EE 308: Communication Systems (Section 1 – Autumn 2018)
Tutorial Problem Set 3b: Random Processes

1. **Mixing of a random process with a sinusoidal signal**

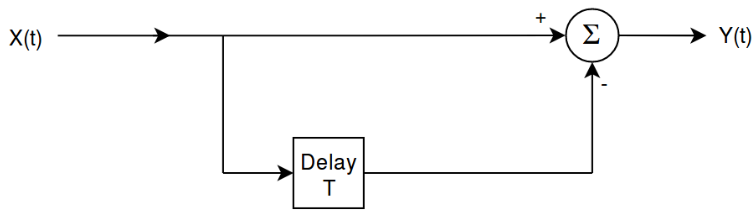
   (a) Consider the random process $Y(t) = X(t)\cos(2\pi f_c t + \Theta)$, where $\Theta \backsim \text{Unif}(0, 2\pi)$. Determine the ACF and PSD of Y(t) ?

   (b) For the complex random process $Z(t) = Z_I(t) + jZ_Q(t)$ where $Z_I(t)$ and $Z_Q(t)$ are real-valued random process given by:-
   $$Z_I(t) = A\cos(2\pi f_1 t + \theta_1)$$
   and
   $$Z_Q(t) = A\cos(2\pi f_2 t + \theta_2)$$
   where $\theta_1, \theta_2 \backsim \text{Unif}(-\pi, \pi)$. (i) Find the ACF of Z(t), (ii) What will be the ACF of Z(t) when $f_1 = f_2$? (iii) What will be the ACF when $\theta_1 = \theta_2 = \theta \backsim \text{Unif}(-\pi, \pi)$, $f_1 \neq f_2$. (iv) Comment on the stationarity of Z(t)?

2. **Relationship among PSD of input and output random process.**

   (a) Consider input random process X(t), with PSD $S_X(f)$. X(t) is passed thru via LTI transfer function H(f) to get output Y(t), with PSD $S_Y(f)$. Prove the relationship $S_Y(f) = |H(f)|^2 S_X(f)$

   (b) **Comb Filter**: Consider the filter below consisting of a delay line and a summing device taking the difference of the signal with the delayed version of the same. (i) Plot the frequency response of the filter, (ii) obtain $S_Y(f)$ in terms of $S_X(f)$ (PSDs of Y(t) and X(t) respectively), (iii) Obtain $S_Y(f)$ for frequencies $f << \frac{1}{T}$ and comment on what does the filter act like for low frequency inputs ? (iv) Repeat the above analysis when taking the sum instead of difference with the delayed version. What does the summing filter look like for low frequency inputs?



3. **Properties of ACF and PSD**

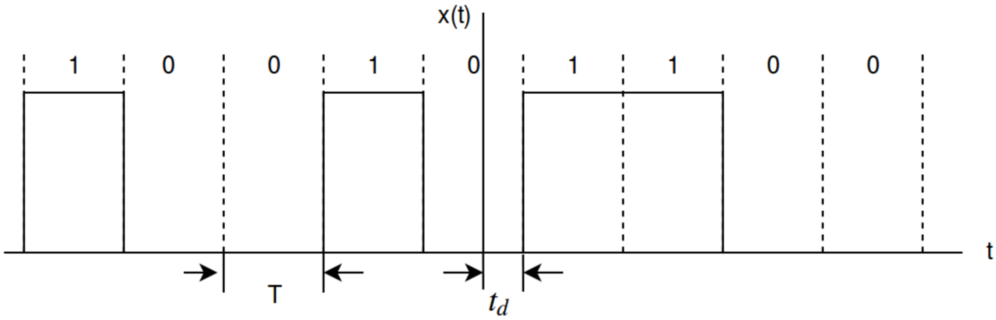
   (a) Consider a pair of wide sense stationary random processes X(t) and Y(t). Show that the cross-correlations $R_{XY}(\tau)$ and $R_{YX}(\tau)$ have the following relations:-
   $$R_{XY}(\tau) = R_{YX}(-\tau)$$
   $$|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

   (b) A wide sense stationary random process X(t) is applied to a LTI filter with impulse response h(t), producing output Y(t).

i. Show that cross-correlation function $R_{YX}(\tau) = \int_{-\infty}^{\infty} h(u)R_X(\tau - u)du$ and $R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u)R_X(\tau + u)du$

ii. Assuming X(t) is a white noise process with zero mean and PSD $N_0/2$, show that $R_{YX}(\tau) = \frac{N_0}{2}h(\tau)$. Comment on the practical significance of the above result.

4. **Random Binary Signal**
The below figure shows a sample function x(t) of a process X(t) consisting of a random sequence of binary symbols 1 and 0. The symbols 1 and 0 are represented by pulses of +A and 0 volts, duration T seconds. The pulses are not synchronized, so that the starting time t_d of the first complete pulse is equally likely to be anywhere between 0 and T seconds.



That is, t_d is the sample value of uniformally distributed random variable T_d, with the following probability density function :-

$$f_{T_d}(t_d) = \begin{cases} \frac{1}{T}, 0 \le t_d \le T \\ 0, \text{elsewhere} \end{cases}$$

During any time interval $(n-1)T < t - t_d < nT$, where n is an integer, the +A or 0 volts amplitude is equally likely. Show that for this random binary signal X(t):-

(a) The autocorrelation function is

$$R_X(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4}\left(1 - \frac{|\tau|}{T}\right), |\tau| < T \\ \frac{A^2}{4}, |\tau| \ge T \end{cases}$$

(b) The power spectral density is

$$S_X(f) = \frac{A^2}{4}\delta(f) + \frac{A^2T}{4}\text{sinc}^2(fT)$$

(c) What is the percentage power contained in the DC component of the binary signal?

5. **Random Processes and Noise**

(a) **Ideal Band-Pass Filtered White Noise**: Consider a white Gaussian noise of zero mean and PSD $N_0/2$, which is passed through an ideal BPF of passband magnitude as unity, mid-band frequency f_c and bandwidth 2B. Determine (i) PSD of the filtered noise n(t) (ii) ACF of filtered noise n(t) (iii) ACF of in-phase and quadrature components of n(t)

(b) **ACF of a Sinusoidal Signal + Noise**: The random process X(t) consists of a sinusoidal signal $A\cos(2\pi f_c t + \Theta)$ and a white Gaussian noise process W(t) of zero mean and PSD $N_0/2$. That is,

$$X(t) = A\cos(2\pi f_c t + \Theta) + W(t)$$

where $\Theta \backsim \text{Unif}(-\pi, \pi)$.

i. Find the ACF of X(t)

ii. Perform the following simulations to confirm the above ACF obtained theoretically:-

A. Take x(t), a sample function of X(t) with $f_c = 0.002$ Hz, $\theta = -\pi/2$ for a finite duration T=1000 seconds, and amplitude to be $\sqrt{2}$ to give unit power, $N_0/2 = 1$. Plot x(t).

B. Compute ACF of x(t) given by $R_x(\tau) = \frac{1}{T}\sum_{-\frac{T}{2}}^{\frac{T}{2}} \left(x^*(t-\tau)x(t)\right)$. Plot $R_x(\tau)$. This is can be interpreted as the time averaged ACF.

C. Repeat the above computation for 500 different x(t)'s by varying θ. Take the average of all ACFs obtained and plot it, which can be interpreted as the ensemble average

Comment on the output plots obtained in the process and infer ergodicity of the process from the plots obtained.

(c) **Gaussian Process**

Let X and Y be statistically independent Gaussian distributed random variables, $X, Y \backsim \mathcal{N}(0,1)$. Define the Gaussian process:-

$$Z(t) = X\cos(2\pi t) + Y\sin(2\pi t)$$

Is the process Z(t) wide-sense stationary? Is it also strictly stationary?