Problem 5.4

This problem is intended to identify a linear filter for satisfying the interpolation formula of Eq. (5.7), albeit in a non-physically realizable manner. Equation (5.7) is based on the premise that the signal $g(t)$ is strictly limited to the band $-W \le f \le W$. With this specification in mind, consider an ideal low-pass filter whose frequency response $H(f)$ is as depicted in Fig. 5.2(c). The impulse response of this filter is defined by (see Eq. (2.25))

$$h(t) = \mathrm{sinc}(2Wt), \qquad -\infty < t < \infty$$

Suppose that the correspondingly instantaneously sampled signal $g_\delta(t)$ defined in Eq. (5.1) is applied to this ideal low-pass filter. With this background, use the convolution integral to show that the resulting output of the filter is defined exactly by the interpolation formula of Eq. (5.7).

Solution

From Eq. (5.5), we have

$$G(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right), \qquad -W < f < W$$

According to this equation, $G(f)$ is low-pass with its frequency content confined to the range $-W < f < W$. Since $G(f)$ is the Fourier transform of $g(t)$, we can also write

$$\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\mathrm{sinc}(2Wt-n) \rightleftharpoons \frac{1}{2W}\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right), \qquad -W < f < W$$

Hence, the reconstruction filter defined by the left-hand side of this Fourier-transform pair is low-pass with its passband confined to the range $W < f < W$.