Problem 4.16

From Fig. 4.20, we see that the envelope detector input is

$$\begin{aligned} v(t) &= s(t) - s(t-T) \\ &= A_c \cos[2\pi f_c t + \phi(t)] - A_c \cos[2\pi f_c (t-T) + \phi(t-T)] \end{aligned}$$

Using a well-known trigonometric identity, we write

$$v(t) = -2A_c \sin\left[\frac{2\pi f_c(2t-T) + \phi(t) + \phi(t-T)}{2}\right] \sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] \qquad (1)$$

For $\phi(t)$, we have

$$\phi(t) = \beta \sin(2\pi f_m t)$$

Correspondingly, the phase difference $\phi(t)$ - $\phi(t - T)$ is given by

$$\begin{aligned} \phi(t) - \phi(t-T) &= \beta \sin(2\pi f_m t) - \beta \sin[2\pi f_m (t-T)] \\ &= \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t)\cos(2\pi f_m T) + \cos(2\pi f_c t)\sin(2\pi f_m T)] \qquad (2) \end{aligned}$$

Using the approximations:

$$\cos(2\pi f_m T) \approx 1$$

$$\sin(2\pi f_m T) \approx 2\pi f_m T$$

we may approximate Eq. (2) as

$$\begin{aligned} \phi(t) - \phi(t-T) &\approx \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t) + 2\pi f_m T \cos(2\pi f_m t)] \\ &= 2\pi \Delta f T \cos(2\pi f_m t) \qquad (3) \end{aligned}$$

where

$$\Delta f = \beta f_m.$$

Therefore, recognizing that $2\pi f_c T = \pi/2$, we may write

$$\begin{aligned} \sin\left(\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right) &\approx \sin(\pi f_c T + \pi \Delta f T \cos(2\pi f_m t)) \\ &= \sin\left(\frac{\pi}{4} + \pi \Delta f T \cos(2\pi f_m t)\right) \\ &= \sqrt{2}\cos(\pi \Delta f T \cos(2\pi f_m t)) + \sqrt{2}\sin(\pi \Delta f T \cos(2\pi f_m t)) \\ &= \sqrt{2} + \sqrt{2}\pi \Delta f T \cos(2\pi f_m t) \end{aligned}$$

where we have made use of the fact that $\pi \Delta f T << 1$. We may therefore rewrite Eq. (1) as

$$v(t) \approx -2\sqrt{2}A_c(1 + \pi \Delta f T \cos(2\pi f_m t)) \sin\left(\pi f_c(2t-T) + \frac{\phi(t) + \phi(t-T)}{2}\right) \qquad (4)$$

Accordingly, the envelope detector output is the envelope of $v(t)$, namely,

$$a(t) \approx 2\sqrt{2}A_c(1 + \pi \Delta f T \cos(2\pi f_m t))$$

which, except for a bias term, is proportional to the modulating wave.