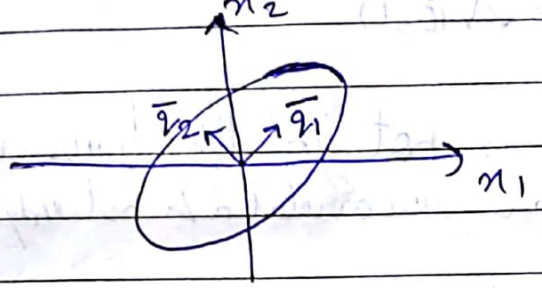
$\hookrightarrow K_{\bar{V}} = \bar{Q}^T K_{\bar{Z}} Q = Q^T Q \Lambda Q^T Q = \Lambda$

$$\therefore f_{\bar{V}}(\bar{v}) = \frac{1}{(2\pi)^{n/2}\sqrt{\lambda_1 \ldots \lambda_n}} \exp\left(-\sum_{i=1}^{n} \frac{v_i^2}{2\lambda_i}\right)$$

ie. $V_1, \ldots V_n$ are independant r.v.s.

$\hookrightarrow$ Consider $n=2$. The equiprobable contours in $f_{\bar{Z}}(\bar{x})$ are given by $\bar{x}^T K_Z^{-1} \bar{x} = \text{const}^n$.



$x^T K_{\bar{Z}}^{-1} \bar{x} = \text{const}^n$

$\bar{v}^T \Lambda^{-1} \bar{v} = \text{const}^n$

$\bar{V} = Q^T \bar{Z} \Rightarrow \bar{Z} = Q\bar{V}$

23/10

$\bar{Z} \sim N(\bar{0}, K_{\bar{Z}})$, $K_{\bar{Z}} = Q\Lambda Q^T$

$\det(K_{\bar{Z}}) = \det(\Lambda) \neq 0$

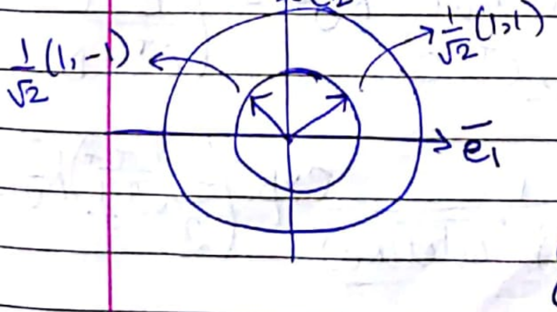
Let $\bar{V} = Q^T \bar{Z}$

$$f_{\bar{V}}(\bar{v}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Lambda}} \exp.\left(-\sum_{i=1}^{n} \frac{v_i^2}{2\lambda_i}\right) = f_{V_i}(v_i)$$

ie. $V_1, \ldots V_n$ are independant.

$\hookrightarrow$ $Q^T \Lambda^{-1/2} \bar{Z}$ will be white.

(compare with $Q \Lambda^{-1/2} Q^T \bar{Z}$)



$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$Q\bar{W} \sim N(0, I_n)$

$\hookrightarrow QQ^T = I_n$

Page No.:
Date:

↳ $\bar{Z} \sim \mathcal{N}(\bar{m}, K_{\bar{Z}})$

$$\bar{Z} = \bar{m} + (Q\Lambda^{-1/2}Q^T)\bar{W}$$

↳ Let $\bar{Y}$ is jointly Gaussian. Let $Y_1, \ldots Y_n$ be uncorrelated. Then $Y_1, \ldots, Y_n$ are indepⁿ

↳ $X \sim \mathcal{N}(0,1)$, $B \sim$ equiprobable on $\{-1, +1\}$

$\boxed{Y = BX} \sim \mathcal{N}(0,1)$

→ But (X,Y) is not jointly Gaussian. $(X+Y)$ is not gaussian

Though (X,Y) are uncorrelated & not indepⁿ

Conditional distribution

Let $\bar{U}$ be $(m+n)$-dim zero mean jointly Gaussian r.v.

Let $\bar{U} = (\bar{X}^T, \bar{Y}^T)^T$ where $\bar{X}$ is m-dim & $\bar{Y}$ is n-dim.

Let $K_{\bar{U}}$ be non-singular.

Then,

$$K_{\bar{U}} = \left(\begin{array}{c|c} K_{\bar{X}} & K_{\bar{X}\bar{Y}} \\ \hline K_{\bar{Y}\bar{X}} & K_{\bar{Y}} \end{array}\right), \text{ where } K_{\bar{X}\bar{Y}} = E(\bar{X}\bar{Y}^T) = K_{\bar{Y}\bar{X}}^T$$

cross covariance

↳ It can be shown that $K_{\bar{U}}^{-1} = \left(\begin{array}{c|c} B & C \\ \hline C^T & D \end{array}\right)$

$\boxed{B^T = B,\ D^T = D,\ C^T \neq C}$

Then,

$$f_{\bar{X},\bar{Y}}(\bar{x},\bar{y}) = \underbrace{\frac{1}{(\sqrt{2\pi})^{m+n}\sqrt{\det(K_{\bar{U}})}}}_{h_1} \exp\left(-\frac{1}{2}(\bar{x}^T \bar{y}^T) K_{\bar{U}}^{-1} \begin{pmatrix}\bar{x} \\ \bar{y}\end{pmatrix}\right)$$

$$= h_1 \exp\left(-\frac{1}{2}(\bar{x}^T B \bar{x} + \bar{x}^T C \bar{y} + \bar{y}^T C^T \bar{x} + \bar{y}^T D \bar{y})\right)$$

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Page No.:
Date:

and $f_{\bar{Y}}(\bar{y}) = \dfrac{1}{(\sqrt{2\pi})^{n} \sqrt{\det K_{\bar{Y}}}} \exp\left(\dfrac{-\bar{y}^T K_{\bar{Y}}^{-1} \bar{y}}{2}\right)$

Then, pdf of $\bar{X} | (\bar{Y} = \bar{y})$ is,

$$f_{\bar{X}|\bar{Y}}(\bar{x}|\bar{y}) = \frac{f_{\bar{X},\bar{Y}}(\bar{x},\bar{y})}{f_{\bar{Y}}(\bar{y})}$$

$$= h_2(\bar{y}) \exp\left(\frac{-1}{2}\left(\bar{x}^T B \bar{x} + \bar{x}^T C \bar{y} + \bar{y}^T C^T \bar{x}\right)\right)$$

Since $\int_{\mathbb{R}^n} f_{\bar{X}|\bar{Y}}(\bar{x}|\bar{y}) d\bar{x} = 1$, $h_2(\bar{y})$ can be found at the end.

$$f_{\bar{X}|\bar{Y}}(\bar{x}|\bar{y}) = h_3(\bar{y}) . \exp\left(\frac{-1}{2}\left(\bar{x} + B^{-1} C \bar{y}\right)^T B \left(\bar{x} + B^{-1} C \bar{y}\right)\right)$$

$$h_3(\bar{y}) = h_2(\bar{y}) . \exp\left(+\frac{1}{2} \bar{y}^T C^T B^{-1} C \bar{y}\right)$$

$$\left(\begin{array}{c|c} B & C \\ \hline C^T & \Delta \end{array}\right) = K_{\bar{U}}^{-1} = \left(\begin{array}{c|c} K_{\bar{X}} & K_{\bar{X}\bar{Y}} \\ K_{\bar{X}\bar{Y}}^T & K_{\bar{Y}} \end{array}\right)$$

indep^n of $\bar{y}$ (depends on distn of $\bar{Y}$)

Since $K_{\bar{U}}$ is symm., $B^T = B$ & $\Delta^T = \Delta$

$\therefore$ Observe that $\bar{X} | \bar{Y} = \bar{y} \sim \mathcal{N}(-B^{-1} C \bar{y}, B^{-1})$

& therefore $h_3(\bar{y}) = \dfrac{1}{(2\pi)^{n/2} \sqrt{\det B^{-1}}}$

25/10

$\bar{U} = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix}$, $K_{\bar{U}} \rightarrow$ non-singular

jointly gaus.

$$K_{\bar{U}} = \left(\begin{array}{c|c} K_{\bar{X}} & K_{\bar{X}\bar{Y}} \\ \hline K_{\bar{X}\bar{Y}}^T & K_{\bar{Y}} \end{array}\right), \quad K_{\bar{U}}^{-1} = \left(\begin{array}{c|c} B & C \\ \hline C^T & \Delta \end{array}\right)$$

B & Δ are +ve definite & symmetric

What is B & C = ?

Page No.:
Date:

$$\begin{pmatrix} B & C \\ C^T & D \end{pmatrix}\begin{pmatrix} K_{\bar{X}} & K_{\bar{X}\bar{Y}} \\ K_{\bar{X}\bar{Y}}^T & K_{\bar{Y}} \end{pmatrix} = I$$

$$\therefore \; BK_{\bar{X}} + CK_{\bar{X}\bar{Y}}^T = I_m$$

$\hookrightarrow \bar{X} = G\bar{Y} + \bar{V}$ (Since $\bar{X}|\bar{Y}=y$ has a mean which is linear in $\bar{y}$ & cov. indep. of $\bar{y}$)

$\bar{Y}$ & $\bar{V}$ are indep^n.

co-variance of LHS must be same as that of RHS (like power conserv^n.)

$$\therefore E(\bar{V}) = E(\bar{X}) - GE(\bar{Y}) = 0$$

$$K_{\bar{X}} = E(X\cancel{X}^T) = GK_{\bar{Y}}G^T + K_{\bar{V}} \quad \left\{\begin{matrix}\text{Since} \\ K_{\bar{V}\bar{Y}} = 0\end{matrix}\right\}$$

$$K_{\bar{X}\bar{Y}} = E(\bar{X}\bar{Y}^T) = GK_{\bar{Y}} + 0 = GK_{\bar{Y}}$$

$$\therefore \boxed{G = K_{\bar{X}\bar{Y}}K_{\bar{Y}}^{-1}}$$

$$\therefore K_{\bar{V}} = K_{\bar{X}} - K_{\bar{X}\bar{Y}}K_{\bar{Y}}^{-1}K_{\bar{Y}\bar{X}}$$

$\therefore \bar{X} = (K_{\bar{X}\bar{Y}}K_{\bar{Y}}^{-1})\bar{Y} + \bar{V}$; $\bar{V}$ is indep^n of $\bar{Y}$.

Similarly, $\bar{Y} = (K_{\bar{Y}\bar{X}}K_{\bar{X}}^{-1})\bar{X} + \bar{W}$; $\bar{W}$ is indep. of $\bar{X}$

$$W \sim N(\bar{0}, K_{\bar{Y}} - K_{\bar{Y}\bar{X}}K_{\bar{X}}^{-1}K_{\bar{X}\bar{Y}})$$

Observe, $(\bar{X} - G\bar{Y})$ is indep^n. of $\bar{Y}$.

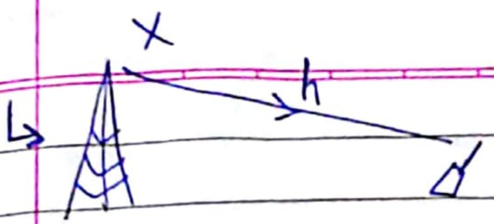
$\hookrightarrow \bar{X}, \bar{Y}$ are jointly Gaussian $\left\{ K_{\bar{U}} = \begin{pmatrix} K_{\bar{X}} & 0 \\ 0 & K_{\bar{Y}} \end{pmatrix} \right.$

$\bar{0}$ matrix

Block diagonal

$\Rightarrow \bar{X} \perp\!\!\!\perp \bar{Y}$

is indep. of



$$hX + W = Y$$

Gaussian   Gaussian

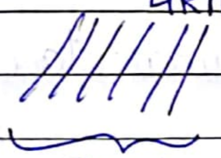
$$\bar{Y} = H\bar{X} + \bar{W}$$

signal   noise   – communication
predictive   innovation   – signal processing

$$\left.\begin{aligned} \bar{X}_{n+1} &= H\bar{X}_n + \bar{W}_n \\ \bar{X}_{n+2} &= H\bar{X}_{n+1} + \bar{W}_{n+1} \end{aligned}\right\} \text{filtering}$$

<u>Mpeg</u>

4kHD } 8 MPixels   (10MB)



30 images × /s
(8 Mpixels)

30fps

29/10 $(\bar{X}, \bar{Y}) \rightarrow$ jointly Gaussian r.v.

↳ $\bar{X} = G\bar{Y} + \bar{V}$, $\bar{V}$ & $\bar{Y}$ are independant.

$$\bar{X} \,|\, \bar{Y} = \bar{y} \sim \mathcal{N}(G\bar{y}, K_{\bar{V}}), \quad G = K_{XY} K_Y^{-1}$$

$$K_{\bar{V}} = K_{\bar{X}} - K_{XY} K_Y^{-1} K_{YX}$$

↳ $\bar{Y} = H\bar{X} + \bar{W}$

<u>Random Processes</u>

$X : \Omega \rightarrow \mathbb{R}$; $F_X(x)$
$\bar{X} : \Omega \rightarrow \mathbb{R}^n$; $F_{\bar{X}}(\bar{x})$
$X(t)$? random funct^n ?

noise process in comm. receivers
Bernoulli arrival process on a server

↳ Many physical processes are well modeled by a random funct^n.

Defⁿ. (Index set)

A subset of $\mathbb{R}$ is an index set, which is the domain of a random process. Denote it by T.

ex: T includes $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}^+$, $\mathbb{R}^+$, $[0,1]$ ....

↳ $\{X(t), t \in T\}$ is a random process that we want to formalise.

There are 3 ways to understand $\{X(t), t \in T\}$

1) $X(t)$ is a r.v. for each $t \in T$
2) Ω (sample space) ; $\omega \in \Omega \longrightarrow$ fn(ω).
   $\{X(t,\omega), t \in T\}$ is a waveform/signal/functn
3) X_T :

30/10

$X_T : \Omega \times T \longrightarrow \mathbb{R}$

ex: Phase error in commⁿ. receiver.

$A_c \cos(2\pi f_c t + \theta)$ ⇒ In a communication receiver, the received (modulated) signal can be modeled as:-

$$X(t) = A_c \cos(2\pi f_c t + \theta)$$

where $\theta \sim \text{Unif.}[0, 2\pi)$

$\{X(t), t \in R\}$ is a random process whose distribution properties depend only on Θ.

ex: Cumulative avg.

Let $X_1, \dots X_n$ be iid r.v's. (Observe that $\{X_n, n \in Z^+\}$ is a random process in itself)

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, observe that $\{\bar{X}_n, n \in Z\}$ is a random process. (discrete)

↳ $t \in T$, $X(t)$ is a random ~~process~~ variable

↳ Random processes are often understood/characterised by their mean & autocorrelation functions.

Defn. (Mean & autocorrelation)

The mean of a random process is :-

$$\boxed{m_x(t) = E(X(t))}, \quad t \in T$$

The autocorrelation funct^n of a random process $\boxed{R_x(t_1, t_2) = E(X(t_1)X(t_2))}$ for every pair (t_1, t_2) in the index set T.

↳ There is also an autocovariance function

$$C_x(t_1, t_2) = R_x(t_1, t_2) - m_x(t_1)m_x(t_2)$$
$$= E\left((X(t_1) - m_x(t_1))(X(t_2) - m_x(t_2))\right)$$

$$R_x(t_1, t_2) \leq \frac{E(X^2(t_1)) + E(X^2(t_2))}{2}$$

Page No.:
Date:

If $m_x(t) = 0 \ \forall \ t \in T$, the process is called a zero mean.

↳ $\{X(t) - m_x(t), t \in T\}$ is a zero-mean random process.

↳ Defn. (stationary process For any set of points $\{t_1, \ldots, t_n\}$ in T, the finite dimensional distribution (fdd), of $\{X(t), t$ is given by :- (of order n)

$$F_{X(t_1)\ldots X(t_n)}(x_1, \ldots x_n) = \mathbb{P}(X(t_1) \leq x_1, \ldots\ldots X(t_n) \leq x$$

Stationary processes

A process $\{X(t) \in T\}$ is (strict sense) stationary if its fdd of any order is invariant to shifts.

ie. for each n, for any $(t_1, \ldots t_n, \tau)$ $\in T$ & for all $(x_1, \ldots x_n) \in \mathbb{R}^n$

$$F_{X(t_1)\ldots X(t_n)}(x_1, \ldots x_n) = F_{X(t_1+\tau)\ldots X(t_n+\tau)}(x_1, \ldots x_n)$$

↳ $T = \mathbb{R}$ & $T = \mathbb{Z}$ are the prominent choices for stationary process studies.

↳ If $\{X(t), t \in T\}$ is stationary, then,

$$F_{X(t)}(x) = F_{X(0)}(x) \ \forall \ x \in \mathbb{R}$$

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↳ For stationary process, also,

$$F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(0),X(t_2-t_1)}(x_1,x_2)$$
for all $x_1, x_2 \in R$

↳ That is, the auto-correlation of a stationary process only depends on (t_2-t_1) and the mean of a stationary process is a constant

<u>Defn (autocorrelation for a stationary process)</u>

Let $\{X(t), t \in \underset{(Z)}{R}\}$ be a stationary process.

Its autocorrelation is defined as:-

$$R_X(\tau) = E(X(t).X(t+\tau))$$
$$(R_X[n] = E(X(k).X(k+n)))$$

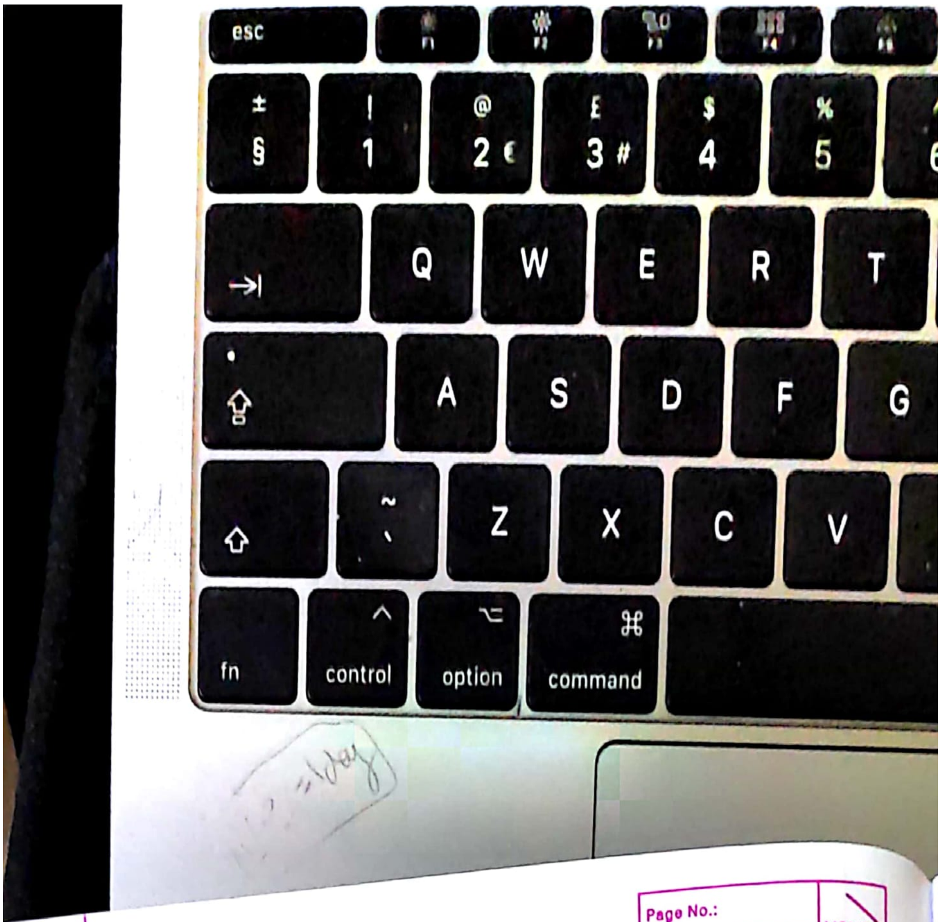
<u>Joint stationality</u>

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$
filter

- $X(t)$ is stationary
- $h(t)$ is an LTI filter

Then $\{Y(t), t \in R\}$ is stationary. Also, $\{X(t), t \in R\}$ & $\{Y(t), t \in R\}$ are jointly stationary. <u>ie</u> for each n, for any $\{t_1, \ldots t_n, \tau\} \in R$, & for all $(x_1, \ldots x_n, y_1, \ldots y_n) \in R^{2n}$.

$$F_{X(t_1),\ldots X(t_n),Y(t_1)\ldots Y(t_n)}(x_1, x_2 \ldots x_n, y_1, \ldots y_n) = F_{X(t_1+\tau),\ldots X(t_n+\tau),Y(t_1+\tau)\ldots Y(t_n+\tau)}(x_1, \ldots x_n, y_1, \ldots y_n)$$



Page No.:
Date:

↳ For an LTI system, if $X(t)$ is the input and $Y(t)$ is the output, then $\{X(t), t \in R\}$ and $\{Y(t), t \in R\}$ are jointly stationary

$Y(t) = X(t) + X(t-1)$

$\{h(t) = \delta(t) + \delta(t-1)\}$

$Y(t)$ is a linear combn. of $\{X(t), t \in R\}$

↳ $X(t,\omega) \longrightarrow \boxed{h(t)} \longrightarrow X(t,\omega) * h(t)$
$= Y(t,\omega)$