

Digital Image Processing

Fourier Analysis – 1

Series, Transform, Properties

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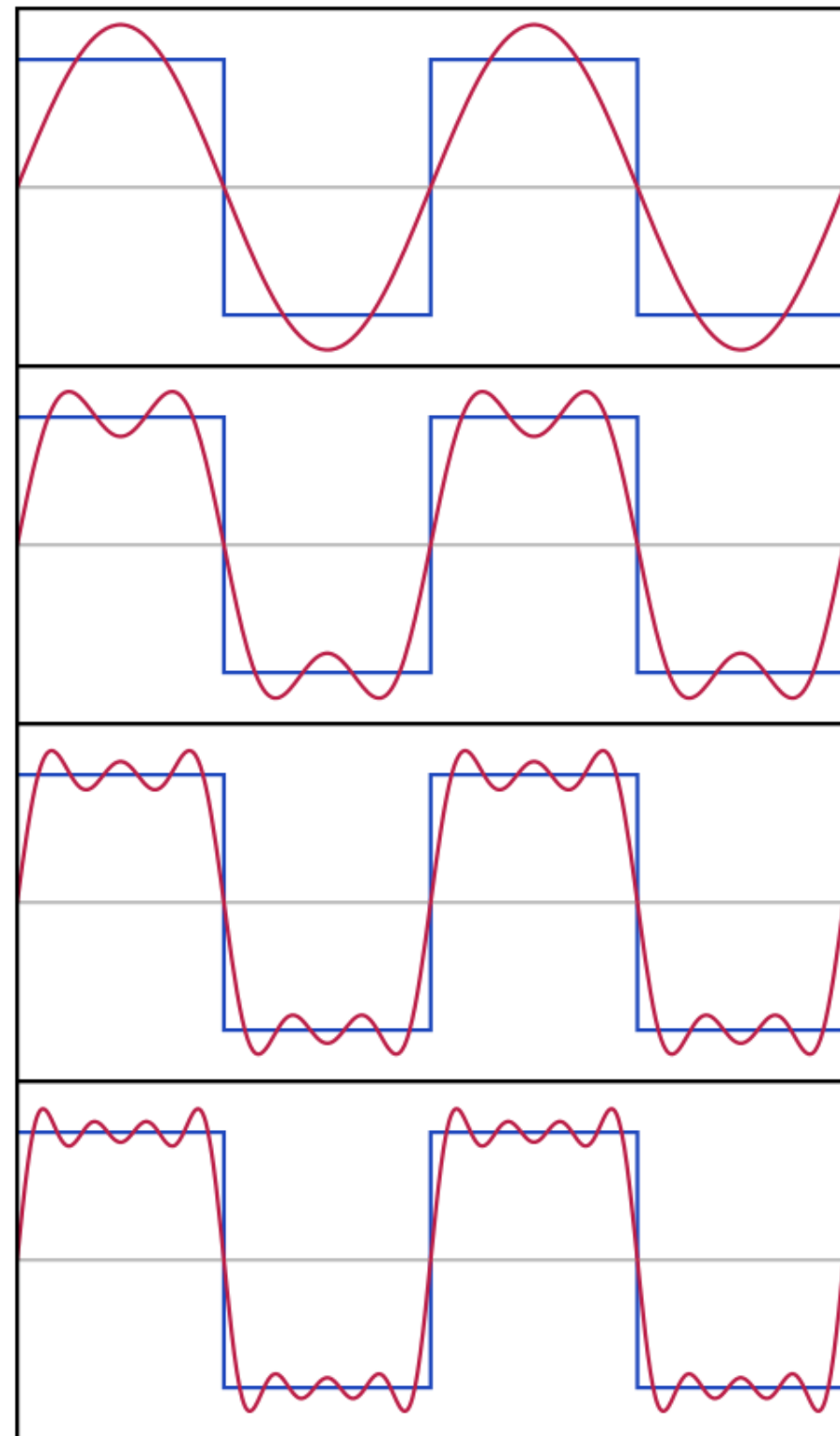
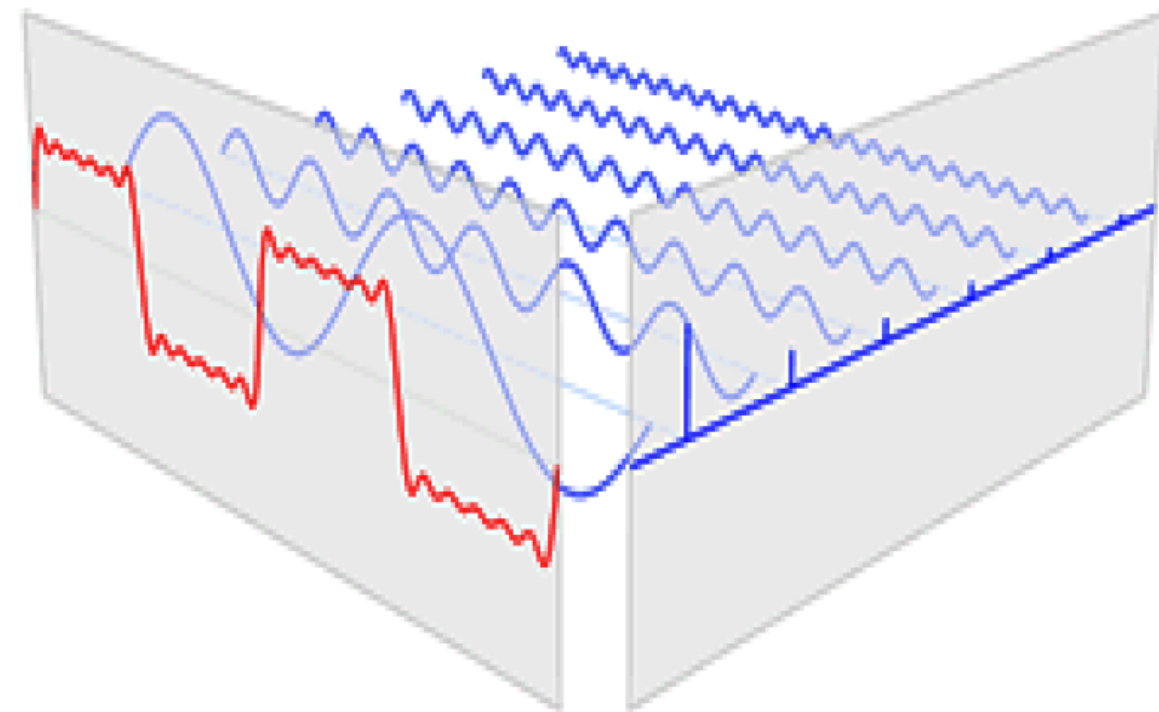
Fourier Analysis

- Joseph Fourier
 - Mathematician and physicist
 - 1768 – 1830
 - Invented Fourier series for studying heat transfer
 - PhD Guide : Lagrange
 - Took part in French revolution
- First application of Fourier analysis to digital images in 1960s



Fourier Analysis

- Fourier Series
 - Represent a function as a linear combination (scaling + superposition) of sinusoidal functions



Fourier Analysis

- Sinusoidal waves
 - $f(t) = \cos(\omega t)$
 - $f(t) = \sin(\omega t)$
 - Period = $2\pi/\omega$
 - Frequency = $1 / \text{period} = \omega/(2\pi)$
 - Larger $\omega \rightarrow$ shorter period and higher frequency

Fourier Analysis

- Complex-valued sinusoidal waves
 - Complex-valued waves e^{int}
 - $n = \text{integer}$
 - **Harmonic frequencies**
 - **Fundamental** frequency : ($n = 1$)
 - 1st harmonic
 - This isn't the lowest frequency
 - Lowest frequency : ($n = 0$) $\rightarrow \exp(i 0 t) = 1 = \text{constant "wave"}$
 - Other frequencies = **integer** multiples of fundamental frequency
 - $m = \dots, -2, -1, 0, 1, 2, \dots$

Fourier Analysis

- Assume a function $f(t)$ defined on domain $= [0, 2\pi]$
- Assume that $f(t)$ can be represented as a linear combination of complex sinusoidal waves of harmonic frequencies
$$f(t) := \sum_{n \in I} c_n e^{int}, \text{ where } c_n \text{ is complex and } 0 \leq t \leq 2\pi$$
 - Frequencies $= n / 2\pi$
- c_n = coefficients (complex)
- Problem
 - Given $f(t)$
 - Find coefficients c_n

Fourier Analysis

- Problem
 - Given $f(t)$
 - Find coefficients c_n
- Important Observation 1
 - Take integral of complex wave e^{int} with e^{-imt} , s.t. $m \neq n$

$$\begin{aligned}\int_0^{2\pi} e^{int} e^{-imt} dt &= \int_0^{2\pi} e^{i(n-m)t} dt \\&= \frac{e^{i(n-m)t}}{i(n-m)} \Big|_0^{2\pi} = \frac{1}{i(n-m)} (e^{i(n-m)2\pi} - e^0) \\&= \frac{1}{i(n-m)} (1 - 1) = 0\end{aligned}$$

Fourier Analysis

- Problem
 - Given $f(t)$
 - Find coefficients c_n
- Important Observation 2
 - Take integral of complex wave e^{int} with e^{-imt} , s.t. $m = n$

$$\int_0^{2\pi} e^{int} e^{-int} dt = \int_0^{2\pi} e^0 dt = 2\pi$$

Fourier Analysis

- Problem
 - Given $f(t)$
 - Find coefficients c_n
- Solution ?
 - We can get c_n by integrating product of $f(t)$ with e^{-int}

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt$$

Fourier Analysis

- Simpler example to gain insights
 - Consider 3D Euclidean space
 - Consider 3 unit vectors orthogonal to each other
 - $\langle \bar{i}, \bar{j}, \bar{k} \rangle$ is an orthogonal basis
 - Basis = Set of linearly-independent vectors that can be used to represent any other vector as a linear combination
 - Coordinate system
 - Consider an arbitrary vector in 3D that can be represented as a linear combination of basis vectors
$$\bar{x} = x_1 \bar{i} + x_2 \bar{j} + x_3 \bar{k}$$
 - Problem
 - Given x
 - How do we find x_1, x_2, x_3 ?

Fourier Analysis

- Simpler example to gain insights
 - Consider an arbitrary vector in 3D that can be represented as a linear combination of basis vectors
$$\bar{x} = x_1\bar{i} + x_2\bar{j} + x_3\bar{k}$$
 - Problem
 - Given x
 - How do we find x_1, x_2, x_3 ?
 - Solution
 - Take dot product of x with each basis vector

Fourier Analysis

- Dot product = inner product
 - Summation of component-wise products of vector values
- How does inner product generalize to the space of functions ?
 - Integration of products of function values
 - If functions are **real** valued: $\langle u, v \rangle = \int_a^b u(x)v(x)dx$
 - If functions are **complex** valued: $\langle \psi, \chi \rangle = \int_a^b \psi(x)\overline{\chi(x)}dx$
 - Bar denotes conjugate

Fourier Analysis

- Inner product of e^{int} with e^{imt}
 - Integral of complex wave e^{int} with e^{-imt}
 - Constant when $m = n$
 - 0 with $m \neq n$
- **Set of complex waves $\{ e^{int} \}$ = an orthogonal set**
 - Actually: An orthogonal basis in Hilbert space of square-integrable complex functions defined on $[0, 2\pi]$
- **Hilbert space**
= Vector space of functions + defined inner product

Fourier Analysis

- **Vector space** defines operations of addition, scaling on elements of this space (called vectors)

Axiom	Meaning
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the <i>zero vector</i> , such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the <i>additive inverse</i> of \mathbf{v} , such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ ^[nb 2]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the <i>multiplicative identity</i> in F .
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Fourier Analysis

- **Inner-product** definition must satisfy 3 conditions :

- (1) Conjugate Symmetry : $\langle f, g \rangle = \langle g, f \rangle^*$

- (2) Linearity in first argument :

$$\langle a_1 f_1 + a_2 f_2, g \rangle = a_1 \langle f_1, g \rangle + a_2 \langle f_2, g \rangle$$

- (3) Positive definite : $\langle f, f \rangle \geq 0$ with equality iff $f = 0$

- A zero function is a function that is almost-everywhere zero

- Check that the inner product on complex-valued functions satisfies these 3 conditions

Fourier Analysis

- Assume that $f(t)$ can be represented as a linear combination of complex sinusoidal waves of harmonic frequencies
$$f(t) := \sum_{n \in I} c_n e^{int}, \text{ where } c_n \text{ is complex and } 0 \leq t \leq 2\pi$$
- Problem
 - Given $f(t)$
 - How do we find coefficients c_n ?
- Solution
 - Inner product of $f(t)$ with e^{int}
 - = integral of product of $f(t)$ with **conjugate of e^{int}**
 - = integral of product of $f(t)$ with e^{-int}

Fourier Analysis

- **Fourier Series**

- Orthogonal basis

- Inner product of real-valued functions

- Integral $\int_{-\pi}^{\pi} f(x) g(x) dx$

- Set of all **sine waves**

- Integral $\int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx$

- π ; if integer $m = n$

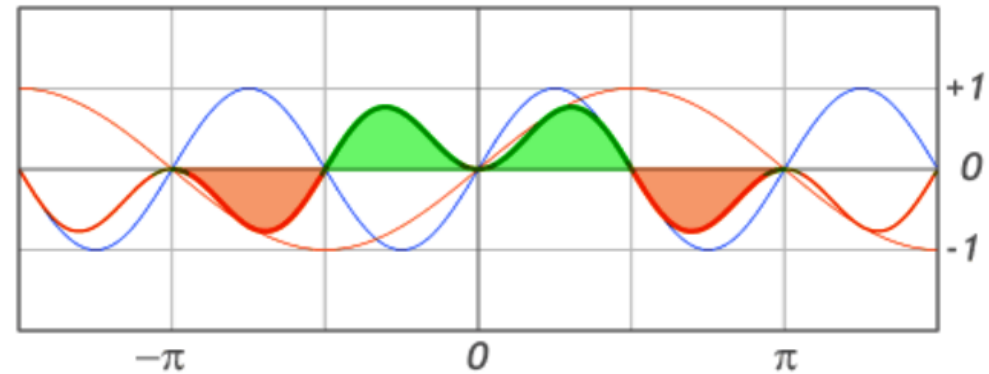
- 0; otherwise

- Set of all **cosine waves**

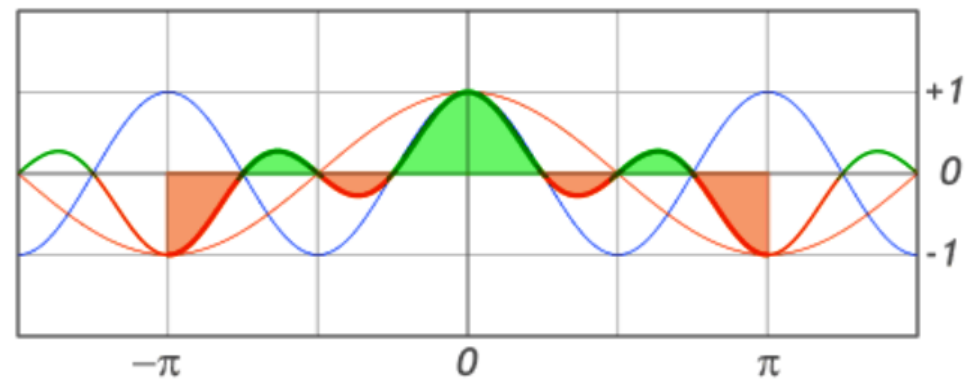
- Integral $\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx$

- π ; if integer $m = n$

- 0; otherwise



$$\int_{-\pi}^{+\pi} \sin(2x) \sin(1x) dx = 0$$



$$\int_{-\pi}^{+\pi} \cos(2x) \cos(1x) dx = 0$$

Fourier Analysis

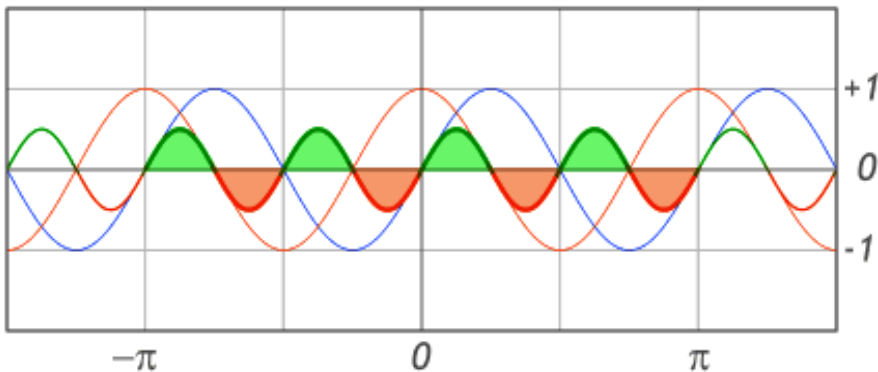
- **Fourier Series**

- Orthogonal basis

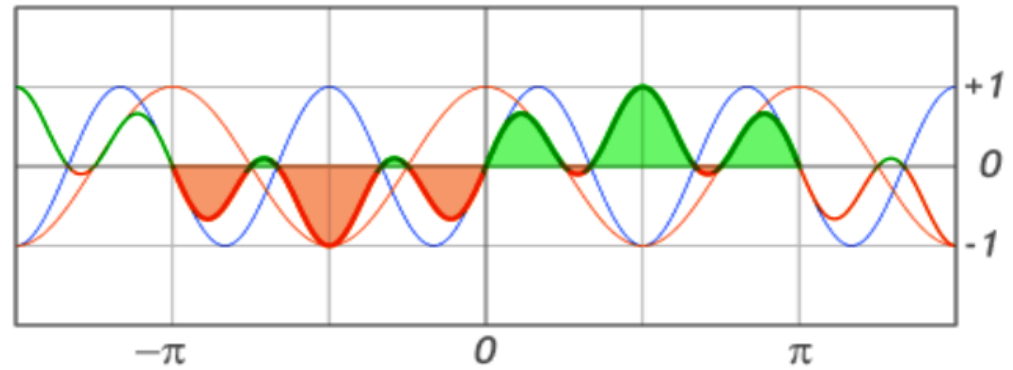
- Set of all **sine waves and cosine waves**

- Integral $\sin(mx) \cos(nx)$

- 0; for any integer m, n



$$\int_{-\pi}^{+\pi} \sin(2x) \cos(2x) dx = 0$$



$$\int_{-\pi}^{+\pi} \sin(3x) \cos(2x) dx = 0$$

Fourier Analysis

- Do we need **both** sine waves **and** cosine waves ?
 - Yes
 - Because we need to represent functions with non-zero phase

$$\sin(\omega t + \phi) := \sin(\omega t) \cos(\phi) + \cos(\omega t) \sin(\phi)$$

- Observe: The set of all sine waves cannot represent a cosine wave !
 - Why not ?
 - What is the intuition ?
 - What is an algebraic argument / proof ?

Fourier Analysis

- **Fourier Series**

- Orthogonal basis

- In case of complex functions

- Inner product uses conjugate

- Integral $f(x) g^*(x)$

- Set of all **complex waves**

- Integral of $\exp(i n x) \exp(-i m x)$

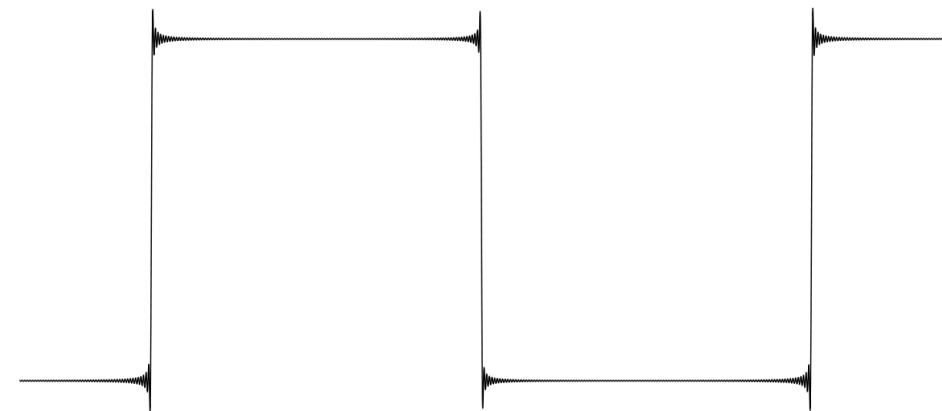
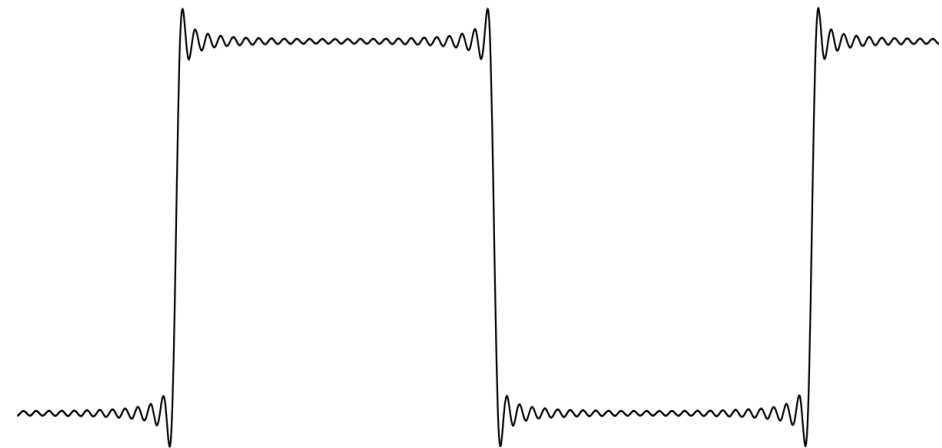
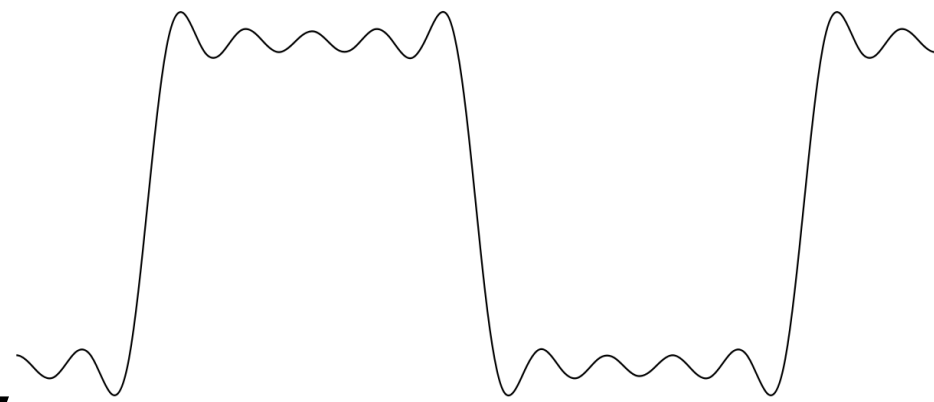
- = Integral of $(\cos(nx) + i \sin(nx)) (\cos(mx) - i \sin(mx))$

- 2π ; if $m = n$

- 0; otherwise

Fourier Analysis

- **Fourier Series**
 - Functions with **jump (or step) discontinuity**
 - Left limit exists, finite
 - Right limit exists, finite
 - These are unequal
 - **Gibbs Phenomenon**
 - (1) **Overshoot and undershoot** around jump discontinuity
 - Approx. 9% of jump magnitude
 - (2) **Oscillations** around discontinuity → “**ringing**”



Fourier Analysis

- **Fourier Series**

- **Gibbs Phenomenon**

- Concerns

Convergence of the sequence of approximated functions (as a Fourier series using waves of increasing frequencies), to the original function $f(x)$

- Series for $f(x)$ converges to $f(x)$, at all x , **except at jump discontinuities**

- Original function : $f(x-) \rightarrow a$. $f(x+) \rightarrow b > a$.

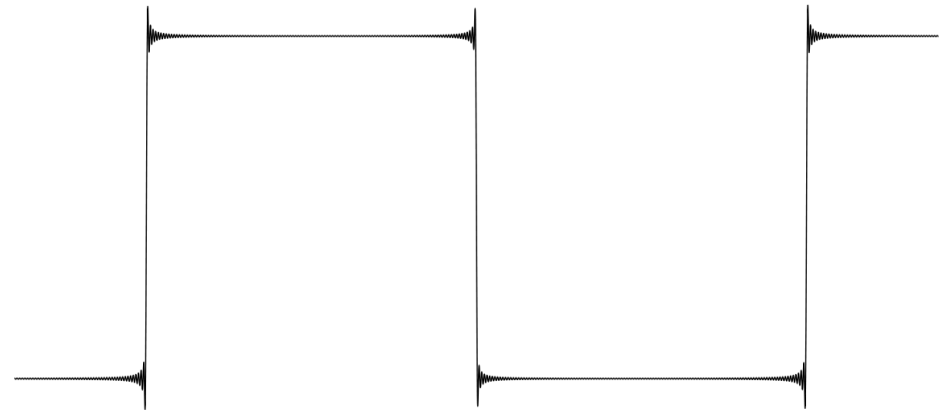
- Series : $f(x-) \rightarrow a - 0.09(b-a)$. $f(x+) \rightarrow b + 0.09(b-a)$. $f(x) = (a+b) / 2$

- What does this mean in practice ?

- As $n \rightarrow \infty$, **integral of squared error** $\rightarrow 0$

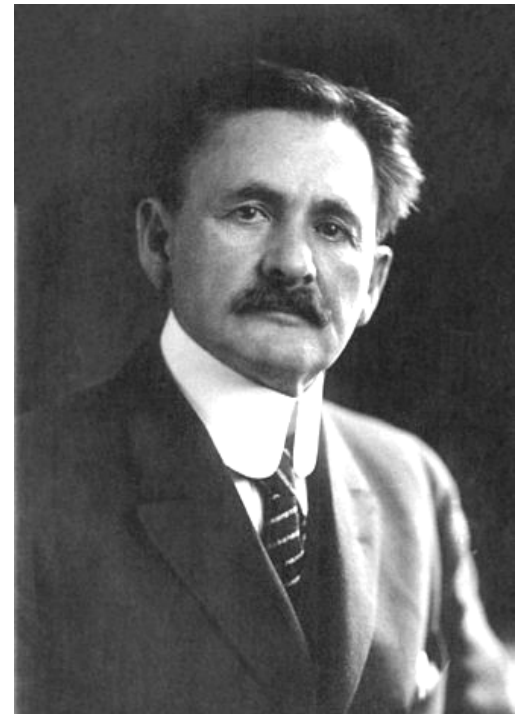
- Mismatches at isolated points \rightarrow don't change practical system behavior

- But, convergence is “infinitely slow” \rightarrow large 'n' needed



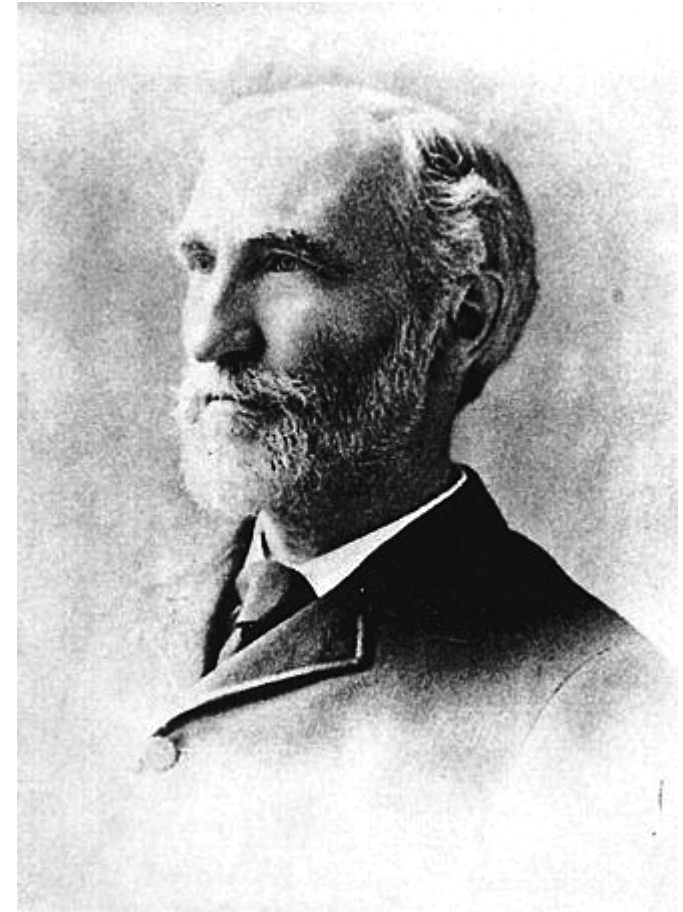
Fourier Analysis

- Albert Michelson (1852 – 1931)
 - In 1898, built machine to show Fourier series representation using large (finite) 'n'
 - Perhaps didn't see ringing and over/under shooting because of low quality of graphs output by machine
 - Experimental physicist
 - Speed of light
 - Relativity
 - Nobel Prize in Physics in 1907
 - First American to receive Nobel Prize in sciences



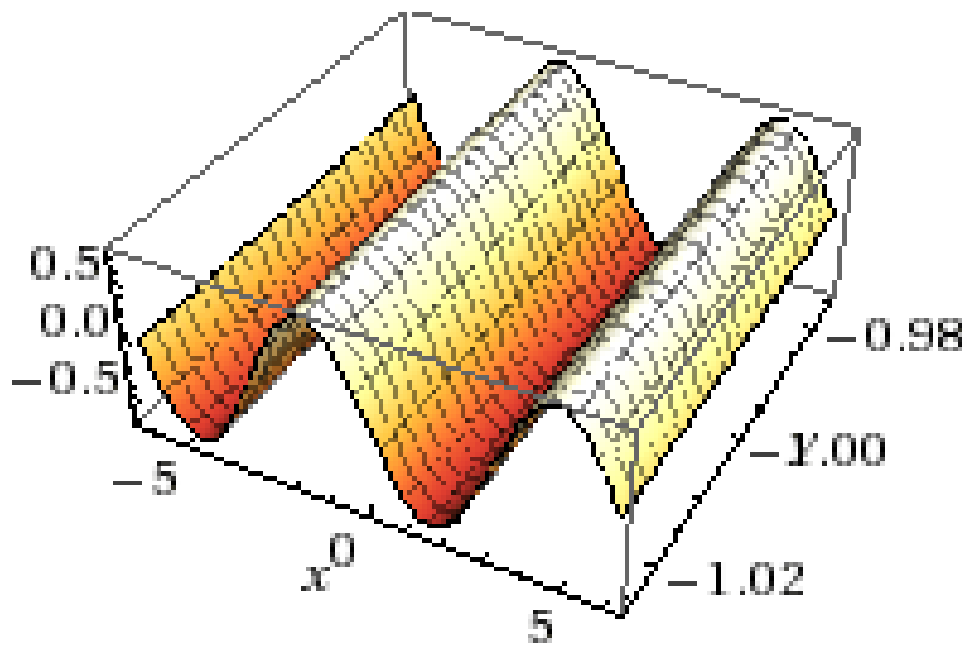
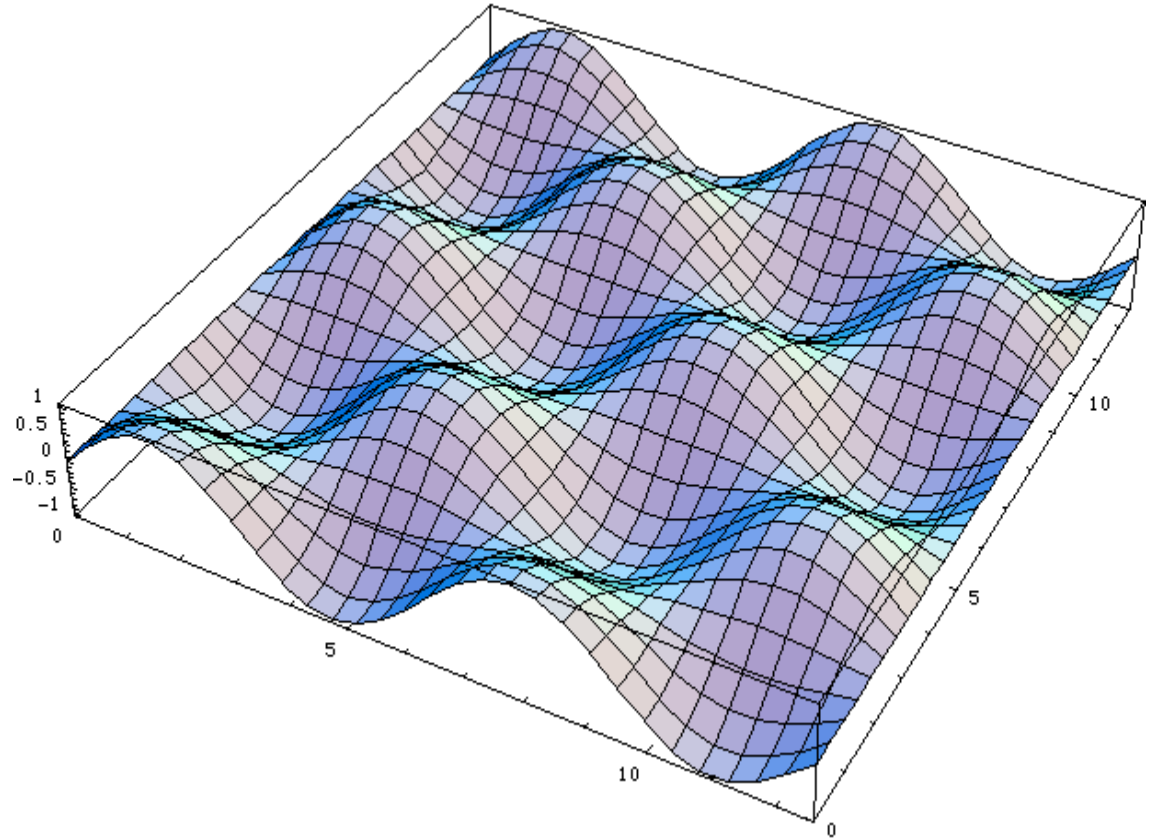
Fourier Analysis

- Josiah Gibbs (1839 – 1903)
 - Explains Fourier-series convergence phenomenon mathematically (1899)
 - Einstein called him “Greatest mind in American history”
 - First American PhD in engineering; at Yale University
 - Scientist
 - Physics, chemistry, mathematics
 - Thermodynamics
 - Gibbs free energy
 - Statistical mechanics
 - Invented vector calculus



Fourier Analysis

- Fourier series (2D)
 - A 2D function \rightarrow
 - A basis element



Fourier Analysis

- **Periodic functions**

- If a function defined on $0 \leq t \leq 2\pi$ has the form

$$f(t) := \sum_{n \in I} c_n e^{int}$$

where c_n is complex

and $|c_1| > 0$ or $|c_{-1}| > 0$

Then, $f(t)$ is periodic with **period 2π**

- Inner-product integral limits will be
 - $0 \rightarrow 2\pi$
- **Fundamental frequency: $1 / 2\pi$**
- Frequencies = ..., -2, -1, 0, 1, 2, ...
- **Separation** between harmonic frequencies: **$1 / 2\pi$**

Fourier Analysis

- Periodic functions
 - **Period can be modified by (re)scaling time axis**
e.g., $f(t) = \sum_{n \in I} c_n e^{-int2\pi/T}$
with $|c_1| > 0$ or $|c_{-1}| > 0$
has **period** $2\pi / (2\pi/T) = \mathbf{T}$
 - Inner-product integral limits can be
 - $0 \rightarrow T$
 - **Fundamental frequency = $1/T$**
 - Frequencies = ..., $-2/T$, $-1/T$, 0 , $1/T$, $2/T$, ...
 - **Separation** between harmonic frequencies = $1/T$

Fourier Analysis

- Periodic functions
 - **Period interval can be modified by shifting time axis**
e.g., $t' \leftarrow t - t_0$
 - Then, interval that contains one period changes
from $[0, T]$
to $[t_0, t_0 + T]$
- **Fourier series decomposes signals that are :
defined on interval $[t_0, t_0 + T]$,
defined periodically outside interval**
 - Inner-product integral limits become t_0 and $t_0 + T$
 - Inner product of wave with itself = T
 - Coefficients obtained by scaling down integral by T

Fourier Analysis

- **Going from Fourier Series \rightarrow Fourier Transform**
 - Think of a **Fourier series** where **period $T \rightarrow \infty$**
 - Separation between harmonic frequencies $\rightarrow 0$
 - **Fundamental frequency (1st-harmonic frequency) $\rightarrow 0$**
- **Fourier transform extends Fourier series :**
 - Allows **all real frequencies** (“n” needn't be integer)
 - **Doesn't assume** signal to be **periodic**

Fourier Analysis

- **Fourier Transform**

- Definition:

Fourier transform of an absolutely-integrable function $f(x)$ is defined for each **real number w** as :

$$Ff(w) := \int_{x=-\infty}^{\infty} f(x)e^{-iwx} dx$$

where

- w is frequency
 - $Ff(w)$ = amplitude of complex wave having frequency w
 - Absolute integrability is a sufficient (not necessary) condition for existence of $Ff(w)$

Fourier Analysis

- **Linearity** of the Fourier transform

- $F(f + g)(w) = Ff(w) + Fg(w)$

- Proof follows from definition

- $F(af)(w) = aFf(w)$

- Proof follows from definition

Fourier Analysis

- **Inverse Fourier Transform**

- For a function $h(w)$,
the inverse Fourier transform is defined
for each real number x as :

$$F^{-1}h(x) := \frac{1}{2\pi} \int_{w=-\infty}^{\infty} h(w)e^{iwx} dw$$

Fourier Analysis

- **Fourier Inversion Theorem**

- If $f(x)$ is continuous : $\forall x, F^{-1}(Ff)(x) := f(x)$
 - Start with $f(x) \rightarrow$ Define $Ff(w)$ via FT \rightarrow Define $g(y)$ via IFT
 - Then, $g(y) = f(y)$, for all y

If $Ff(w) := \int_{x=-\infty}^{\infty} f(x)e^{-iwx}dx$ Fourier Transform

And If $g(y) := \int_{w=-\infty}^{\infty} Ff(w)e^{iwy}dw$ Inverse Fourier Transform

$$= \int_{w=-\infty}^{\infty} \left(\int_{x=-\infty}^{\infty} f(x)e^{-iwx}dx \right) e^{iwy}dw$$

$$= \int_{w=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f(x)e^{-iw(x-y)}dx dw$$

Then, $g(y) = f(y), \forall y$

Fourier Analysis

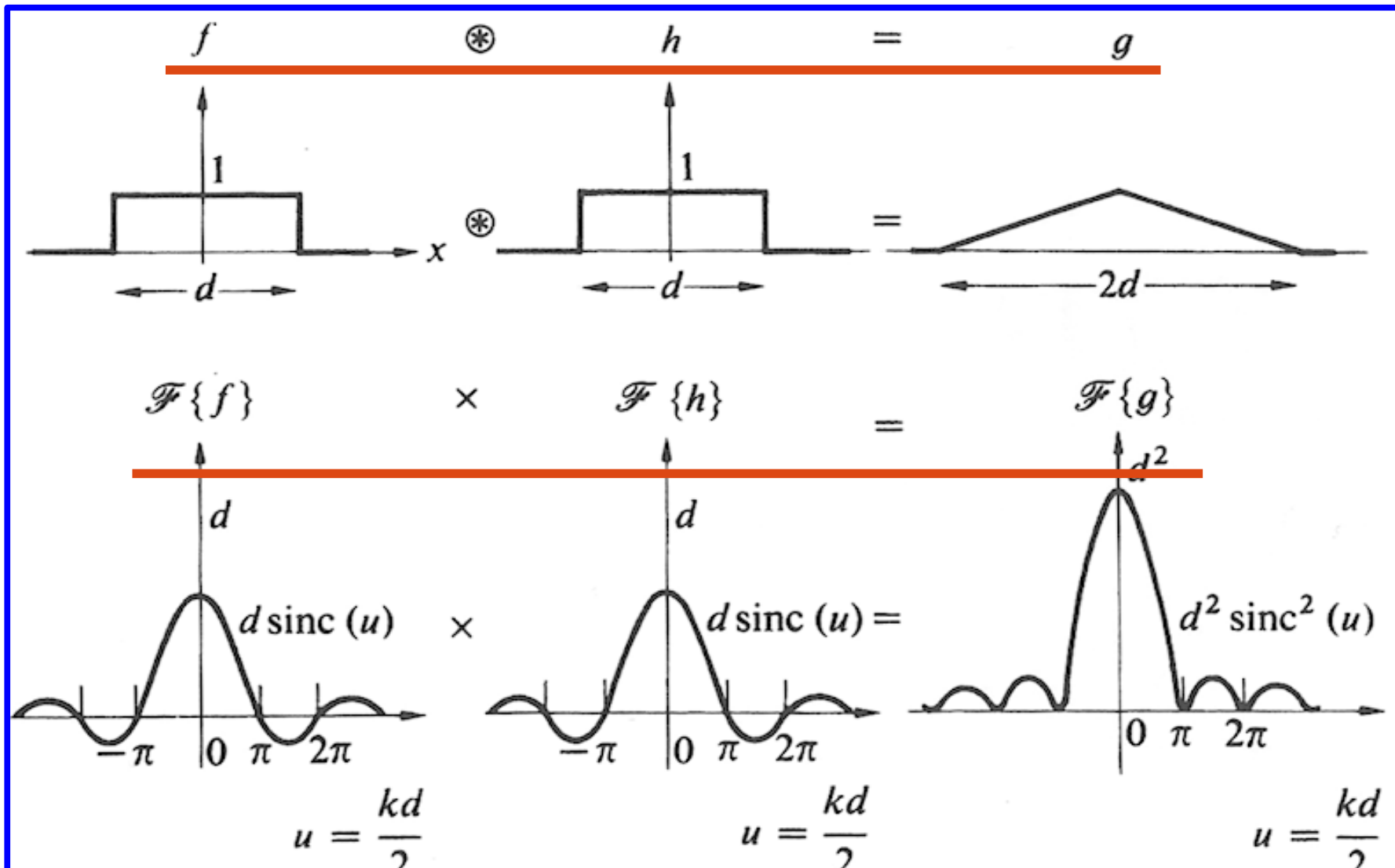
- **Fourier Transform and Convolution**

- Let $f(x)$ and $g(x)$ be 2 functions with Fourier transforms $Ff(w)$ and $Fg(w)$
- **Theorem:**
Product of Fourier transforms of $f(\cdot)$ and $g(\cdot)$
= Fourier transform of convolution of $f(\cdot)$ and $g(\cdot)$
$$Ff(w) \cdot Fg(w) = F(f * g)(w)$$
- Illustration and Proof ... next

Fourier Analysis

- Fourier Transform and Convolution

$$Ff(w) \cdot Fg(w) = F(f * g)(w)$$



Fourier Analysis

- Fourier Transform and Convolution
 - Product of Fourier transforms of $f(\cdot)$ and $g(\cdot)$
= Fourier transform of convolution of $f(\cdot)$ and $g(\cdot)$

$$\begin{aligned} Ff(w) \cdot Fg(w) &:= \int_{x=-\infty}^{\infty} f(x)e^{-iwx} dx \int_{y=-\infty}^{\infty} g(y)e^{-iwy} dy \\ &= \int_{x=-\infty}^{\infty} f(x)e^{-iwx} dx \int_{s=-\infty}^{\infty} g(s-x)e^{-iw(s-x)} ds \quad \text{Substitute } s := y + x \\ &= \int_{s=-\infty}^{\infty} \left(\int_{x=-\infty}^{\infty} f(x)g(s-x) dx \right) e^{-iws} ds \quad \text{Rearranging terms} \\ &= \int_{s=-\infty}^{\infty} (f * g)(s) e^{-iws} ds \quad \text{Definition of convolution} \\ &= F(f * g)(w) \quad \text{Definition of Fourier transform} \end{aligned}$$

Fourier Analysis

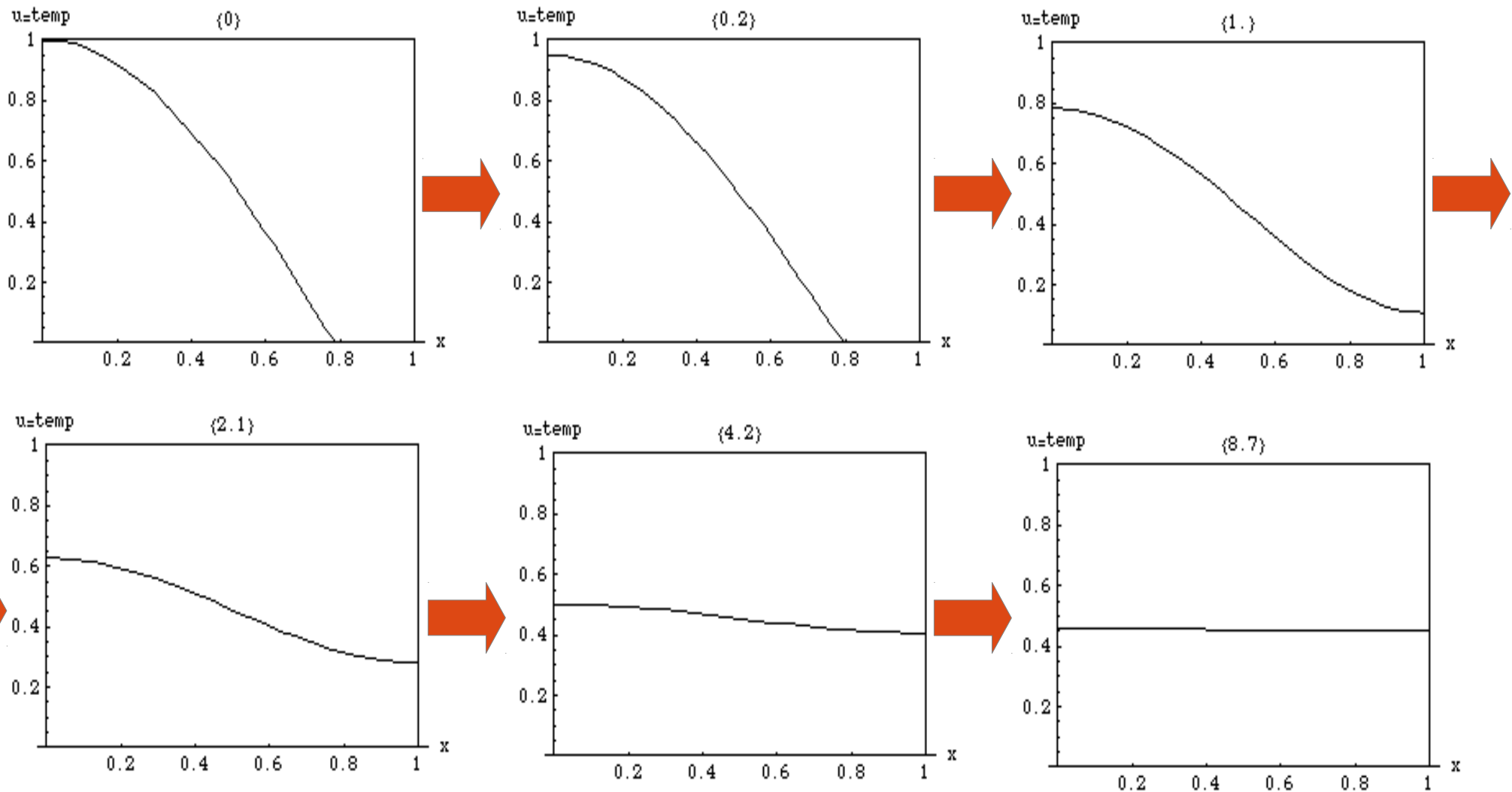
- Fourier Transform and Convolution

(1) $Ff(w) \cdot Fg(w) = F(f * g)(w)$

(2) Similarly, $F(f(x) \cdot g(x)) = (Ff * Fg)(w)$

Fourier Analysis

- Why was Fourier analysis invented ?
 - To study heat flow (diffusion of thermal energy)
 - The Analytic Theory of Heat. J Fourier. 1822.



Fourier Analysis

- Heat equation
 - Consider a function over space and time : $f(x, t)$
 - where x = location, t = time
 - e.g., temperature distribution within an object over time
 - Rate of change (in time) of $f(x)$ is proportional to second spatial derivative (Laplacian) of $f(x)$:

$$f_t(x, t) = \alpha f_{xx}(x, t)$$

- **Problem**
 - **Given:** $f(x, t = 0)$, for all x
 - **Find:** $f(x, t = T)$, for all x
 - **Strategy:** Analyze in frequency domain !

Fourier Analysis

- Heat equation

$$f_t(x, t) = \alpha f_{xx}(x, t)$$

- **Fourier transform** (over x) of the **left hand side** is:

$$\begin{aligned} F f_t(w, t) &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} f(x, t) e^{-iwx} dx \\ &= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} f(x, t) e^{-iwx} dx \\ &= \frac{\partial}{\partial t} F f(w, t) \end{aligned}$$

- Switching order of integral (w.r.t. ' x ') and derivative (w.r.t. ' t ')

Fourier Analysis

- Heat equation

$$f_t(x, t) = \alpha f_{xx}(x, t)$$

- **Fourier transform of the right hand side = ?**

- Simple way to find Fourier transform of derivatives of $f(x)$

$$f(x) = \int_{w=-\infty}^{\infty} F(w) e^{iwx} dw \text{ Fourier inversion theorem}$$

$$f_x(x) = \int_{w=-\infty}^{\infty} (iw) F(w) e^{iwx} dw \text{ Differentiate both sides w.r.t. } x$$

$$f_{xx}(x) = \int_{w=-\infty}^{\infty} (iw)^2 F(w) e^{iwx} dw \text{ Differentiate both sides w.r.t. } x$$

Thus, the Fourier transform of f_{xx} is:

$$\int_{x=-\infty}^{\infty} f_{xx}(x) e^{-iwx} dx = (iw)^2 F(w)$$

Fourier Analysis

- Heat equation $f_t(x, t) = \alpha f_{xx}(x, t)$
 - In the Fourier domain, the heat equation is:

$$\frac{\partial F f(w, t)}{\partial t} = \alpha(-w^2) F f(w, t)$$

$$\frac{\partial F f(w, t)}{F f(w, t)} = \alpha(-w^2) \partial t$$

- Now integrate RHS from $t = 0$ to $t = T$,
and integrate LHS from $F f(w, 0)$ to $F f(w, T)$

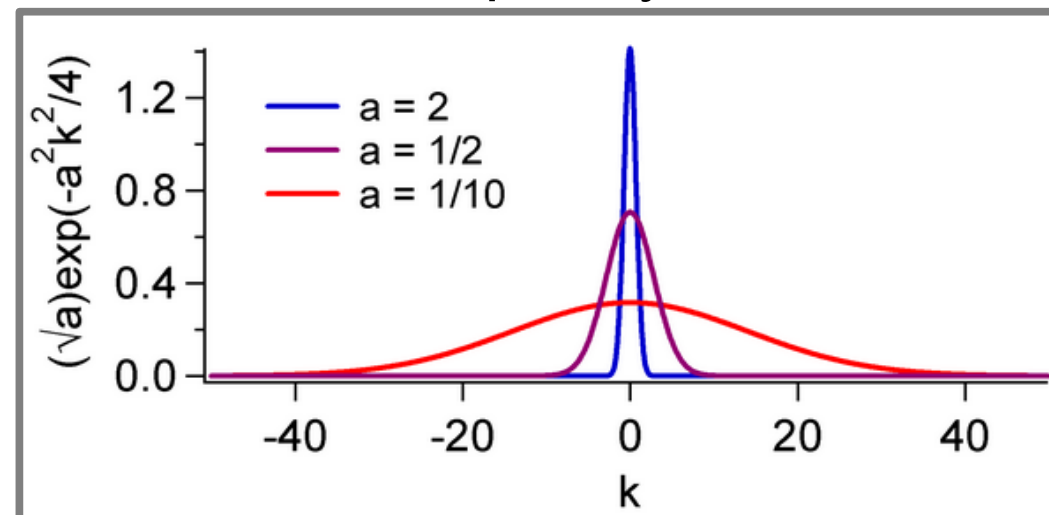
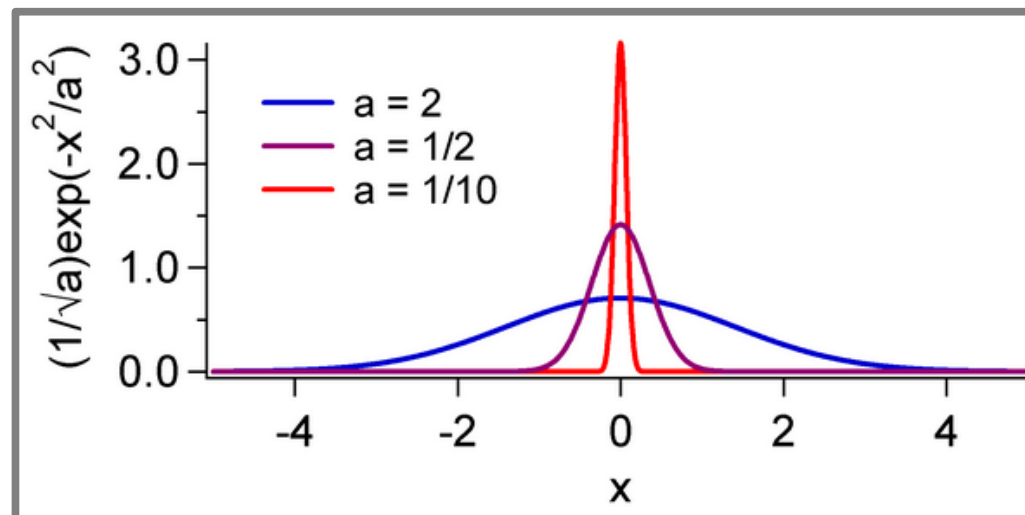
$$[\log F f(w, t)]_{t=0}^T = -\alpha w^2 t \Big|_{t=0}^T = -\alpha w^2 T$$

$$F f(w, t = T) = e^{-\alpha w^2 T} F f(w, t = 0)$$

- What does this tell us ?

Fourier Analysis

- Heat equation $f_t(x, t) = \alpha f_{xx}(x, t)$
 - In Fourier domain $F f(w, t = T) = e^{-\alpha w^2 T} F f(w, t = 0)$
 - **Fourier** transform of function at time $t=T$ equals Fourier transform of initial function, at $t=0$, **multiplied** by Gaussian (scaled)
 - Variance of the Gaussian proportional to $(1 / T)$
 - In **spatial** domain, function at time $t=T$ equals **convolution** of initial function, at $t=0$, with ... what ?
 - Inverse Fourier transform of Gaussian in frequency domain = ?



Fourier Analysis

- Heat equation $f_t(x, t) = \alpha f_{xx}(x, t)$
 - In Fourier domain
$$F f(w, t = T) = e^{-\alpha w^2 T} F f(w, t = 0)$$
 - **Fourier** transform of function at time $t=T$ equals Fourier transform of initial function, at $t=0$, **multiplied** by Gaussian (scaled)
 - Variance of the Gaussian proportional to $(1 / T)$
 - In **spatial** domain, function at time $t=T$ equals **convolution** of initial function, at $t=0$, with Gaussian
 - Variance of Gaussian proportional to T
 - Thus, a function evolving based on the heat equation → function undergoing increasing Gaussian smoothing

Fourier Analysis

- Heat equation
 - The heat equation is a partial differential equation (PDE) that defines “isotropic” diffusion on the function
 - Isotropic → same in all directions
 - NOT edge preserving
 - **Linear** filter (convolution in spatial domain)
 - “Anisotropic” diffusion
 - Diffuse / average intensities along the edge, NOT across the edge
 - **Nonlinear** filtering in spatial domain

Fourier Analysis

- Anisotropic diffusion
 - Pietro Perona and Jitendra Malik. 1987.
- Coherence-enhancing diffusion
 - Joachim Weickert. 1998.

Fourier Analysis

- Anisotropic diffusion. Perona and Malik. 1987.

- Key idea
$$\frac{\partial I}{\partial t} = \operatorname{div} (c(x, y, t) \nabla I) = \nabla c \cdot \nabla I + c(x, y, t) \Delta I$$

- $c(x, y, t)$ = diffusion coefficient

- For heat equation, $c(x, y, t) = \alpha$
- $c(\cdot)$ controls rate of diffusion
- Chosen as a function of the image gradient

- Perona-Malik
$$c(\|\nabla I\|) = e^{-(\|\nabla I\|/K)^2} \quad c(\|\nabla I\|) = \frac{1}{1 + \left(\frac{\|\nabla I\|}{K}\right)^2}$$

- Diffusion is really “isotropic”
- But, quantity of diffusion reduces as pixel gets close to an edge

Fourier Analysis

- Heat equation

- How to discretize Perona-Malik PDE ?
- Replace derivatives by finite differences

- For time derivative → use forward difference

$$\Delta_h[f](x) = f(x+h) - f(x)$$

- For space derivative → use central difference

$$\delta_h[f](x) = f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h)$$

$$\frac{\delta_h^2[f](x)}{h^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

- Forward-Time-Central-Space (FTCS) scheme

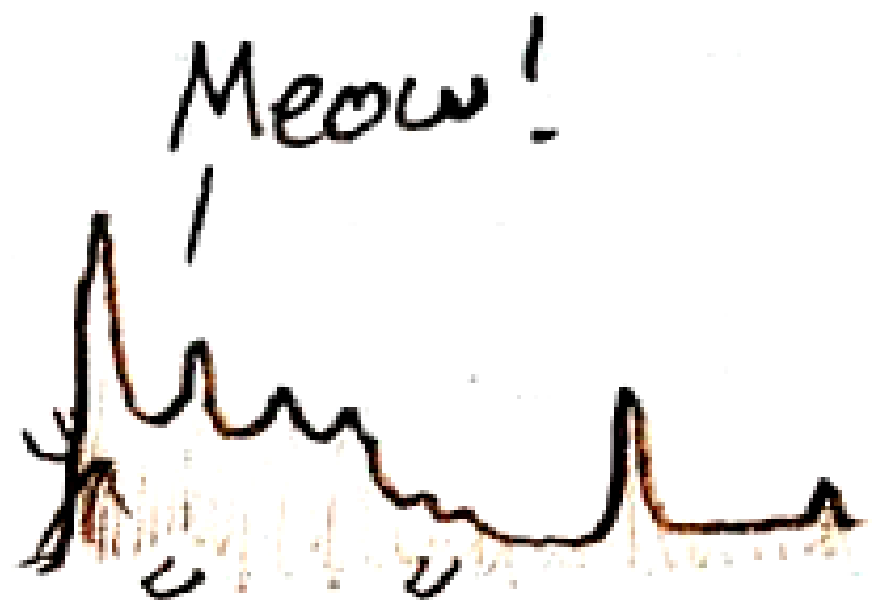
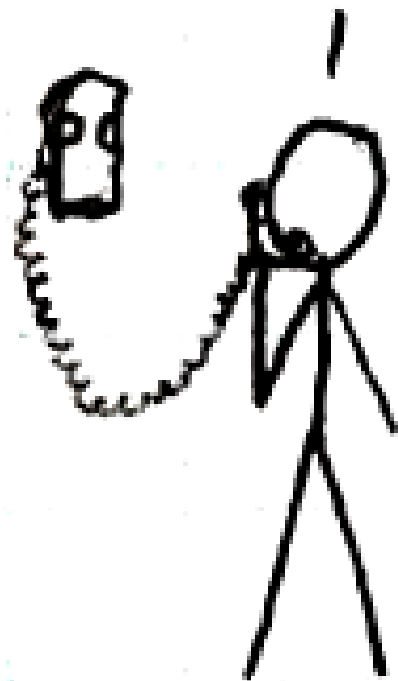
$$I_{i,j}^{t+1} = I_{i,j}^t + \lambda [I_{i-1,j}^t + I_{i+1,j}^t + I_{i,j-1}^t + I_{i,j+1}^t - 4I_{i,j}^t]$$

- Choose time-step lambda within [0, 0.25] for stability

Fourier Transform of Dubstep



Hi, Dr. Elizabeth?
Yeah, uh... I accidentally took
the Fourier transform of my cat...

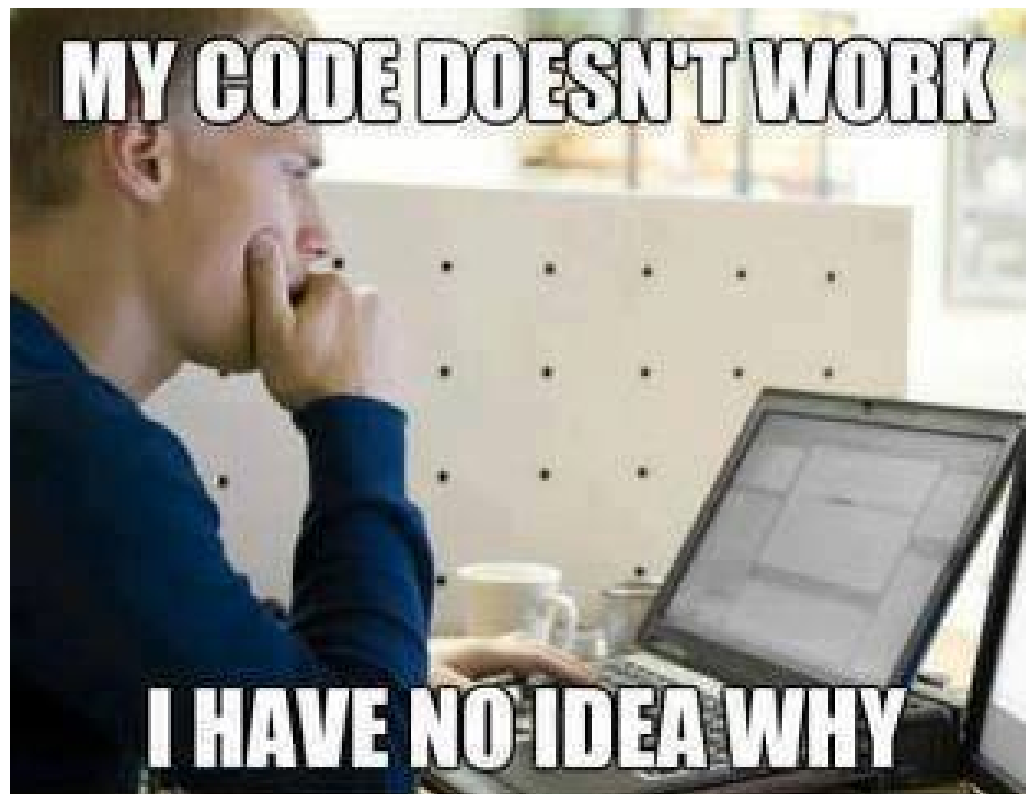




Solve this for
WiFi Password

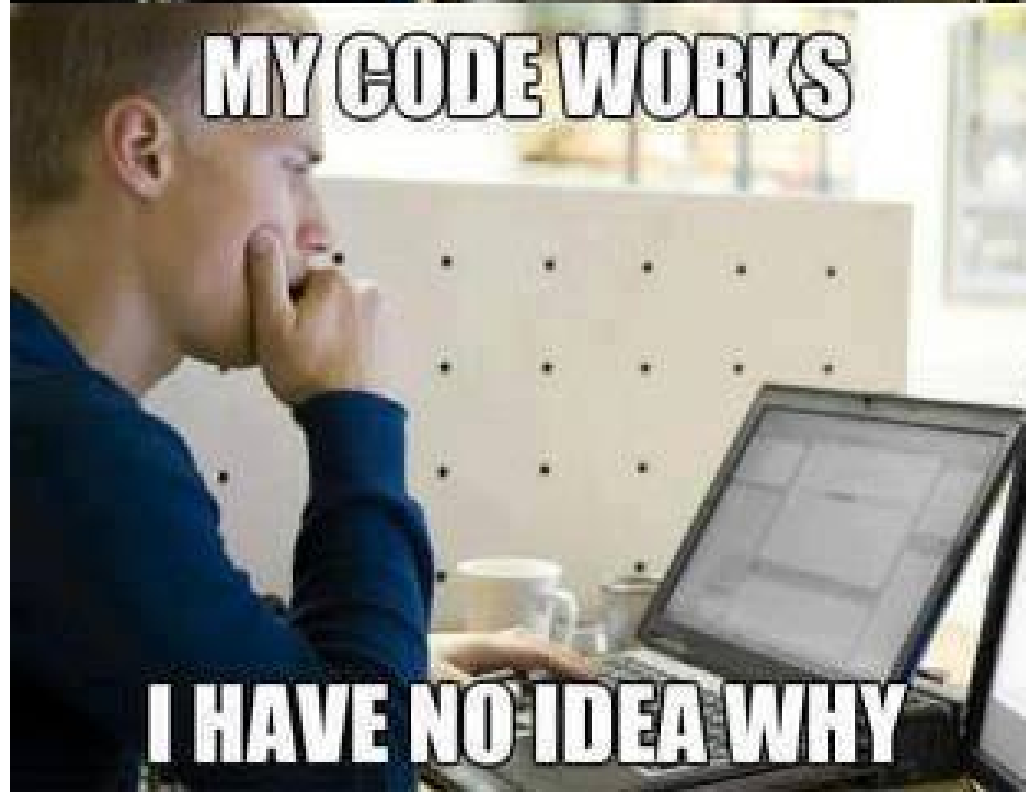
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

MY CODE DOESN'T WORK



I HAVE NO IDEA WHY

MY CODE WORKS



I HAVE NO IDEA WHY