Problem 4.7

Using the linearized model of Fig. 4.15(a), show that the model is approximately governed by the integro-differential equation

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0 \int_{-\infty}^{\infty} \phi_e(\tau) h(t-\tau) d\tau \approx \frac{d\phi_1(t)}{dt}$$

Hence, derive the following two approximate results in the frequency domain:

(a) $\Phi_e(f) = \dfrac{1}{1+L(f)}\Phi_1(f)$

(b) $V(f) = \dfrac{jf}{k_v} \dfrac{L(f)}{1+L(f)}\Phi_1(f)$

where

$$L(f) = K_0 \frac{H(f)}{jf}$$

is the open-loop transfer function. Finally, show that when $L(f)$ is large compared with unity for all frequencies inside the message band, the time-domain version of the formula in part (b) reduces to the approximate form in Eq. (4.68).

Solution

According to condition 1 stated on p.178 of the text, the frequency of the VCO is set equal to the carrier frequency f_c. According to condition 2 on the same page, the VCO output has a 90º phase shift with respect to the unmodulated carrier. In light of these two conditions, we note starting with the equation

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0 \int_{-\infty}^{\infty} \phi_e(\tau) h(t-\tau) d\tau \approx \frac{d\phi_1(t)}{dt}$$

the integral in the left-hand side of the equation is the convolution of $\phi_e(t)$ and $h(t)$. Therefore, applying the Fourier transform to this equation and using two properties of the fourier transform pertaining to differentiation and convolution, we get

$$j2\pi f \Phi_e(f) + 2\pi K_0 \Phi_e(f) H(f) \approx j2\pi f \Phi_1(f) \qquad (1)$$

where

$$\Phi_e(f) = \mathbf{F}[\phi_e(t)] \quad \text{and} \quad \Phi_1(f) = \mathbf{F}[\phi_1(t)]$$

(a) Solving Eq. (1) for $\Phi_e(f)$, we get

$$\begin{aligned} \Phi_e(f) &\approx \frac{j2\pi f}{j2\pi f + 2\pi K_0 H(f)}\Phi_1(f) \\ &= \frac{1}{1 + K_0 \dfrac{H(f)}{jf}}\Phi_1(f) \\ &= \frac{1}{1+L(f)}\Phi_1(f) \qquad (2) \end{aligned}$$

where $L(f) = \dfrac{H(f)}{jf}$

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Problem 4-7 continued

(b) Next, from Eq. (4.63) we have

$$e(t) = \frac{K_0}{k_v}\phi_e(t)$$

Therefore

$$E(f) = \frac{K_0}{k_v}\Phi_e(f)$$

And, from Eq. (4.65) we have

$$v(t) = \int_{-\infty}^{\infty} e(\tau)h(t-\tau)d\tau$$

Therefore

$$V(f) = E(f)H(f)$$

Eliminating $E(f)$ between these two transform-related equations, we get

$$V(f) = \frac{K_0}{k_v}H(f)\Phi_e(f) \tag{3}$$

Eliminating $\Phi_e(f)$ between Eqs. (1) and (3), we get

$$V(f) = \frac{K_0}{k_v}H(f) \cdot \frac{1}{1+L(f)}\Phi_1(f)$$

Since

$$L(f) = K_0\frac{H(f)}{jf}$$

then

$$\frac{K_0}{k_v}H(f) = \frac{jf}{k_v}L(f)$$

and so we get the desired result

$$V(f) \approx \frac{jf}{k_v}\ \frac{L(f)}{1+L(f)}\Phi_1(f) \tag{4}$$

Finally, when $L(f) >> 1$ for all f, Eq. (4) simplifies further as

$$(f) \approx \frac{jf}{k_v}\Phi_1(f) \approx \frac{j2\pi f}{2\pi k_v}\Phi_1(f)$$

The time-domain version of this formula reads as follows

$$v(t) \approx \frac{1}{2\pi k_v}\ \frac{d\phi_1(t)}{dt}$$

which is a repeat of Eq. (4.67).