Problem 2.32

(a) The integrator output is

$$y(t) = \int_{t-T}^{t} x(\tau)d\tau$$

Let $x(t) \rightleftharpoons X(f)$; then

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

Therefore,

$$y(t) = \int_{t-T}^{t}\left[\int_{-\infty}^{\infty} X(f)\exp(j2\pi f\tau)df\right]d\tau$$

Interchanging the order of integration:

$$y(t) = \int_{-\infty}^{\infty} X(f)\left[\int_{t-T}^{t} \exp(j2\pi f\tau)d\tau\right]df$$

$$= \int_{-\infty}^{\infty} [TX(f)\operatorname{sinc}(fT)\exp(-j\pi fT)]\exp(j2\pi ft)df$$

The Fourier transform of the integrator output is therefore

$$Y(f) = TX(f)\operatorname{sinc}(fT)\exp(-j\pi fT) \qquad (1)$$

Equation (1) shows that $y(t)$ can be obtained by passing the input signal $x(t)$ through a linear filter whose transfer function is equal to $T\operatorname{sinc}(fT)\exp(-j\pi fT)$.

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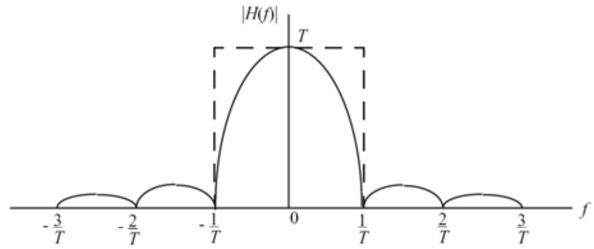
Problem 2-32 continued

(b) The amplitude response of this filter is shown in Fig. 1:

Figure 1



The approximation with an ideal low-pass filter of bandwidth $1/T$, gain T, and delay $T/2$, is shown dashed in Fig. 1. The response of this ideal filter to a unit step function applied at $t = 0$ is given by

$$y_{\text{ideal}}(t) = \frac{T}{\pi}\int_{-\infty}^{\frac{2\pi}{T}\left(t-\frac{T}{2}\right)} \frac{\sin\lambda}{\lambda}d\lambda$$

At time $t = T$. we therefore have

$$\begin{aligned} y_{\text{ideal}}(t) &= \frac{T}{\pi}\int_{-\infty}^{\pi} \frac{\sin\lambda}{\lambda}d\lambda \\ &= \frac{T}{\pi}\left[\int_{-\infty}^{0} \frac{\sin\lambda}{\lambda}d\lambda + \int_{0}^{\pi} \frac{\sin\lambda}{\lambda}d\lambda\right] \\ &= \frac{T}{\pi}[\text{Si}(\infty) + \text{Si}(\pi)] \\ &= \frac{T}{\pi}\left(\frac{\pi}{2} + 1.85\right) \\ &= 1.09T \end{aligned} \tag{2}$$

On the other hand, the output of the ideal integrator to a unit step function, evaluated at time $t = T$, is given by

$$\begin{aligned} y(T) &= \int_{0}^{T} u(\tau)d\tau \\ &= T \end{aligned} \tag{3}$$

Thus, comparing Eqs. (2) and (3) we see that the ideal low-pass filter output exceeds the ideal integrator output by only nine percent for $T = 1$.