Problem 2.7
Prove the following properties of the convolution process:

(a) The commutative property:
$g_1(t) \star g_2(t) = g_2(t) \star g_1(t)$

Proof:

$$g_1(t) \star g_2(t) = \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} g_2(t-\tau) g_1(\tau) d\tau$$

Replace t - τ with λ. That is, $\tau = t - \lambda$. Hence

$$g_1(t) \star g_2(t) = -\int_{+\infty}^{-\infty} g_2(\lambda) g_1(t-\lambda) d\lambda$$

$$= \int_{-\infty}^{\infty} g_2(\lambda) g_1(t-\lambda) d\lambda$$

$$= g_2(t) \star g_1(t)$$

(b) The associative property:
$g_1(t) \star [g_2(t) \star g_3(t)] = [g_1(t) \star g_2(t)] \star g_3(t)$

Proof:
.
Let

$$x(t) = g_2(t) \star g_3(t)$$

$$= \int_{-\infty}^{\infty} g_2(\tau) g_3(t-\tau) d\tau$$

Hence

$$I(t) = g_1(t) \star \underbrace{[g_2(t) \star g_3(t)]} = \int_{-\infty}^{\infty} g_1(\lambda) x(t-\lambda) d\lambda$$

$$= \int_{-\infty}^{\infty} g_1(\lambda) \int_{-\infty}^{\infty} g_2(\tau) g_3(t-\tau-\lambda) d\tau d\lambda \qquad (1)$$

Replace $\tau + \lambda$ with μ; that is, $\tau = \mu - \lambda$. Hence, keeping λ fixed, we may write

$$I(t) = \int_{-\infty}^{\infty} g_1(\lambda) \int_{-\infty}^{\infty} g_2(\mu-\lambda) g_3(t-\mu) d\mu d\lambda \qquad (2)$$

With μ fixed, the integral $\int_{-\infty}^{\infty} g_1(\lambda) g_2(\mu-\lambda) d\lambda$ is recognized as the convolution of $g_1(\mu)$ and $g_2(\mu)$, as shown by

$$g_{12}(\mu) = \int_{-\infty}^{\infty} g_1(\lambda) g_2(\mu-\lambda) d\lambda = g_1(\mu) \star g_2(\mu)$$

We may therefore rewrite Eq. (1) as

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Problem 2.7 continued

$$I(t) = \int_{-\infty}^{\infty} g_{12}(\mu) g_3(t-\mu) d\mu = g_{12}(t) \bigstar g_3(t) = [g_1(t) \bigstar g_2(t)] \bigstar g_3(t)$$

(c) The distributive property:

$$g_1(t) \bigstar [g_2(t) + g_3(t)] = g_1(t) \bigstar g_2(t) + g_1(t) \bigstar g_3(t)$$

Proof:

$$g_1(t) \bigstar [g_2(t) + g_3(t)] = \int_{-\infty}^{\infty} g_1(\tau)[g_2(t-\tau) + g_3(t-\tau)] d\tau$$

$$= \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau + \int_{-\infty}^{\infty} g_1(\tau) g_3(t-\tau) d\tau$$

$$= g_1(t) \bigstar g_2(t) + g_1(t) \bigstar g_3(t)$$