EE 325

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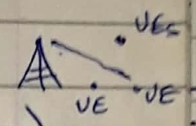
Random Variable (r.v.)

A r.v models uncertainty in engineering systems. It typically represents a scalar, whose value is pre-determined.

ex: i) received signal strength in UEs 

ii) internet speed b/w two ports (real no.)

iii) number of e^-/charge stored in a flash memory cell (discrete)

iv) Signal to interference ratio between a radio transmitter & a receiver (real no.)

X is a r.v. if:-

1) $P(X \in (-\infty, \infty)) = 1$, ie X is finite with prob 1

2) $P(X \in [a, b))$ is well defined for any $-\infty < a < b < \infty$

3) $P(X \in \bigcup_{i=1}^{\infty} [a_i, b_i)) = \sum_{i=1}^{\infty} P(X \in [a_i, b_i))$ if the intervals $[a_i, b_i)$ are disjoint.

Ex **Uniform r.v.**

A uniform r.v. (Unif [0,1]) is defined as

$P(U \in [a, b)) = (b - a)$ for $0 \le a < b \le 1$

The prob. values for X lying in various intervals can be obtained from a single f^n called as cumulative distribution f^n.

c.d.f The function $F_X(x) = P(\underset{\text{r.v.}}{X} \le \underset{\text{scalar}}{x}) = P(X \in (-\infty, x])$ is called the c.d.f of X.

$\Rightarrow F_X(-\infty) = 0$, $F_X(\infty) = 1$ & $F_X(x)$ is non-decreasing

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if $x_1 < x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$

$\therefore P(X \in (x_1, x_2]) = F_X(x_2) - F_X(x_1)$

One can show that $F_X(x)$ is right continuous,

ie. $\lim_{\epsilon \downarrow 0} P(X \le x + \epsilon) = P(X \le x)$

or $\lim_{\epsilon \downarrow 0} F_X(x+\epsilon) = F_X(x)$

↳ $P(X \in [a,b))$ is well defined. Note that $P(X \in (a,b))$, $P(X \in a,b)$, $P(X \in a,b)$ are all well defined.

$P(X \in [a,b]) = F_X(?) - F_X(?)$

$[a,b] \cup (-\infty, a) = (-\infty, b]$

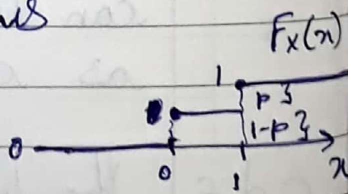
$\therefore P(X \in [a,b] \cup (-\infty,a)] = P(X \in (-\infty, b))$

$\therefore P(X \in [a,b]) + P(X \in (-\infty, a)) = F_X(b)$

$\therefore \boxed{P(X \in [a,b]) = F_X(b) - F_X(a^-)}$

Example (A switch)

Consider a r.v. X which is



$X = 1$ w.p p
$\;\;= 0$ w.p $(1-p)$

X represents a switch, X is a Bernoulli (p) r.v., or simply Ber(p)

Verify that cdf is right contⁿ. at $x=0$ & $x=1$

cdf of r.v $\iff$ A cad lag fⁿ.
starts at 0, ↲
ends in 1, non-decreasing, right contⁿ.

Discrete r.v

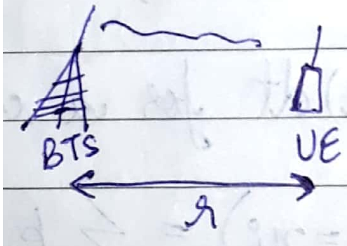
If X takes finite or countable set of values with prob. 1, then X is a discrete r.v.

Defn. Contⁿ. r.v

If $F_X(x)$ has a finite derivative as a fⁿ. of x, then X is a contⁿ. r.v & its derivative is called as p.d.f - prob. density functⁿ.

Ex. (phase)

In a wireless system, the trans, rec. are separated.



$A_c P(r) \cos(2\pi f_c t + \Theta)$

At receiver → ⊗ with local oscill. funcⁿ. $(\cos(2\pi f_c t))$

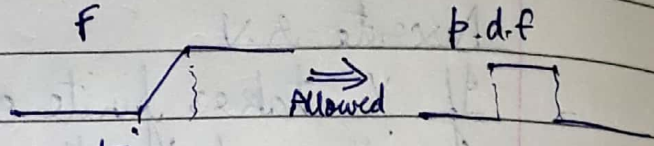
The phase offset is well modelled by a Uniform $[0, 2\pi)$ distribution.

Its pdf is given by,
$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & ; 0 \le \theta < 2\pi \\ 0 & \text{o.w} \end{cases}$$

Its cdf, $F_\Theta(\theta) = \begin{cases} 0 & \theta < 0 \\ \theta/2\pi & \theta \geq 0, \theta < 2\pi \\ 1 & \theta \geq 2\pi \end{cases}$

24/07 For a contn. r.v., its pdf may have discontinuity

~~You~~ We can look at one sided derivative.



For a cont^n. r.v X, p.d.f $f_X(x) = \dfrac{dF_X(x)}{dx}$

For a discrete r.v., ie there are countable set of pts $\{x_i, i \in \mathbb{Z}^+\}$ such that

$$P(X = x_i) \neq 0 \ \& \ \sum_{i=1}^{\infty} P(X = x_i) = 1$$

Some r.v. can be of the mixed type (ie. neither continuous nor discrete)

Finally, $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ for a cont^n r.v.

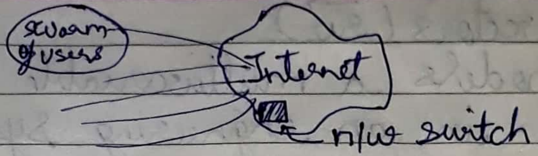
and

$$F_X(x) = \sum_{x_i \leq x} P(X = x_i) = \sum_{x_i \leq x} p_X(x_i)$$

for a discrete r.v.

Poisson distribution

It models the no. of ~~people~~ arrivals in a queue in time $[0, t]$



↳ Let X be the r.v. which models the no. of packets arriving at ~~an~~ n/w switch in time [0,T]

The poisson model

$$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!} \;;\; \lambda > 0,\; k = 0,1,2,\ldots$$

where λ is the parameter that models the rate of arrival.

↳ The prob of no arrival in [0,T] is
$P(X=0) = e^{-\lambda}$

↳ This distribution models many counting and arrival process.

↳ The cdf of a Poisson(λ) r.v is not known in closed form.

Boltzmann distn.

In statiscal mechanics or in electronic devices, this distribution is useful (in particle equil. sys)

Let $e_1, e_2, \ldots e_m$ be m energy states in which the particle can be present. Then,

$$P(X=i) = \frac{e^{-e_i/kT}}{\sum_{j=1}^{m} e^{-e_j/kT}},$$

where X=i means particle is in state i corresponding to energy e_i

$0.25 \cdot 0.15$ $n = 10$ $\frac{1}{2} \times 0.7$ $\frac{1}{2} \times 0.2$

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Random vectors ($\vec{rv}$)

A $\vec{rv}$ models a multivariable uncertain parameter in an engineering system.

Let $\vec{X} = (X_1, \dots X_n)$. Then $\vec{X}$ is a rv. if

1) $P(\vec{X} \in (-\infty, \infty)^n) = 1$, that is all of the entries of $\vec{X}$ are finite

2) $P(\vec{X} \in [a_1, b_1) \times \dots \times [a_n, b_n))$ is well defined for every $-\infty < a_i < b_i < \infty$, $i = 1, \dots$

3) $P(\vec{X} \in \bigcup_{i=1}^{\infty} S_i) = \sum_{i=1}^{\infty} P(\vec{X} \in S_i)$ where $S_1, \dots S_\infty$ are disjoint subsets of R^n

cdf. of a $\vec{rv}$

The cdf of a $\vec{rv}$ $\vec{X} = (X_1, \dots X_n)^T$ is defined as:-

$$F_{\vec{X}}(\vec{x}) \underset{\text{is defined as}}{:=} P(X_1 \le x_1, X \le x_2, \dots X_n \le x_n)$$

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Properties

joint cdf.

1) $\lim_{x_i \to -\infty} F_X(\vec{x}) = 0$ (subset ineq)

marginal cdf. 2) $F_{X_i}(x_i) = F_X(+\infty, \dots x_i \dots +\infty)$

3) $P(X \in A) \le P(X \in B)$ for $A \subseteq B \in R$

$X \in B = \underset{\text{disjoint}}{(X \in A) \cup (X \in B \cap A^c)}$

$\therefore P(X \in B) = P(X \in A) + P(X \in B \cap A^c) \to \ge 0$

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$(X_i \le -\infty) \supseteq (X_i \le -\infty, X_1 \le x_1, \ldots X_n \le x_n)$

4) $F_{X_1,X_2}(x_1,x_2) = F_{\bar{X}}(x_1,x_2,+\infty,\ldots,+\infty)$

5) For a cont^n. r.v., the pdf is defined as

$$f_{\bar{X}}(\bar{x}) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F_{\bar{X}}(\bar{x})$$

6) Independance

~~Two r~~ Two r.v. are statiscally indep. if

$$F_{X_1,X_2}(x_1,x_2) = F_{X_1}(x_1) F_{X_2}(x_2)$$

for every $x_1, x_2 \in \mathbb{R}$

ie. the joint cdf of $\bar{X}$ should be a product of the marginal cdfs

$$F_{\bar{X}}(\bar{x}) = F_{X_1}(x_1) \ldots F_{X_n}(x_n)$$

for all $(x_1, \ldots x_n) \in \mathbb{R}^n$

Joint indp. $\Rightarrow$ Pair wise indep. & not vice versa.
(marginal indep.)

For cont^n. r.v., statiscally indep holds iff

$$f_{\bar{X}}(\bar{x}) = f_{X_1}(x_1) \ldots f_{X_n}(x_n)$$

For discrete r.v. $\bar{X}$, statistically indep. is equivalent to $p_{\bar{X}}(\bar{x}) = p_{X_1}(x_1) \ldots p_{X_n}(x_n)$

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Conditional prob. distribut^n.

There is a pair of r.v (X, Y) which have the joint cdf. as $F_{X,Y}(x,y)$

The value of Y is revealed after realization (or prob. exp.) but X is not revealed.

The initial distribution of X was $F_X(x)$. With $Y = y$, can we obtain / compute a better distributn. for X ?

$$P(X=x \mid Y=y) := \frac{P(X=x, Y=y)}{P(Y=y)}$$

whenever $P(Y=y) > 0$

For cont^n. r.v $P(Y=y) = \int_{y^-}^{y^+} f_Y(y)\,dy = 0$

$\therefore$ This defn. can't be used for cont^n. r.v.

The conditional distr^n of a cont^n. rv. can be understood using the conditional pdf defined as :-

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{whenever } f_Y(y) > 0$$

Note :- $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = 1 = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}{f_Y(y)}$

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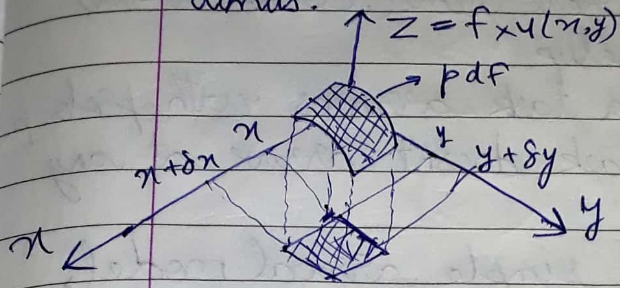
Interpretation

One way to interpret this formula is using limits.



- $P\left(X \in [x, x+\delta x), Y \in [y, y+\delta y)\right)$
  $\approx f_{XY}(x,y)\, \delta x\, \delta y$

- $P\left(X \in [x, x+\delta x) \mid Y \in [y, y+\delta y)\right)$
  $\times P\left(Y \in [y, y+\delta y)\right)$
  $\approx f_Y(y)\, \delta y\, P\left(X \in [x, x+\delta x) \mid y \le Y < y+\delta y\right)$

Taking ratios,

$$\frac{P\left(X \in [x+\delta x), Y \in [y, y+\delta y)\right)}{P\left(Y \in [y, y+\delta y)\right)} \approx \frac{f_{XY}(x,y)\, \delta x}{f_Y(y)}$$

Let $\delta y \to 0$,

$$\therefore P\left(X \in [x, x+\delta x) \mid Y = y\right) \approx \frac{f_{XY}(x,y)\, dx}{f_Y(y)}$$

Thus we can interpret conditional density

Independance

$X_1, \ldots X_n$ are independ. if

$$F_{X_1, \ldots X_n}(x_1, \ldots x_n) = F_{X_1}(x_1) \ldots F_{X_n}(x_n)$$

Indep^n & identically distributed r.v (IID)

$X_1, X_2, \ldots X$ are iid rv if $X_1, \ldots X_n$ are indp^n & $X_i \sim F_X(x)$ → is distributed as

Example (clocked arrivals)

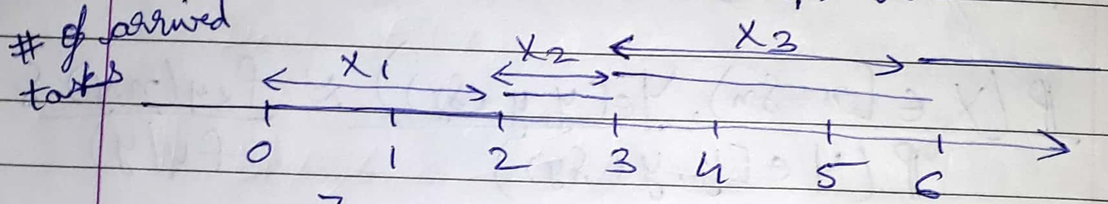
Consider a discrete time (clock-based) task arrival setup

↳ At each time, a task arrives with prob p. More than one task doesn't arrive at any clock instance

↳ This is a very simple arrival model,

let, $Z_i = 1$ if tasks arrives at time i ...
$= 0$ if " doesn't arrive " "

Then ~~let P(Zi)~~ let $P(Z_i = 1) = p$. Let $Z_1, \ldots Z_n$ be iid. Ber(p) r.v.



$Z_1 = 0 \quad Z_2 = 1 \quad Z_3 = 1 \quad Z_4 = 0 \quad Z_5 = 0 \quad Z_6 = 1$

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For analysis purposes, we might be interested in

(i) inter-arrival time - distn.

(ii) # of tasks arrived till time n.

Let S_n = # of jobs arrived till time n

Let $X_1, X_2 \ldots$ be the inter-arrival times

Verify that $X_1, X_2, X_3 \ldots$ are iid.

X is a discrete r.v.

$p_{X_1}(1) = P(Z_1 = 1) = p$

$p_{X_1}(2) = P(Z_1 = 0, Z_2 = 1) = P(Z_1 = 0) . P(Z_2 = 1)$

$= (1-p) . p$

In general $\boxed{p_{X_1}(k) = (1-p)^{k-1} p}$ → Geometric distribn with parameter $(1-p)$

↳ What about S_n?

a) Through X_i's (b) Through Z_i's)

Note that,

$$\boxed{S_n = \sum_{i=1}^{n} Z_i} \Rightarrow S_n \text{ are not indep.}$$

$$S_n = S_{n-1} + Z_n$$

$$p_{S_n}(k) = 0 \text{ if } k<0 \text{ or } k>n$$

$$= \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \leq k \leq n$$

We say that $S_n \sim \underbrace{\text{Binomial}(n,p)}_{\text{Bin}(n,p)}$

31/07 $P(\text{Bin}(n,p) \approx np) \approx 1 \Rightarrow$ conv. in prob.

Expectation of a r.v

Let $X \to$ r.v. & let $E(X)$ denote its expectation

If X is discrete,

$$E(X) := \sum_{x_i} x_i \, p_X(x_i)$$

If X is cont^n.,

$$E(X) := \int_{-\infty}^{\infty} x f_X(x) \, dx$$

In both cases, it is assumed that $E(X)$ is well defined.

Existence of E(X)

Let X be any r.v. Then define

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{if } X < 0 \end{cases}$$

$$X^- = \begin{cases} -X & \text{if } X \leq 0 \\ 0 & \text{if } X > 0 \end{cases}$$

Then $\boxed{X^+ - X^- = X}$ and both X^+ & X^- are non-negative r.v.

$\therefore \quad E(X) = E(X^+) - E(X^-)$

since E(X) is a linear operator

If $E(X^+) < \infty$ & $E(X^-) < \infty$, then E(X) exists (as a finite real no.)

If $E(X^+) = \infty$	& $E(X^-) < \infty$,	then $E(X) = \infty$
$< \infty$	$= \infty$	$-\infty$
$= \infty$	$= \infty$	undefined

Finding expectation using cdf

Let $F_X^c(x) = 1 - F_X(x)$

$$= \mathbb{P}(X > x)$$

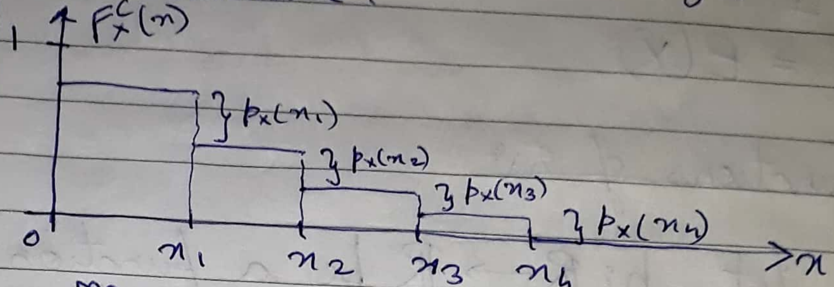
$$= \mathbb{P}(X \in (x, \infty))$$

If $X \geq 0$ (ie. a non-negative r.v.)

Then,

$$E(X) = \int_0^\infty F_X^c(x)\,dx = \int_0^\infty \mathbb{P}(X > x)\,dx$$

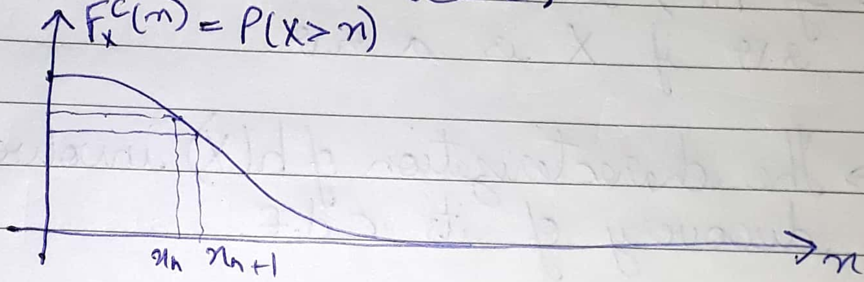
For the discrete case, a graphical proof is available.



$$\therefore \int_0^\infty F_X^c(x)\,dx = x_4 p_X(x_4) + x_3 p_X(x_3) + x_2 p_X(x_2) + x_1 p_X(x_1)$$

$\therefore$ It's true ~~of~~ for discrete.

For continuous case,



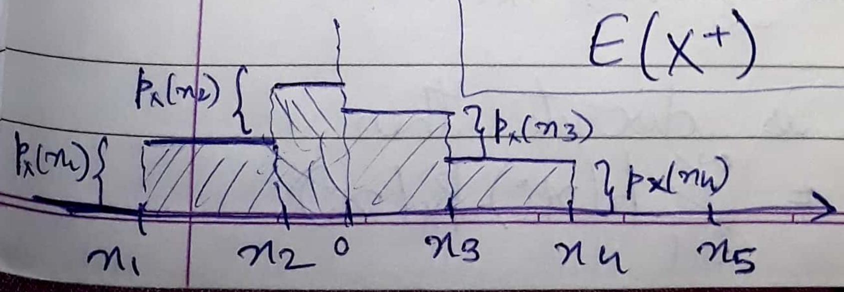
$$\therefore x_n P(X \in [x_n, x_{n+1})) \le \int_{x_n}^{x_{n+1}} F_X^c(x)\,dx \le x_{n+1} P(X \in [x_n, x_{n+1}))$$

2/08 # $X \geq 0$,

$$E(X) = \int_0^\infty F_X^c(x)\,dx$$

For a two sided rv,

$$E(X) = \underbrace{\int_0^\infty F_X^c(x)\,dx}_{E(X^+)} - \underbrace{\int_{-\infty}^0 F_X(x)\,dx}_{E(X^-)}$$



$$\left[p_x(x_4)x_4 + p_x(x_3)x_3\right] - \left(-x_1 p_x(x_1) - x_2 p_x(x_2\right.$$
$$= E(X)$$

Functions of r.v

↳ Let $h: \mathbb{R} \to \mathbb{R}$ be a function. Then we want to examine if $h(X)$ is a r.v i.e. we will need to ensure the 3 properties of r.v

↳ If $h(\cdot)$ is continuous, then $h(X)$ is a r.v if X is a r.v.

↳ The characterization of $h(X)$ involves the discovery of its c.d.f.

↳ The simplest characterization of $h(X)$ is finding its expectation

$E(h(X)) \longrightarrow$ Find the distribution of $h(X)$ from X, & evaluate $E(h(X))$

easier way using the distbn. of X?

↳ If X is continuous, then

$$E(h(X)) = \int_{-\infty}^{\infty} h(x)\, f_x(x)\, dx$$

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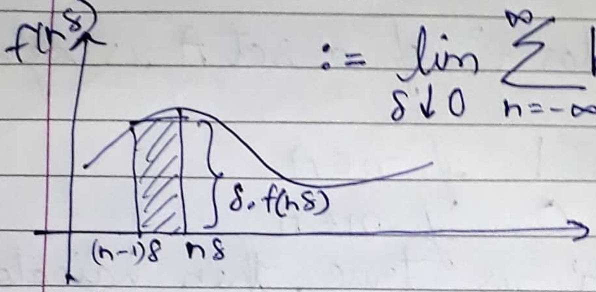
$E(h(X)) \longrightarrow$ Find the cdf of $Y = h(X)$
Find $E(Y)$

$\hookrightarrow E(h(X))$ can also be found using the cdf of X.

$$E(h(X)) = \int_{-\infty}^{\infty} h(x)\, dF_X(x)$$

$$:= \lim_{\delta \downarrow 0} \sum_{n=-\infty}^{\infty} h(n\delta)\left[F_X(n\delta) - F_X((n-1)\delta)\right]$$



If X is continuous, then
$$E(h(X)) = \int_{-\infty}^{\infty} h(x) f_X(x)\, dx$$

If X is discrete, then
$$E(h(X)) = \sum_{i=1}^{\infty} h(x_i)\, p_X(x_i)$$

Example (mean & variance)

Recall that the mean is $E(X)$. Often we denote $\mu = E(X)$
Let $Y = X - \mu$ (centered r.v) be another r.v. One measure of "closeness" of X to μ is $\sigma_X^2 = E((X-\mu)^2)$

$$\sigma_X^2 = E(X^2 - 2X\mu + \mu^2)$$
$$= E(X^2) - 2\mu E(X) + \mu^2$$
$$= E(X^2) - \mu^2 = E(X^2) - (E(X))^2$$

variance of X

$$\therefore \sigma_X^2 = E(X^2) - [E(X)]^2$$

σ_X small $\Rightarrow$ X is "closer" to its avg. value $E(X)$

Example (Indicator functions)

The indicator f^n of a set A is defined as

$$\mathbb{1}(x \in A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

If 'x in A' is true, then indicator is 1.

All cont^n maps are measurable but vice-versa not true, ex: Indicator f^n.

$$\boxed{E(\mathbb{1}(X \in (-\infty, b])) = F_X(b)}$$

$$= \int_{-\infty}^{\infty} \mathbb{1}(x \in (-\infty, b])\, dF_X(x)$$

$$= \int_{-\infty}^{b} 1 \, dF_X(x)$$

$$= F_X(b)$$

$$\boxed{E(\mathbb{1}(X \in A)) = P(X \in A)}$$

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Example (pdf of a 1-1 function of contn. r.v.)

Let X be a contn. r.v. with pdf $f_X(x)$ & cdf $F_X(x)$.
Let $h: \mathbb{R} \to \mathbb{R}$ where $h(x)$ is increasing (one-to-one) function. That is $h^{-1}(x)$ is well defined.

Then, $Y = h(X)$ has a cdf as below:-

$$F_Y(y) = P(Y \leq y) = P(h(X) \leq y)$$
$$= P(X \leq h^{-1}(y))$$
$$= F_X(h^{-1}(y))$$

Taking derivative w.r.t. y,

$$\boxed{f_Y(y) = \frac{f_X(h^{-1}(y))}{h'(h^{-1}(y))}}$$

(As $\frac{d}{dy} h^{-1}(y) = \frac{1}{h'(h^{-1}(y))}$

$h(x)$ is assumed to be monotonically $\uparrow$ or $\downarrow$ so that denom $\neq 0$.

Example (Sum of indep. r.v)

Let X & Y be 2 indep. r.v with cdf $F_X(x)$ & $F_Y(y)$.

(*) Indep. implies that $\mathbb{P}(Y \leq y \mid X \in A) = \mathbb{P}(Y \leq y)$
where $\mathbb{P}(X \in A) \neq 0$

∴ Let $Z = X + Y$, then,

$$F_Z(z) = \mathbb{P}(X+Y \leq z)$$

$$= \int_{-\infty}^{\infty} \underbrace{\mathbb{P}((X \leq z-y) | Y=y)\, dF_Y(y)}_{\text{Total prob. rule (law/theorem)}}$$

$\{$ X are ... p^n.

$$= \int_{-\infty}^{\infty} \mathbb{P}(X \leq z-y)\, dF_Y(y)$$

$$= \int_{-\infty}^{\infty} F_X(z-y)\, dF_Y(y)$$

Like convolution.

$\{$ symm. & Y.

$$= \int_{-\infty}^{\infty} F_Y(z-x)\, dF_X(x)$$

For the continuous case, this results in,

$$f_Z(z) = \frac{d F_Z(z)}{dz} = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy$$

$$\boxed{f_Z(z) = f_X(z) * f_Y(z)}$$

For the discrete case, when pmf. of X & Y are non-zero only on $\mathbb{Z}$, then

$$p_Z(z) = \sum_{x \in \mathbb{Z}} p_X(x)\, p_Y(z-x)$$

g :-
$$\frac{d}{dx} \int_{0}^{g(x)} f(u)\, du = f(g(x))\, g'(x)$$

$i=1$

↳ For any r.v. X, the expectation of $h(X)$ is understood as

$$E(h(X)) := \int_{-\infty}^{\infty} h(x)\, dF_X(x)$$

$$:= \lim_{\delta \downarrow 0} \sum_{n=-\infty}^{\infty} h(n\delta)\left[F_X(n\delta) - F_X((n-1)\delta)\right]$$

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Total probability rule

Let (X, Y) be a pair of r.v.

$$F_X(x) = \underbrace{\mathbb{P}(X \le x)}_{F_X(x)} = \underbrace{\mathbb{P}(X \le x, Y \le \infty)}_{F_{XY}(x, \infty)}$$

Break $Y \le \infty$ into infi small intervals dy.

$$= \int_y \mathbb{P}(X \le x, y \le Y \le y + dy)$$

$= Y = y$.

$$\boxed{F_X(x) = \int_{-\infty}^{\infty} \mathbb{P}(X \le x \mid Y = y)\, dF_Y(y)}$$

If Y is continuous, then.

$$F_X(x) = \int_{-\infty}^{\infty} \mathbb{P}(X \le x \mid Y = y)\, f_Y(y)\, dy$$

and if Y is discrete, then,

$$F_X(x) = \sum_{i \in \mathbb{Z}} \mathbb{P}(X \le x \mid Y = y_i)\, p_Y(y_i)$$

If $h: \mathbb{R} \to \mathbb{R}$ is strictly increasing, then

$$f_Y(y) = \frac{f_X(h^{-1}(y))}{h'(h^{-1}(y))}$$

) When $h: \mathbb{R} \to \mathbb{R}$ is not one to one, then

$$(h(x) \leq y) \not\Longleftrightarrow (x \leq h^{-1}(y))$$

$$\mathbb{P}(Y \in (y, y+dy)) = \mathbb{P}(X \in (x_1, x_1+dx_1))$$
$$\approx f_Y(y)\,dy \qquad + \mathbb{P}(X \in (x_2 - dx_2, x_2))$$
$$+ \ldots\ldots$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(X \in (x_i, x_i + dx_i))$$

$$= \sum_{i=1}^{\infty} f_X(x_i)\,|dx_i| \qquad \Big|\ x_i\text{s are all the soln. of } y = h(x)$$

$$\therefore\ f_Y(y) = \sum_{i=1}^{\infty} \frac{f_X(x_i)}{\left|\frac{dy}{dx_i}\right|}$$

$$\Rightarrow f_Y(y) = \frac{f_X(x_1)}{|h'(x_1)|} + \frac{f_X(x_2)}{|h'(x_2)|} + \ldots\ldots$$

(At max. & min, we use $h''(x_1)$)

In a wireless channel, we encounter,

$$Y = \overset{\text{channel gain (fading coeff.)}}{H} X + W \rightarrow \text{noise}$$

received symbol → transmitted symbols

$G = H^2$ represents the channel gain & determines the signal power at the receiver.

In a class of channel models (Rayleigh channels),

$$f_H(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} \; ; \quad x \geq 0$$

Note that $G = h(H)$; then $h(x) = x^2$
& $h'(x) = 2x$
Since $x \geq 0$ & $h(x)$ is monotonic in R^+,

$$f_G(y) = \frac{\sqrt{y}}{\sigma^2} \cdot \frac{e^{(-y/2\sigma^2)}}{2\sqrt{y}} = \frac{1}{2\sigma^2} e^{(-y/2\sigma^2)}, \quad y \geq 0$$

∴ G has an exponential distribⁿ.