Lie Groups

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1 4th January 23

One can study Lie Groups from several points of view. The course is aimed to understand the structure of Lie Groups.

Definition 1.1. A smooth manifold M is a Hausdorff space which is locally Euclidean with a smooth atlas i.e. (i) given any $x \in M$, $\exists$ a chart (U, ϕ), $x \in U \subset M$ with $\phi : U \to \phi(U)$ open in $\mathbb{R}^m$.

(ii) We have collection $\{(U, \phi)\}$ of charts such that

$$\phi(U \cap V) \xrightarrow{\psi \circ \phi^{-1}} \psi(U \cap V)$$

is a diffeomorphism.

Suppose $f : M \to N$ is a continuous map between manifolds. We say that f is smooth if for $(U, \phi) \in \amalg(M)$, $(V, \psi) \in \amalg(N)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}$ is smooth.

TO DO : Construction of tangent bundle and vector bundle

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Definition 2.1. G is a Lie group if

1. G is a smooth manifold
2. G is also a group s.t

$$\begin{aligned} \mu : G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

and

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth maps.

Definition 2.2. A real (or complex) vector space V together with a bilinear map

$$[,] : V \times V \to V$$

is called a **Lie Algebra** if

1. $[X, Y] = -[Y, X]$ - skew symmetry
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ - Jacobi identity

Example.
1. $(\mathbb{R}, +), (\mathbb{C}, +)$, V any f.d vector space over $\mathbb{R}$ or $\mathbb{C}$.
2. $(\mathbb{R}^\times, \cdot), (\mathbb{C}^\times, \cdot)$
3. $S^1 = \{z \in \mathbb{C}^\times || z| = 1\}$
4. $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$
5. $\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}^n/\mathbb{Z}^n) \cong (S^1)^n$
6. Suppose $\Gamma \subset V$ is a discrete subgroup. Then V/Γ is a Lie group.
7. $N =$ unipotent upper triangular matrices, $B =$ upper triangular matrices. As manifolds $N \cong \mathbb{R}^{\binom{n}{2}}$ and $B \cong (\mathbb{R}^\times)^n \times N$.
8. $\mathrm{SL}_n(\mathbb{R}) = \{X \in \mathrm{GL}_n(\mathbb{R}) | \det X = 1\}$, $\mathrm{SL}_n(\mathbb{C})$.
9. $O(n)$, $SO(n)$.
10. $U(n)$, $SU(n)$.
11. $\mathbb{H}^\times$, S^3 with quaternion multiplication.
12. $Sp(n) = \{X \in \mathrm{GL}_n(\mathbb{R}) | X$ preserves quaternion structure as a subset of $\mathrm{Aut}_{\mathbb{H}} \mathbb{H}^n\}$

Problem. $V/\Gamma \cong \mathbb{R}^k \times (S^1)^{n-k}$ for n-dimensional vector space V.

Theorem 2.1. Suppose G is a compact, connected, simple Lie group. Then G is locally isomorphic to

1. $SU(n), n \geq 2$ denoted by A_{n-1}
2. $SO(2n+1), n \geq 2$ denoted by B_n
3. $Sp(n), n \geq 1$ denoted by C_n
4. $SO(2n), n \geq 2$ denoted by D_n

or one of the following exceptional Lie group G_2, F_4, E_6, E_7, E_8.

Problem. Prove that $\mathrm{SL}_n(\mathbb{R})$ and $O(n)$ are smooth manifold, hence Lie groups.

Examples of Lie algebra -

Example. 1. $(V, [\cdot,\cdot] \equiv 0)$ is called trivial Lie algebra.

2. $(\mathfrak{gl}_n(\mathbb{R}), [A, B] = AB - BA)$, $\mathfrak{gl}_n(\mathbb{C})$

3. $\mathfrak{sl}_n(\mathbb{R})$ ($\mathfrak{sl}_n(\mathbb{C})$) is the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ ($\mathfrak{gl}_n(\mathbb{C})$) consisting of trace 0.

4. $\mathfrak{so}_n$ is Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ consisting of skew-symmetric matrices.

Definition 2.3. A vector field X on a Lie group G is called left invariant if $(L_g)_*(X_h) = X_{gh}$

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Recall $\mathbb{H} = \{a + bi + cj + dk : (a, b, c, d) \in \mathbb{R}^4,\ i^2 = -1, j^2 = -1, k^2 = -1, ij = k, jk = l, ki = j\}$ is the quaternion division algebra with the norm

$$||a + bi + cj + dk||^2 = a^2 + b^2 + c^2 + d^2$$

which satisfies $||q_1 \cdot q_2|| = ||q_1|| \cdot ||q_2||$

We can put this multiplication on $S^3 \cong SU(2)$ to get a compact Lie group. To get the isomorphism $SU(2) \cong S^3$, we define a map

$$\begin{aligned} \phi : S^3 &\to SU(2) \\ (a, b, c, d) &\mapsto \begin{bmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{bmatrix} \end{aligned}$$

which is an algebra isomorphism.

Definition 3.1. The Lie algebra of G is the space of all left-invariant vector fields on G.

We have an isomorphism

$$\begin{aligned} \mathfrak{g} = \mathrm{Lie}(G) &\to T_e G \\ X &\mapsto X_e \end{aligned}$$

Example. Let $G = \mathbb{R}^n$, with identity element $0 \in \mathbb{R}^n$ and left-invariant vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$. Then the Lie bracket is

$$[\cdot,\cdot] \equiv 0$$

Remark. In general for any abelian Lie group G, the Lie bracket is $[\cdot,\cdot] \equiv 0$.

Theorem 3.1. Let G be a connected Lie group. Then

1. $\mathrm{Lie}(G) = \mathfrak{g}$ is isomorphic as a vector space to $T_e(G)$.

2. Left-invariant vector fields are smooth.

3. Lie(G) is closed under Lie bracket.

Proof. 1. Let X be a left-invariant vector field on G. We need to show that Xf is smooth for each $f \in C^\infty(G)$.

$$\begin{aligned}(Xf)(g) &= X_g f \\ &= (d\lambda_g X_e)f \\ &= X_e(f \circ \lambda_g)\end{aligned}$$

To show that Xf is smooth, it suffices to show that $X_e(f \circ \lambda_g)$ is smooth. We realize $X_e(f \circ \lambda_g)$ as evaluation of a smooth function on a smooth function.

Let Y be a smooth vector field on G such that $Y_e = X_e$

$$Y_e(f \circ \lambda_g) = X_e(f \circ \lambda_g)$$

We look at λ_g as the composition of

$$\begin{gathered} G \xrightarrow{i_g^2} G \times G \xrightarrow{\mu} G \\ x \mapsto (g, x) \mapsto gx \end{gathered}$$

Regard Y as the vector field $(0, Y)$ on $G \times G$. Now

$$\begin{aligned}(0, Y)(f \circ \mu) \circ i_e^1(g) &= (0, Y)_{(g,e)}(f \circ \mu) \\ &= 0_g(f \circ \mu \circ i_g^1) + Y_e(f \circ \mu \circ i_g^2) \\ &= Y_e(f \circ \lambda_g)\end{aligned}$$

which proves the smoothness.

2. Let X, Y left-invariant vector fields on G. We must show that $[X, Y]$ is a left-invariant vector field.

$$\begin{aligned}d\lambda_g([X, Y]_e)f &= [X, Y]_g f \\ &= [X, Y]_e(f \circ \lambda_g) \\ &= X_e(Y(f \circ \lambda_g)) - Y_e(X(f \circ \lambda_g)) \\ &= X_e(d\lambda_g(Yf)) - Y_e(d\lambda_g(Yf))\end{aligned}$$

□

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Lemma 4.1. Suppose $\psi : M \to N$ is a smooth map. Let X_1, X_2 be vector fields on M, Y_1, Y_2 be vector fields on N such that X_i is ψ-related to Y_i. Then $[X_1, X_2]$ is ψ-related to $[Y_1, Y_2]$.

Proof. Notice that

$$\begin{aligned}d\psi[X_1, X_2](f) &= [X_1, X_2](f \circ \psi) \\ &= X_1(X_2 f \circ \psi) - X_2(X_1 f \circ \psi) \\ &= X_1(d\psi X_2 f) - X_2(Y_1 f \circ \psi) \\ &= X_1(Y_2 f \circ \psi) - X_2(Y_1 f \circ \psi) \\ &= d\psi X_1(Y_2 f) - d\psi X_2(Y_1 f) \\ &= Y_1 Y_2 f \circ \psi - Y_2 Y_1 f \circ \psi \\ &= [Y_1, Y_2](f) \circ \psi\end{aligned}$$

$\square$

This lemma proves that the set of left-invariant vector fields forms a Lie algebra.

Consider the Lie group $\mathrm{GL}_n(\mathbb{R})$. We want to verify the Lie algebra structure on $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ with the isomorphism

$$\begin{aligned} Lie(\mathrm{GL}_n(\mathbb{R})) &\to \mathfrak{gl}_n(\mathbb{R}) \\ X &\overset{\beta}{\mapsto} X_e \end{aligned}$$

Lemma 4.2.

$$\beta([X,Y]) = [\beta(X), \beta(Y)]$$

Proof. Evaluating the bracket on coordinate function x_{ij}.

$$[X,Y]_e(x_{ij}) = X_e(Y(x_{ij})) - Y_e(X(x_{ij})) \tag{1}$$

Now

$$\begin{aligned} Y(x_{ij})(g) &= d\lambda_g Y_e(x_{ij}) \\ &= Y_e(x_{ij} \circ \lambda_g) \\ &= \sum_k x_{ik}(g) Y_e(x_{kj}) \end{aligned}$$

Considering the above as function of g and substituting this in Eq. (1) we get

$$\begin{aligned} [X,Y]_e(x_{ij}) &= X_e Y_e(x_{ij}) - Y_e X_e(x_{ij}) \\ &= \sum_k \{X_e(x_{ik})Y_e(x_{kj}) - Y_e(x_{ik})X_e(x_{kj})\} \\ &= [X_e, Y_e](x_{ij}) \end{aligned}$$

$\square$

Definition 4.1. A **Lie subgroup** H of a Lie group G is a submanifold $H \xrightarrow{\alpha} G$ where α is smooth and a group homomorphism.

We say that H is closed Lie subgroup if it is Lie subgroup such that $H \to \alpha(H)$ is a diffeomorphism.

Example. Consider the map $\mathbb{R} \to S^1 \times S^1$ given by

$$t \mapsto (e^{2\pi i t}, e^{2\pi i \sqrt{2} t})$$

The image is a Lie subgroup of $S^1 \times S^1$ but it is not a closed Lie subgroup. It is also known as "Skew-line" in the torus.

Definition 4.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $f : \mathfrak{g} \to \mathfrak{h}$ be a vector space homomorphism. Then we say that f is a Lie algebra homomorphism if

$$f([X,Y]) = [f(X), f(Y)]$$

Theorem 4.3. Suppose that $\psi : G \to H$ is a Lie group homomorphism. Let X be a left-invariant vector field on G. Extend $d\psi(X_e) = Y_e \in T_e H$ to a left-invariant vector field Y on H. Then X and Y are ψ-related. This implies $d\psi_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd} TG \arrow[r, "d\psi"] \arrow[d] & TH \arrow[d] \\ G \arrow[r, "\psi"] \arrow[u, bend left, "X"] & H \arrow[u, bend right, "Y"'] \end{tikzcd}$$

We want to show that $Y \circ \psi = d\psi \circ Y$. Now

$$\lambda_{\psi(g)} \circ \psi = \psi \circ \lambda_g$$

so

$$\begin{aligned} Y_{\psi(g)} &= d\lambda_{\psi(g)} Y_e \\ &= d\lambda_{\psi(g)} d\psi X_e \\ &= d(\lambda_{\psi(g)} \circ \psi)(X_e) \\ &= d(\psi \circ \lambda_g)(X_e) \\ &= d\psi d\lambda_g(X_e) \\ &= d\psi X_g \end{aligned}$$

□

Theorem 4.4. Let G, H be Lie groups with G connected. Let

$$\phi, \psi : G \to H$$

be homomorphism of Lie groups such that

$$d\phi = d\psi : T_e G \to T_e H$$

Then $\phi = \psi$.

5 23 Jan 2023

I missed the class. Regardless here are some definitions from Warner covered on this day.

Definition 5.1. Let M be a smooth d-dimensional manifold. For any integer $1 \le c \le d$, a **c-dimensional distribution** $\mathscr{D}$ on manifold is a choice of c-dimensional subspace $\mathscr{D}_p \subset T_pM$. $\mathscr{D}$ is smooth if for each $p \in M$ there is an open neighborhood U of p and there are c smooth vector fields $X_1, \ldots, X_c$ on U which span $\mathscr{D}_m$ for each $p \in U$.

We say $\mathscr{D}$ is **involutive** if $[X, Y] \in \mathscr{D}$ whenever $X, Y \in \mathscr{D}$.

Definition 5.2. A submanifold (N, ϕ) of M is an integral manifold of a distribution $\mathscr{D}$ if

$$d\phi(N_p) = \mathscr{D}_{\phi(p)}$$

Suppose there exists an integral manifold N for a distribution $\mathscr{D}$, then for the points on N the distribution $\mathscr{D}$ is necessarily involutive. Frobenius theorem states that it is sufficient condition for a distribution to be integral.

Theorem 5.1. **(Frobenius)** Let $\mathscr{D}$ be a c-dimensional involutive smooth distribution on M. Then there exists an integral manifold of $\mathscr{D}$ passing through each point of M.

Differential Ideals

Let $E^*(M) = \bigoplus_{i=0}^{\infty} E^i(M)$ denote the graded algebra of smooth differential forms over manifold M.

Definition 5.3. Let $\mathscr{D}$ be a smooth p-dimensional distribution on M. A q-form ω is said to **annihilate** $\mathscr{D}$ if for each $x \in M$

$$\omega_x(v_1, \ldots, v_q) = 0 \quad \text{whenever } v_1, \ldots, v_q \in \mathscr{D}_x$$

A form $\omega \in E^*(M)$ is said to annihilate $\mathscr{D}$ if each of the homogenous components of ω annihilate $\mathscr{D}$. Define

$$\mathscr{I}(\mathscr{D}) \doteqdot \{\omega \in E^*(M) : \omega \text{ annihilates } \mathscr{D}\}$$

Definition 5.4. An ideal $\mathscr{I} \in E^*(M)$ is called a **differential ideal** if it is closed under exterior differentiation; i.e.

$$d(\mathscr{I}) \subset \mathscr{I}.$$

Theorem 5.2. A smooth distribution $\mathscr{D}$ on M is involutive if and only if the ideal $\mathscr{I}(\mathscr{D})$ is a differential ideal.

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Theorem 6.1. If $\phi : H \to G$ is a homomorphism of Lie groups and if ω is a left-invariant differential form on G, then $\phi^*(\omega)$ is again a left-invariant form on H.

Suppose that $\phi : H \to G$ is a homomorphism of Lie groups. Let $\omega_1, \ldots, \omega_d$ be a basis for $E^1_{\text{inv}}(G)$. Then

$$\mathcal{I}_\phi = \langle \{\pi_1^* \phi^*(\omega_j) - \pi_2^*(\omega_j)\} : 1 \leq j \leq d \rangle$$

is a left-invariant differential ideal of $H \times G$.

Lemma 6.2. Suppose $X_1, \ldots, X_d$ is a basis of $\mathfrak{g}$ dual to $\omega_1, \ldots, \omega_d$. Suppose the Lie bracket is given by

$$[X_i, X_j] = \sum c_{ij}^k X_k$$

Then the C^∞ functions c_{ij}^k are constant. Further,

$$d\omega_i = -c_{kj}^i \omega_k \wedge \omega_j$$

Proof. Notice that

$$\begin{aligned} d\omega_k(X_i, X_j) &= -\omega_k([X_i, X_j]) \\ &= -c_{ij}^k \end{aligned}$$

which is a constant because a left-invariant 1-form evaluated on a left-invariant vector field is a constant. $\square$

Remark. The c_{ij}^k are called the structural constants of G with respect to the basis $\{X_i\}$ of $\mathfrak{g}$.

Proof. Theorem 4.4. Notice that $\mathcal{I}_\psi = \mathcal{I}_\phi$ since $d\phi = d\psi$ and these are invariant differential ideals; hence integral manifolds of $\mathcal{I}_\phi$ and $\mathcal{I}_\psi$ passing through (e, e) are the same. Thus, $\phi = \psi$. $\square$

Lemma 6.3. Suppose G is any Hausdorff topological group which is connected. Suppose $e \in U \subset G$ is any open set. Then

$$G = \bigcup_{n \geq 1} U^n$$

where $U^n = \{x_1 \cdots x_n | x_i \in U\}$

Proof. Since $e \in U$ is open, $U^{-1} = \{x^{-1} | x \in U\}$ is also an open neighborhood of e. Let $V = U \cap U^{-1}$. Note that

$$H \doteqdot \bigcup_{n \geq 1} V^n$$

is a subgroup of G, and it is open. Since the cosets gH are also open it follows that $G = \cup_g H$ being connected must be H.

$\square$

Theorem 6.4. Let G be a Lie group and $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists connected Lie subgroup H of G such that $T_e H = \mathfrak{h}$.

Proof. Consider the distribution $\mathscr{D}$ defined as

$$\mathscr{D}_g = \{X_g | X \in \mathfrak{h}\}$$

on G. Suppose $X_1, \ldots, X_c$ is a basis of $\mathfrak{h}$. Then $\mathscr{D}$ is generated by $X_1, \ldots, X_c$ and $\mathscr{D}$ is involutive. □

Corollary. (a) There is a one-to-one correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.

(b) Suppose $(H, i) \leftrightarrow \mathfrak{h} \subset \mathfrak{g}$. Then (H, i) is an embedded manifold if and only if H is closed.

Theorem 6.5. Suppose that $A \subset G$ is an abstract subgroup of G and if A has a manifold structure such that $(A, i) \to G$ is a submanifold. Then the manifold structure is unique, A is a Lie group and hence (A, i) is a Lie subgroup of G.

Theorem 6.6. (Adó) Suppose that $\mathfrak{g}$ is a finite dimensional Lie algebra. Then $\mathfrak{g}$ can be realized as a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

Given any connected Lie group G, it has a universal cover $\tilde{G} \xrightarrow{\pi} G$. Choose $\tilde{e} \in \pi^{-1}(e) \in \tilde{G}$ such that the following diagram

$$\begin{tikzcd}
\tilde{G} \times \tilde{G} \arrow[r] \arrow[d, "\pi \times \pi"'] & \tilde{G} \arrow[d, "\pi"] \\
G \times G \arrow[r] & G
\end{tikzcd}$$

commutes.

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Lemma 7.1. Suppose that G is a connected Lie group. Then $\pi_1(G)$ is abelian.

Proof. Suppose $\sigma, \tau : I \to G$ be two loops. Define $\sigma \cdot \tau$ by

$$(\sigma \cdot \tau)(s) = \sigma(s) \cdot \tau(s)$$

Then we have

$$\sigma * \tau \cong \sigma \cdot \tau$$

where $*$ denote the product in the fundamental group $\pi_1(G)$ (given by concatenation) and $\cong$ denotes equivalent in homotopy. Also,

$$\sigma \cdot \tau \cdot \sigma^{-1} \cong \tau$$

which implies $\sigma\tau \cong \tau \cdot \sigma$ □

Theorem 7.2. Suppose that G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ with G simply connected. Let $\tilde{\phi}:\mathfrak{g}\to\mathfrak{h}$ be a Lie algebra homomorphism. Then there exists a Lie group homomorphism

$$\phi : G \to H$$

such that $d\phi_e : T_e(G) = \mathfrak{g} \to \mathfrak{h} = T_e H$ is equal to $\tilde{\phi}$.

Proof. Let $\{\omega_i\}$ be a basis for invariant differential forms in $E^1(H)$. Let $\mathscr{I}$ be the ideal generated by $\{\pi_1^*\tilde{\phi}^*(\omega_j) - \pi_2^*(\omega_j) | 1 \le j \le d\}$. Then $\mathscr{I}$ is an invariant differential ideal of $G\times H$, so it comes from vanishing of an integrable submanifold of $G \times H$ passing through (e,e).

Then M is a Lie subgroup of $G \times H$ and $M \xrightarrow{p} G$ obtained by restriction of π_1 is a group homomorphism and also a local diffeomorphism. So $p : M \to G$ is a covering projection but G is simply connected so p is a diffeomorphism

$$G \xrightarrow{p^{-1}} M \hookrightarrow G \times H \to H.$$

□

Corollary.

1. Suppose $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras and G and H are simply connected. Then $G \cong H$ as Lie groups.
2. There exists a one-to-one correspondence between (finite dimensional) Lie algebras and simply connected Lie groups.
3. The differential structure of a Lie group is determined by its Lie algebra.

If G is a topological group which is locally Euclidean, does it support a Lie group structure? The answer is yes but the proof is quite difficult.

Exponential map

Let X be a left-invariance vector field on G. We have a Lie algebra homomorphism

$$\begin{aligned} \mathrm{Lie}(\mathbb{R}) \cong \mathbb{R} &\to \mathfrak{g} \\ c\frac{d}{dt} &\to cX \end{aligned}$$

This yields a Lie group homomorphism

$$\begin{aligned} \mathbb{R} &\xrightarrow{\exp_X} G \\ x &\mapsto \exp_X(x) \end{aligned}$$

then $d\exp_X(c\frac{d}{dt}) = cX$. The map

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\exp} G \\ X &\mapsto \exp_X(1) \end{aligned}$$

is called the **exponential map**.

Theorem 7.3. Let $X \in \mathrm{Lie}(G)$. Then

1. $\exp(tX) = \exp_X(t)$
2. $\exp(t_1 X_1 + t_2 X) = \exp(t_1 X) \cdot \exp(t_2 X)$
3. $\exp(-tX) = (\exp(tX))^{-1}$
4. $\exp : \mathfrak{g} \to G$ is smooth and $d\exp : T_0\mathfrak{g} \to T_e G = \mathfrak{g}$ is the identity map
5. $\lambda_g \circ \exp_X : \mathbb{R} \to G$ is the unique integral curve of X which is based at g.
6. The left-invariant vector fields are complete, i.e. their integral curves exist for all time.
7. The one-parameter group of diffeomorphism $\psi_{X,t}$ for $t \in \mathbb{R}$ is given by

$$\psi_{X,t} = \rho_{exp_X(t)}$$

where ρ_g denote right-multiplication by g.

Theorem 7.4. Suppose $\psi : H \to G$ is a Lie group homomorphism. Then

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "d\psi"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ H \arrow[r, "\psi"] & G \end{tikzcd}$$

commutes.

[DO THIS COMMUTATIVE DIAGRAM.]

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Theorem 8.1. Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra where $\mathrm{Lie}(G)$. Let $A \subset G$ an abstract subgroup such that there exists a neighbourhood $0 \in V \subset \mathfrak{g}$ such that

$$\exp(V \cap \mathfrak{h}) = U \cap H$$

for some neighborhood $e \in U \subset G$. Then H has a unique manifold structure such that $(H, i) \hookrightarrow G$ is an embedded submanifold of G and H is closed in subset topology.

Remark. Lines with irrational slope in torus doesn't satisfy the hypothesis.

Matrix exponentiation

Recall that $\mathfrak{gl}(n, \mathbb{R})$ denotes the Lie algebra of $n \times n$ matrices over $\mathbb{R}$ and similarly for $\mathfrak{gl}(n, \mathbb{C})$.

Definition 8.1. Define a map

$$\begin{aligned}\mathfrak{gl}(n, \mathbb{C}) &\to \mathrm{GL}(n, \mathbb{C}) \\ A &\mapsto e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}\end{aligned}$$

It can be proved that the series is convergent with sup norm and further we have a lemma

Lemma 8.2. If $AB = BA$ then

$$e^{A+B} = e^A e^B$$

which can be used to prove that $e^A \in \mathrm{GL}(n, \mathbb{C})$, so the definition makes sense.

Fix A and consider the function

$$\mathbb{R} \ni t \mapsto e^{tA} \in \mathrm{GL}(n, \mathbb{C})$$

then its derivative is

$$\left.\frac{d}{dt}\right|_{t=0} e^{tA} = A$$

because we can differentiate term by term in uniform convergence. This confirms Theorem 7.3 4th part.

The left-invariant vector field given by $A \in \mathfrak{gl}(n, \mathbb{C})$ is just multiplication by A on the right. Thus, $t \mapsto e^{tA}$ is the integral curve associated to the vector field $A \in \mathfrak{gl}(n, C)$ based at I. Hence, this is the exponential map in the cases of $\mathrm{GL}(n, \mathbb{C})$.

Theorem 8.3. The exponential map $\exp : \mathfrak{g} \to G$ is smooth.

Proof. Let $X \in \mathfrak{g}$ and consider the map

$$\begin{aligned}V : G \times \mathfrak{g} &\to TG \times \mathfrak{g} \\ (g, X) &\mapsto (X_g, 0)\end{aligned}$$

then V is smooth. Also, V is left-invariant on $G \times \mathfrak{g}$. Consider the integral curve γ based at (g, X) of V. Then

$$\gamma_V(t) = (g \exp_X(t), X)$$

because of left invariance so

$$\gamma_V(1) = (g \exp(X), X)$$

$$\begin{aligned}G \times \mathfrak{g} &\xrightarrow{\gamma_V(1)} G \times \mathfrak{g} \xrightarrow{\pi} G \\ (e, X) &\mapsto \gamma_V(1) \to \exp(X)\end{aligned}$$

□

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Note that exponential map commutes with Lie group homomorphisms. Using Ado's theorem we get that for any Lie group

$$\begin{tikzcd} G \arrow[r, "\psi"] & GL(n,\mathbb{C}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r] & \mathfrak{gl}(n,\mathbb{C}) \arrow[u, "\exp"'] \end{tikzcd}$$

Consider the Lie group $\mathrm{SL}(n,\mathbb{C}) = \{X \in \mathrm{GL}(n,\mathbb{C}) | \det(X) = 1\}$, for any $A \in \mathfrak{gl}(n,\mathbb{C})$ upper triangular with diagonal entries $\lambda_1, \ldots, \lambda_n$ then

$$\det(e^A) = e^{\lambda_1 + \ldots + \lambda_n} = e^{\mathrm{tr}(A)}$$

Now $\mathfrak{sl}(n,\mathbb{C}) = \{A \in \mathfrak{gl}(n,\mathbb{C}) | \mathrm{tr}(A) = 0\}$, then $\mathfrak{sl}(n,\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ and exponential maps $\mathfrak{sl}(n,\mathbb{C})$ to the Lie subgroup $\mathrm{SL}(n,\mathbb{C})$. As $\mathrm{SL}(n,\mathbb{C})$ is a closed subgroup of $\mathrm{GL}(n,\mathbb{C})$ and dimension $2(n^2-1)$. Using Theorem 8.1 on an appropriate neighborhood we can complete the proof.

Lie subgroup	Lie subalgebra $\mathfrak{gl}(n,\mathbb{C})$
$U(n) \longleftrightarrow$	$u(n) =$ skew-Hermitian matrices
$\mathrm{SU}(n) \longleftrightarrow$	$su(n) =$ skew-Hermitian + trace $= 0$

Prove the above given correspondence using this lemma (TO DO).

Lemma 9.1. Suppose that $P \in \mathrm{GL}(n,\mathbb{C})$ and $A \in \mathfrak{gl}(n,\mathbb{C})$, then

$$Pe^A P^{-1} = e^{PAP^{-1}}.$$

Theorem 9.2 (Baker-Campbell-Hausdorff formula)**.** Let $\mathfrak{g}$ be a Lie algebra corresponding to a connected Lie group G. Then in a neighborhood U of the identity, the multiplication $U \times U \to G$ is completely determined by Lie algebra structure of $\mathfrak{g}$. There is a formula for $Z = Z(X,Y)$, $X, Y \in V \subset \mathfrak{g}$, where $e^X \cdot e^Y = e^Z$

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \ldots$$

Consider

$$\begin{aligned} e^{tX} \cdot e^{tY} &= \left(\sum \frac{t^k X^k}{k!}\right)\left(\sum \frac{t^l Y^l}{l!}\right) \\ &= \sum_{m\geq 0}\left(\sum_{k+l=m} \frac{X^k Y^l}{k!l!}\right) t^m \end{aligned}$$

Suppose $Z = tZ_1 + t^2Z_2 + t^3Z_3 \ldots$, then

$$\begin{aligned} e^Z &= 1 + (tZ_1 + t^2Z_2 + \ldots) + \frac{(tZ_1 + t^2Z_2 +)}{2!} + \ldots \\ &= 1 + t(Z_1) + t^2\left(Z_2 + \frac{Z_1^2}{2!}\right) \end{aligned}$$

So we get $Z_1 = X + Y$,

$$\begin{aligned} \frac{X^2}{2!} + XY + \frac{Y^2}{2!} &= Z_2 + \frac{Z_1^2}{2!} \\ &= Z_2 + \frac{1}{2}\left(X^2 + XY + YX + Y^2\right) \end{aligned}$$

so $Z_2 = XY - \frac{1}{2}(XY + YX) = \frac{1}{2}(XY - YX) = \frac{1}{2}[X, Y]$

Theorem 9.3. Suppose that $\psi : R \to G$ is a continuous homomorphism. The ψ is smooth.

Proof. It is enough to show that ψ is smooth at 0. Let U be a star-like neighborhood of $0 \in \mathfrak{g}$ such that $\exp|_U : U \to G$ is a diffeomorphism onto $\exp(U)$. Let $U' = \{\frac{X}{2} | X \in U\}$. Choose $Y \in U'$ and let $\psi(t_0) = \exp(Y)$. Choose $t_0 > 0$ such that

$$\psi([-t_0, t_0]) \subset \exp(U')$$

Let $n \geq 2$, and suppose that $X \in U'$ such that $\exp(X) = \psi(\frac{t_0}{n})$. Claim $nX = Y$ □

10 6 Feb

11 8 Feb

12 13 Feb

Definition 12.1. Let $\mathfrak{a} \in \mathfrak{g}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$. We say that $\mathfrak{a}$ is an **ideal** in $\mathfrak{g}$ if $[X, Y] \in \mathfrak{a}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$.

Theorem 12.1. Suppose $A \subset g$ is a connected Lie subgroup of a connected Lie group G. Then A is normal in G if and only if $\mathfrak{a} = \mathrm{Lie}(A)$ is an ideal in $\mathfrak{g}$.

Proof. Suppose that $\mathfrak{a} \subset \mathfrak{g}$ is an ideal. Let $g \in G$, $h \in A$. We must show that $ghg^{-1} \in A$, to do this it is enough to show this for g in a neighborhood of e and h in a neighborhood of e in A. So we may write $g = \exp X, h = \exp Y$

$$\begin{aligned} ghg^{-1} &= \exp \circ \mathrm{Ad}_g(Y) \\ &= \exp \mathrm{Ad}_{\exp(X)}(Y) \\ &= (\exp(\exp(id_X))) \\ &= \exp\left(I + \mathrm{ad}_X + \frac{\mathrm{ad}_X^2}{2!} + \ldots\right)(Y) \\ &= \exp\left(Y + [X, Y] + \frac{[X, [X, Y]]}{2!} + \ldots\right) \in A \end{aligned}$$

Now assume A is normal in G. Let $X \in \mathfrak{g}$, $Y \in \mathfrak{a}$. Write $g_t = \exp tX$. We know that

$$\begin{aligned} A \ni g_t(\exp(sY))g_t^{-1} &= \exp(\mathrm{Ad}_{g_t}(sY)) \\ &= \exp(s\,\mathrm{Ad}_{g_t}) \\ &= \exp(s \exp \mathrm{ad}_{tX}(Y)) \end{aligned}$$

This implies $\exp \mathrm{ad}_{tX}(Y) \in \mathfrak{a}$ so $Y + t[X,Y] + \frac{t^2}{2!}[X,[X,Y]] + \dots$ and using $\left.\frac{d}{dt}\right|_{t=0} \exp \mathrm{ad}_{tX}(Y) = [X,Y] \in \mathfrak{a}$. ☐

Definition 12.2. The center of a Lie algebra $\mathfrak{g}$ is the vector space $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \,|\, [X,Y] = 0 \,\forall\, Y \in \mathfrak{g}\}$.

Remark. Note that $\mathfrak{z}$ is trivial Lie subalgebra of $\mathfrak{g}$.

Theorem 12.2. Let $Z = Z(G)$ be the center of G. Then $Z(G) = \ker(\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}))$.

Proof. If $\mathfrak{g} \in Z(G)$, then $i_g : G \to G = \mathrm{id}_G$ where i_g is the conjugation map. Taking the differential, this implies $A_g : \mathfrak{g} \to \mathfrak{g}$ is identity, hence $g \in \ker(\mathrm{Ad})$.

Suppose that $g \in \ker(\mathrm{Ad})$, so $\mathrm{Ad}_g(X) = X$. Let $X \in \mathfrak{g}$ then

$$\begin{aligned} \exp tX &= \exp(t\,\mathrm{Ad}_g(X)) \\ &= g \exp(tX) g^{-1} \end{aligned}$$

so g commutes with elements $\exp(tX)$ in a neighborhood of e, but that is enough since elements of the form $\exp tX$ for any $t \in \mathbb{R}, X \in \mathfrak{g}$ generate G. Therefore, $g \in Z(G)$. ☐

Proposition 12.3. If $X, Y \in \mathfrak{g}$ are such that $[X,Y] = 0$. Then

$$\exp(X+Y) = \exp(X)\exp(Y).$$

Proof. Let $\mathfrak{a} = \mathbb{R}X + \mathbb{R}Y$. Then $\mathfrak{a}$ is abelian subalgebra of $\mathfrak{g}$. Then the corresponding Lie subgroup A is abelian. Define $\alpha : \mathbb{R} \to G$ such that

$$\alpha(t) = \exp(tX)\exp(tY) \in A$$

It follows that $\alpha(s+t) = \alpha(s)\alpha(t)$ since A is abelian. Now $\alpha(t) = \exp(tZ)$ for some $Z \in \mathfrak{g}$ where $Z = \left.\frac{d}{dt}\right|_{t=0} \alpha(t)$.

$$\begin{aligned} \frac{d}{dt}\alpha(t) &= \left.\frac{d}{dt}\right|_{t=0} \exp(tX) + \left.\frac{d}{dt}\right|_{t=0} \exp(tY) \\ &= X_e + Y_e \end{aligned}$$

So $Z_e = X_e + Y_e$ and $\exp(tZ) = \exp(tX)\exp(tY)$ for all $t \in \mathbb{R}$. ☐

13 15 Feb

Motivation. We will try to look into automorphism group of Lie group now and the expectation is that it is a Lie group itself.

Let $\psi : V \otimes V \to V$ be a linear map. Consider the sets

$$A_\psi(V) = \{\alpha \in \mathrm{GL}(V) | (\alpha u, \alpha v) = \alpha((u,v))\},$$

i.e. the diagram commutes

$$\begin{tikzcd} V \otimes V \arrow[r, "\psi"] \arrow[d, "\alpha\otimes\alpha"] & V \arrow[d, "\alpha"] \\ V \otimes V \arrow[r, "\psi"] & V \end{tikzcd}$$

and

$$Dev_\psi(V) = \{f \in \mathrm{End}(V) | f(\psi(u,v)) = \psi(f(u),v) + \psi(u, f(v))\}$$

Proposition 13.1.
1. $A_\psi(V)$ is a closed subgroup of $\mathrm{GL}(V)$.
2. $Dev_\psi(V)$ is a Lie subalgebra of $\mathfrak{g}(V)$.

Proof. TO DO ☐

Theorem 13.2. Lie algebra of $A_\psi(V)$ equals $Dev_\psi(V)$.

Proof. Let $\mathfrak{a} = Lie(A_\psi(V)) \subset \mathfrak{g}(V) = \mathrm{End}(V)$. We must show that $\mathfrak{a} = Dev_\psi(V)$. Suppose that $f \in \mathfrak{a}$, then $\exp(tf) \in A_\psi(V)$ for all t. We need to show that

$$f \circ \psi = \psi \circ (f \otimes 1 + 1 \otimes f)$$

To do this, let $u, v \in V$, then

$$\begin{aligned} \exp tf(u,v) &= (\exp tf(u), \exp tf(v)) \\ &= (u,v) + (tf(u),v) + (u, tf(v)) + \text{higher powers of } t \end{aligned}$$

so

$$f(u,v) = \frac{d}{dt}\bigg|_{t=0} \exp tf(u,v) = (f(u),v) + (u, f(v))$$

so $f \in Dev_\psi(V)$.

Let $f \in Dev_\psi(V)$, we must show that

$$\begin{aligned} &\exp(tf)(u,v) = (\exp(tf)u, \exp(tf)v) \\ i.e \qquad &\exp(tf) \circ \psi = \psi \circ (\exp(tf) \otimes \exp(tf)) \qquad \forall u, v \in V \text{ and } \forall t \in \mathbb{R} \end{aligned}$$

As $f \in Dev_\psi(V)$, we have

$$\begin{aligned} f \circ \psi &= \psi \circ (f \otimes 1 + 1 \otimes f) \\ f^2 \circ \psi &= f \circ f \circ \psi \\ &= f \circ \psi \circ (f \otimes 1 + 1 \otimes f) \\ &= \psi \circ (f \otimes 1 + 1 \otimes f)^2 \end{aligned}$$

By induction,

$$f^n \circ \psi = \psi \circ (f \otimes 1 + 1 \otimes f)$$

and $f \otimes 1, 1 \otimes f : V \otimes V \to V \otimes V$ commutes. It follows that

$$\begin{aligned}
\exp(tf) \circ \psi &= \sum \left(\frac{t^k f^k}{k!} \circ \psi \right) \\
&= \sum \frac{t^k}{k!} \psi \circ (f \otimes 1 + 1 \otimes f)^k \\
&= \psi \circ \sum \frac{t^k}{k!} (f \otimes 1 + 1 \otimes f)^k \\
&= \psi \circ \exp(tf \otimes 1 + 1 \otimes tf) \\
&= \psi \circ (tf \otimes 1) \circ \exp(1 \otimes tf) \\
&= \psi \circ \exp(tf \otimes tf) \\
&= \psi(\exp(tf) \otimes \exp(tf))
\end{aligned}$$

□

Let $V = \mathfrak{g} = \mathrm{Lie}(G)$ and $\psi = [\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ be the Lie bracket. Then

$$A_\psi(V) = \mathrm{Aut}_{Lie}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$$

and

$$\mathrm{Der}_\psi(V) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$$

by the theorem. Note that $G \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathfrak{g})$ factors through $G \to \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ and $\mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathfrak{g})$.

Let V be a finite dimensional vector space. Consider a bilinear form

$$B : V \times V \to F,$$

equipped with a linear map

$$V \otimes V \to F$$

An element $g \in \mathrm{GL}(V)$ is B-invariant if

$$(u, v) = (gu, gv) \qquad \forall u, v \in V$$

An element $f \in \mathrm{End}(V)$ is B-invariant if

$$(fu, v) + (u, fv) = 0$$

Then $O_B(V) = \{g \in \mathrm{GL}(V) | g \text{ is } B\text{-invariant}\}$ is a closed Lie subgroup of $\mathrm{GL}(V)$ with Lie algebra B-invariant linear map endomorphisms of V.

Example. Take $V = \mathbb{R}^n$ and B is the standard inner product. Then $O_B(V) = O(n)$.

14 1 March

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16 8 March

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17 13 March

Fundamental group of Lie groups

Reference - Hall (?)

Complexification

Let V be a real vector space. Then the complexification is the vector space $V \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}$. If V is a Lie algebra, then $V_{\mathbb{C}}$ is a Lie algebra where the bracket operates on $V_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of that on V. It is given by

$$[X + iY, X' + iY'] = [X, X'] - [Y, Y'] + i\left([X, Y'] + [X', Y]\right)$$

for all $X, Y, X', Y' \in V$. Suppose that V is a real Lie algebra and W is a complex Lie algebra. Suppose $f : V \to W$ is a Lie algebra homomorphism where W is regarded as a $\mathbb{R}$-Lie algebra. Then f extends to a unique complex Lie algebra homomorphism

$$f_{\mathbb{C}} : V \otimes \mathbb{C} \to W$$

Suppose that $W = V + iV$ as $\mathbb{C}$ vector space and where $V \cap iV = 0$ (internal direct sum). Then we say that V is a real form of W.

Suppose W is a complex Lie algebra and V is a real Lie subalgebra contained in W which is a real form of W. Then

$$V_{\mathbb{C}} \equiv W$$

as $\mathbb{C}$-Lie algebra.

Q. Given a Lie algebra, when is it the Lie algebra of a compact Lie group?

A. Something about Killing form and non-degeneracy of complexified Lie algebra and semisimple Lie algebra.

18 20 March

Suppose that $\psi : H \to G$ is a Lie algebra homomorphism into a connected $\mathbb{C}$-Lie group G. Then $d\psi : \mathfrak{h} \to \mathfrak{g}$ extends to a complex Lie algebra homomorphism

$$\mathfrak{h}_{\mathbb{C}} \xrightarrow{d\psi \otimes \mathbb{C}} \mathfrak{g}.$$

Definition 18.1. We say that $\psi : H \to G$ is a complexification of H if for any complex Lie group L and any real Lie group homomorphism $f : H \to L$, there exists a unique complex

Lie group homomorphism $\phi : G \to L$ such that

$$f = \phi \circ \psi.$$

Also,

Definition 18.2. A homomorphism of Lie groups $\psi : G \to L$ is a complex Lie group homomorphism if G, L are complex and $d\psi : \mathfrak{g} \to \mathfrak{l}$ is a complex Lie algebra homomorphism.

Idea : Like we can complexify a real Lie algebra, we would like to have a concept of complexification of a Lie group, but we may not be able to do so for all real Lie groups.

Given a complex Lie group G, any (H, ψ) whose complexification is G will be called a real form. For a given complex Lie group there can be more than one real form. E.g. consider $SU(n) \subset \mathrm{SL}(n, \mathbb{C})$, and it can be proven that $SU(n)$ is a real form of $\mathrm{SL}(n, \mathbb{C})$ (also called compact form since $SU(n)$ is compact) by dimension analysis.

Consider the diagram

$$\begin{tikzcd} SU(n) \arrow[r, "f"] \arrow[d, hook, "\psi"] & L \\ SL(n, \mathbb{C}) \arrow[ur, "\exists ? \phi"'] & \end{tikzcd}$$

where L is a complex Lie group. At the Lie algebra level

$$\begin{tikzcd} \mathfrak{su}(n) \arrow[r, "df"] \arrow[d, hook] & L \\ \mathfrak{sl}(n, \mathbb{C}) \arrow[ur, "\theta"'] & \end{tikzcd}$$

where $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) + i\mathfrak{su}(n)$, $\exists\, \mathrm{SL}(n, \mathbb{C}) \xrightarrow{\phi} L$ such that $d\phi = \theta$ since $\mathrm{SL}(n, \mathbb{C})$ is simply connected. Then ϕ restricts to f since $d\phi|_{\mathfrak{su}(n)} = df$.

Theorem 18.1. Let K be a compact connected Lie group. Then there exists a complex Lie group $K_{\mathbb{C}}$ and a Lie group homomorphism $f : K \to K_{\mathbb{C}}$ such that

1. $f_* : \pi_1(K) \to \pi_1(K_{\mathbb{C}})$ is an isomorphism.
2. $\mathrm{Lie}(K_{\mathbb{C}}) = \mathrm{Lie}(K) \otimes \mathbb{C}$.
3. K_C is the compactification of K.

Theorem 18.2. Suppose that G is a complex linear connected semisimple Lie group. Then any maximal compact Lie subgroup $K \subset G$ is a real form of G.

19 22 March

Let β be a symmetric bilinear form on V, where V is a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Let Q be the associated quadratic form

$$Q : V \to \mathbb{R} \qquad \text{or} \qquad Q : V \to \mathbb{C}$$

$$Q(\lambda v) = \lambda^2 v$$

We have $Q(V) = \beta(v, v)$, and $\beta(u, v) = \frac{Q(u+v)-Q(u)-Qv}{2}$. Suppose that (V, β), (V, β') are quadratic spaces. Then we say that $(V, \beta), (V, \beta')$ are equivalent if there exists $T : V \to V$ such that

$$\beta'(u, v) = \beta(Tu, TV) \quad \forall u, v \in V$$

Suppose that $v_1, \dots, v_n$ is a basis for V. Then the matrix of β is $B = (\beta(v_i, v_j))$.

Let B, B' be the matrices of β, β'. Then $(V, \beta), (V, \beta')$ are equivalent if there exists $T \in M_n(F)$ such that

$$B = {}^tTBT$$

where ^{t} denotes transpose. Now if $x = (x_1 \dots x_n)^t$, $y = (y_1, \dots, y_n)^t$ are vectors in $F^n \equiv V$, then

$$x^t B y = \beta(x, y)$$

and

$$\begin{aligned}
\beta'(x, y) &= \beta(Tx, Ty) \\
&= x^t T^t B T y \\
&= x^t B' y
\end{aligned}$$

which proves the statement. Suppose $E_1 \subset E$, (E, β) is a quadratic space. Then $(E_1, \beta|_{E_1})$ is a quadratic space.

$$E_1^{\perp} = \{x \in E : \beta(x, y) = 0 \, \forall y \in E_1\}$$

Lemma 19.1. Suppose that $E_1 \subset E$ and $(E, \beta|_{E_1})$ is non-degenerate. Then

$$E = E_1 \oplus E_1^{\perp} = E_1 \perp E_1^{\perp}$$

If (E, β) is non-degenerate, then $(E_1^{\perp}, \beta_{E_1^{\perp}})$ is also non-degenerate.

Proof. TO DO. □

Example - Consider the quad space (H, β) where $H = \mathbb{R}^2$ and $Q((x, y)) = x^2 - y^2$. Then $(H, \beta) \cong (H, \beta')$ where $Q'((x, y)) = xy$. One can calculate that

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$B' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

so the forms are non-degenerate and similar using transformation $T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Suppose (V, β) is non-singular, if $\beta|_E \equiv 0$, then $\dim E \leq \frac{1}{2} \dim V$. Further

Lemma 19.2. If (V, β) is non-singular then

$$V = V_1 \oplus \cdots \oplus V_n$$

where each V_i is 1-dimensional and $(V_i, \beta|_{V_i})$ is non-degenerate, $V_i \perp V_j$ if $i \neq j$, i.e. there exists a basis of V with respect to the matrix B of β is diagonal.

Proof. The proof is by induction on dimension. First suppose that $v \in V$ is non-zero then choose $V_1 = Fv$ then

$$V = V_1 \oplus V_1^\perp$$

and $(V_1^\perp, \beta|_{V_1^\perp})$ is non-degenerate. Apply induction to $(V_1^\perp, \beta|_{V_1^\perp})$.

Suppose $\beta(v, v) = 0$. Choose by non-degeneracy of β a vector $V \in V$ such that $\beta(u, v) = 0$. Notice that $\beta(u + v, u + v) = 2\beta(u, v) \neq 0$ which lands us in earlier case. $\square$

Now suppose that (V, β) is arbitrary. Let $V_0 = \text{rad}(\beta) = \{x \in V : \beta(x, y) = 0 \forall y \in V\}$ Consider the quotient $(\frac{V}{V_0}, \overline{\beta})$ with

$$\overline{\beta}(u + V_0, v + V_0) = \beta(u, v)$$

and $\text{rad}(\beta) = 0$, so $(\frac{V}{V_0}, \overline{\beta})$ is non-degenerate. Main theorem

Theorem 19.3. Over $\mathbb{R}$ any non-degenerate β is equivalent to the bilinear form with basis

$$\begin{bmatrix} I_k & \\ & -I_l \end{bmatrix}$$

with $k + l = n$. Moreover, k, l are uniquely determined by β.

Definition 19.1. Let (V, Q) be a quadratic space. The **Clifford algebra** $C(Q)$ associated to it is an algebra over F with a homomorphism $\theta : V \to C(Q)$ such that

1. $\theta(x)^2 = Q(x)$

2. $C(Q)$ is universal with respect to 1st property, i.e. if $\psi : V \to A$ is any vector space homomorphism to an F-algebra such that

$$\psi(x)^2 = Q(x)$$

then there exists a unique algebra homomorphism f such that

$$\begin{tikzcd} C(Q) \arrow[rr, "f"] & & A \\ & V \arrow[ul, "\theta"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes.

We can construct the Clifford algebra by

$$C(Q) = \frac{T(V)}{\langle x \otimes x - \psi(x) \rangle}$$

where $T(V)$ is the tensor algebra of V.

Example. 1. $V = \mathbb{R}$, $Q(x) = -x^2$ then

$$T(V) = \mathbb{R} \oplus \mathbb{R}e_1 \oplus \mathbb{R}(e_1 \otimes e_1) \oplus \dots$$

and $C(Q) = \mathbb{R} \oplus \mathbb{R}e_1$, $e_1^2 = -1$ so $C(Q) \cong \mathbb{C}$.

2. $V = \mathbb{R}$, $Q'(x) = x^2$, then

$$C(Q') = \mathbb{R} \oplus \mathbb{R}e_1$$

with $e_1^2 = 1$ so it is the polynomial ring $\frac{\mathbb{R}[x]}{(x^2-1)}$.

20 27 March

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21 29 March

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22 3 April

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23 5 April

Lemma 23.1 (Schur's lemma)**.** Suppose that G is a compact Lie group. Let V_0, V_1 be a finite dimensional irreducible representation over $\mathbb{C}$. Then any G-homomorphism $\psi : V_0 \to V_1$ is either 0 or an isomorphism. Moreover, any G-homomorphism $V_0 \to V_0$ is a scalar multiple of the identity.

Proof. If V is any irreducible representation, then V is simple i.e. the only subrepresentation of V are 0 and V. Now $\operatorname{im}(\psi) \subset V_1$ is a subrepresentation. Assume $\psi \neq 0$. Then $\operatorname{im}(\psi) = V_1$.

Also, $\ker \psi \subset V_0$ is a subrepresentation. If $\ker \psi = V_0$, then $\psi = 0$ therefore $\ker \psi \neq V_0$ which implies $\ker \psi = 0$. Since V_0 is irreducible, the map ψ is one-one hence ψ is an isomorphism.

For the second part, suppose $\phi : V_0 \to V_0$ is a G-homomorphism. Let λ be an eigenvalue of ϕ. Then $(\lambda I - \phi)$ is singular and is a G-homomorphism. By previous part we get $\lambda I - \phi \equiv 0$ or $\phi = \lambda I$. □

Representation ring of G

Let $[V]$ denote the isomorphism class of finite dimensional G-representation $V/\mathbb{C}$. Consider the free abelian group A with basis $\{[V] : V \text{ is a } G\text{-representation}\}$. We consider the subgroup of elements of the form

$$S = \{[V_0 \oplus V_1] - [V_0] - [V_1] : V_0, V_1 \text{ are } G - \text{representations}\}$$

then $RG \doteqdot A/S$ is an abelian group. Further we can define multiplication by

$$[V] \cdot [W] = [V \otimes W]$$

Distributivity follows from $(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W$

Remark : Given two representations (V, π) and (W, σ) the tensor $(V \otimes W, \rho)$ is also G-representation via

$$\rho(g)(a \otimes b) = \pi(g)a \otimes \sigma(g)b$$

i.e. $g \cdot (a \otimes b) = ga \otimes gb$.

This makes RG a ring generated by the classes of irreducible representations of G.

Example. Any irreducible representation of S^1 is one-dimensional. Let

$$\begin{aligned} \chi_n : S^1 &\to U(1) = S^1 \\ z &\mapsto z^n \end{aligned}$$

If $V_n = (\mathbb{C}, \chi_n)$, then $V_m \otimes V_n = \mathbb{C}$ as a vector space.

$$g(u_1 \otimes u_2) = gu_1 \otimes gu_2 = g^m u_1 \otimes g^n u_2 = g^{m+n} u_1 \otimes u_2$$

Further calculations gives $RS^1 \cong \mathbb{Z}[\chi_1, \chi_1^{-1}]$

Let V be a G-representation over $\mathbb{C}$ endowed with a G-invariant. Fix $u, v \in V$, we have a function $\psi_{\pi,u,v} : G \to \mathbb{C}$ given by

$$\psi_{\pi,u,v}(g) = \langle \pi(g)u, v \rangle .$$

This is called a matrix coefficient of G. Then $\psi_{\pi,u,v} \in L^2(G)$.

Remark. Matrix coefficients form a dense subset of $L^2(G)$ but we will not prove it.

Given a representation (V, π) of G, we have a function

$$\begin{aligned} \chi_\pi &: G \to \mathbb{C} \\ \chi_\pi(g) &= \operatorname{tr}(\pi(g)). \end{aligned}$$

This is called the characteristic function of V. Properties

1. $\chi_\pi = \chi_\sigma$ if $\pi \cong \sigma$.
2. $\chi_{\pi \oplus \sigma} = \chi_\pi + \chi_\sigma$
3. $\chi_{\pi \otimes \sigma} = \chi_\pi \cdot \chi_\sigma$

Lemma 23.2. The characteristic function χ_π is a matrix coefficient.

Proof. Let $v_1, \dots, v_n$ be a Hermitian basis, i.e. $\langle v_i, v_j \rangle = \delta_{ij}$. Then

$$\pi(g) = (\langle \pi(g)v_i, v_j \rangle)_{i,j}$$

therefore

$$\chi_\pi(g) = \sum_{i=1}^{n} \langle \pi(g)v_i, v_j \rangle$$

Now it is enough to show that sum of two matrix coefficients is again a matrix coefficient. Suppose ρ_1, ρ_2 are G-representation and $u_i, v_i \in V_i$,

$$\psi_{\rho_1,u_1,v_1}(g) + \psi_{\rho_2,u_2,v_2}(g) = \psi_{\rho_1\oplus\rho_2,(u_1,u_2),(v_1,v_2)}(g)$$

on $V_{\rho_1\oplus\rho_2} = V_{\rho_1} \oplus V_{\rho_2}$. □

Theorem 23.3 (Schur orthogonality). If (V_1, ρ_1) and (V_2, ρ_2) are irreducible representations over $\mathbb{C}$ of a compact Lie group G, then

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \begin{cases} 0 \text{ if } V_1 \neq V_2 \\ 1 \text{ if } V_1 \cong V_2 \end{cases}$$

Let $\mathrm{Ch}(G)$ or χG denote the ring given by characteristic of representation of G.

$$\begin{aligned} RG &\xrightarrow{\chi} \chi G \\ [V_\pi] &\mapsto \chi_\pi \end{aligned}$$

is a ring homomorphism.

Theorem 23.4. $RG \cong \chi(G)$

Proof. We need only show that χ is a monomorphism. Suppose

$$a = \sum a_i[V_i]$$

where V_i are irreducible such that $\chi(a) = 0$. So

$$\sum a_i \chi_{V_i} = 0$$

this implies

$$\sum a_i \delta_{ij} = \sum a_i \langle \chi_{V_i}, \chi_{V_j} \rangle = 0$$

for all j. Thus, $a_j = 0$ hence $a = 0$. □

Suppose that $g \sim h$ in G, so $g = xhx^{-1}$ for some $x \in G$. Then $\chi_\pi(g) = \chi_\pi(h)$, i.e. χ_π is constant on conjugacy classes.

Suppose $T \subset G$ is torus and G is compact connected. We say that T is a maximal torus if

$$T \subset T'$$

and T' a torus implies $T' = T$.

Lemma 23.5. Any $g \in G$ is contained in a maximal torus.

Theorem 23.6. Fix any maximal torus $T \subset G$. Then

$$G = \bigcup_{x \in G} xTx^{-1}.$$

$$\begin{tikzcd} RG \arrow[r, "res"'] \arrow[d, rightsquigarrow] & RT \arrow[d, rightsquigarrow] \\ \chi(G) \arrow[r, "res"] & \chi(T) \end{tikzcd}$$

where zigzag lines denote isomorphism. Further

$$R(G \times H) = RG \otimes RH$$

$$R(T^n) = \mathbb{Z}[\chi_1, \chi_1^{-1}, \ldots, \chi_n, \chi_n^{-1}]$$

24 10 April

Lemma 24.1. Suppose that $\langle \cdot, \cdot \rangle$ is a G-invariant Hermitian inner product on V_1 where G is compact. Let $v_i \in V_i$. Then we obtain a linear transformation $T : V_1 \to V_2$ defined by

$$T(\omega) = \int_G \langle \pi_1(g)w, v_1 \rangle \, \pi_2(g^{-1})v_2 dg \in V_2$$

where dg is a Haar measure (unimodular here because G is compact). Then T is a G-equivariant, i.e. $T(\pi_1(h)\omega) = \pi_2(h)T(\omega)$.

Proof.

$$\begin{aligned} T(\pi_1(h)\omega) &= \int_G \langle \pi_1(g)\pi_1(h)\omega, v_1 \rangle \, \pi_2(g^{-1})(v_2)dg \\ &= \int_G \langle \pi_1(gh)\omega, v_1 \rangle \, \pi_2(g^{-1})v_2 dg \end{aligned}$$

Put $gh = x$, then $g = xh^{-1} = \rho_h(x)$ and $dg = dx$. So

$$\begin{aligned} T(\pi_1(h)\omega) &= \int_G \langle \pi_1(x)\omega, v_1 \rangle \, \pi_2(h)\pi_2(x^{-1})v_2 dx \\ &= \pi_2(h) \int_G \langle \pi_1(x)\omega, v_1 \rangle \, \pi_2(x^{-1})v_2 dx \\ &= \pi_2(h)T(\omega) \end{aligned}$$

□

Recall

Lemma 24.2 (Schur's ortho). Suppose that (π_1, V_1) and (π_2, V_2) are irreducible. Then every matrix coefficient $\psi_{\pi_1,u,v}$ is orthogonal to $\psi_{\pi_2,u',v'}$ or (π_1, V_1) is isomorphic to (π_2, V_2).

Now

Proof. continuing Assume $\psi_{\pi_1,u,v}$ and $\psi_{\pi_2,u',v'}$ are not orthogonal. So

$$\begin{aligned} 0 &\neq \int_G \langle \pi_1(g)u, v\rangle \overline{\langle \pi_2 u', v'\rangle} dg \\ &= \int_G \langle \pi_1(g)u, v\rangle \left\langle v', \pi_2(g)u', v'\right\rangle dg \\ &= \int_G \langle \pi_1(g)u, v\rangle \left\langle \pi_2(g^{-1})v', u'\right\rangle dg \end{aligned}$$

which is $\langle T(u), u'\rangle$ hence T is non-zero so by T is an isomorphism by Schur's lemma. □

Let T be a subgroup of G, then we know that there is a map

$$RG \xrightarrow{Res} RT$$

Basic fact : If (π, V) is an irreducible representation of a torus T. Then V is one-dimensional.

Proof. Let $t \in T$. Consider $\pi(t) : V \to V$. Because T is abelian, $\pi(t)$ is T-linear, i.e.

$$\pi(ts)(v) = \pi(t)(\pi(s)v) = \pi(s)\pi(t)v = \pi(st)(v)$$

hence by Schur's lemma

$$\pi(t)v = \chi(t)v$$

for all v where $\chi : T \to C^\times$ so

$$\chi(st) = \chi(s)\chi(t)$$

holds. Now

$$\begin{aligned} \chi(st)v = \pi(st)v &= \pi(s)\pi(t)v \\ &= \pi(s)(\chi(t)v) = \chi(t)\pi(s)v \\ &= \chi(t)\chi(s)v \end{aligned}$$

Since every non-zero subspace of V is a T-representation (as $\pi(t) = \chi(t)I$) we must have $\dim V = 1$ as V is irreducible.

□

Example. Let $G = SU(2)$ with torus

$$T = \left\{ \begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix} : 0 \leq \theta \leq 2\pi \right\}$$

where T is maximal since the only matrices in $SU(2)$ which commute with every $\begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix}$ is itself diagonal and hence in T.

Std : $V = \mathbb{C}^2 = V_1 \oplus V_2$ be irreducible where $V_i = \mathbb{C}e_i$ and

$$\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} e_1 = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{i\theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

thus

$$\chi_1 \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} = e^{i\theta}$$

similarly

$$\chi_2 \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} = e^{-i\theta}$$

Let $S^k(V)$ be the k-th symmetric power of V which is same as polynomials of degree k in e_1, e_2. The characters of S^k are

$$\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} e_1^j e_2^{k-j} = e^{ij\theta} e^{-i(k-j)\theta} e_1^j e_2^{k-j} = e^{i(2j-k)\theta} e_1^j e_2^{k-j}$$

$$RSU(2) \xrightarrow{Res} RT$$
$$S^k \mapsto V_k \oplus V_{k-2} \oplus \cdots \oplus V_{-k}$$

where $V_i \leftrightarrow x_j$

Theorem 24.3. The S^k are the only irreducible representations of $SU(2)$.

Known as SL(2) theory.

Let G be a compact connected Lie group. Let T be a maximal torus. Then we define the Weyl group $W = W(G,T)$ of G with respect to T as

$$W = N_G(T)/T$$

where $N_G(T) = \{g \in G : gTg^{-1} = T\}$ and $N_G \subset \mathrm{Aut}(T)$ via conjugation. Hence, W acts in T via automorphism.

Theorem 24.4. W is a finite group.

Proof. W acts on Lie(T) as linear map. Consider the projection map

$$\mathbb{R}^n \cong \mathrm{Lie}(T) \xrightarrow{p} T \cong \mathbb{R}^n/\mathbb{Z}^n$$
$$(t_1, \dots, t_n) \mapsto (e^{2\pi i t_1}, \dots, e^{2\pi i t_n})$$

$$\begin{tikzcd} Lie(T) \arrow[r, "w"] \arrow[d, "p"'] & Lie(T) \arrow[d, "p"] \\ T \arrow[r, "w"'] & T \end{tikzcd}$$

where $w(\mathbb{Z}^n) = \mathbb{Z}^n$ for all $w \in W$. Now $N_G(T)$ is closed in G and hence compact. So W is compact and W is finite since $W \subset \mathrm{GL}(n, \mathbb{Z})$ which is discrete. $\square$

Example. Take $G = U(n)$ and $T = \left\{ \begin{pmatrix} t_1 & \dots & \\ & \ddots & \\ & \dots & t_n \end{pmatrix} : t_i \in S^1 \right\}$ Noe that $U(n) = \bigcup_{g \in U(n)} gT^{-1}g^{-1}$ since given any $x \in U(n)$, there exists a unitary basis $\mathcal{U} = u_1, \dots, u_n$ of $\mathbb{C}^n$ such that the matrix of x with respect to $\mathcal{U}$ is diagonal. Take g to be such that $g(e_i) = u_i$ for all i. Then $(g^{-1}xg)(e_i) = g^{-1}x(u_i) = \lambda_i g^{-1}u_i = \lambda_i e_i$

On the other hand if $gTg^{-1} = T$, then choose

$$g \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} g^{-1} \in T$$

where $\lambda_1, \dots, \lambda_n$ are pairwise distinct. This implies $ge_i = z_i e_{\sigma(i)}$ for some j,for some $\sigma \in S_n$. Thus $N(T)$ is a monomial matrix which implies $N(T)/T \cong S_n$.