Problem 2.22

(a) $$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt$$
$$= \int_{-\infty}^{0} g(t)\exp(-j2\pi ft)dt + \int_{0}^{\infty} g(t)\exp(-j2\pi ft)dt$$
$$= \int_{-\infty}^{0} g(t)\cos(2\pi ft)dt - \int_{-\infty}^{0} jg(t)\sin(2\pi ft)dt$$
$$+ \int_{0}^{\infty} g(t)\cos(2\pi ft)dt - \int_{0}^{\infty} jg(t)\sin(2\pi ft)dt$$

If $g(t)$ is even, then $g(t) = g(-t)$. Hence,

$$\int_{-\infty}^{0} g(t)\cos(2\pi ft)dt = \int_{0}^{\infty} g(t)\cos(2\pi ft)dt$$
$$\int_{-\infty}^{0} g(t)\sin(2\pi ft)dt = -\int_{0}^{\infty} g(t)\sin(2\pi ft)dt$$

and so

$G(f) = 2\int_{0}^{\infty} g(t)\cos(2\pi ft)dt$. which is purely real.

If, on the other hand, $g(t)$ is odd, $g(t) = -g(-t)$. Hence,

$$\int_{-\infty}^{0} g(t)\sin(2\pi ft)dt = \int_{0}^{\infty} g(t)\sin(2\pi ft)dt$$
$$\int_{-\infty}^{0} g(t)\cos(2\pi ft)dt = -\int_{0}^{\infty} g(t)\cos(2\pi ft)dt$$

and thus

$G(f) = -2j\int_{0}^{\infty} g(t)\sin(2\pi ft)dt$ which is purely imaginary.

(b) The Fourier transform of $g(t)$ is defined by

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt$$

Differentiating both sides of this relation n times with respect to f:

$$\frac{d^n G(f)}{df^n} = (-j2\pi)^n \int_{-\infty}^{\infty} t^n \exp(-j2\pi ft)dt \qquad (1)$$

That is,

$$t^n g(t) \rightleftharpoons \left(\frac{j}{2\pi}\right)^n \frac{d^n G(f)}{df^n}$$

(c) Putting $f = 0$ in Eq. (1), we get

$$\int_{-\infty}^{\infty} t^n g(t)dt = \left(\frac{j}{2\pi}\right)^n G^{(n)}(0)$$

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Problem 2.22 continued

where $G^{(n)}(f) = \dfrac{d^n G(f)}{d\,fn}$

(d) Since

$$g_2^*(t) \rightleftharpoons G_2^*(-f)$$

it follows that

$$g_1(t)g_2^*(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda - f)d\lambda$$

From this result we deduce the Fourier transform

$$\mathbf{F}[g_1(t)g_2^*(t)] = \int_{-\infty}^{\infty} g_1(t)g_2^*(t)\exp(-j2\pi ft)dt$$

$$= \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda - f)d\lambda \qquad (2)$$

Setting $f = 0$ in Eq. (2), we get the desired relation

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t) = \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda)d\lambda$$