Problem 3.30

The multiplexed signal is defined by

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

Therefore, the spectrum of $s(t)$ is

$$S(f) = \frac{A_c}{2}[M_1(f-f_c)+M_1(f+f_c)] + \frac{A_c}{2j}[M_2(f-f_c) - M_2(f+f_c)]$$

where $M_1(f) = \mathbf{F}(m_1(t)]$ and $M_2(f) = \mathbf{F}(m_2(t)]$. The spectrum of the received signal is therefore

$$R(f) = H(f)S(f)$$

$$= \frac{A_c}{2}H(f)\left[M_1(f-f_c)+M_1(f+f_c)+\frac{1}{j}M_2(f-f_c)-\frac{1}{j}M_2(f+f_c)\right]$$

To recover $m_1(t)$, we multiply $r(t)$ [i.e., the inverse Fourier transform of $R(f)$] by $\cos(2\pi f_c t)$ and then pass the resulting output through a low-pass filter, which is designed to have a cutoff frequency equal to the message bandwidth W. The signal produced at the filter output has the following spectrum

$$\mathbf{F}[r(t)\cos(2\pi f_c t)] = \frac{1}{2}[R(f-f_c)+R(f+f_c)]$$

$$= \frac{A_c}{4}H(f-f_c)[M_1(f-2f_c)+M_1(f)+\frac{1}{j}M_2(f-2f_c)-\frac{1}{j}M_2(f)]$$

$$+\frac{A_c}{4}H(f+f_c)\left[M_1(f)+M_1(f+2f_c)+\frac{1}{j}M_2(f)-\frac{1}{j}M_2(f+2f_c)\right] \quad (1)$$

The condition $H(f_c+f) = H^*(f_c - f)$ is equivalent to $H(f+f_c) = H(f-f_c)$; this follows from the fact that for a real-valued impulse response $h(t)$, we have $H(-f) = H^*(f)$. Hence, substituting this condition in Eq. (1), we get

$$\mathbf{F}[r(t)\cos(2\pi f_c t)] = \frac{A_c}{2}H(f-f_c)M_1(f)$$

$$+\frac{A_c}{4}H(f-f_c)\left[M_1(f-2f_c)+\frac{1}{j}M_2(f-2f_c)+M_1(f+2f_c)-\frac{1}{j}M_2(f+2f_c)\right]$$

The low-pass filter output therefore has a spectrum equal to $(A_c/2)H(f-f_c)M_1(f)$.

Similarly, to recover $m_2(t)$, we multiply $r(t)$ by $\sin(2\pi f_c t)$, and then pass the resulting signal through a low-pass filter. In this case, we get an output with a spectrum equal to $(A_c/2)H(f-f_c)M_2(f)$.