Problem 2.35

Applying the formula for the autocorrelation function

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t)g^*(t-\tau)dt$$

to the specified signal

$$g(t) = \frac{1}{t_0}\exp\left(-\frac{\pi t^2}{t_0^2}\right), \qquad -\infty < t < \infty$$

we get

$$R_g(\tau) = \int_{-\infty}^{\infty} \frac{1}{t_0^2}\exp\left[\frac{\pi}{t_0^2}(t^2 + (t-\tau)^2)\right]dt$$

$$= \frac{1}{t_0^2}\int_{-\infty}^{\infty} \exp\left[\left(-\frac{\pi}{t_0^2}\right)(2t^2 - 2t\tau + \tau^2)\right]dt$$

$$= \frac{1}{t_0^2}\int_{-\infty}^{\infty} \exp\left[\left(-\frac{\pi}{t_0^2}\right)\left(\sqrt{2}t - \frac{\tau}{\sqrt{2}}\right) - \frac{\pi}{t_0^2}\ \frac{\tau^2}{2}\right]dt$$

$$= \frac{1}{t_0^2}\exp\left(-\frac{\pi\tau^2}{2t_0^2}\right)\int_{-\infty}^{\infty} \exp\left[\left(-\frac{\pi}{t_0^2}\right)\left(\sqrt{2}t - \frac{\tau}{\sqrt{2}}\right)^2\right]dt \qquad (1)$$

Let $x = \frac{1}{t_0}\left(\sqrt{2}t - \frac{\tau}{\sqrt{2}}\right)$, and therefore (for fixed τ)

$$dt = \frac{t_0}{\sqrt{2}}dx$$

We may then rewrite Eq. (1) as

$$R_g(\tau) = \frac{1}{\sqrt{2}t_0}\exp\left(-\frac{\pi\tau^2}{2t_0^2}\right)\int_{-\infty}^{\infty} \exp(-\pi x^2)dx \qquad (2)$$

Recognizing that

$$\int_{-\infty}^{\infty} \exp(-\pi x^2)dx = 1$$

we find that Eq. (2) simplifies to

$$R_g(\tau) = \frac{1}{\sqrt{2}t_0}\exp\left(-\frac{\pi\tau^2}{2t_0^2}\right)$$

which has the same form as the bell-shaped Gaussian curve: