Lie Groups

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1 4th January 23

One can study Lie Groups from several points of view. The course is aimed to understand the structure of Lie Groups.

Definition 1.1. A smooth manifold M is a Hausdorff space which is locally Euclidean with a smooth atlas i.e. (i) given any $x \in M$, $\exists$ a chart (U, ϕ), $x \in U \subset M$ with $\phi : U \to \phi(U)$ open in $\mathbb{R}^m$.

(ii) We have collection $\{(U, \phi)\}$ of charts such that

$$\phi(U \cap V) \xrightarrow{\psi \circ \phi^{-1}} \psi(U \cap V)$$

is a diffeomorphism.

Suppose $f : M \to N$ is a continuous map between manifolds. We say that f is smooth if for $(U, \phi) \in \amalg(M)$, $(V, \psi) \in \amalg(N)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}$ is smooth.

TO DO : Construction of tangent bundle and vector bundle

2 9th Jan 2023

Definition 2.1. G is a Lie group if

1. G is a smooth manifold

2. G is also a group s.t

$$\begin{aligned} \mu : G \times G &\to G \\ (g,h) &\mapsto gh \end{aligned}$$

and

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth maps.

Definition 2.2. A real (or complex) vector space V together with a bilinear map

$$[,] : V \times V \to V$$

is called a **Lie Algebra** if

1. $[X,Y] = -[Y,X]$ - skew symmetry
2. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ - Jacobi identity

Example.

1. $(\mathbb{R},+), (\mathbb{C},+)$, V any f.d vector space over $\mathbb{R}$ or $\mathbb{C}$.
2. $(\mathbb{R}^\times, \cdot), (\mathbb{C}^\times, \cdot)$
3. $S^1 = \{z \in \mathbb{C}^\times | |z| = 1\}$
4. $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$
5. $\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}^n/\mathbb{Z}^n) \cong (S^1)^n$
6. Suppose $\Gamma \subset V$ is a discrete subgroup. Then V/Γ is a Lie group.
7. N = unipotent upper triangular matrices, B = upper triangular matrices. As manifolds $N \cong \mathbb{R}^{\binom{n}{2}}$ and $B \cong (\mathbb{R}^\times)^n \times N$.
8. $\mathrm{SL}_n(\mathbb{R}) = \{X \in \mathrm{GL}_n(\mathbb{R}) | \det X = 1\}$, $\mathrm{SL}_n(\mathbb{C})$.
9. $O(n)$, $SO(n)$.
10. $U(n)$, $SU(n)$.
11. $\mathbb{H}^\times$, S^3 with quaternion multiplication.
12. $Sp(n) = \{X \in \mathrm{GL}_n(\mathbb{R}) | X \text{ preserves quaternion structure as a subset of } \mathrm{Aut}_{\mathbb{H}} \mathbb{H}^n\}$

Problem. $V/\Gamma \cong \mathbb{R}^k \times (S^1)^{n-k}$ for n-dimensional vector space V.

Theorem 2.1. Suppose G is a compact, connected, simple Lie group. Then G is locally isomorphic to

1. $SU(n), n \geq 2$ denoted by A_{n-1}
2. $SO(2n+1), n \geq 2$ denoted by B_n
3. $Sp(n), n \geq 1$ denoted by C_n
4. $SO(2n), n \geq 2$ denoted by D_n

or one of the following exceptional Lie group G_2, F_4, E_6, E_7, E_8.

Problem. Prove that $\mathrm{SL}_n(\mathbb{R})$ and $O(n)$ are smooth manifold, hence Lie groups.

Examples of Lie algebra -

Example.
1. $(V, [\cdot,\cdot] \equiv 0)$ is called trivial Lie algebra.
2. $(\mathfrak{gl}_n(\mathbb{R}), [A,B] = AB - BA)$, $\mathfrak{gl}_n(\mathbb{C})$
3. $\mathfrak{sl}_n(\mathbb{R})$ ($\mathfrak{sl}_n(\mathbb{C})$) is the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ ($\mathfrak{gl}_n(\mathbb{C})$) consisting of trace 0.
4. $\mathfrak{so}_n$ is Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ consisting of skew-symmetric matrices.

Definition 2.3. A vector field X on a Lie group G is called left invariant if $(L_g)_*(X_h) = X_{gh}$