EE 308: Communication Systems (Section 1 – Autumn 2018)
Tutorial Problem Set 3b, Solutions: Random Processes

1. **Mixing of a random process with a sinusoidal signal**

   (a) Consider the random process $Y(t) = X(t)\cos(2\pi f_c t + \Theta)$, where $\Theta \backsim \text{Unif}(0, 2\pi)$. Determine the ACF and PSD of Y(t) ?

   (b) For the complex random process $Z(t) = Z_I(t) + jZ_Q(t)$ where $Z_I(t)$ and $Z_Q(t)$ are real-valued random process given by:-

   $$Z_I(t) = A\cos(2\pi f_1 t + \theta_1)$$

   and

   $$Z_Q(t) = A\cos(2\pi f_2 t + \theta_2)$$

   where $\theta_1, \theta_2 \backsim \text{Unif}(-\pi, \pi)$. (i) Find the ACF of Z(t), (ii) What will be the ACF of Z(t) when $f_1 = f_2$? (iii) What will be the ACF when $\theta_1 = \theta_2 = \theta \backsim \text{Unif}(-\pi, \pi)$, $f_1 \neq f_2$. (iv) Comment on the stationarity of Z(t)? theta 1 nd 2 are independent

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Solution, Part (a)

Since $\Theta \backsim \text{Unif}(0, 2\pi)$ the time origin of the sine wave is arbitary.

$$\begin{aligned}
R_Y(\tau) &= \mathbb{E}\big(Y(t+\tau)Y(t)\big) \\
&= \mathbb{E}\big(X(t+\tau)X(t)\cos(2\pi f_c(t+\tau)+\Theta)\cos(2\pi f_c t+\Theta)\big) \\
&= \frac{R_X(\tau)}{2}\mathbb{E}\big(\cos(2\pi f_c\tau) + \cos(4\pi f_c t + 2\pi f_c\tau + 2\Theta)\big) \\
&= \frac{R_X(\tau)}{2}\cos(2\pi f_c\tau)
\end{aligned}$$

Take Fourier Transform of $R_Y(\tau)$ to obtain $S_Y(f) = frac14\big(S_X(f-f_c) + S_X(f+f_c)\big)$ Observe the $\frac{1}{2}$ factor which is there since the average power of cosine signal is $\frac{1}{2}$, and $R_X(0)$ represents the power of the signal. **End of Part (a)**

Solution, Part (b)

For complex signals we have $R_Z(t_1, t_2) = \mathbb{E}[Z(t_1)Z^*(t_2)]$. Let $\omega_1 = 2\pi f_1$, $\omega_2 = 2]pi f_2$.

$$\begin{aligned}
R_Z(t_1,t_2) &= A^2\mathbb{E}[\big(\cos(\omega_1 t_1+\theta_1) + j\cos(\omega_2 t_1+\theta_2)\big)\big(\cos(\omega_1 t_2+\theta_1) + j\cos(\omega_2 t_2+\theta_2)\big)] \\
&= \mathbb{E}\Big[\cos(\omega_1 t_1+\theta_1)\cos(\omega_1 t_2+\theta_1) - \cos(\omega_2 t_1+\theta_2)\cos(\omega_2 t_2+\theta_2) \\
&+ j\big(\cos(\omega_1 t_1+\theta_1)\cos(\omega_2 t_2+\theta_2) + \cos(\omega_2 t_1+\theta_2)\cos(\omega_1 t_2+\theta_1)\big)\Big] \\
&= \frac{A^2}{2}\mathbb{E}\Big[\cos(\omega_1(t_1-t_2)) + \cos(\omega_1(t_1+t_2)+2\theta_1) + \cos(\omega_2(t_1-t_2)) + \cos(\omega_2(t_1+t_2)+2\theta_2)\Big] \\
&+ j\frac{A^2}{2}\mathbb{E}\Big[\cos(\omega_1 t_1 - \omega_2 t_2 + \theta_1 - \theta_2) + \cos(\omega_1 t_1 + \omega_2 t_2 + \theta_1 + \theta_2)\Big] \quad \text{Text} \\
&+ j\frac{A^2}{2}\mathbb{E}\Big[\cos(\omega_2 t_1 - \omega_1 t_2 + \theta_2 - \theta_1) + \cos(\omega_2 t_1 + \omega_1 t_2 + \theta_1 + \theta_2)\Big] \\
&= \frac{A^2}{2}\big(\cos(\omega_1(t_1-t_2)) + \cos(\omega_2(t_1-t_2))\big) \because \text{Terms like } \mathbb{E}(\theta) \text{ will be zero, End of Part (i)}
\end{aligned}$$

If $\omega_1 = \omega_2 = \omega$, we get $R_Z(t_1, t_2) = A^2 cos(\omega(t_1 - t_2))$ End of Part (ii)

Observe that if $\theta_1 = \theta_2$, the imaginary terms also come in play, with

$$\begin{aligned}
R_Z(t_1, t_2) = & \frac{A^2}{2}\mathbb{E}\Big[\cos(\omega_1(t_1 - t_2)) + \cos(\omega_1(t_1 + t_2) + 2\theta_1) + \cos(\omega_2(t_1 - t_2)) + \cos(\omega_2(t_1 + t_2) + 2\theta_2)\Big] \\
& + j\frac{A^2}{2}\mathbb{E}\Big[\cos(\omega_1 t_1 - \omega_2 t_2 + \theta_1 - \theta_2) + \cos(\omega_1 t_1 + \omega_2 t_2 + \theta_1 + \theta_2)\Big] \\
& + j\frac{A^2}{2}\mathbb{E}\Big[\cos(\omega_2 t_1 - \omega_1 t_2 + \theta_2 - \theta_1) + \cos(\omega_2 t_1 + \omega_1 t_2 + \theta_1 + \theta_2)\Big] \\
& = \frac{A^2}{2}\Big(\cos(\omega_1(t_1 - t_2)) + \cos(\omega_2(t_1 - t_2)) + j\big(\cos(\omega_2 t_1 - \omega_1 t_2) + \cos(\omega_1 t_1 - \omega_2 t_2)\big)\Big)
\end{aligned}$$

End of Part (iii)

Observe that for Parts (i), (ii) since the ACF was dependent only on time difference $t_1 - t_2$, these show stationarity. However, when $\theta_1 = \theta_2$, the direct dependence on t_1, t_2 is lost and process becomes non-stationary. End of Part (iv), Question 1

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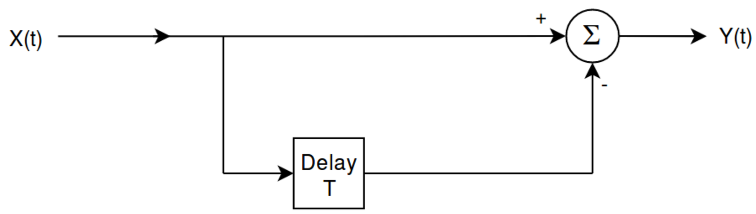
2. **Relationship among PSD of input and output random process.**

   (a) Consider input random process X(t), with PSD $S_X(f)$. X(t) is passed thru via LTI transfer function H(f) to get output Y(t), with PSD $S_Y(f)$. Prove the relationship $S_Y(f) = |H(f)|^2 S_X(f)$

   (b) **Comb Filter**: Consider the filter below consisting of a delay line and a summing device taking the difference of the signal with the delayed version of the same. (i) Plot the frequency response of the filter, (ii) obtain $S_Y(f)$ in terms of $S_X(f)$ (PSDs of Y(t) and X(t) respectively), (iii) Obtain $S_Y(f)$ for frequencies $f << \frac{1}{T}$ and comment on what does the filter act like for low frequency inputs ? (iv) Repeat the above analysis when taking the sum instead of difference with the delayed version. What does the summing filter look like for low frequency inputs?



Solution, Part (a)

$$\begin{aligned}
R_Y(\tau) &= \mathbb{E}(Y(t)Y^*(t+\tau)) \\
&= \mathbb{E}\big(\big(\int_{-\infty}^{\infty} H(\tau_1)X(t-\tau_1)d\tau_1\big)\big(\int_{-\infty}^{\infty} H(\tau_2)X(t+\tau-\tau_2)d\tau_2\big)\big) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\tau_1)H(\tau_2)\mathbb{E}(X(t-\tau_1)X(t+\tau-\tau_2))d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\tau_1)H(\tau_2)R_X(\tau+\tau_1-\tau_2)d\tau_1 d\tau_2
\end{aligned}$$

$$
\begin{aligned}
S_Y(f) &= \int_{-\infty}^{\infty} R_Y(\tau)e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\tau_1)H(\tau_2)R_X(\tau+\tau_1-\tau_2)e^{-j2\pi f\tau}d\tau d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\tau_1)H(\tau_2)R_X(\tau_0)e^{-j2\pi f(\tau_0-\tau_1+\tau_2)}d\tau_0 d\tau_1 d\tau_2, \text{ Substitute } \tau_0 = \tau+\tau_1-\tau_2 \\
&= \int_{-\infty}^{\infty} H(\tau_1)e^{j2\pi f\tau_1}\int_{-\infty}^{\infty} H(\tau_2)e^{-j2\pi f\tau_2}\int_{-\infty}^{\infty} R_X(\tau_0)e^{-j2\pi f\tau_0} \\
&= H^*(f)H(f)S_X(f) \\
&= |H(f)|^2 S_X(f)
\end{aligned}
$$

End of Part (a)

Solution, Part (b)

$H_m(f) = 1 - e^{-j2\pi fT} \implies |H_m(f)|^2 = 4\sin^2(\pi fT)$

|H|^2 = (H)x(H*)

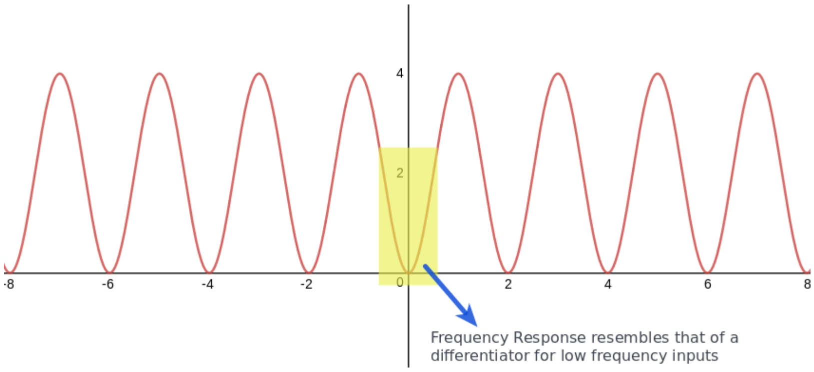


Figure 1: Plotting $|H_m(f)|^2$ for T=0.5

Observe the comb like frequency response of the above figure. Recall that $S_Y(f) = |H_m(f)|^2 S_X(f)$ from part (a). Here we will get $S_Y(f) = 4\sin^2(\pi fT)S_X(f)$. Since $|H_m(f)|^2$ looks like a comb, it will selectively give low power at some frequencies, and hence this filter can be used to 'comb' out certain frequencies. When $f << \frac{1}{T} \implies fT << 1$ we get $S_Y(f) = 4(\pi fT)^2 S_X(f)$, and the filter acts like a differentiator/high pass filter.

Now consider $H_p(f) = 1 + e^{-j2\pi fT} \implies |H_p(f)|^2 = 4\cos^2(\pi fT)$

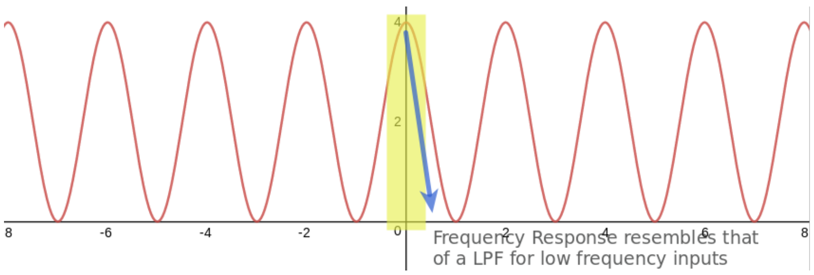


Figure 2: Plotting $|H_p(f)|^2$ for T=0.5

From the figure, observe that the filter will look like a low pass filter for low frequency inputs. End of Part (b), Question 2

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3. **Properties of ACF and PSD**

(a) Consider a pair of wide sense stationary random processes X(t) and Y(t) . Show that the cross-correlations $R_{XY}(\tau)$ and $R_{YX}(\tau)$ have the following relations:-

These are real signals

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

(b) A wide sense stationary random process X(t) is applied to a LTI filter with impulse response h(t), producing output Y(t).

i. Show that cross-correlation function $R_{YX}(\tau) = \int_{-\infty}^{\infty} h(u) R_X(\tau - u) du$ and $R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u) R_X(\tau + u) du$

ii. Assuming X(t) is a white noise process with zero mean and PSD $N_0/2$, show that $R_{YX}(\tau) = \frac{N_0}{2} h(\tau)$. Comment on the practical significance of the above result.

Solution, Part (a)

$$\begin{aligned} R_{XY}(\tau) &= \mathbb{E}(X(t+\tau)Y(t)) \\ R_{XY}(-\tau) &= \mathbb{E}(X(t-\tau)Y(t)) \\ &= \mathbb{E}(X(u)Y(u+\tau)) \\ &= R_{YX}(\tau) \end{aligned}$$

For the next relation, consider

$$\begin{aligned} \mathbb{E}[(X(t+\tau) \pm Y(t))^2] &= \mathbb{E}(X(t+\tau)^2 \pm 2X(t+\tau)Y(t) + Y(t)^2) \geq 0 \\ &\implies R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \geq 0 \\ &\implies |R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)] \end{aligned}$$

End of Part (a)

Solution, Part (b)

$$\begin{aligned} R_{YX}(\tau) &= \mathbb{E}(Y(t+\tau)X(t)) \\ Y(t) &= \int_{-\infty}^{\infty} X(u)h(t-u)du \\ \therefore R_{YX}(\tau) &= \mathbb{E}\Big(\int_{-\infty}^{\infty} X(u)X(t)h(t+\tau-u)\Big) \\ &= \int_{-\infty}^{\infty} \mathbb{E}(X(u)X(t))h(t+\tau-u) \\ &= \int_{-\infty}^{\infty} R_X(u-t)h(t+\tau-u) \\ &= \int_{-\infty}^{\infty} R_X(\tau-\alpha)h(\alpha), \text{ where} \alpha = t+\tau-u \end{aligned}$$

Use Relation (2) of part (a) to prove the $R_{XY} = \int_{-\infty}^{\infty} h(u)R_X(\tau - u)du$ result.

When X(t) white gaussian with PSD $\frac{N_0}{2}$, we get $R_X(\tau-u) = \frac{N_0}{2}\delta(\tau-u)$, which gives $R_{YX} = \int_{-\infty}^{\infty} h(u)\frac{N_0}{2}\delta(\tau-u)du$. Using the sifting property of delta function we get $R_{YX} = \frac{N_0}{2}h(\tau)$.
This gives $h(\tau) = \frac{2}{N_0}R_{YX}(\tau)$. This means that we may measure the impulse response of the filter by applying a white noise of PSD $\frac{N_0}{2}$ to the filter input, cross-correlating the filter output with input and scaling the result by $\frac{2}{N_0}$ End of Part (b), Question 3

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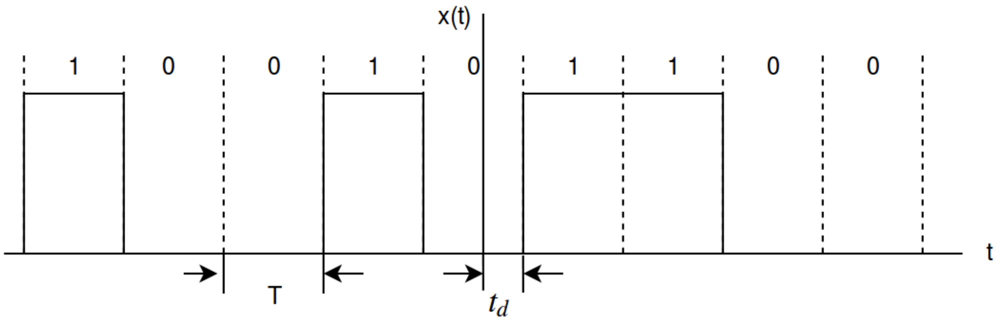
4. **Random Binary Signal**

The below figure shows a sample function x(t) of a process X(t) consisting of a random sequence of binary symbols 1 and 0. The symbols 1 and 0 are represented by pulses of +A and 0 volts, duration T seconds. The pulses are not synchronized, so that the starting time t_d of the first complete pulse is equally likely to be anywhere between 0 and T seconds.



That is, t_d is the sample value of uniformally distributed random variable T_d, with the following probability density function :-

$$f_{T_d}(t_d) = \begin{cases} \frac{1}{T}, 0 \le t_d \le T \\ 0, \text{elsewhere} \end{cases}$$

During any time interval $(n-1)T < t - t_d < nT$, where n is an integer, the +A or 0 volts amplitude is equally likely. Show that for this random binary signal X(t):-

(a) The autocorrelation function is

$$R_X(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4}\left(1 - \frac{|\tau|}{T}\right), |\tau| < T \\ \frac{A^2}{4}, |\tau| \ge T \end{cases}$$

(b) The power spectral density is

$$S_X(f) = \frac{A^2}{4}\delta(f) + \frac{A^2T}{4}\text{sinc}^2(fT)$$

(c) What is the percentage power contained in the DC component of the binary signal?

Solution

When $|\tau| > T$, $X(t), X(t+\tau)$ are independent and correspond to two different coin tosses. Hence

$$\mathbb{E}[X(t)X(t+\tau)] = \mathbb{E}[X(t)]\mathbb{E}[X(t+\tau)] = \frac{A^2}{4}$$

for $|\tau| \le T$, we get the same pulse if $t_d < T - |\tau|$ (i.e. $t_d + \tau < T$, we have the same amplitude of the 2 samples).

Hence, $\mathbb{E}[X(t)X(t+\tau)] = \begin{cases} A^2 \times \frac{1}{2} + 0^2 \times \frac{1}{2}, t_d < T - |\tau| \\ \frac{A^2}{4}, \text{otherwise} \end{cases}$ both obs at same pulse

both obs at diff pulse

Hence $R_X(\tau) = \int_0^{T-|\tau|} \frac{A^2}{2T} dt_d + \int_{T-|\tau|}^{T} \frac{A^2}{4T} dt_d$
Solving and rearranging we get

$$R_X(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4}\left(1 - \frac{|\tau|}{T}\right), |\tau| < T \\ \frac{A^2}{4}, |\tau| \geq T \end{cases}$$

End of Part (a)

Use the fact that for $g(\tau) = \begin{cases} 1 - \frac{|\tau|}{T}, |\tau| \leq T \\ 0, \text{otherwise} \end{cases}$ has the fourier transform $G(f) = T\text{sinc}^2(fT)$ to obtain

$$S_X(f) = \frac{A^2}{4}\delta(f) + \frac{A^2T}{4}\text{sinc}^2(fT)$$

End of Part (b)

Observe that $R_X(0) = \frac{A^2}{2}$. DC component of signal has amplitude $\frac{A}{2}$ and thus power proportional to $\frac{A^2}{4}$. Hence, the DC component of the signal carries half the power of the signal. End of Part (b), Question 3

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5. **Random Processes and Noise**

   (a) **Ideal Band-Pass Filtered White Noise**: Consider a white Gaussian noise of zero mean and PSD $N_0/2$, which is passed through an ideal BPF of passband magnitude as unity, mid-band frequency f_c and bandwidth 2B. Determine (i) PSD of the filtered noise n(t) (ii) ACF of filtered noise n(t) (iii) ACF of in-phase and quadrature components of n(t)

   (b) **ACF of a Sinusoidal Signal + Noise**: The random process X(t) consists of a sinusoidal signal $A\cos(2\pi f_c t + \Theta)$ and a white Gaussian noise process W(t) of zero mean and PSD $N_0/2$. That is,

   $$X(t) = A\cos(2\pi f_c t + \Theta) + W(t)$$

   where $\Theta \backsim \text{Unif}(-\pi, \pi)$.

   i. Find the ACF of X(t)

   ii. Perform the following simulations to confirm the above ACF obtained theoretically:-

      A. Take x(t), a sample function of X(t) with $f_c = 0.002$ Hz, $\theta = -\pi/2$ for a finite duration T=1000 seconds, and amplitude to be $\sqrt{2}$ to give unit power, $N_0/2 = 1$. Plot x(t).

      B. Compute ACF of x(t) given by $R_x(\tau) = \frac{1}{T}\sum_{-\frac{T}{2}}^{\frac{T}{2}} \big(x^*(t-\tau)x(t)\big)$. Plot $R_x(\tau)$. This is can be interpreted as the time averaged ACF.

      C. Repeat the above computation for 500 different x(t)'s by varying θ. Take the average of all ACFs obtained and plot it, which can be interpreted as the ensemble average

   Comment on the output plots obtained in the process and infer ergodicity of the process from the plots obtained.

   (c) **Gaussian Process**

   Let X and Y be statistically independent Gaussian distributed random variables, $X, Y \backsim \mathcal{N}(0,1)$. Define the Gaussian process:-

   $$Z(t) = X\cos(2\pi t) + Y\sin(2\pi t)$$

Is the process Z(t) wide-sense stationary? Is it also strictly stationary?

Solution, Part (a)

The PSD of the filtered noise n(t) will be as shown in Fig. 3. The autocorrelation function n(t) will be the inverse fourier transform of the PSD,

$$R_n(\tau) = \int_{-f_c-B}^{-f_c+B} \frac{N_0}{2} e^{j2\pi ft} df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} e^{j2\pi ft} df = 2N_0 B \text{sinc}(2B\tau) cos(2\pi f_c t)$$

The PSD of the in-phase and quadrature-phase components will be given as in Fig. 3. Hence the autocorrelation function $R_{n_I}(\tau) = R_{n_Q}(\tau) = 2N_0 B\text{sinc}(2B\tau)$

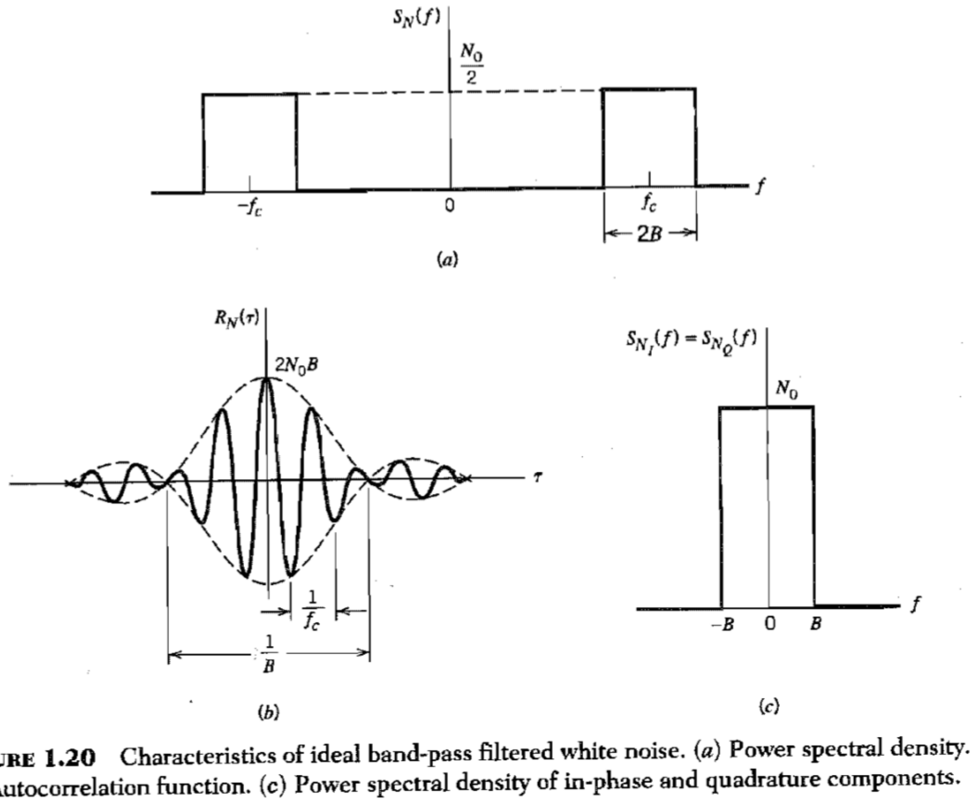

Figure 3: PSD and ACF of gaussian noise after an ideal BPF

End of Part (a)

Solution, Part (b)

ACF of X(t), $R_X(\tau) = A^2 cos(2\pi f_c \tau) + \frac{N_0}{2}\delta(\tau)$. Go through and Run the attached code. Observe that the time averaged and ensemble averaged plots nearly match, which indicate the ergodicity of the process. **End of Part (b)**

Solution, Part (c)

$Z(t) = X\cos(2\pi t) + Y\sin(2\pi t)$. Observe that at any times t_1, t_2, $\mathbb{E}(Z(t_1)) = \mathbb{E}(Z(t_2)) = 0$.

$$
\begin{aligned}
R_Z(t_1, t_2) &= \mathbb{E}\big(Z(t_1)Z(t_2)\big) \\
&= \mathbb{E}\Big(\cos(2\pi t_1)\cos(2\pi t_2)X^2 + (\cos(2\pi t_1)\sin(2\pi t_2) + \cos(2\pi t_2)\sin(2\pi t_1))XY \\
&+ \sin(2\pi t_1)\sin(2\pi t_2)Y^2\Big) \\
&= \cos(2\pi t_1)\cos(2\pi t_2)\mathbb{E}(X^2) + (\cos(2\pi t_1)\sin(2\pi t_2) + \cos(2\pi t_2)\sin(2\pi t_1))\mathbb{E}(XY) \\
&+ \sin(2\pi t_1)\sin(2\pi t_2)\mathbb{E}(Y^2) \\
&= \cos(2\pi t_1)\cos(2\pi t_2) + \sin(2\pi t_1)\sin(2\pi t_2) \\
&= \cos(2\pi(t_1 - t_2)), \because \mathbb{E}(X^2) = \mathbb{E}(Y^2) = 1, \mathbb{E}(XY) = 0
\end{aligned}
$$

Since $R_Z(t_1, t_2)$ depends only on the time difference $t_1 - t_2$, Z(t) is weakly stationary. Since Z(t) is a Gaussian process it is also strictly stationary.