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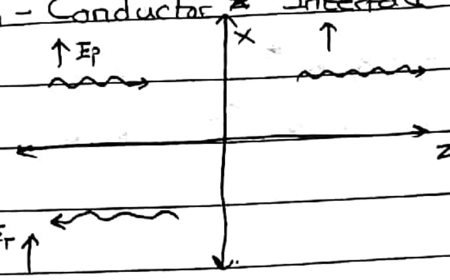
A] $\longrightarrow$ Lossless Medium – Conductor Interface



ϵ_1, μ_1 $\qquad$ $\epsilon_2, \mu_2, \sigma$ $\qquad$ (σ is very high)

– For medium 2, $\gamma = \alpha + j\beta = (1+j)\sqrt{\dfrac{\mu\omega\sigma}{2}}$

$$\eta_2 = \frac{j\omega\mu}{\gamma}$$

– For $z < 0$, (last lecture)

$$S^- = \hat{z}\,\frac{|E_0|^2}{\eta_1^*}\left[1 - |\Gamma|^2 + \Gamma - \Gamma^*\right] = (\bar{E}_p + \bar{E}_r) \times (\bar{H}_i + \bar{H}_r)^*$$

For $z > 0$

$$S^+ = \bar{E}_t \times \bar{H}_t^* = \cancel{\hat{z}\,\frac{|E_0|^2|T|^2 e^{-\gamma z}}{\eta_2^*}} \quad \hat{z}\,\frac{|E_0|^2\,|T|^2\,e^{-2\alpha z}}{\eta_2^*}$$

– Note :- $\dfrac{|T|^2}{\eta_2^*} = \dfrac{1}{\eta_1}\left(1 - |\Gamma|^2 + \Gamma - \Gamma^*\right)$

Proof :- LHS :- $\quad T = \dfrac{2\eta_2}{\eta_1 + \eta_2}$

$$= \frac{2(a + jb)}{a + \eta_1 + jb}$$

$$|T|^2 = \frac{4(a^2 + b^2)}{(a + \eta_1)^2 + b^2}$$

LHS = RHS

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$$\therefore \quad S^+ = \hat{z} \frac{|E_0|^2}{\eta_1} [\underbrace{1 - |\Gamma|^2}_{\text{real}} + \underbrace{\Gamma - \Gamma^*}_{\text{imaginary}}] e^{-2\alpha z}$$

- At $z=0$, $S^+ = S^-$
- For $z>0$, the power is getting attenuated exponentially with distance.
- $\boxed{\text{Real Power (time-averaged)} = \frac{1}{2} \text{Re}(\vec{S} \cdot \hat{z})}$

$$= \frac{1}{2} \times \frac{|E_0|^2}{\eta_1} [1 - |\Gamma|^2]$$

→ For $z > 0$, $\bar{J}_t = \sigma \bar{E}_t = \hat{x} E_0 T \sigma e^{-\gamma z}$

– Real Power dissipated $= \frac{1}{2} \int_{vol} \vec{E}_t \cdot \bar{J}_t^* \, dv$

$$= \frac{1}{2} \int_0^1 \int_0^1 \int_0^\infty \hat{x} E_0 T e^{-\gamma z} \cdot \hat{x} E_0^* T^* \sigma e^{-\gamma^* z} dz dy dx$$

considering a 1×1 area in XY plane

$$= \frac{1}{2} \int_0^\infty |E_0|^2 |T|^2 \sigma e^{-2\alpha z} dz$$

$$= \frac{\sigma |E_0|^2 |T|^2}{4\alpha}$$

– Real Power entering $z > 0$, $= \frac{1}{2} \text{Re}(S^+ \cdot \hat{z})$ at $z = 0$.

Now, $S^+ = \hat{z} \frac{|E_0|^2 |T|^2}{\eta_2^*} e^{-2\alpha z}$

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$$\eta_2 = (1+j)\frac{\alpha}{\sigma} \Rightarrow \frac{1}{\eta_2^*} = \frac{\sigma}{2\alpha}(1-j)$$

$$\therefore S^+ = \hat{z}\,|E_0|^2\,|T|^2\,e^{-2\alpha z} \times \frac{\sigma}{2\alpha}(1+j)$$

$\therefore$ Power entering $= \frac{1}{2}\,\mathrm{Re}\,(S^+ \cdot \hat{z}) = \frac{|E_0|^2\,|T|^2\,\sigma}{4\alpha}$ at $z=0$

- $\therefore$ Real Power entering = Real Power dissipating ✓

→ Perfect Conductor $(\sigma \to \infty)$

$\delta_s \to 0, \quad \alpha \to \infty, \quad \eta_2 \to 0.$

- $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1$ ---- perfect reflection

- Because $\alpha \to \infty$, power in second medium dissipates instantaneously in a very very short distance.

- $$\bar{E} = \bar{E}_i + \bar{E}_r = \hat{x}\,E_0\left[e^{-jk_0 z} - e^{+jk_0 z}\right]$$
$$= -\hat{x}\,2j\,E_0 \sin(k_0 z)$$

$$\bar{H} = \bar{H}_i + \bar{H}_r = \hat{y}\,\frac{E_0}{\eta_1}\left[e^{-jk_0 z} + e^{jk_0 z}\right]$$

$$= \hat{y}\,\frac{E_0}{\eta_1}\,2\cos(k_0 z)$$

- $$S^- = \bar{E} \times \bar{H}^* = \hat{z}\,j\,\frac{|E_0|^2}{\eta_1}\,4\cos(k_0 z)\sin(k_0 z)$$

- $Re(S^-) = 0$:-
  No real power is being transmitted.

- Surface current density $= \overline{J_s} = \hat{n} \times \overline{H}^*$

$$= -\hat{z} \times \hat{y} \frac{E_0}{\eta_1} 2\cos(k_0 z)$$

$$= \hat{x} \frac{2E_0}{\eta_1} \quad \text{at } z = 0.$$

Standing Waves

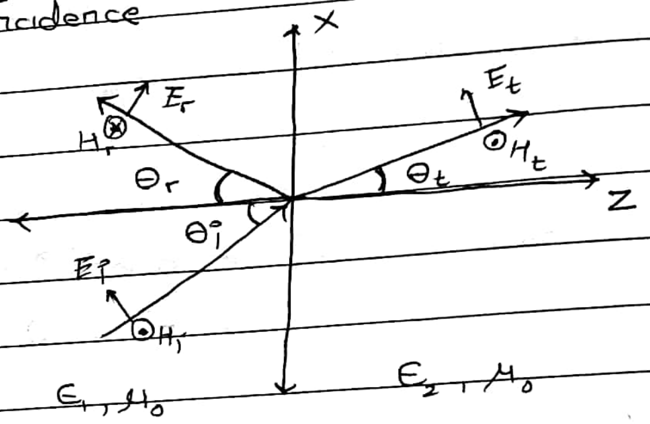
$$\overline{E} = \overline{E_i} + \overline{E_r} = \hat{x} E_0 e^{-jk_0 z} + \hat{x} \Gamma E_0 e^{jk_0 z}$$

$$= \hat{x} \underbrace{E_0 e^{-jk_0 z}}_{\text{Travelling Wave}} \underbrace{\left(1 + \Gamma e^{+j2k_0 z}\right)}_{\text{Amplitude modulation of travelling wave}}$$

- The variation of amplitude of travelling wave is the phenomenon of standing waves

- Standing Wave Ratio $= \dfrac{\text{Max Amplitude}}{\text{Min Amplitude}} = \dfrac{1 + |\Gamma|}{1 - |\Gamma|}$

B] Oblique Incidence



$\epsilon_1 < \epsilon_2$ (Why ?)

~~Case I~~

Case I Parallel Polarization

$\bar{E}$ is in the plane of reflection-transmission

- $\bar{E_i} = E_0 (\hat{x} \cos\theta_i - \hat{z} \sin\theta_i) e^{-jk_1 (\hat{x} \sin\theta_i + z\cos\theta_i)}$

$\vec{k_i} = k_1 (\hat{x} \sin\theta_i + \hat{z} \cos\theta_i)$

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$k_1 = \omega \sqrt{\mu_0 \epsilon_1}$

$\bar{H_i} = \frac{E_0}{\eta_1} \hat{y} \, e^{-jk_1 (x\sin\theta_i + z\cos\theta_i)}$

- $\bar{E_r} = E_0 \Gamma (\hat{x} \cos\theta_r + \hat{z} \sin\theta_r) e^{-jk_1 (x\sin\theta_r - z\cos\theta_r)}$

$\bar{H_r} = \frac{E_0 \Gamma}{\eta_1} (-\hat{y}) \, e^{-jk_1 (x\sin\theta_r - z\cos\theta_r)}$

- $\bar{E_t} = E_0 T (\hat{x} \cos\theta_t - \hat{z} \sin\theta_t) e^{-jk_2 (x\sin\theta_t + z\cos\theta_t)}$

$\bar{H_t} = \frac{E_0 T}{\eta_2} (\hat{y}) \, e^{-jk_2 (x\sin\theta_t + z\cos\theta_t)}$

- At $z=0$, Continuity equations :- $E_{ix} + E_{rx} = E_{tx}$.

$$H_{iy} + H_{ry} = H_{ty}$$

i. $E_0 \cos\theta_i \, e^{-jk_1 x \sin\theta_i} + \Gamma E_0 \cos\theta_r \, e^{-jk_1 x \sin\theta_r} = E_0 T \cos\theta_t \, e^{-jk_2 x \sin\theta_t}$

ii. $\dfrac{E_0}{\eta_1} e^{-jk_1 x \sin\theta_i} - \dfrac{E_0 \Gamma}{\eta_1} e^{-jk_1 x \sin\theta_r} = \dfrac{E_0 T}{\eta_2} e^{-jk_2 x \sin\theta_t}$

- For i & ii to be always valid, the exponents are equal.

$$\theta_i = \theta_r$$

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

- Solve for Γ and T

$$T = \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \qquad \Gamma = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

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- The value of θ_i for which $\Gamma = 0$ is called Brewster's Angle (θ_b)

$$\cos\theta_b = \frac{\eta_2 \cos\theta_t}{\eta_1}$$

Also use $k_1 \sin\theta_i = k_2 \sin\theta_t$

$$= \sqrt{\frac{\epsilon_1}{\epsilon_2}} \sqrt{1 - \sin^2\theta_t}$$

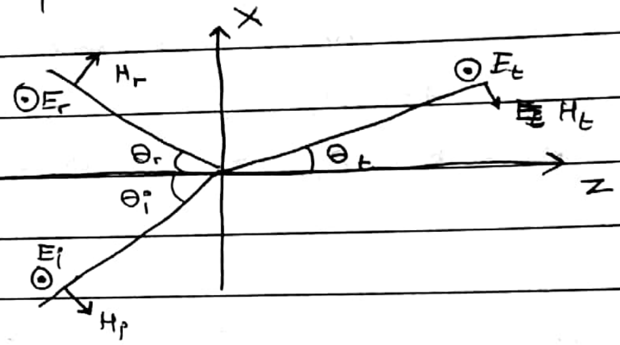
$$= \sqrt{\frac{\epsilon_1}{\epsilon_2}} \sqrt{1 - \frac{k_1^2}{k_2^2} \sin^2\theta_b}$$

$$\cos\theta_b = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \sqrt{1 - \frac{\epsilon_1}{\epsilon_2} \sin^2\theta_b}$$

$$\boxed{\sin\theta_b = \frac{1}{\sqrt{1 + \epsilon_1/\epsilon_2}}}$$

Note :- After reflection, the sense of direction of $\bar{E}$ (leftwards or rightwards) changes.
This will not happen in case of perpendicular polarization

Case 2 Perpendicular Polarization
$\bar{H}$ is in the plane of reflection-transmission.



☆ • In perpendicular polarization, the direction of circular polarization reverses after reflection (RHCP ↔ LHCP)

- $\bar{E_i} = \hat{y}\, E_0\, e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$

  $\bar{H_i} = \frac{E_0}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)\, e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$

- $\bar{E_r} = \Gamma E_0\, \hat{y}\, e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$

  $\bar{H_r} = \frac{\Gamma E_0}{\eta_1}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)\, e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$

- $\bar{E_t} = \hat{y}\, E_0 T\, e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$

  $\bar{H_t} = \frac{E_0 T}{\eta_2}(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\, e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$

- Continuity at $z=0$:- i) $E_{iy} + E_{ry} = E_{ty}$
  ii) $H_{ix} + H_{rx} = H_{tx}$

- Results :-

i) $k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$

ii) $\theta_i = \theta_r$

iii) $\Gamma = \dfrac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}$, $T = \dfrac{2 n_2 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t}$

- Brewster's Angle :-

$$n_2 \cos\theta_i = n_1 \cos\theta_t$$

$$\vdots$$

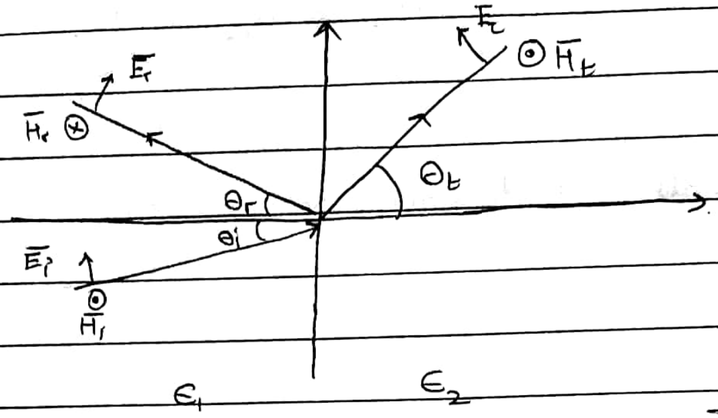
$$\sin^2\theta_i \underbrace{(n_1^2 k_1^2 - n_2^2 k_2^2)}_{\text{always zero}} = (n_1^2 - n_2^2) k_2^2$$

Brewster's angle does not exist.

* Polarization lenses :-

☆ At Brewster's angle, the wave is polarized only in one direction (parallel, not perpendicular).
Hence, reflected wave ~~do~~ is not received by the lens

Case 3 Total (Internal) Reflection.



$\boxed{\epsilon_1 > \epsilon_2}$

- For critical angle $\theta_i = \theta_c$, $\theta_t = \frac{\pi}{2}$

Since $k_1 \sin\theta_i = k_2 \sin\theta_t$, $\boxed{\sin\theta_c = \frac{k_2}{k_1}} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$

- Consider $\theta_i > \theta_c$

$\Rightarrow \quad \sin\theta_t > 1$

$\Rightarrow \quad \cos\theta_t = \sqrt{1 - \sin^2\theta_t}$ ----- imaginary

Let $k_2 \sin\theta_t = \beta$ and $k_2 \cos\theta_t = j\alpha$.

- $\bar{E}_t = E_0 T \left(\frac{j\alpha}{k_2} \hat{x} - \frac{\beta}{k_2} \hat{z} \right) e^{-j\beta x} e^{\alpha z}$

$\bar{H}_t = \frac{E_0 T}{\eta_2} \left(\hat{y} \right) e^{-j\beta x} \underbrace{e^{\alpha z}}$

α is negative

$\Rightarrow$ Attenuation of $\bar{E}_t$, $\bar{H}_t$ in medium 2.

- Use Helmholtz's Equation :- $\nabla^2 \bar{H}_t + k_2^2 \bar{H}_t = 0$

$\left(-\beta^2 + \alpha^2 + k_2^2\right) \bar{H}_t = 0$

$\therefore \quad \alpha^2 - \beta^2 + k_2^2 = 0$

The expressions for E_i, H_i, E_r, H_r remain as seen in case 1

- Use continuity at $z = 0$

- $k_1 \sin\theta_i = k_2 \sin\theta_r = \beta = k_2 \sin\theta_t$ .... equating exponents

$\theta_i = \theta_r$ .... as before

- Two simultaneous equations for Γ and T.

$$\frac{E_0}{\eta_1} - \frac{E_0 \Gamma}{\eta_1} = \frac{E_0 T}{\eta_2} \qquad E_0 \cos\theta_i + E_0 \Gamma \cos\theta_r = \frac{E_0 T (j\alpha)}{k_2}$$

Solve :- $\Gamma = \dfrac{\eta_2 \left(\frac{j\alpha}{k_2}\right) - \eta_1 \cos\theta_i}{\eta_2 \left(\frac{j\alpha}{k_2}\right) + \eta_1 \cos\theta_i}$, $T = \dfrac{2\eta_2 \cos\theta_i}{\eta_2 \left(\frac{j\alpha}{k_2}\right) + \eta_1 \cos\theta_i}$

Note :- $|\Gamma| = 1$

- Now, $\bar{S_i} = \bar{E_i} \times \bar{H_i}^* = \dfrac{|E_0|^2}{\eta_1} \left(\hat{z} \cos\theta_i + \hat{x} \sin\theta_i\right)$

$\bar{S_r} = \bar{E_r} \times \bar{H_r}^* = -\dfrac{|E_0|^2 |\Gamma|^2}{\eta_1} \left(\hat{z} \cos\theta_i - \hat{x} \sin\theta_i\right)$

Real Power transmitted $= Re\left(\left(\bar{E_i} + \bar{E_r}\right) \times \left(\bar{H_i} + \bar{H_r}\right)^*\right)$

$= \bar{S_i} + \bar{S_r}$ (seen previously)

$= \dfrac{|E_0|^2}{\eta_1} 2 \sin\theta_i \, \hat{x}$

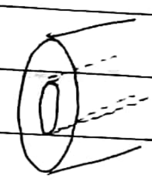
- No power is being transmitted along Z, into medium 2

- Also, $\bar{S_t} = \bar{E_t} \times \bar{H_t}^*$

$$= \frac{|E_0|^2 |T|^2}{\eta_2} e^{2\alpha z} \left(\frac{j\alpha}{k_2} \hat{z} + \frac{\beta}{k_2} \hat{x} \right)$$

- z component is imaginary. ⇒ No real power flowing
  - Standing wave along z direction.

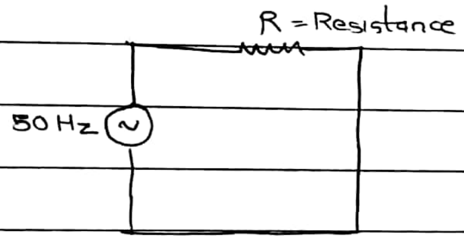
* Coaxial Cables



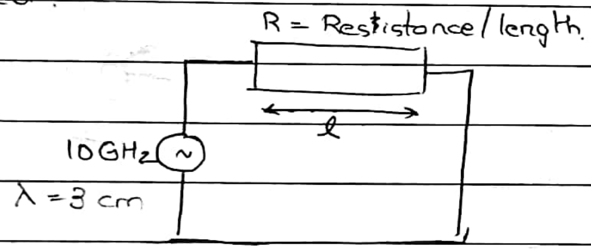
$\overline{E}$ is in radial direction.
$\overline{H}$ is in tangential (concentric circular) direction

* Distributed Circuits

When wavelengths of EM waves become ~~large~~ comparable to circuit size we cannot assume circuit elements to be localized.



Lumped-element circuit



Distributed circuit

- When studying distributed circuits, we cannot assume two points connected by a conductor to be at same potential.

Define L, C, R, G to be inductance, capacitance, resistance, conductance per unit length.

- This happens in transmission lines, which run for several kilometres, because wavelength is comparable to length.

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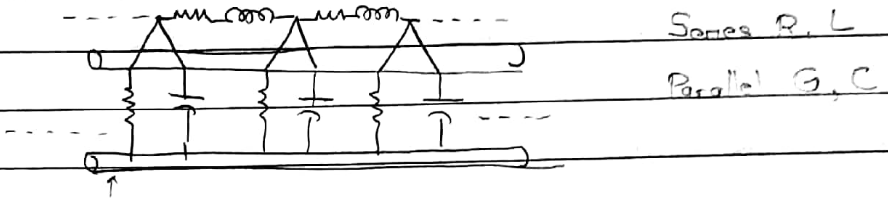
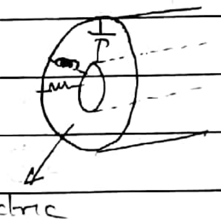
→ Transmission Lines

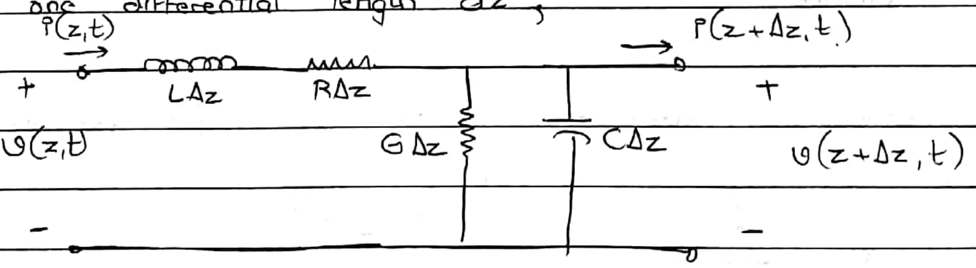
- R, L, C are all distributed. There are some R, L, C between ~~two wires as well~~, outer and inner conductors of coaxial cables



- Coaxial Cables have C, G along radial direction



- For one differential length dz,



~~$v(z,t) - i(z,t)R\Delta z =$~~

- L, R, G, C ~ H/m, Ω/m, ℧/m, F/m

- $v(z,t) - i(z,t)R\Delta z - L\Delta z \dfrac{\partial i(z,t)}{\partial t} = v(z+\Delta z, t)$ ---- KVL

$$\lim_{\Delta z \to 0} \frac{v(z+\Delta z, t) - v(z,t)}{\Delta z} = -R\,i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

$$\therefore \boxed{\frac{\partial v(z,t)}{\partial z} = -R\,i(z,t) - L\frac{\partial i(z,t)}{\partial t}}$$

• $-v(z+\Delta z,t)G - C\dfrac{\partial v(z+\Delta z,t)}{\partial t} = \lim_{\Delta z \to 0} \dfrac{i(z+\Delta z,t) - i(z,t)}{\Delta z}$

$\therefore$ $$\boxed{\frac{\partial i(z,t)}{\partial z} = -G\,v(z,t) - C\frac{\partial v(z,t)}{\partial t}}$$ ..... KCL.

• $v(z,t) = V(z)e^{j\omega t}$ and $v(z,t) = V(z)\cos\omega t$ are both solutions for the voltage.
Similarly, $i(z,t) = I(z)e^{j\omega t}$ and $i(z,t) = I(z)\cos\omega t$.

- Substitute in KVL.
$$\frac{\partial V(z)}{\partial z} = -(R+j\omega L)\,I(z)$$

- Substitute in KCL
$$\frac{\partial I(z)}{\partial z} = -(G+j\omega C)\,V(z)$$

• These equations are similar to Maxwell's equations. Hence, we also have counterparts of Helmholtz's equation.

i $$\frac{\partial^2 V(z)}{\partial z^2} = -(R+j\omega L)\frac{\partial I(z)}{\partial z}$$
$$= (R+j\omega L)(G+j\omega C)\,V(z)$$

Write $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \alpha + j\beta$ = 'Propagation constant'

$\therefore$ $$\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2 V(z)$$

$$V(z) = \underbrace{V_0^+ e^{-\gamma z}}_{\text{Voltage wave travelling forward}} + \overbrace{V_0^- e^{\gamma z}}^{\text{Voltage wave travelling backwards.}}$$

$$\frac{\partial^2 I(z)}{\partial z^2} = \gamma^2 I(z)$$

$$\therefore \; I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

- Substitute these in previous equations.

$$\frac{\partial V(z)}{\partial z} = -(R+j\omega L)\, I(z)$$

$$\Rightarrow -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} = -(R+j\omega L)\left(I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}\right)$$

$$\therefore \; -\gamma V_0^+ = -(R+j\omega L) I_0^+ \;, \quad \gamma V_0^- = -(R+j\omega L) I_0^-$$

$$\therefore \; I_0^+ = \frac{V_0^+ \gamma}{R+j\omega L} \qquad I_0^- = \frac{-V_0^- \gamma}{R+j\omega L}$$

Define 'characteristic impedance' $= Z_0 = \sqrt{\dfrac{R+j\omega L}{G+j\omega C}}$

$$\therefore \; I_0^+ = \frac{V_0^+}{Z_0} \;, \quad I_0^- = \frac{-V_0^-}{Z_0}$$

* $\gamma = \alpha + j\beta$

Lossless medium :- $\alpha = 0$

$\Rightarrow R = G = 0$

$\Rightarrow \gamma = j\beta = j\omega\sqrt{LC}$

$$\beta = \omega\sqrt{LC} \;, \quad Z_0 = \sqrt{L/C}$$

Wavelength $= \lambda = \dfrac{2\pi}{\beta}$, Phase velocity $= v_p = \dfrac{1}{\sqrt{LC}}$

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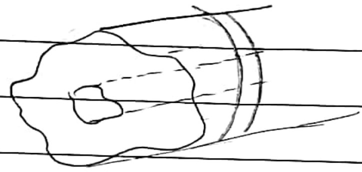
I | CALCULATING G.R.L.C

Consider an arbitrary cable.



Total Magnetic energy $= \frac{1}{2}\int_{vol} \vec{B}\cdot H^* \, dv$ ← C.S volume (length l, area S)

$= \frac{\mu}{2}\int_{vol} \bar{H}\cdot\bar{H}^* \, dv$

Time averaged Magnetic energy stored per unit length $= \frac{\mu}{4}\int_S \bar{H}\cdot\bar{H}^* \, ds$

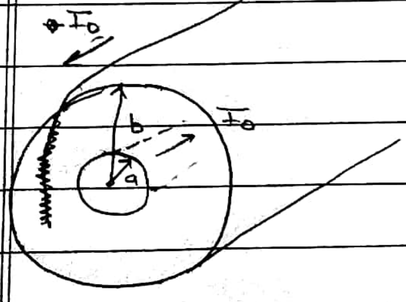
$\frac{1}{2}$ comes from $\cos^2$ from $\bar{H}\cdot\bar{H}^*$

This should be equal to $\frac{1}{2}\left(\frac{1}{2} L |I_0|^2\right)$

↳ because I is sinusoidal

$$\therefore \boxed{L = \frac{\mu}{|I_0|^2}\int_S \bar{H}\cdot\bar{H}^* \, ds}$$

*



Voltage between $\rho = a$, $\rho = b$

$$\left.\begin{aligned}\bar{E} &= \frac{V_0 e^{-\gamma z}\hat{\rho}}{\rho \ln(b/a)} \\ \bar{H} &= \frac{I_0 e^{-\gamma z}\hat{\phi}}{2\pi\rho}\end{aligned}\right\} \text{ for } a<\rho<b$$

--- Proof later

$\rho < a$ is solid metal

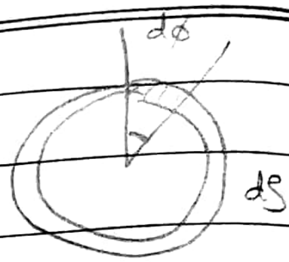
$\rho = b$ is hollow metal

Outside $\rho = b$, $E = H = 0$

$H = 0$ by Ampere's law

P.T.O

($I_{enc} = 0$, because equal and opposite currents are flowing)



$ds = (\rho\, d\phi)\, d\rho$

Find $L = \dfrac{\mu}{|I_0|^2} \displaystyle\int_s \bar{H} \cdot \vec{H}^* \, ds$

$\hookrightarrow I_0 = I_0 e^{-\gamma z}$

$$= \frac{\mu}{\cancel{I_0^2 e^{-2\alpha z}}} \int_a^b \int_0^{2\pi} \frac{\cancel{I_0^2 e^{-2\alpha z}}}{4\pi^2 \rho^2} \, \rho \, d\rho \, d\phi$$

$$= \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$