \times U \rightarrow G$  is completely determined by Lie algebra structure of  $\mathfrak{g}$ . There is a formula for  $Z = Z(X, Y)$ ,  $X, Y \in V \subset \mathfrak{g}$ , where  $e^X \cdot e^Y = e^Z$

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \dots$$

Consider

$$\begin{aligned} e^{tX} \cdot e^{tY} &= \left( \sum \frac{t^k X^k}{k!} \right) \left( \sum \frac{t^l Y^l}{l!} \right) \\ &= \sum_{m \geq 0} \left( \sum_{k+l=m} \frac{X^k Y^l}{k!l!} \right) t^m \end{aligned}$$

Suppose  $Z = tZ_1 + t^2Z_2 + t^3Z_3 \dots$ , then

$$\begin{aligned} e^Z &= 1 + (tZ_1 + t^2Z_2 + \dots) + \frac{(tZ_1 + t^2Z_2 + \dots)^2}{2!} + \dots \\ &= 1 + t(Z_1) + t^2 \left( Z_2 + \frac{Z_1^2}{2!} \right) \end{aligned}$$

So we get  $Z_1 = X + Y$ ,

$$\begin{aligned} \frac{X^2}{2!} + XY + \frac{Y^2}{2!} &= Z_2 + \frac{Z_1^2}{2!} \\ &= Z_2 + \frac{1}{2}(X^2 + XY + YX + Y^2) \end{aligned}$$

so  $Z_2 = XY - \frac{1}{2}(XY + YX) = \frac{1}{2}(XY - YX) = \frac{1}{2}[X, Y]$

**Theorem 9.3.** Suppose that  $\psi : R \rightarrow G$  is a continuous homomorphism. The  $\psi$  is smooth.

**Proof.** It is enough to show that  $\psi$  is smooth at 0. Let  $U$  be a star-like neighborhood of  $0 \in \mathfrak{g}$  such that  $\exp|_U : U \rightarrow G$  is a diffeomorphism onto  $\exp(U)$ . Let  $U' = \{\frac{X}{2} | X \in U\}$ . Choose  $Y \in U'$  and let  $\psi(t_0) = \exp(Y)$ . Choose  $t_0 > 0$  such that

$$\psi([-t_0, t_0]) \subset \exp(U')$$

Let  $n \geq 2$ , and suppose that  $X \in U'$  such that  $\exp(X) = \psi(\frac{t_0}{n})$ . Claim  $nX = Y$   $\square$