Problem 2.30

The transfer function $H(f)$ and impulse response $h(t)$ of a linear time-invariant filter are related by

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt$$

Applying a special form of Schwarz's inequality (see Appendix 5), we may write

$$|H(f)| \le \int_{-\infty}^{\infty} |h(t)\exp(-j2\pi ft)|dt$$

Since $|\exp(-j2\pi ft)| = 1$, we may simplify this relation as

$$|H(f)| \le \int_{-\infty}^{\infty} |h(t)|dt$$

If the filter is stable, the impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

Therefore, the amplitude response of a stable filter is bounded for every value of the frequency f, as shown by

$$|H(f)| = \infty$$

According to Rayleigh's energy theorem, the energy of the input signal $x(t)$ is given by

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and the energy of the output signal $y(t)$ is

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |Y(f)|^2 df$$

The Fourier transforms $Y(f)$ and $X(f)$ are related by

$$Y(f) = H(f)X(f)$$

Therefore,

$$E_y = \int_{-\infty}^{\infty} |H(f)|^2 |X(f)|^2 df \qquad (1)$$

For a stable filter, we may express $|H(f)|$ in the form $K|H_n(f)|$ where K is a scaling factor equal to the maximum value of $|H(f)|$ and $|H_n(f)| \le 1$ for all f. Thus, we may rewrite Eq. (1) in the form:

$$E_y = K^2 \int_{-\infty}^{\infty} |H_n(f)|^2 |X(f)|^2 df$$

Since $|H_n(f)| \le 1$ for all f, it follows that

$$\int_{-\infty}^{\infty} |H(f)|^2 |X(f)|^2 df \le \int_{-\infty}^{\infty} |X(f)|^2 df$$

or equivalently

$$E_y \le K^2 \int_{-\infty}^{\infty} |X(f)|^2 df$$

If the input signal has finite energy, we then have

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \infty$$

Accordingly, we find that $E_y < \infty$, which means that the output signal $y(t)$ also has finite energy.