Problem 2.18

In an implicit sense, Eq. (2.153) embodies *Parseval's power theorem*, which states that for a *periodic signal* $x(t)$ we have

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |X(nf_0)|^2$$

where T is the period of the signal, f_0 is the fundamental frequency, and $X(nf_0)$ is the Fourier transform of $x(t)$ evaluated at the frequency nf_0. Prove this theorem.

Solution

Adapting Eq. (2.86) to the problem at hand, we may write

$$x_T(t) = f_0 \sum_{n=-\infty}^{\infty} X(nf_0)\exp(j2\pi n f_0 t) \tag{1}$$

where

$$x_T(t) = \begin{cases} x(t), & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$f_0 = \frac{1}{T}$$

and $X(nf_0)$ is the Fourier transform of $g(t)$, evaluated at the frequency $f = nf_0$. Using Eq. (1) to evaluate the integral

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$

we write

$$I = f_0\int_{-T/2}^{T/2}\left(\sum_{n=-\infty}^{\infty} X(nf_0)\exp(j2\pi n f_0 t)\right)\left(\sum_{m=-\infty}^{\infty} X^*(mf_0)\exp(-j2\pi m f_0 t)\right)dt$$

$$= f_0 \sum_{n=-\infty}^{\infty} X(nf_0)X^*(mf_0)\int_{-T/2}^{T/2}\exp(j2\pi(n-m)f_0 t)dt \tag{2}$$

To evaluate the integral on the right-hand side of Eq. (2), we write

$$\int_{-T/2}^{T/2}\exp(j2\pi(n-m)f_0 t)dt = \frac{1}{j2\pi(n-m)f_0}\exp(j2\pi(n-m)f_0 t)\Big|_{t=-T/2}^{T/2}$$

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Problem 2-18 continued

$$= \frac{1}{j2\pi(n-m)f_0}[\exp((j\pi(n-m)) - \exp-j\pi(n-m))]$$

$$= \frac{1}{\pi(n-m)f_0}\sin(\pi(n-m)) \qquad (3)$$

Whenever the indices n and m are assigned different integer values, Eq. (3) assumes the value zero. On the other hand, whenever the indices are assigned the same integer value, the integral in Eq. (3) assumes the limiting value

$$\frac{1}{f_0}\lim_{n=m}\frac{\sin(\pi(n-m))}{\pi(n-m)} = \frac{1}{f_0}$$

Accordingly, we may simplify Eq. (3) as

$$\int_{-T/2}^{T/2}\exp(j2\pi(n-m)f_0 t)dt = \begin{cases} \frac{1}{f_0}, & n = m \\ 0, & \text{otherwise} \end{cases} \qquad (4)$$

Hence, substituting Eq. (4) into (2), we get

$$I = \sum_{n=-\infty}^{\infty}|X(nf_0)|^2 \qquad (5)$$

We finally write

$$\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt = \sum_{n=-\infty}^{\infty}|X(nf_0)|^2$$

which is the desired result.