Homework 1: random variables

EE 325 (DD): Probability and Random Processes, Autumn 2018
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Instructions: Some of these questions will be asked in a quiz in the class next week. The quiz will not be held on Monday (30/07/2018). *If you have queries, then meet the instructor or the TA during office hours.*

1. Let $X \sim \text{Uniform}[-2, 2]$ random variable and Y be obtained by clipping X. That is,

$$\begin{aligned} Y &= X, \text{ if } |X| \leq 1 \\ &= 1, \text{ if } X > 1 \\ &= -1, \text{ if } X < -1. \end{aligned}$$

   What are the values of $\mathbb{P}(Y = 1)$, $\mathbb{P}(Y = -1)$, and $\mathbb{P}(Y = 0)$? Is Y continuous or discrete? Give reasons for your answer.

2. Using the cdf $F_X(x)$ of a random variable X, and the definition of a random variable, how will compute $\mathbb{P}(1 \leq X \leq 2)$, $\mathbb{P}(3 \leq X < 4)$, and $\mathbb{P}(\{1 \leq X \leq 2\} \cup \{3 \leq X \leq 4\})$? Your answers should be explicit formulas, with reasoning, in terms of $F_X(x)$.

3. Let $F(x, y)$ be the joint cdf of two random variables (X, Y). Show that

$$F(2, 2) + F(1, 1) \geq F(2, 1) + F(1, 2).$$

   How can this inequality be generalized?

4. Sketch the cdf of the following random variables:

   (a) A Poisson random variable with the parameter $\lambda = 2$.

   (b) A Cauchy random variable with the pdf as follows:

$$f_X(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \quad x \in \mathbb{R}.$$

5. Let (X, Y, Z) be independent random variables. Show that any two subset of random variables, for example (X, Y), are also independent. How will your result generalize to more than three random variables?

6. Let k and n be non-negative integers, and $0 < p < 1$. A random variable X has a geometric distribution if its pmf is given by $p_X(k) = (1 - p)p^k$. Define the residual lifetime distribution function as, $l_X(k, n) := \mathbb{P}(X \geq n + k | X \geq n)$.

   (a) Show that $l_X(k, n) = \mathbb{P}(X \geq k)$ independent of n, i.e., the geometric distribution satisfies the memoryless property.

   (b) Assume that $Y \geq 0$ is any other discrete integer-valued distribution which exhibits memoryless property, i.e., $l_Y(k, n) = \mathbb{P}(Y \geq k)$. Show that $l_Y(k, n)$ has to be of the form α^k for some $0 < \alpha < 1$.

   (c) Using (b), show that if Y satisfies the memoryless property, then it has a geometric distribution.