Problem 7.28

The idea of quadrature multiplexing rests on the following premise: Two signals can be transmitted over a common channel, provided that two conditions are satisfied:
(i) The two signals are orthogonal to each other.
(ii) They both occupy the same bandwidth.
This principle is satisfied by quadriphase-shift keying (QPSK), as demonstrated next.

Consider the QPSK signal defined by

$$s(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t), & \text{dibit 00} \\ -\sqrt{\frac{2E}{T}}\sin((2\pi f_c t),) & \text{dibit 01} \\ \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{dibit 11} \\ -\sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{dibit 10} \end{cases} \tag{1}$$

This signal can be decomposed into the sum of two BPSK signals, defined as follows:

$$s_1(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t), & \text{dibit 00} \\ \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{dibit 11} \end{cases} \tag{2}$$

and

$$s_2(t) = \begin{cases} -\sqrt{\frac{2E}{T}}\sin(2\pi f_c t), & \text{dibit 01} \\ -\sqrt{\frac{2E}{T}}\sin(2\pi f_c t + \pi), & \text{dibit 10} \end{cases} \tag{3}$$

In light of Eqs. (1) through (3), we may write

$$s(t) = s_1(t) + s_2(t) \tag{4}$$

which means that $s_1(t)$ and $s_2(t)$ can be transmitted simultaneously on a common channel and be detected separately at the receiver. This statement is justified on two accounts:

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(i) Both $s_1(t)$ and $s_2(t)$ occupy exactly the same bandwidth, as their magnitude spectra are identical.

(ii) They are orthogonal over the symbol period T, as shown by

$$\int_0^T s_1(t)s_2(t) = \int_0^T \sqrt{\frac{2E}{T}}\cos(2\pi f_c t)\left(-\sqrt{\frac{2E}{T}}\right)\sin(2\pi f_c t)dt$$

$$= -\frac{E}{T}\int_0^T \sin 4(\pi f_c t)dt$$

which is zero by the band-pass assumption, provided that the carrier frequency f_c is high enough.

The assertion embodied in Eq. (4) holds for any clockwise or counterclockwise rotation of the QPSK constellation defined in Eq. (1).

Consider next the 8-PSK defined by

$$s'(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t), & \text{symbol } 000 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{4}\right), & \text{symbol } 001 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{2}\right), & \text{symbol } 101 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{3\pi}{4}\right), & \text{symbol } 111 \\ \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{symbol } 011 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{5\pi}{4}\right), & \text{symbol } 010 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{3\pi}{2}\right), & \text{symbol } 110 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{7\pi}{4}\right), & \text{symbol } 100 \end{cases} \quad (5)$$

Following what we did with the QPSK signal of Eq. (1), we may decompose the 8-PSK of Eq. (5) as follows:

$s(t) = s'_1(t) + s'_2(t)$

whose constituents are defined by

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$$s'_1(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi f_c t), & \text{symbol } 000 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{2}\right), & \text{symbol } 101 \\ \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \pi), & \text{symbol } 011 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{3\pi}{2}\right), & \text{symbol } 110 \end{cases} \tag{6}$$

and

$$s'_2(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{4}\right), & \text{symbol } 001 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{3\pi}{4}\right), & \text{symbol } 111 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{5\pi}{4}\right), & \text{symbol } 010 \\ \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{7\pi}{4}\right), & \text{symbol } 100 \end{cases} \tag{7}$$

Basically, the signal $s'_1(t)$ is a rewrite of the QPSK signal $s(t)$ of Eq. (1). The signal $s'_2(t)$ is a rotated version of $s(t)$. The two constituent QPSK signals $s'_1(t)$ and $s'_2(t)$ satisfy the common bandwidth requirement (i). However, they fail to satisfy requirement (ii). To demonstrate this failure, let us test the first components of $s'_1(t)$ and $s'_2(t)$ for orthogonality by writing

$$\begin{aligned} \int_0^T s'_1(t)s'_2(t) &= \int_0^T \sqrt{\frac{2E}{T}}\cos(2\pi f_c t) \cdot \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{\pi}{4}\right)dt \\ &= \frac{2E}{T}\int_0^T \cos(2\pi f_c t)\cos\left(2\pi f_c t + \frac{\pi}{4}\right)dt \\ &= \frac{E}{T}\int_0^T \left[\cos\left(\frac{\pi}{4}\right) + \cos\left(4\pi f_c t + \frac{\pi}{4}\right)\right]dt \\ &= \frac{E}{T} \cdot \frac{T}{\sqrt{2}} + \frac{E}{T}\int_0^T \cos\left(4\pi f_c t + \frac{\pi}{4}\right)dt \end{aligned} \tag{8}$$

The integral term of Eq. (8) may be set equal to zero under the band-pass assumption, provided that the carrier frequency f_c is high enough. But the first term, namely, $E/\sqrt{2}$ is nonzero. We therefore conclude that the orthogonality requirement is violated by the two QPSK signals $s'_1(t)$ and $s'_2(t)$. Hence, The "conquer and divide" approach theorem cannot be exploited beyond the QPSK signal.