EE 308: Communication Systems (Section 1 – Autumn 2018)
Tutorial Problem Set 3: Noise in Analog Modulation

1. A DSB-SC modulated signal is transmited over a noisy channel, with the power spectral density of the noise being as shown in Figure 1. The message bandwidth is 4 kHz and the carrier frequency is 200 kHz. Assuming that the average power of the modulated wave is 10 watts, determine the output SNR of the receiver. **(Simon Haykins, "Communications Systems" 5th Edition, Q 6.3, Page 235)**

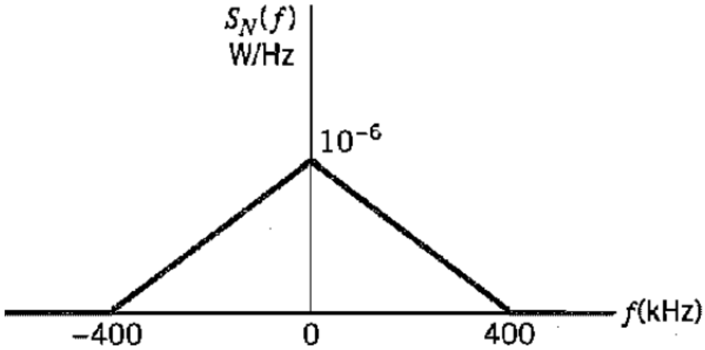


Figure 1: Power spectral density of noise for Q1

Solution
After passing through the received signal through the band pass filter centered on 200 kHz and having bandwidth 8 kHz (DSB), we get

$$x(t) = A_c m(t)\cos(2\pi f_c t) + n(t) = A_c m(t)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) + n_I(t)\sin(2\pi f_c t)$$

Coherent detection of x(t) will yield $y(t) = A_c m(t) + n_I(t)$ ::::::::should be a factor of 0.5
Average power of modulated wave is given to be 10W. To calculate average power of the in-phase noise component $n_I(t)$ refer to the spectra in Figure 2.

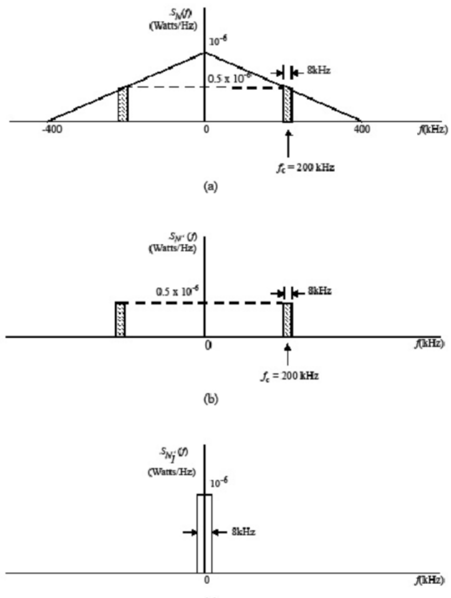


Figure 2: Power spectral density plots for average noise power calculation, Q1

Part (a) of Figure 2 shows the PSD of n(t) and a superposition of the frequency response of BPF
Part (b) of Figure 2 shows the PSD of $n_I(t)$ at filter output
Part (c) of Figure 2 shows the PSD of $n_I(t)$ after coherent detection
Note that since bandwidth of the filter is small compared to carrier frequency, spectral characteristics of $n_I(t)$ is assumed to be flat.
Hence, average power of in-phase noise component is $(10^{-6}$ watts/Hz$)(8 * 10^3$Hz$) = 0.008$ watts
Hence, output SNR is $\frac{10}{0.008} = 1250$ should be 2500 **End of Question 1**

2. **Noise in FM system**

argument detector and then differentiate

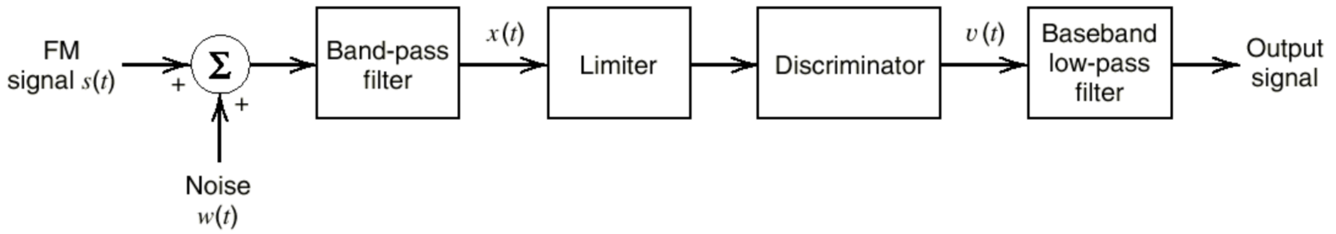


Figure 3: Block Diagram of FM receiver

Assume that FM signal s(t) is given by $s(t) = A_c \cos(2\pi f_c t + \phi(t))$, where $\phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau$. Also noise after BPF is given by $n(t) = n_i(t)\cos(2\pi f_c t) - n_q(t)\sin(2\pi f_c t)$, assume $n_i(t), n_q(t)$ to be white gaussian. Let v(t), the output of discriminator be $v(t) = k_f m(t) + n_d(t)$ where $n_d(t)$ is the noise in the system after discriminator.

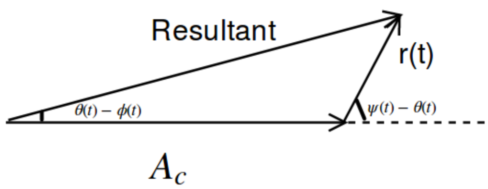
(a) Prove that $n_d(t) \approx \frac{1}{2\pi A_c}\frac{d}{dt}(n_q(t))$,
assuming (i) High SNR $A_c >> |n(t)|$, (ii) $\tan^{-1}(x) \approx x$ and (iii) Considering noise phase to be uniformly distributed in $(0, 2\pi)$, noise signal to be independent of the modulating signal

(b) Determine the power spectral density of $n_d(t)$ and henceforth underline the reasons for noise quieting effect and pre-emphasis in FM systems

Solution
FM signal s(t) is given by $s(t) = A_c \cos(2\pi f_c t + \phi(t))$, where $\phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau$. Represent noise after BPF in terms of it's envelope and phase as
$n(t) = r(t)\cos(2\pi f_c t + \psi(t))$, where $r(t) = [n_I^2(t) + n_Q^2(t)]^{1/2}$, $\psi(t) = \tan^{-1}[\frac{n_Q(t)}{n_I(t)}]$
We have, x(t), i.e. the signal after band pass filter (Ref. Figure 3), as $x(t) = s(t) + n(t)$ and hence

$$x(t) = A_c \cos(2\pi f_c t + \phi(t)) + r(t)\cos(2\pi f_c t + \psi(t))$$

From the phasor diagram below, it is clear that the phase of x(t) given by $\theta(t)$ is given by



$$\theta(t) = \phi(t) + \tan^{-1}(\frac{r(t)sin(\psi(t) - \phi(t))}{A_c + r(t)cos(\psi(t) - \phi(t))})$$

Now, assume that (i) $A_c >> |r(t)|$ and (ii) $\tan^{-1}(x) \approx x$ to get $\theta(t) = \phi(t) + \frac{r(t)}{A_c}\sin[\psi(t) - \phi(t)]$ The discriminator output $v(t) = \frac{\theta'(t)}{2\pi}$. Hence $v(t) \approx k_f m(t) + n_d(t)$, where $n_d(t)$ is noise in system after the discriminator, given by

$$n_d(t) = \frac{1}{2\pi A_c}\frac{d}{dt}\{r(t)\sin[\psi(t) - \phi(t)]\}$$

Now, we use the third assumption, i.e. we consider noise phase($\psi(t)$) to be uniformly distributed in (0, 2π) and noise signal to be independent of the modulating signal. This implies that $\psi(t) - \phi(t)$ is also uniformly distributed in (0, 2π) and hence $[\psi(t) - \phi(t)]$ equidistributed to $\psi(t)$. This simplifies $n_d(t)$ to $\frac{1}{2\pi A_c}\frac{d}{dt}\{r(t)\sin[\psi(t)]\}$. Recall $\psi(t)$ was the phase of the noise and hence $r(t)\sin[\psi(t)]$ is nothing but $n_Q(t)$. Hence, we get

$$n_d(t) \approx \frac{1}{2\pi A_c}\frac{dn_Q(t)}{dt}$$

(**End of Part a**)

Now, since the differentiation of a function wrt time corresponds to multiplication of it's fourier transform bu $j2\pi f$, hence, we can obtain $n_d(t)$ by passing $n_Q(t)$ through a linear filter with transfer function equal to $\frac{j2\pi f}{2\pi A_c} = \frac{jf}{A_c}$. Now, we use the following property of power spectral density

$$Y(f) = H(f)X(f) => S_y(f) = |H(f)|^2 S_x(f)$$

Hence, $S_{N_d}(f) = \frac{f^2}{A_c^2}S_{N_Q}(f)$. Now $N_Q(f)$ is white and gaussian, and hence it has the spectral density given in Figure 4. This is assuming if the band pass filter in receiver model of Figure 3 had ideal frequency response characterized by bandwidth B_T with mid band frequency f_c. Hence, we will have

$$S_{N_d}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2}, & \text{if } |f| \leq \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

After passing through the post detection low pass filter (Let $n_o(t)$ be the output noise of the receiver), which will further restrict the frequencies to message bandwidth W, we will get.

$$S_{N_o}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2}, & \text{if } |f| \leq W \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

In wide band FM, usually W is smaller than B_T. B_T is usually set to the transmission bandwidth of FM signal.

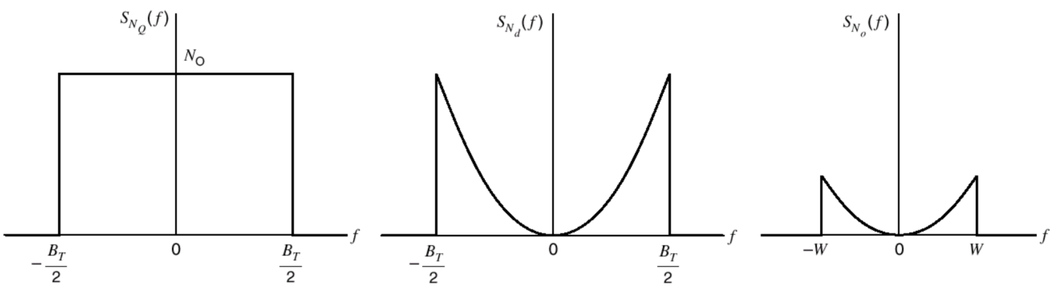


Figure 4: Power spectral densities of $n_Q(t), n_d(t)$ and $n_o(t)$

Motivation towards Noise quieting effect: Since PSD is inversely proportional to A_c^2, increasing carrier amplitude helps to quiten the noise drastically

Motivation towards Emphasis: Since PSD is proportional to f^2, high frequency components will brandish higher noise amplitudes, and hence, we need to pre-emphasise the higher frquency components in the message to mitigate for this effect. **End of Question 2**

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3. **Validity of AM approximation**

Consider the output of an envelope detector $y(t) = \{[A_c + A_c k_a m(t) + n_I(t)]^2 + n_Q(t)^2\}^{\frac{1}{2}}$

(a) Assume that the probability of the event

$$|n_Q(t)| > \epsilon A_c |1 + k_a m(t)|$$

is equal to or less than δ_1, where $\epsilon << 1$. What is the probability that the effect of quadrature component of noise is negligible

(b) Now, suppose that k_a is adjusted relative to the message signal m(t) such that the probability of the event

$$A_c[1 + k_a m(t)] + n_I(t) < 0$$

is equal to δ_2. What is the probability that the approximation

$$y(t) \approx A_c[1 + k_a m(t)] + n_I(t)$$

is valid ?

(c) Comment on the significance of the result in part (b) for the case when δ_1 and δ_2 are both small compared to unity **(Simon Haykins, "Communications Systems" 5th Edition, Q 6.7, Page 236)**

Solution

Part a

If the probability $P(|n_Q(t)| > \epsilon A_c |1 + k_a m(t)|) \leq \delta_1$, then clearly with a probability of $1 - \delta_1$, we may say that

$$y(t) \approx \left\{[A_c + A_C k_a m(t) + n_I(t)]\right\}$$

That is, the probability that the quadrature component $n_Q(t)$ is negligibly small is greater than $1 - \delta_1$

Part b

Note that if $k_a m(t) < -1$ then we get overmodulation, so that even in absence of noise, the envelope detector is distorted. Hence, k_a is adjusted relative to message signal such that the probability

$$P(A_c[1 + k_a m(t)] + n_I(t) < 0) \leq \delta_2$$

Hence, the probability of the event $y(t) \approx A_c[1 + k_a m(t)] + n_I(t)$ becomes greater than $(1 - \delta_1)(1 - \delta_2)$ since the adjustment made in k_a is independent of the quadrature phase noise as in Part b.

Part c

When δ_1 and δ_2 are both small compared with unity, the probability of the event, $y(t) \approx A_c[1 + k_a m(t)] + n_I(t)$ for nay value of t is very close to unity. **End of Question 3**

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4. **Comparing FM and PM systems performance towards noise**

(a) Consider a **PM** system with the modulated wave defined by

$$s(t) = A_c \cos(2\pi f_c t + k_p m(t))$$

The additive noise at the phase detector input is $n(t) = n_i(t)\cos(2\pi f_c t) - n_q(t)\sin(2\pi f_c t)$ as usual. Assuming that carrier to noise ratio at detector input is high as compared to unity, determine (i) output SNR ratio and (ii) Compare results with **FM** system in case of sinusoidal modulation

(b) A **PM** system uses a pair of pre-emphasis and de-emphasis filters defined by the transfer functions

$$H_{pe}(f) = 1 + \frac{jf}{f_0}, H_{de}(f) = \frac{1}{1 + \frac{jf}{f_0}}$$

Show that the improvement in output SNR produced by this pair of filters is

$$I = \frac{W/f_0}{\tan^{-1}(W/f_0)}$$

where W is the message bandwidth. Evaluate this improvement in the case when W = 15 kHz and f_0 = 2.1 kHz, and compare the result with corresponding value for an **FM** system.

(Simon Haykins, "Communications Systems" 5th Edition, Q 6.10,6.15 , Page 236)

Solution
Part a

$s(t) = A_c \cos(2\pi f_c t + k_p m(t))$, $\phi(t) = k_p m(t)$
Using the assumptions and analysis enlisted in Q2, we get

$$\theta(t) = \phi(t) + \frac{r(t)}{A_c} sin(\psi(t)) = k_p m(t) + \frac{n_Q(t)}{A_c}$$

Now, the average signal power in x(t) is given by $k_p^2 P$ where P is the message signal power
Referring to flat power spectrum of $n_Q(t)$ from Figure 4, we can see that the average noise power is given by $2WN_0/A_c^2$, where W is the message bandwidth (one sided). The change from $\frac{B_T}{2}$ to W is due to filtering of noise by the post detection base band low pass filter.
Hence, the output SNR is given by $\text{SNR}_{op} = \frac{A_c^2 k_p^2 P}{2N_0 W}$ **End of Part (a,i)**

Now, for **Part (ii)**, we again use some results from Q2.
Recall that discriminator output is given by $v(t) \approx k_f m(t) + n_d(t)$
Hence, average Power of output signal is $P_s = k_f^2 P$, where P is average power of the message. However, one thing different in FM system is the dicriminator, because of which $n_d(t) \approx \frac{1}{2\pi A_c}\frac{dn_Q(t)}{dt}$
Also, recall from Q2 part b t hat $S_{N_d}(f) = \frac{f^2}{A_c^2} S_{N_Q}(f)$ and hence

$$S_{N_o}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2}, & \text{if } |f| \leq W \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

after the post detection base band low passs filter. Hence, average power of output noise (P_n) will be determined by integrating the power spectral density from -W to W to get

$$P_n = \frac{N_0}{A_c^2}\int_{-W}^{W} f^2 df = \frac{2N_0 W^3}{3A_c^2}$$

Hence, output SNR for FM is $\text{SNR}_{op} = \frac{P_s}{P_n} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}$

For both PM and FM, input SNR is given by $\text{SNR}_{ip} = \frac{A_c^2}{2WN_0}$, since average power in modulated signal is $\frac{A_c^2}{2}$ and average noise power in message bandwidth is WN_0

should be SNR cp

To compare between FM and PM systems, we compare the improvement caused in SNR, which is in indicated by $\frac{\text{SNR}_{op}}{\text{SNR}_{ip}}$

For FM we have $\frac{\text{SNR}_{op}}{\text{SNR}_{ip}} = \frac{3k_f^2 P}{W^2}$ and for PM we have $\frac{\text{SNR}_{op}}{\text{SNR}_{ip}} = k_p^2 P$

In case of sinusoidal modulation, $m(t) = A_m \cos(2\pi f_m t)$. Hence, $P = \frac{A_m^2}{2}, W = f_m$. Also, recall that for PM, $\beta_p = k_p A_m$ and for FM, $\beta_f = \frac{k_f A_m}{f_m}$

Hence, we get for FM and sinusoidal modulation, $\frac{\text{SNR}_{op}}{\text{SNR}_{ip}} = \frac{3\beta_f^2}{2}$ and for PM we have $\frac{\text{SNR}_{op}}{\text{SNR}_{ip}} = \frac{\beta_p^2}{2}$

Assume for fair comparison $\beta_p = \beta_f$, and hence, for a specified phase deviation, FM system performs 3 times better than PM system. **End of Part (a,ii)**

Part b

From the previous question, we know that the power spectral density of noise at the phase discriminator output is approximately constant, since noise term is proportional to $n_Q(t)$ itself, which is white (Ref. PSD of $n_Q(t)$ from Figure 4). Hence, in presence of a de-emphasis filter, the power spectral density will be proportional to $|H_{de}(f)|^2$

In presence of a de-emphasis filter, we get the average output noise power to be $\int_{-W}^{W} k|H_{de}(f)|^2 df$.

Without the de-emphasis filter, average noise power will be $\int_{-W}^{W} k df = 2kW$

We have $H_{de}(f) = \frac{1}{1+\frac{jf}{f_0}}$, and hence $|H_{de}(f)|^2 = \frac{1}{1+\frac{f^2}{f_0^2}}$

Hence, improvement $I = \frac{\text{avg output noise power w/o emphasis}}{\text{avg output noise power with emphasis}} = \frac{2W}{\int_{-W}^{W} |H_{de}(f)|^2 df} = \frac{W}{\int_0^W \frac{1}{1+\frac{f^2}{f_0^2}} df} = \frac{W/f_0}{\tan^{-1}(W/f_0)}$

Proceeding similarly for a FM system, we have average noise power without emphasis to be $\int_{-W}^{W} kf^2 df$ (We have k=$\frac{N_0}{A_c^2}$, writing this as k for succintness and since it will get divided when accounting for improvement factor)

Hence, average noise power with emphasis will be $\int_{-W}^{W} kf^2 |H_{de}(f)|^2 df$

Hence, we get $I = \frac{\int_{-W}^{W} kf^2 df}{\int_{-W}^{W} kf^2 |H_{de}(f)|^2 df} = \frac{\int_{-W}^{W} kf^2 df}{\int_{-W}^{W} k\frac{f^2}{1+\frac{f^2}{f_0^2}} df} = \frac{\int_0^W f^2 df}{\int_0^W \frac{f^2}{1+\frac{f^2}{f_0^2}} df} = \frac{(W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]}$

Substituting values W=15 kHz and f_0= 2.1 kHz, we get

$$I_{fm} = 22 \text{ and } I_{pm} = 5$$

This should be expected since in noise power spectrum of FM we had a squared growth with freqeuncy whereas the noise power spectrum was flat in case of PM. Hence, improvement factor in FM is greater than that of PM. **End of Question 4**

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5. **Threshold effect in FM systems**

   (a) Consider a practical and non ideal FM receiver, which has a non ideal discriminator which instead of differentiating the signal, takes the difference between succesive measurements:

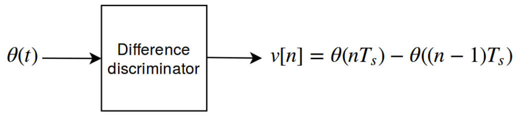

Figure 5: Block model for Q5.a

The input to the block is $\theta(t) = 2\pi\Delta f + k_f \int_0^t m(\tau)d\tau$, where Δf is representative of offset caused by noisy sinusoids and offsets in carrier frequency at receiver. The block samples the input at sampling time T_s and outputs a discrete signal $v[n] = \theta(nT_s) - \theta((n-1)T_s)$.
What should be the maximum T_s to sample the signal so as to avoid clicks in the demodulated signal, assuming the fact that message signal magnitude is upper bounded by M, i.e. $|m(t)| < M$ $\forall t$? For this question, assume that the system has been designed to meet the Nyquist criteria

(b) Recall that while deriving the noise performance in FM systems, the assumption that carrier to noise ratio was high (i.e. $A_c >> |n(t)|$) had been used. In this question, we explore the effect when the said assumption does not hold.
Consider the FM receiver block diagram as in Q2, Figure 3, but with message signal as 0, i.e. $s(t) = A_c \cos(2\pi f_c t)$ for ease of analysis. Noise after BPF is given by $n(t) = n_i(t)\cos(2\pi f_c t) - n_q(t)\sin(2\pi f_c t)$. Represent the noise as $n(t) = r(t)\cos(2\pi f_c t + \Psi(t))$ where, $r(t) = [n_I^2(t) + n_Q^2(t)]^{1/2}$, $\Psi(t) = \tan^{-1}[\frac{n_Q(t)}{n_I(t)}]$. Derive the output v(t) after the discriminator considering $r(t) = \lambda A_c$ and $\Psi(t) = \pi \sin(t)$. Plot and comment on the output waveform v(t) for $\lambda = 1.05, 0.05$

Part b of this question has been formed from the slides no. 167-169 on "Continuous Wave Modulation" by Prof. Po-Ning Chen, NCTU.

Solution

Part a

$\theta(t) = 2\pi\Delta f + \phi(t)$, where $\phi(t) = k_f \int_0^t m(\tau)d\tau$, and thus we have

$$\theta(nT_s) - \theta((n-1)T_s) = 2\pi\Delta f T_s + \phi(nT_s) - \phi((n-1)T_s) = T_s(2\pi\Delta f + \frac{\phi(nT_s) - \phi((n-1)T_s))}{T_s}$$

Now, assume that T_s is small enough so that we can approximate $\frac{\phi(nT_s)-\phi((n-1)T_s))}{T_s} = \phi'(nT_s)$
The assumption makes sense since usually we send audio signals via FM and those are sampled at around 44kHz (hence $T_s \approx 20 * 10^{-6}s$), which leads to a small enough T_s. We have $\phi'(t) = k_f m(t)$ and hence, we get discriminator output to be

$$v(t) = (2\pi\Delta f + k_f m(nT_s))T_s$$

Since the output is measured in $(-\pi, \pi)$, we get potential 2π jumps when v(t) is in vicinity of π. Hence, to avoid clicks caused by 2π jumps, we should have $|v(t)| < \pi$ and hence

$$|(2\pi\Delta f + k_f m(nT_s))|T_s < \pi => T_s < \frac{\pi}{|2\pi\Delta f - k_f|m(nT_s)||}$$

Since the message signal magnitude is upper bounded by M, we get the following condition on T_s

$$T_s < \frac{\pi}{|2\pi\Delta f - k_f M|}$$

It has already been assumed $F_s > 2W$ and hence $T_s < \frac{1}{2W}$, where W is the message bandwidth. Otherwise, both these conditions should be evaluated and the more stringent of the two conditions should be chosen. **End of part a**

Part b

Note here that message signal is 0. Hence $\phi(t)$ terms in the analysis established in Q2 will vanish. Hence, we obtain $\theta(t) = \tan^{-1}\{\frac{r(t)\sin(\psi(t))}{A_c + r(t)\cos(\psi(t))}\}$

$$v(t) = \frac{1}{2\pi}\theta'(t) = \frac{A_c r(t)\psi'(t)\cos[\psi(t)] + A_c r'(t)\sin[\psi(t)] + r^2(t)\psi'(t)}{2\pi(A_c^2 + 2A_c r(t)\cos[\psi(t)] + r^2(t))}$$

Now substitute $\psi(t) = \pi\sin(t)$ and $r(t) = \lambda A_c$

$$v(t) = \frac{\lambda}{2}\cos(t)\Big\{\frac{\cos[\pi\sin(t)] + \lambda}{1 + 2\lambda\cos[\pi\sin(t)] + \lambda^2}\Big\}$$

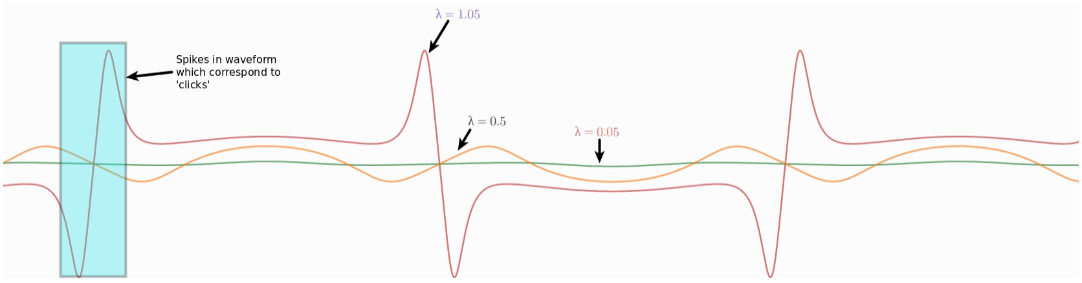

Figure 6: Plots of v(t) vs t for $\lambda = 1.05, 0.5, 0.05$

From the above plot, we can see the clicking phenomena when noise amplitude is of the order of signal amplitude (i.e. $\lambda \approx 1$) and the phase difference between noise and signal is near π.