Digital Image Processing

Fourier Analysis – 2

Sampling

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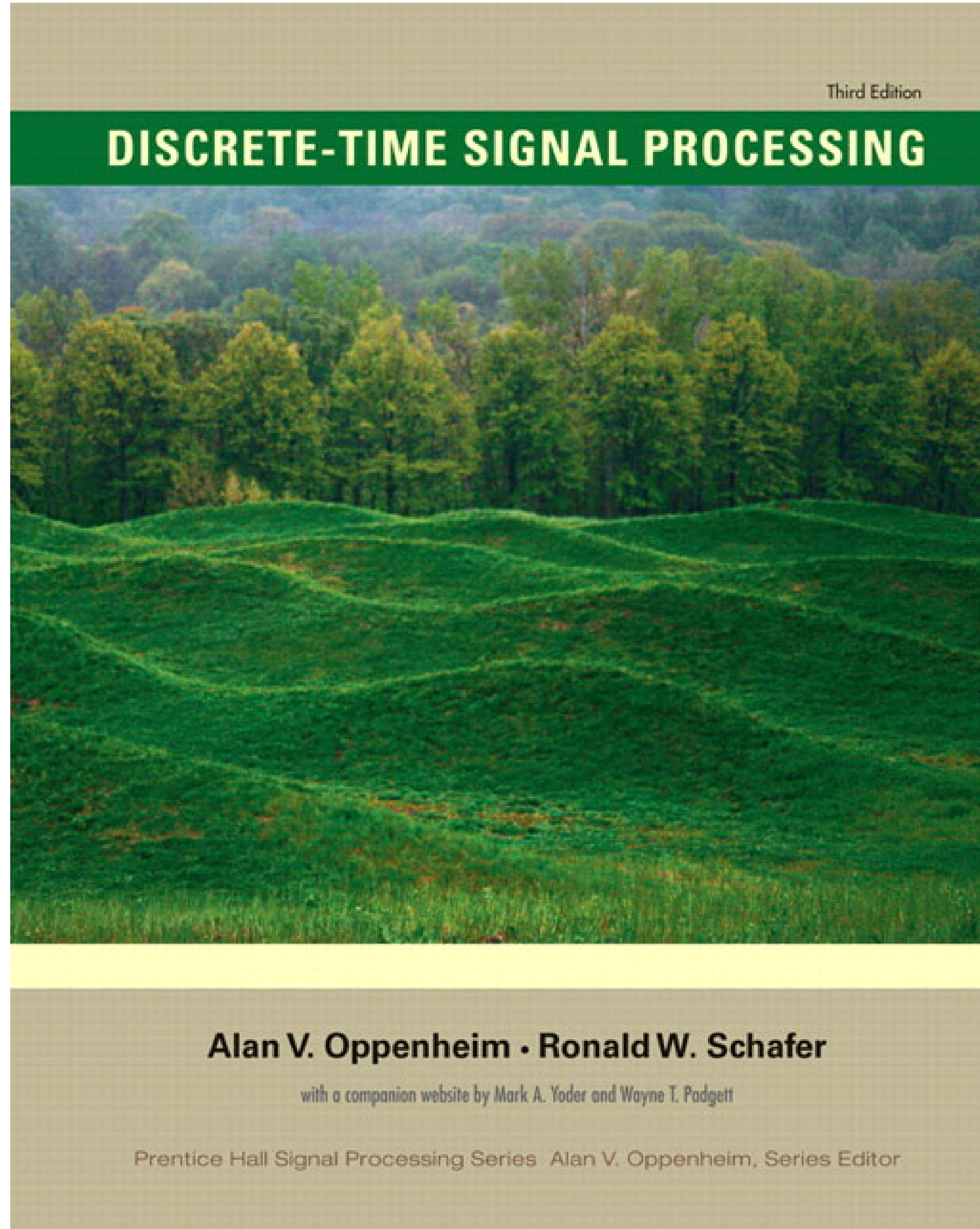
Applications of Fourier Analysis

- Signal processing
  - Image processing
    - Understanding sampling
    - Filtering (and convolution), denoising
    - Design of Gabor filter bank (wavelets) for texture analysis
      - Mimics human brain's processing of visual stimulus
    - Precursor of popular wavelet analysis
  - Speech processing
    - Syllables characterized by frequency spectra
  - Fourier optics
    - Imaging mechanisms measure coefficients in transform domain
  - Medical imaging
    - Imaging mechanisms measure coefficients in transform domain
- Things to do with L1/L2 function spaces. e.g., in ML

Fourier Analysis

- Books
  - Both your textbooks
  - And …



Fourier Analysis

- Mathematical Methods for Computer Science
  - https://www.cl.cam.ac.uk/teaching/1213/MathMforCS
- The Fourier Transform and its Applications
  - https://see.stanford.edu/Course/EE261
  - https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf
  - Reference books (15+): https://see.stanford.edu/materials/lsoftaee261/References.pdf
- The Fourier Transform – A Primer
  - ftp://ftp.cs.brown.edu/pub/techreports/95/cs95-37.pdf
- Image Analysis – 2D Fourier Transforms and Applications
  - http://www.robots.ox.ac.uk/~az/lectures/ia

Fourier Analysis

- **Fourier transform examples**
  - **Delta function**
    - Sifting property of the delta function

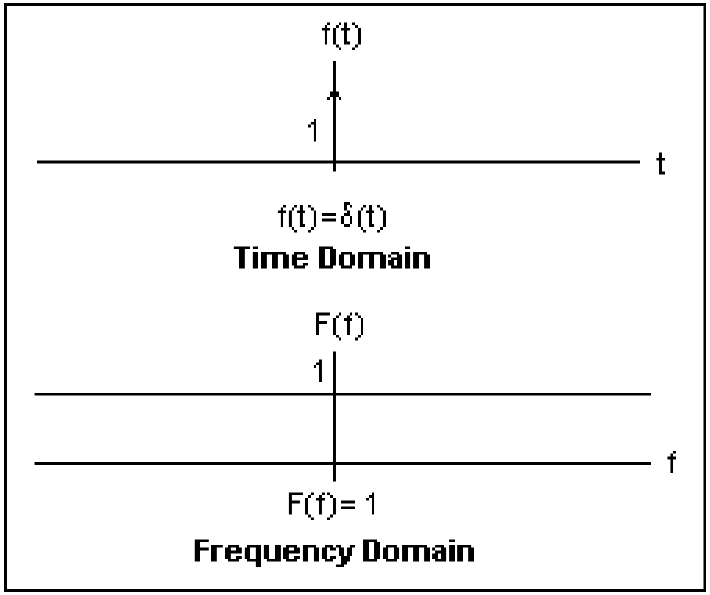
$$\int_{-\infty}^{\infty} f(t)\delta(t-T)\,dt = f(T)$$

    - Fourier transform **(delta function at origin)**

$$F\delta(w) = \int_{-\infty}^{\infty} \delta(x)e^{-iwx}dx = e^{-iwx}|_{x=0} = 1$$

= constant function (independent of 'w')

    - If we superpose all complex waves,
with equal weights,
what function will we get ?



Fourier Analysis

- Fourier transform examples
  - Delta function
    - This also gives the following identity (which we will use later)

$$\delta(x) := F^{-1}(F\delta)(x) \text{ Fourier inversion theorem}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(F\delta)(w)e^{iwx}\,dw \text{ Definition of inverse Fourier transform}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwx}\,dw$$

$$= \frac{-1}{2\pi}\int_{u=+\infty}^{-\infty} e^{-iux}\,du \text{ Substitute u = -w}$$

$$= \frac{1}{2\pi}\int_{u=-\infty}^{\infty} e^{-iux}\,du$$

Fourier Analysis

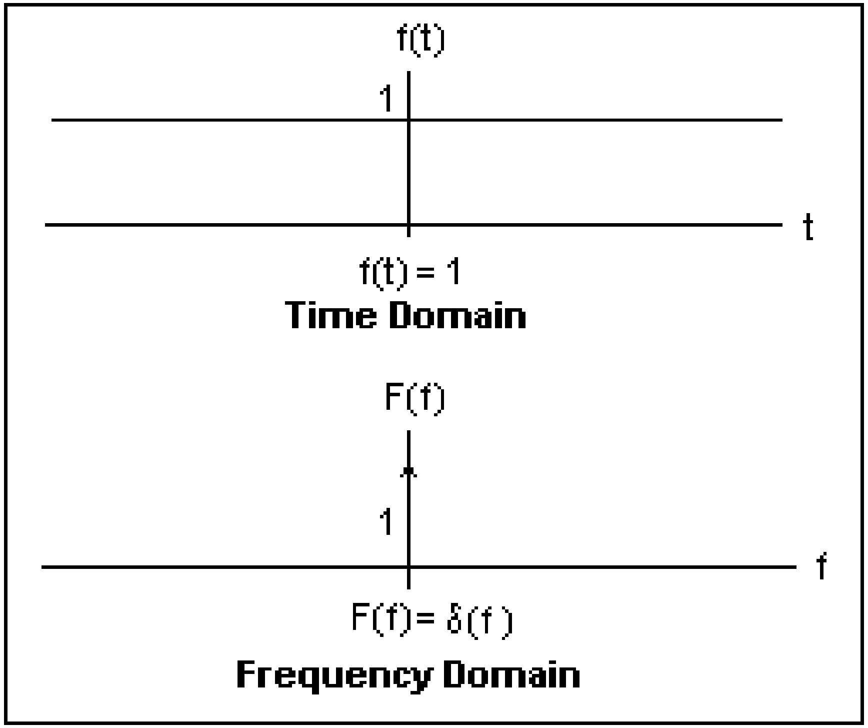
- Unabsolutely integrable functions
  - Periodic functions
    - sine (wt), cosine (wt), step function u(t),
  - Impulse “function”
    - Not a function
      - Zero almost everywhere. How can “integral” still be 1 ?
    - Dirac delta functional → maps each function to its values at 0
  - Impulse train
- Can be handled using the theory of generalized function and generalized Fourier transform

Fourier Analysis

- Fourier transform examples
  - **Constant function**
    - Fourier transform (**constant** funct.n) = **delta function at origin**

$$F(a)(w) = \int_{-\infty}^{\infty} a e^{-iwx} dx = a \int_{-\infty}^{\infty} e^{-iwx} dx$$

$$= 2\pi a \delta(w)$$ Using the previous identity



Fourier Analysis

- Fourier transform examples
  - **Complex sinusoidal wave** $f(x) = e^{iw_0 x}$
    - Fourier transform of a complex sinusoidal wave is a **shifted delta function**

$$Ff(w) = \int_{-\infty}^{\infty} e^{iw_0 x} e^{-iwx} dx = \int_{-\infty}^{\infty} e^{-i(w-w_0)x} dx$$

$$= 2\pi\delta(w - w_0)$$ Using the previous identity

Fourier Analysis

- Fourier transform examples
  - **Shifted Delta function**
    - Fourier transform of a shifted delta function δ(x − x0) is a **complex sinusoidal wave**

$$F(\delta(x-x_0))(w) = \int_{-\infty}^{\infty} \delta(x-x_0)e^{-iwx}dx = e^{-iwx}|_{x=x_0} = e^{-iwx_0}$$

  - Is the Fourier transform shift / time invariant ?
    - Fourier transform of shifted delta function is **NOT** shifted Fourier transform of delta function
    - This implies that the **Fourier transform is NOT shift invariant / space invariant / time invariant**

Fourier Analysis

- Fourier transform examples
  - What we saw so far
    - Notation on this slide : 2 π f = ω

$$\delta(t) \xLeftrightarrow{\mathcal{F}} 1$$

$$\delta(t - t_0) \xLeftrightarrow{\mathcal{F}} e^{-j2\pi f t_0}$$

$$1 \xLeftrightarrow{\mathcal{F}} \delta(f)$$

$$e^{j2\pi f_0 t} \xLeftrightarrow{\mathcal{F}} \delta(f - f_0)$$

Fourier Analysis

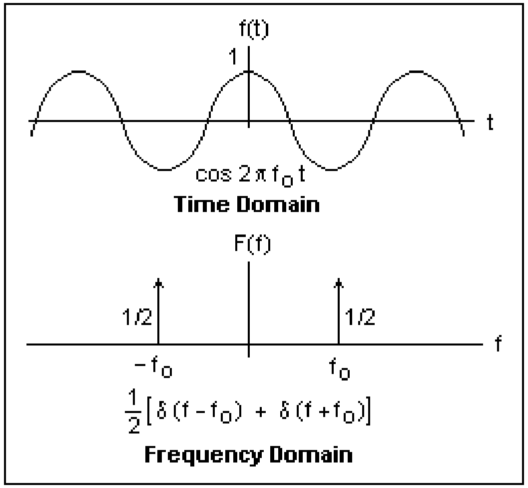
- Fourier transform examples
  - **Real-valued cosine wave** $f(x) = \cos(w_0 x)$

$$Ff(w) = \int_{-\infty}^{\infty} \cos(w_0 x) e^{-iwx} dx$$

$$= \int_{-\infty}^{\infty} 0.5(e^{iw_0 x} + e^{-iw_0 x}) e^{-iwx} dx$$

$$= 0.5 \int_{-\infty}^{\infty} e^{iw_0 x} e^{-iwx} dx + 0.5 \int_{-\infty}^{\infty} e^{-iw_0 x} e^{-iwx} dx$$

$$= \pi\delta(w - w_0) + \pi\delta(w + w_0)$$ Using the previous identity



Fourier Analysis

- Fourier transform examples
  - **Real-valued sine wave** $f(x) = \sin(w_0 x)$

$$Ff(w) = \int_{-\infty}^{\infty} \sin(w_0 x) e^{-iwx} dx$$

$$= \int_{-\infty}^{\infty} 0.5(e^{iw_0 x} - e^{-iw_0 x}) e^{-iwx} dx$$

$$= 0.5 \int_{-\infty}^{\infty} e^{iw_0 x} e^{-iwx} dx - 0.5 \int_{-\infty}^{\infty} e^{-iw_0 x} e^{-iwx} dx$$

$$= \pi\delta(w - w_0) - \pi\delta(w + w_0)$$ Using the previous identity

- Typo: (1 / i) factor missing

Fourier Analysis

- Fourier transform examples
  - **Impulse train** of period Δx $$f(x) := \sum_{n=-\infty}^{\infty} \delta(x - n\Delta X)$$
  - What is Fourier-**series representation** of impulse train ?
  - Let, over 1 period [-Δx/2,Δx/2], $f(x) := \sum_n c_n e^{inx2\pi/\Delta X}$
  - What are the coefficients ?

$$c_n := \frac{1}{\Delta X} \int_{-\Delta X/2}^{\Delta X/2} f(x) e^{-inx2\pi/\Delta X} dx$$

$$= \frac{1}{\Delta X} \int_{-\Delta X/2}^{\Delta X/2} \left(\sum_{n=-\infty}^{\infty} \delta(x - n\Delta X) \right) e^{-inx2\pi/\Delta X} dx$$

$$= \frac{1}{\Delta X} \int_{-\Delta X/2}^{\Delta X/2} \delta(x) e^{-inx2\pi/\Delta X} dx$$ Only one impulse falls within integral limits

$$= \frac{1}{\Delta X} e^0 = \frac{1}{\Delta X}$$

Fourier Analysis

- Fourier transform examples
  - Impulse train of period Δx has Fourier **series represent.n**

$$f(x) := \frac{1}{\Delta X}\sum_n e^{inx2\pi/\Delta X}$$

  - Fourier **transform** of impulse train of period Δx = ?

$$Ff(w) := F\left(\frac{1}{\Delta X}\sum_n e^{inx2\pi/\Delta X}\right)(w)$$

$$= \frac{1}{\Delta X}\sum_n F\left(e^{inx2\pi/\Delta X}\right)(w)$$

$$= \frac{2\pi}{\Delta X}\sum_n \delta\left(w - n\frac{2\pi}{\Delta X}\right)$$ Fourier transform of a complex wave

  - Fourier **transform** of **impulse train of period Δx** = another **impulse train with period 2π / Δx**

Fourier Analysis

- **Parseval's Theorem** (applies to Fourier **series**)
  - Let A(x) and B(x) be periodic functions of period 2 π represented via their Fourier series

$$A(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \qquad B(x) = \sum_{n=-\infty}^{\infty} b_n e^{inx}$$

  - Then,

$$\sum_{n=-\infty}^{\infty} a_n \overline{b_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(x) \overline{B(x)}\, dx$$

Fourier Analysis

- **Parseval's Theorem** (applies to Fourier **transform**)
  - For functions f(x),g(x), with Fourier transforms Ff(s),Fg(s):

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} \mathcal{F}f(s)\overline{\mathcal{F}g(s)}\,ds$$

  - Proof:
    - Invertibility of FT implies: $g(x) = \int_{-\infty}^{\infty} \mathcal{F}g(s)e^{isx}\,ds$
      - Take complex congujate on both sides above
    - LHS

$$= \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} \overline{\mathcal{F}g(s)}e^{-isx}\,ds\right)dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)e^{-isx}\,dx\right)\overline{\mathcal{F}g(s)}\,ds$$

$$= \int_{-\infty}^{\infty} \mathcal{F}f(s)\overline{\mathcal{F}g(s)}\,ds$$

Fourier Analysis

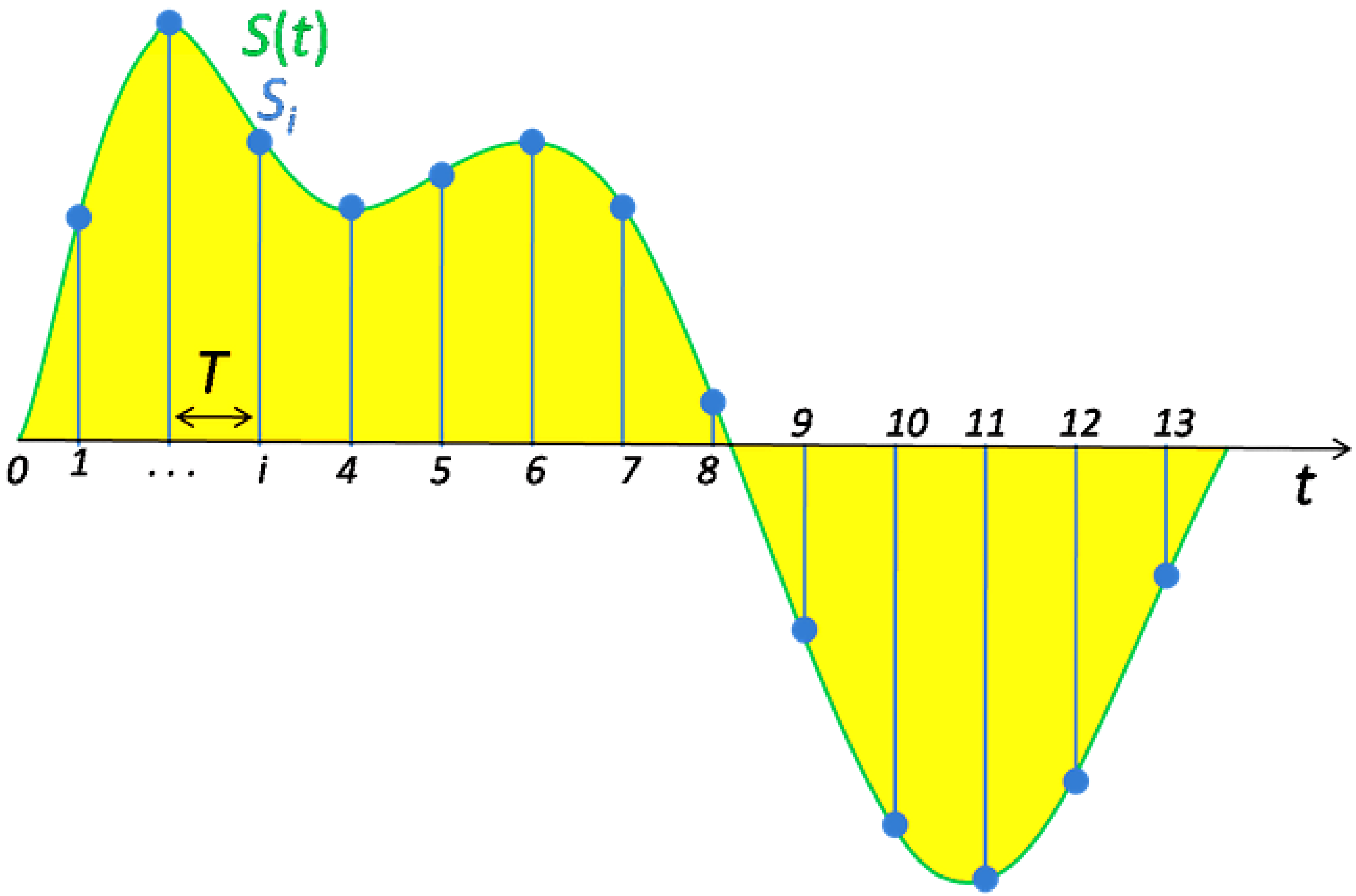
- **Parseval's / Plancherel Theorem**
  - Let f(x) be integrable and square-integrable function with Fourier **transform** Ff(w)
  - Then,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |Ff(w)|^2 dw$$

  - **G(w) := | Ff(w) |^2** is called the **power spectrum** of f(x)
  - What does the theorem imply ?
    - Fourier transform preserves norm of the original function
      - Norm of a function := inner-product of function with itself
    - Norm ↔ "Total Energy" associated with a function
    - Why isn't this surprising ?

Fourier Analysis

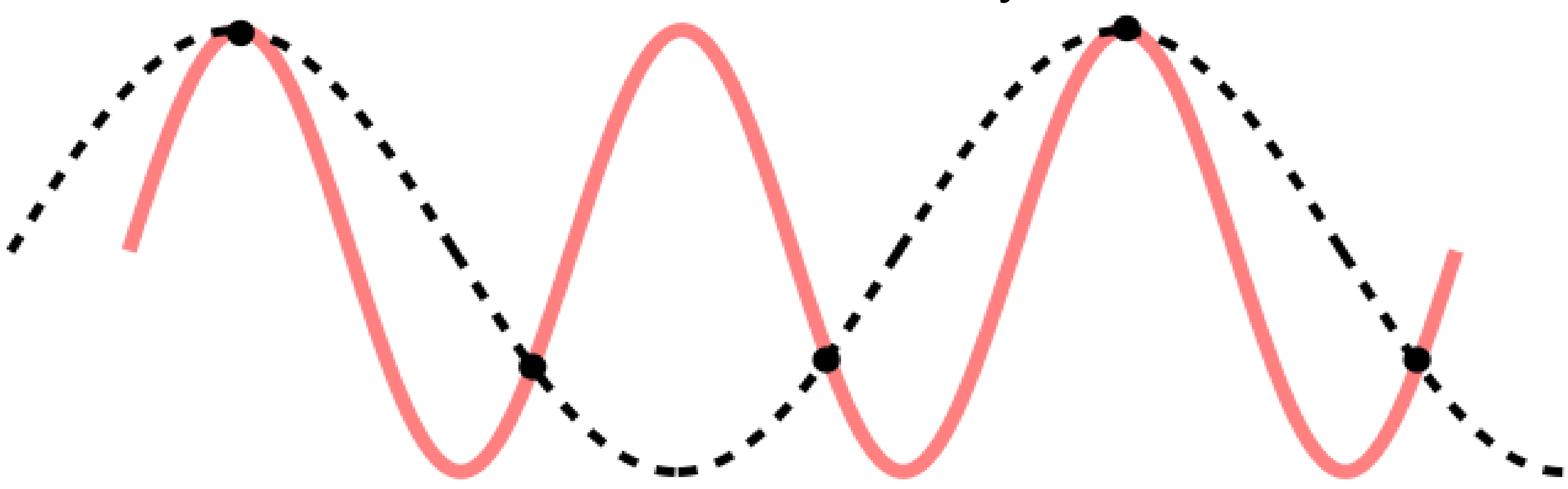
- Sampling



  - What is the optimal rate for sampling a continuous signal ?

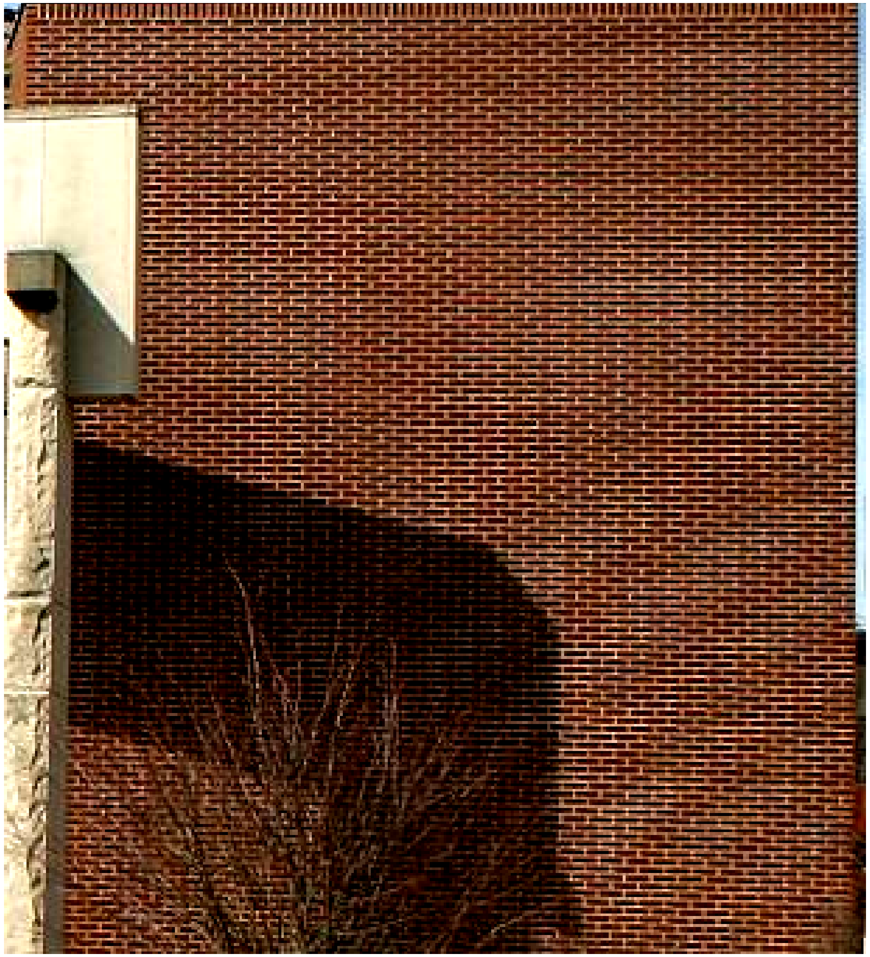
Fourier Analysis

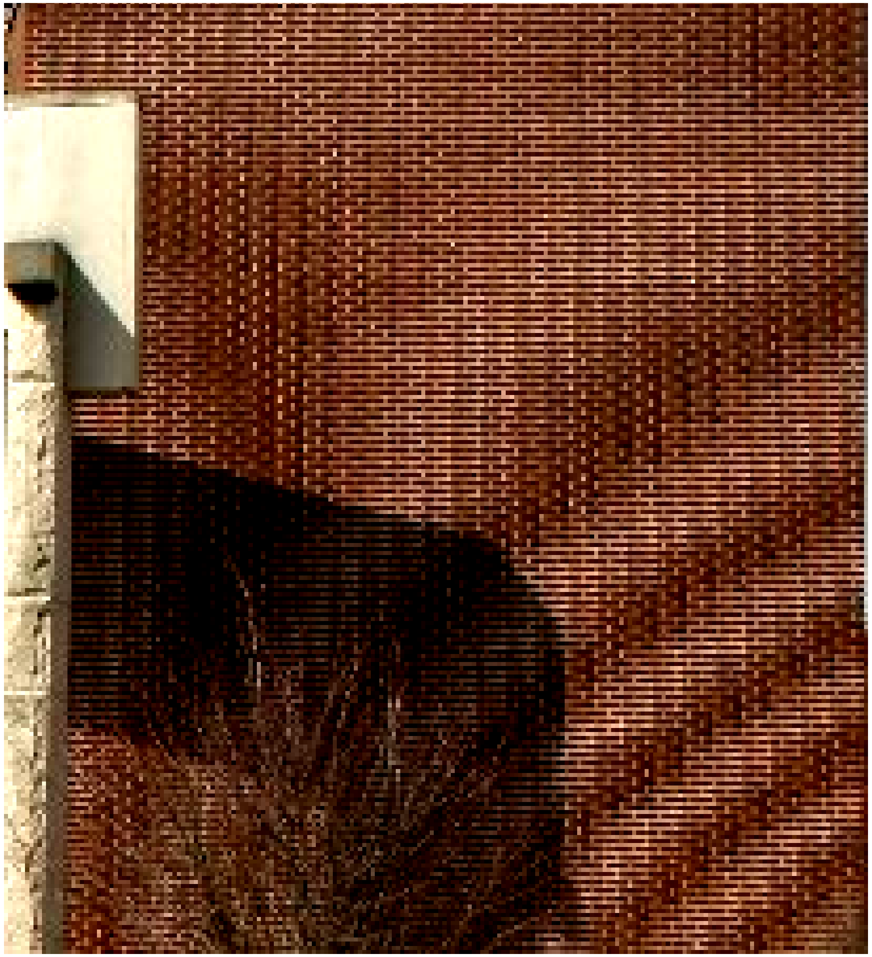
- Sampling
  - Why is sampling rate important ?
    - Samples of several different sine waves can be identical, when at least **one of them** has:
      - Period < 2 * sample spacing (for red sinusoid)
      - Frequency > 0.5 * sampling rate
  - Problem
    - Given : sample points
    - Find : which continuous function did they come from ?



Fourier Analysis

- Sampling
  - Why is sampling rate important ?
    - Artifacts like Moire patterns
    - Image on right is subsampled (coarser resolution)





Fourier Analysis

- Sampling
  - **Suppose** continuous function f(x) is **bandlimited**
    - **Bandlimited = In Fourier transform, the highest frequency that has a non-zero coefficient is finite**
  - Assume Fourier transform is defined (non-zero) on (-L,L)

    i.e., $Ff(w) = 0$ whenever $|w| > L$
  - In practice, this implies:
    - f(x) doesn't have very steep edges
    - Steepest edge is representable via sinusoids of frequency <= L
    - f(x) doesn't have any discontinuities
    - f(x) is “smooth”

Fourier Analysis

- Sampling

  $Ff(w) = 0$ whenever $|w| > L$

  - **Key Idea : Represent frequency-domain function Ff(·) using a *series***
    - We are NOT representing f(.) as a (Fourier) series
    - We are representing Ff(.) as a (Fourier) series
    - *Just to find the series representation*, you may consider the Fourier transform Ff(·) to be *periodic outside [−L,L]*
  - Let us represent the function Ff(·), **within (-L,L)**, as:

    $$Ff(w) := \sum_n B_n e^{inw\pi/L}$$

    - **Outside (-L,L), Ff(w) := 0** (even if sinusoids can be evaluated)
  - Then, what are the coefficients Bn ?

Fourier Analysis

- Sampling

$$Ff(w) := \sum_n B_n e^{inw\pi/L}$$

  - Coefficients Bn can be obtained by taking the (complex) inner product of Ff(·) with the wave $e^{inw\pi/L}$

$$B_n = \frac{1}{2L}\int_{-L}^{L} Ff(w)e^{-inw\pi/L}$$

  - But, what is this value ? Can we simplify further ?
    - Yes, we can. Here is how ...

Fourier Analysis

$$B_n = \frac{1}{2L}\int_{-L}^{L} Ff(w)e^{-inw\pi/L}$$

- Sampling

  (1) What is the inverse Fourier transform of Ff(w) evaluated at nπ / L ?

  - By definition $$F^{-1}(Ff)(n\pi/L) := \frac{1}{2\pi}\int_{-\infty}^{\infty} Ff(w)e^{inw\pi/L}dw$$

  (2) Ff(w) = 0 outside (-L,L)

$$F^{-1}(Ff)(n\pi/L) := \frac{1}{2\pi}\int_{-L}^{L} Ff(w)e^{inw\pi/L}dw = \frac{2L}{2\pi}B_{-n}$$

  (3) Inverse Fourier transform of Ff(w) evaluated at nπ / L equals f (nπ / L)

  - Fourier inversion theorem

$$F^{-1}(Ff)(n\pi/L) = \frac{2L}{2\pi}B_{-n} = f(n\pi/L)$$

$$\Longrightarrow \quad B_{-n} = \frac{\pi}{L}f(n\pi/L)$$

Fourier Analysis

- Sampling

$$Ff(w) := \sum_n B_n e^{inw\pi/L} \qquad\qquad B_{-n} = \frac{\pi}{L} f(n\pi/L)$$

  - What does the simplified value of coefficients B_n tell us ?
    - For a **bandlimited function** f(x),
      Fourier transform Ff(.) within (-L,L)
      can be **represented by a series**
      where the **coefficients** B_n are
      (scaled) f(.) values *<u>sampled</u>* at locations nπ/L

Fourier Analysis

- Sampling
  - Simplified value of the coefficients Bn allows us to **rewrite** the series representation of F f (·) as

$$Ff(w) := \sum_n B_{-n} e^{-inw\pi/L} = \sum_n \frac{\pi}{L} f(n\pi/L) e^{-inw\pi/L}$$

  - Using this result
    - So, what sample points should be "acquired", based on above result ?

    - How can we "reconstruct" the continuous function f(x) given its discretized / sampled version ?
      - See the formula next ...

Fourier Analysis

$f(x) :=$

$:= F^{-1}(Ff)(x)$ Fourier inversion theorem

$= \frac{1}{2\pi} \int_{-L}^{L} Ff(w) e^{iwx} dw$ Definition of Fourier inverse

$= \frac{1}{2\pi} \int_{-L}^{L} \left[\sum_n \frac{\pi}{L} f(n\pi/L) e^{-inw\pi/L} \right] e^{iwx} dw$ Substitute series for $Ff(\cdot)$

$= \frac{1}{2L} \sum_n f(n\pi/L) \int_{-L}^{L} e^{iw(x-n\pi/L)} dw$ Rearranging terms

$$= \frac{1}{2L} \sum_n f(n\pi/L)(2L) \frac{\sin(L(x-n\pi/L))}{L(x-n\pi/L)}$$

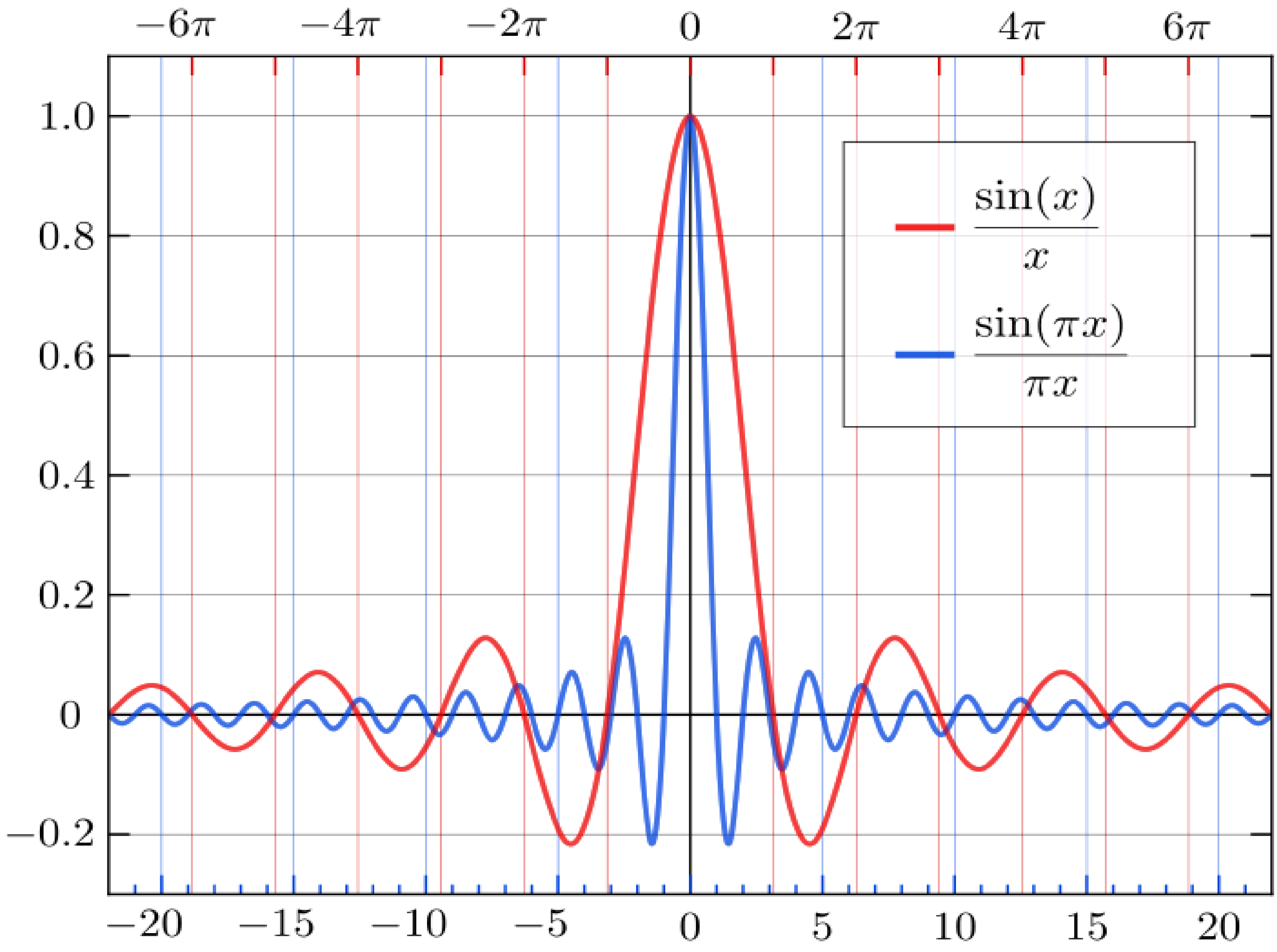
Expand $e^{i\theta} = \cos(\theta) + i\sin(\theta) =$ even + odd function

$$= \sum_n f(n\pi/L) \frac{\sin(L(x-n\pi/L))}{L(x-n\pi/L)}$$

$= \sum_n f(n\pi/L)$ sinc $(L(x-n\pi/L))$ sinc function shifted to $n\pi/L$

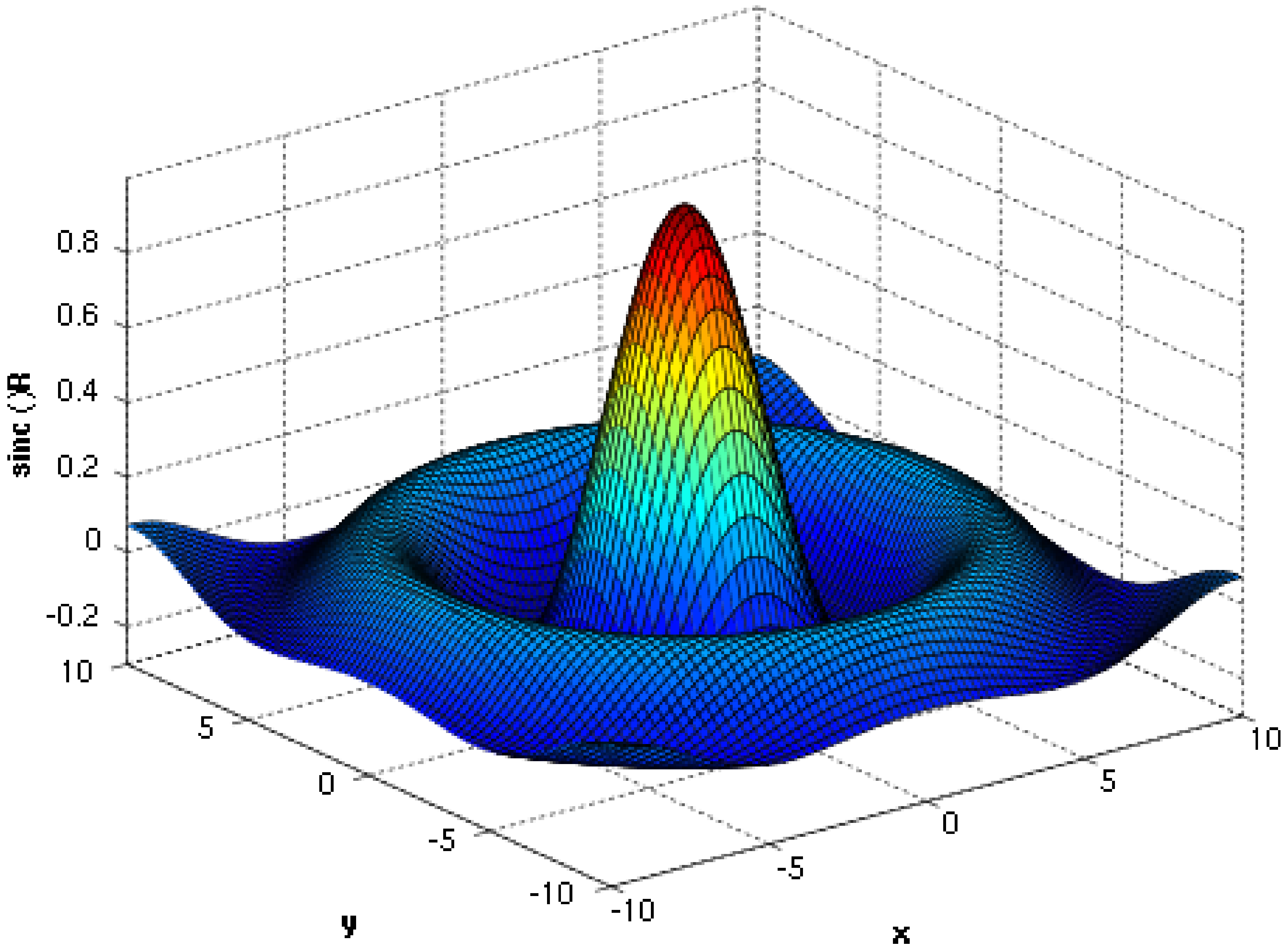
Fourier Analysis

- Sampling: **Sinc** function: sinc(x) := sin(x) / x



Fourier Analysis

- Sinc function over 2D domain



Fourier Analysis

- Sampling
  - Thus, $f(x) = \sum_n f(n\pi/L)\,\text{sinc}\,(L(x - n\pi/L))$
  - This means that :
    for sampling a bandlimited function f(x)
    with highest frequency L in the Fourier transform

    (1) **optimal sample spacing = π / L (or smaller)**

    (2) optimal sampling allows us to **exactly reconstruct** continuous function f(x) using sampled values f (n π/L)
  - This is the Nyquist-Shannon **Sampling theorem**

  - <u>Question</u>: If f(x) has steeper edges than g(x),
    then how should sampling rate vary
    to achieve exact reconstruction ?

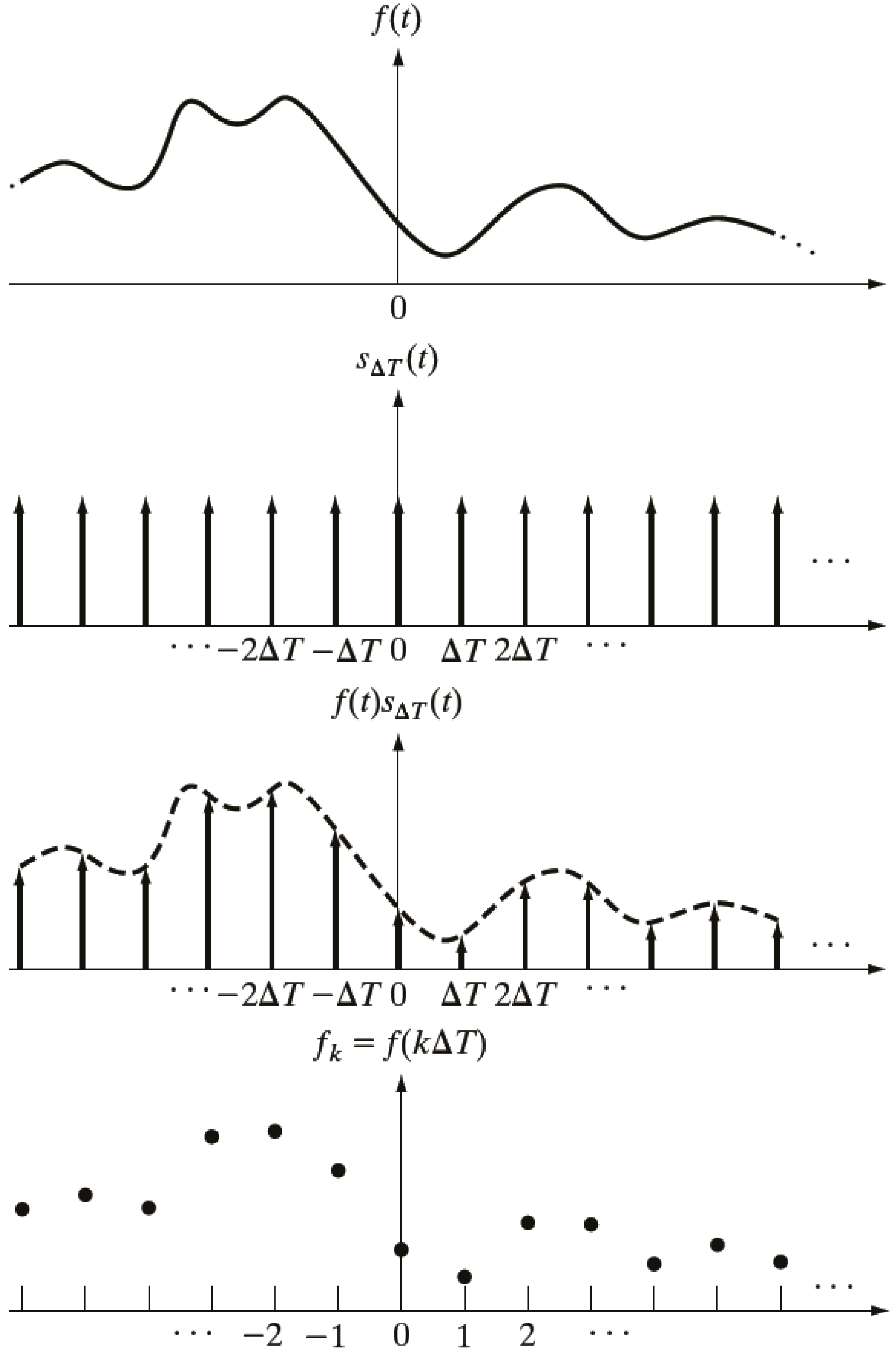
Fourier Analysis

- How to measure f(nπ/L)?
  - Continuous function = f(t)
  - Sampling function = s(t)

$$
\begin{aligned}
\tilde{f}(t) &= f(t)s_{\Delta T}(t) \\
&= \sum_{n=-\infty}^{\infty} f(t)\delta(t-n\Delta T)
\end{aligned}
$$

  - In practical acquisition, sampled values = integrate f(t) x single impulse

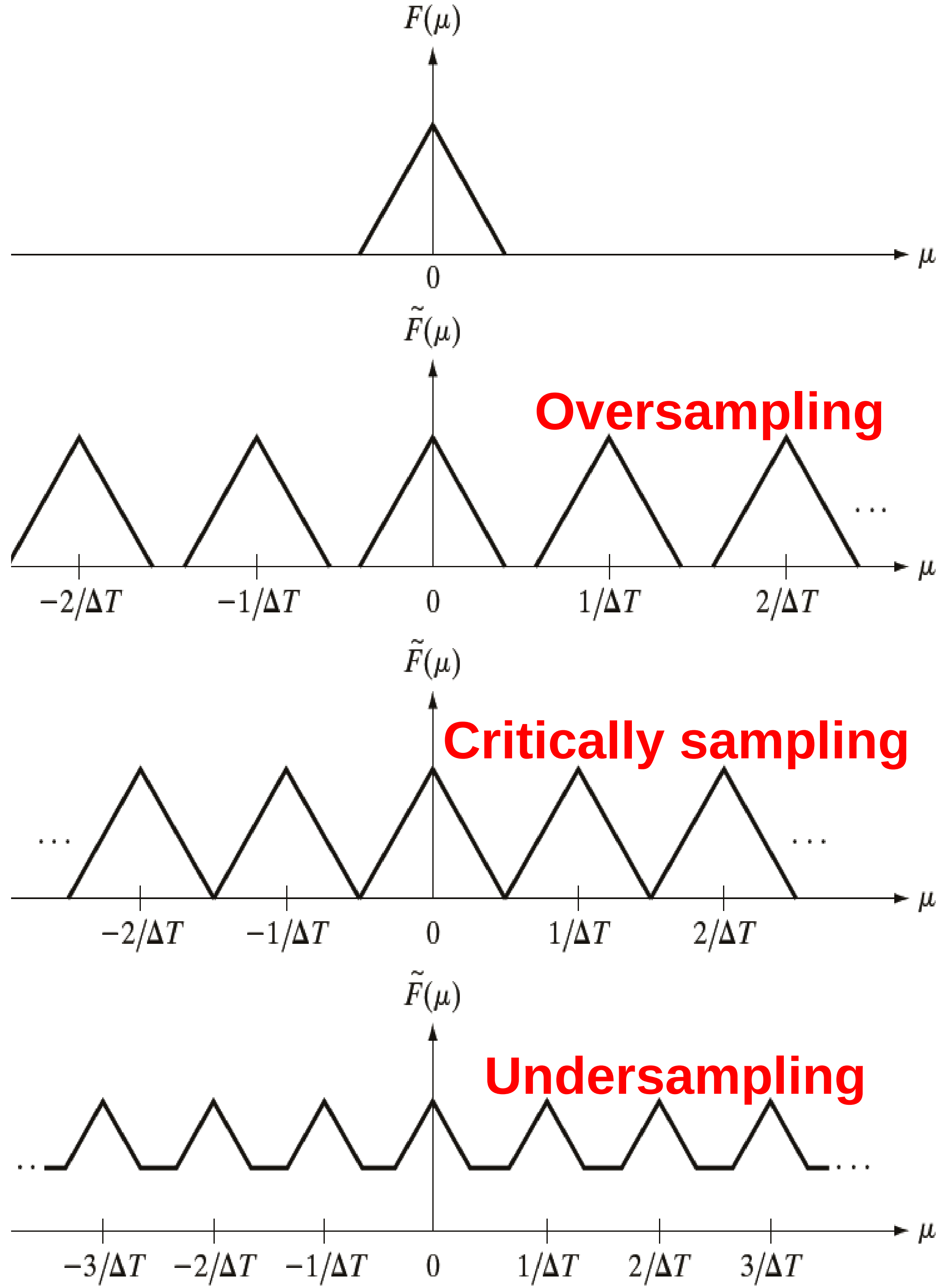
$$
\begin{aligned}
f_k &= f(k\Delta T) \\
&= \int_{-\infty}^{\infty} f(t)\delta(t-k\Delta T)dt
\end{aligned}
$$



Fourier Analysis

- Sampling
  - f(t) is bandlimited
  - Fourier transform of
    [f(t) **x** impulseTrain(t)
    with spacing DeltaT]
    =
    Ff(w) * impulseTrain(w)
    with spacing 1/DeltaT
    - x : multiplication
    - * : convolution

$$
\begin{aligned}
\tilde{F}(\mu) &= \mathcal{F}\{f(t)s_{\Delta T}(t)\} \\
&= F(\mu) \star S(\mu)
\end{aligned}
$$



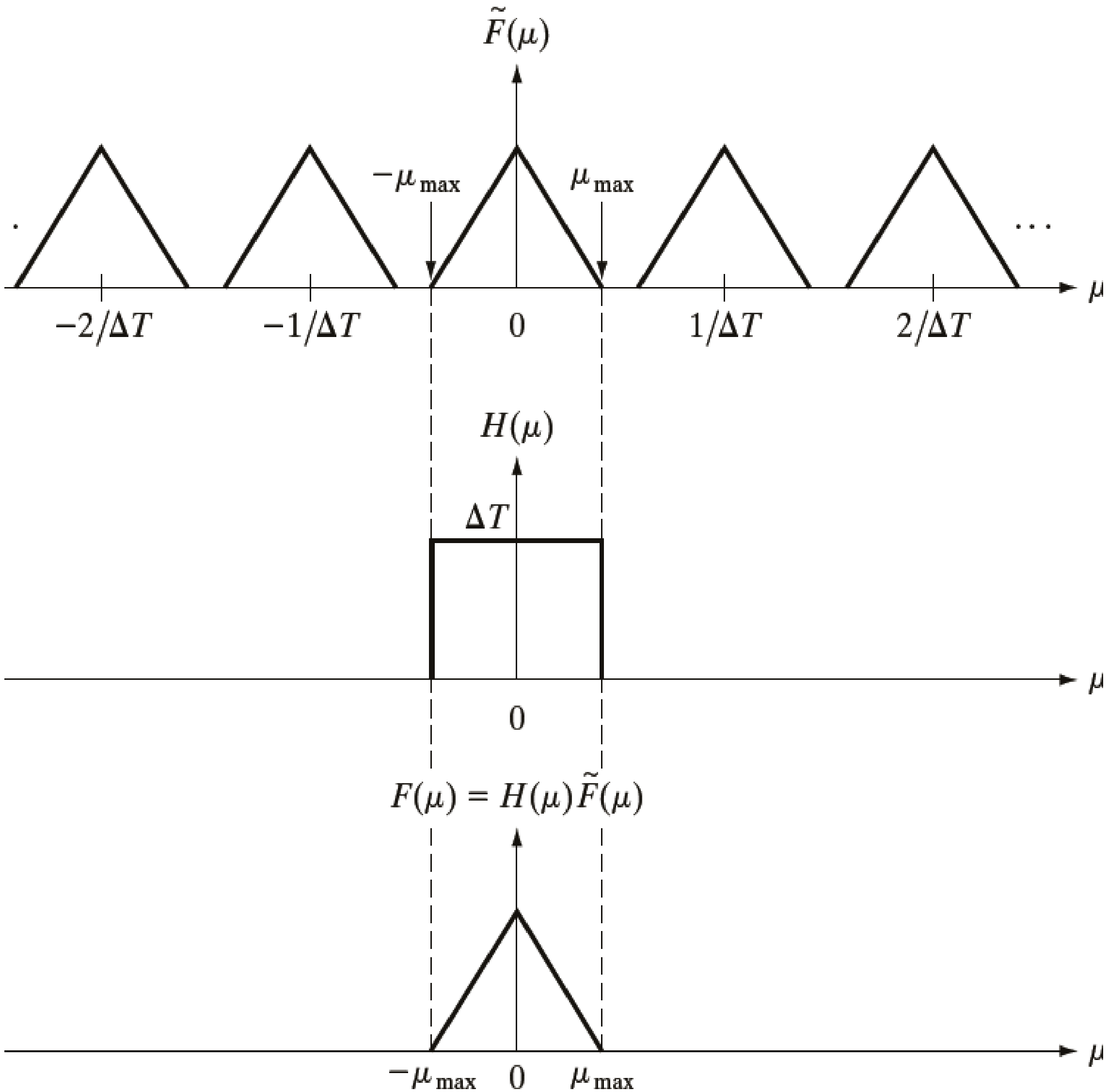
Fourier Analysis

- **Reconstructing** continuous function from its samples

  (1) Get FT
  of sampled function

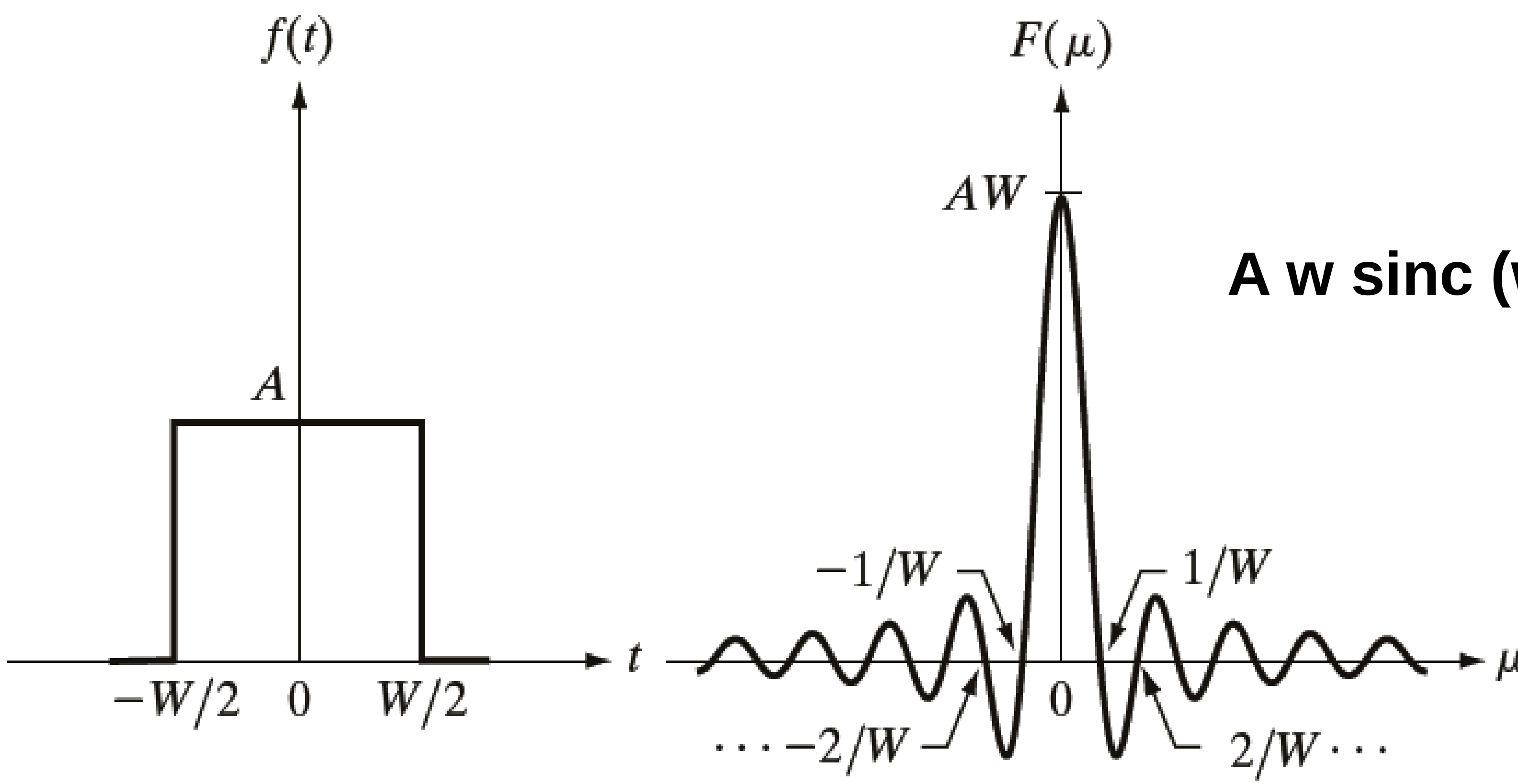
  (2) Extract one period
  in frequency domain

  (3) Take
  inverse FT



Fourier Analysis

- Reconstructing continuous function from its samples
  - Multiplication with box function in Fourier domain = Convolution with sinc function in space domain



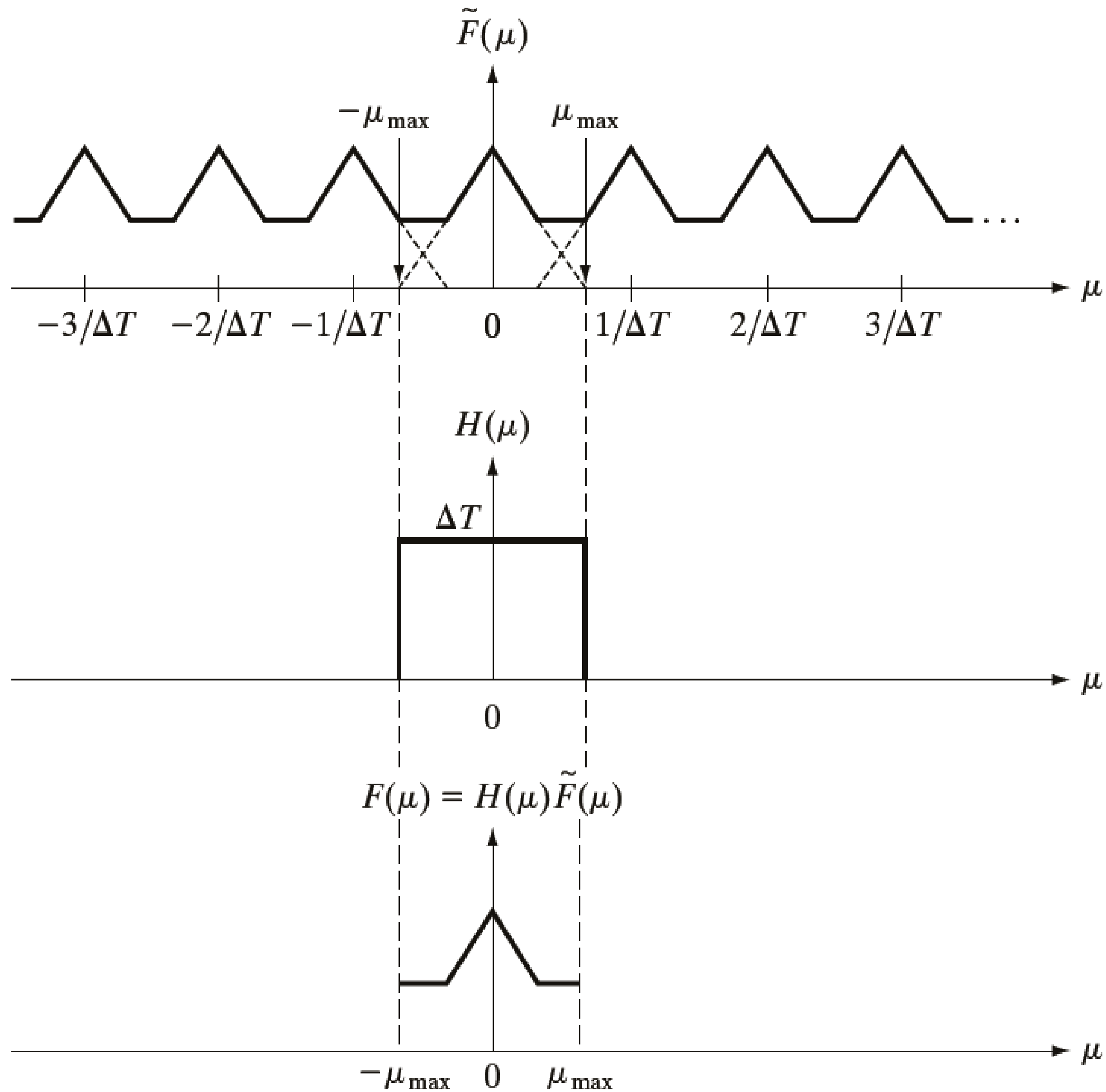
A w sinc (wx/2)

Paul Dirac

- Theoretical physicist [1902-1984]
- Nobel Prize in Physics, 1933
  - Shared with Erwin Schrödinger
  - Describe fermions, predict existence of antimatter
- Personality
  - Didn’t speak much, precise, taciturn
    - Cambridge colleagues joked: 1 unit of "dirac" = 1 word per hour

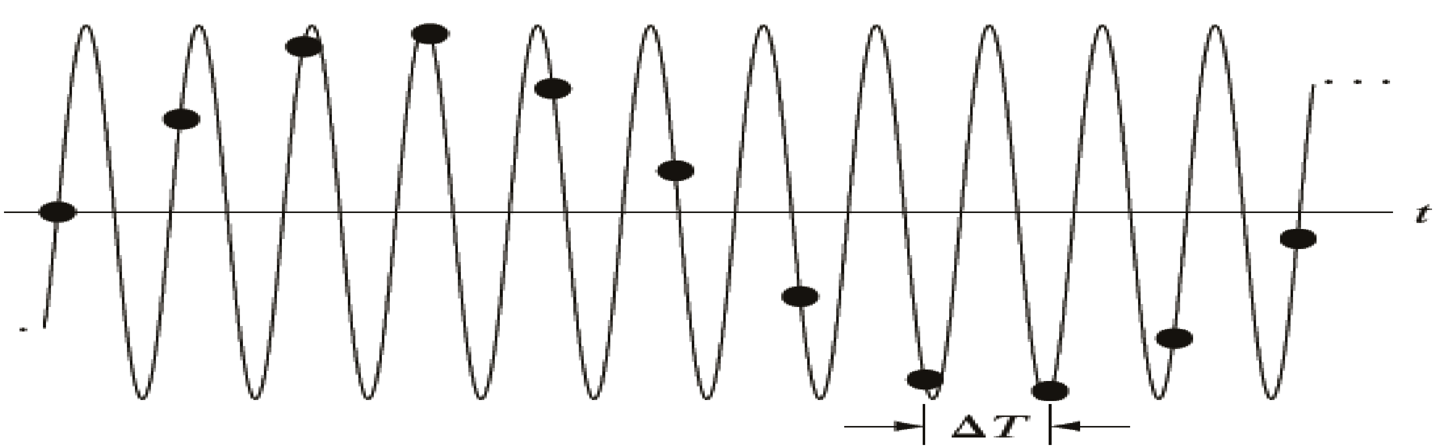
Fourier Analysis

- Reconstructing continuous function from its samples
  - **Undersampled** case
  - High-frequency components of original signal appear as if they are lower-frequency components of the sampled signal
  - **Alias**ing
    = alternate identity



Fourier Analysis

- Sampling : **Aliasing**
  - Here, **sampling rate < Nyquist rate**
  - For f(x) = sin(Lx) → period = 2 π / L; frequency = L / 2 π
    - Ff(w) = 0 outside (-L, L)
    - Nyquist **sample spacing** = π / L → **two samples per period**
    - Nyquist **sampling rate (frequency) = L / π**
      = twice frequency of sinusoid
  - Here, samples of higher-frequency sinusoid appear to have come from a low-frequency sinusoid

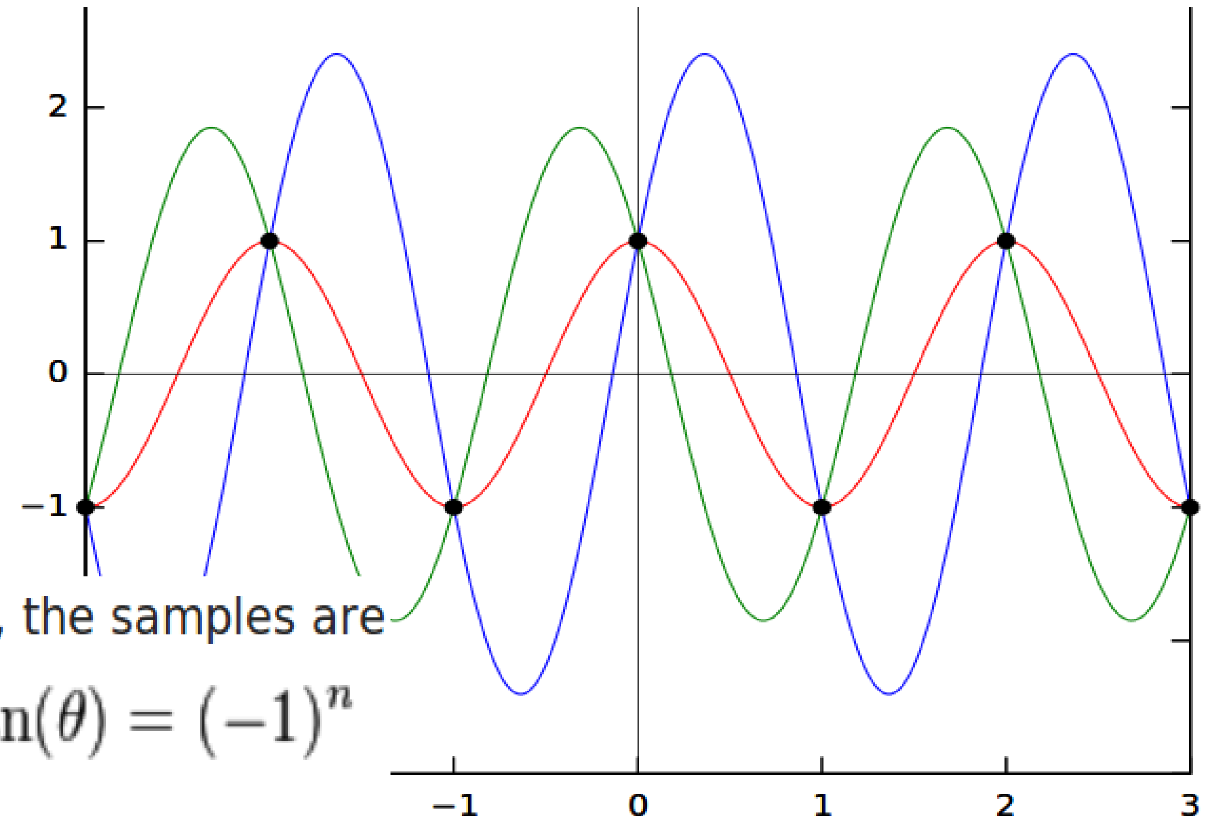


Fourier Analysis

- **Sampling rate in practice**
  - Here, **sampling rate = Nyquist rate** for ALL 3 sinusoids
    - Same frequency, Different phase, Different amplitude
  - Samples **fail to distinguish** between **multiple sinusoids**
  - So, **sampling rate** must be **strictly > Nyquist rate**

$$x(t) = \frac{\cos(2\pi B t + \theta)}{\cos(\theta)}$$

$$-\pi/2 < \theta < \pi/2$$



With $f_s = 2B$ or equivalently $T = 1/(2B)$, the samples are

$$x(nT) = \cos(\pi n) - \underbrace{\sin(\pi n)}_{0}\tan(\theta) = (-1)^n$$

Fourier Analysis

- Sampling
  - Aliasing in 2D

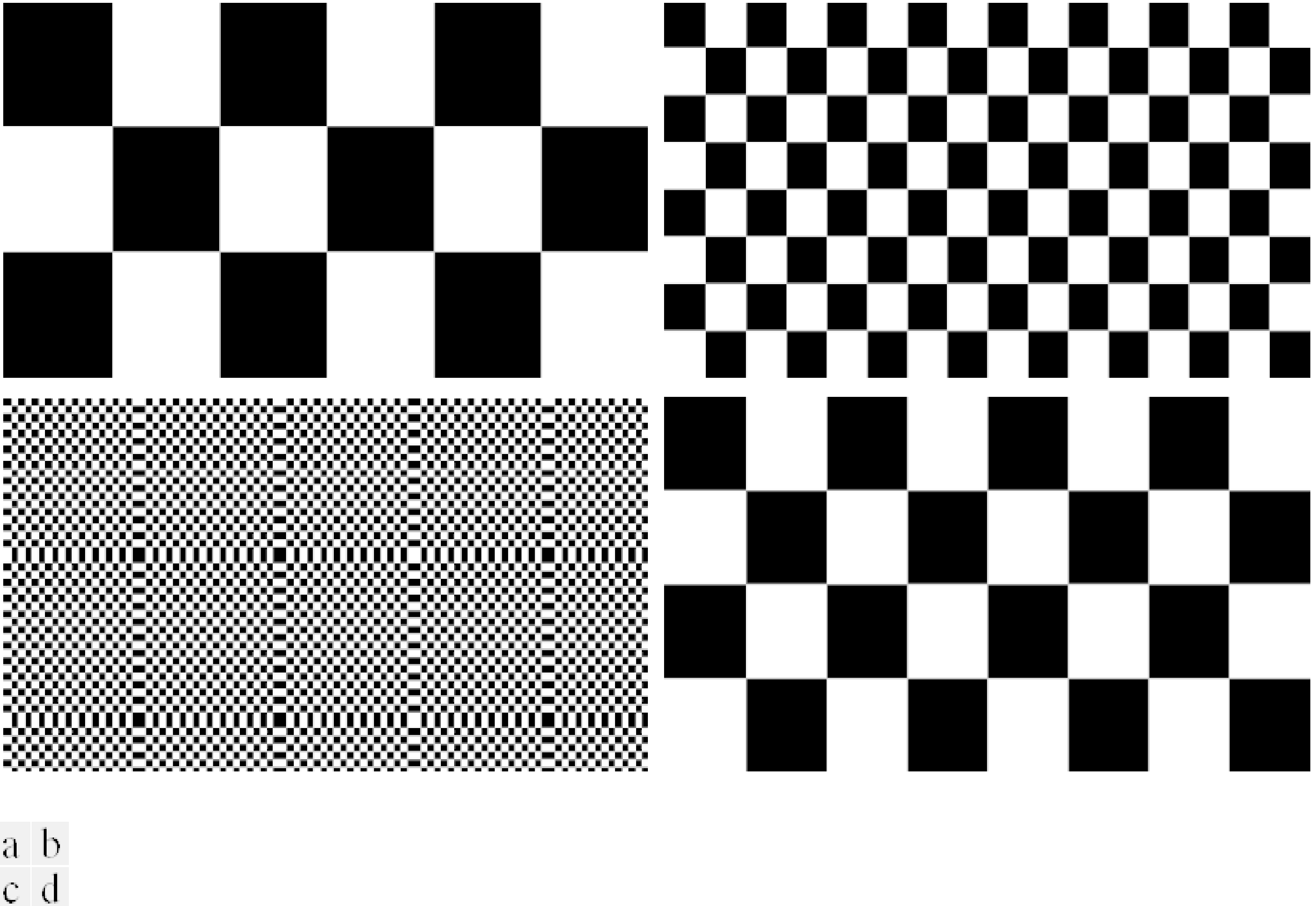


FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

Fourier Analysis

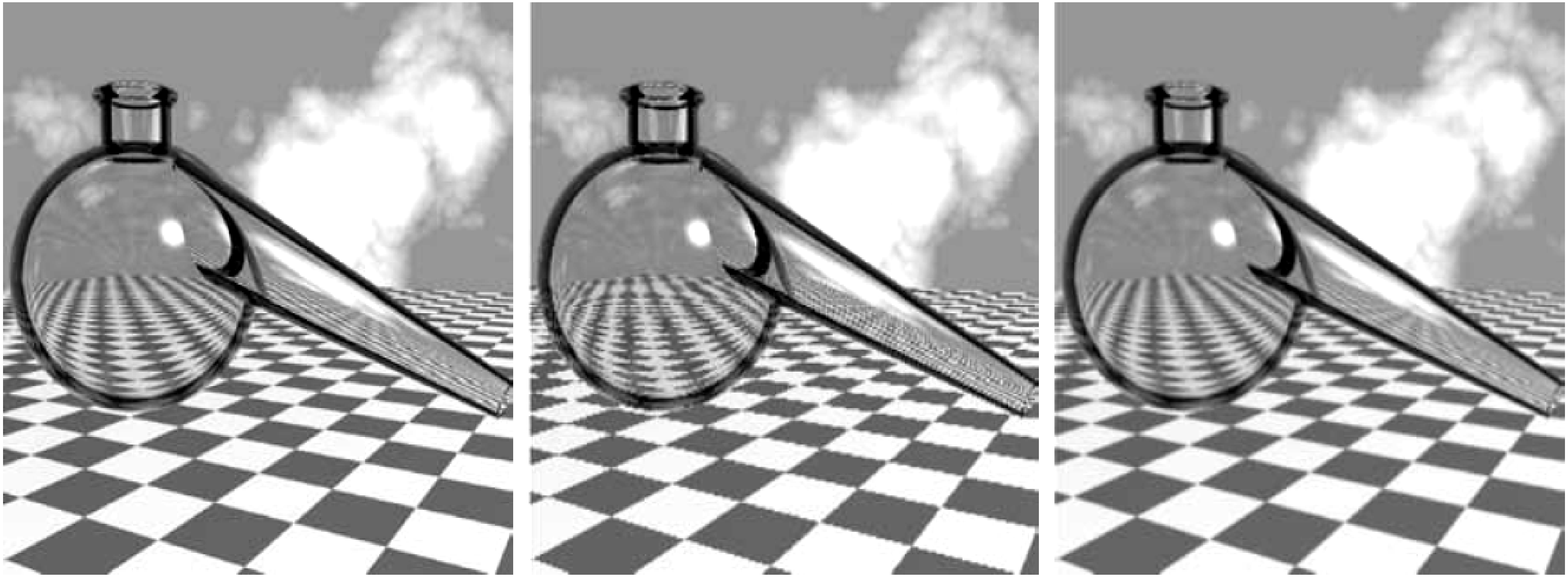
- Sampling
  - Reduce aliasing by blurring image before (sub)sampling

a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3 × 3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Fourier Analysis

- Sampling
  - Reduce aliasing by blurring image before (sub)sampling

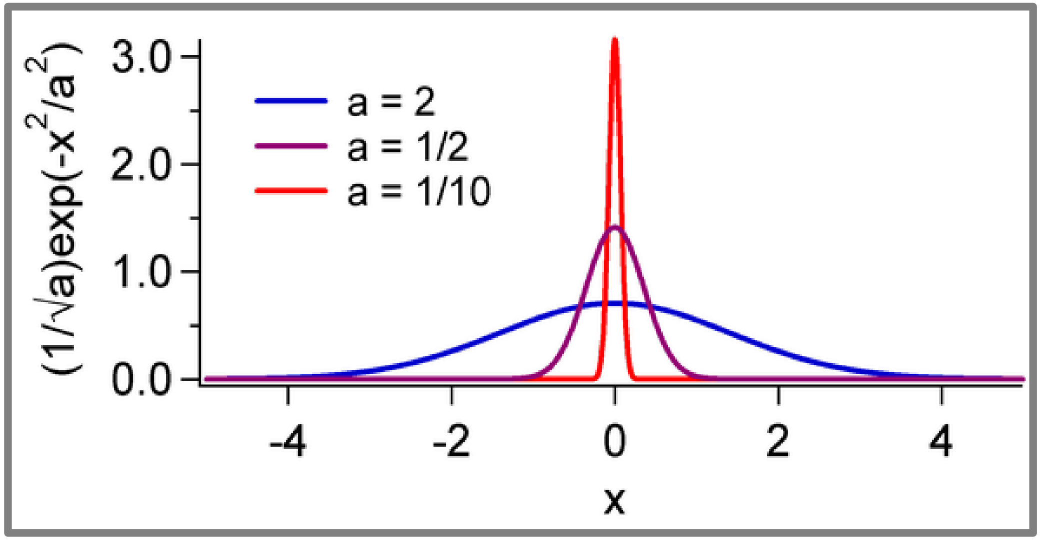


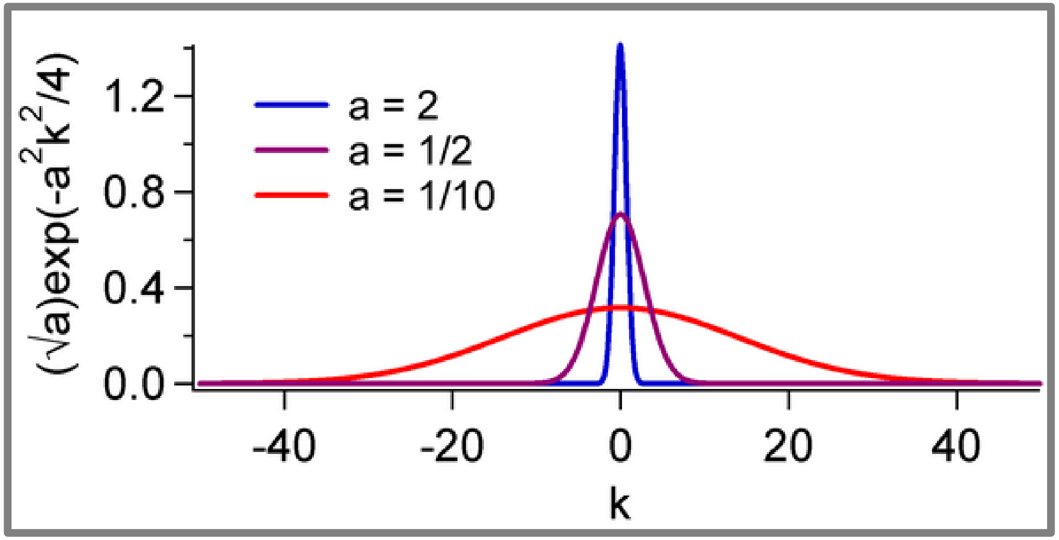
a b c

FIGURE 4.18 Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)

Fourier Analysis

- Sampling : **Why does blurring reduce aliasing ?**
  - What is the effect of convolving with a Gaussian G(x) ?
    - Fourier transform gets multiplied with FG(w)
  - What is FG(w) ?
    - Left: Gaussian in space (e.g., wide blue Gaussian)
    - Right: Gaussian in frequency (e.g., narrow blue Gaussian)
  - **Gaussian convolution attenuates high frequencies**





Fourier Analysis

- Sampling
  - **Aliasing can be avoided only for bandlimited signals**
    - **By sampling them at the Nyquist rate**

  - **Can aliasing be avoided in the real world ?**

    OR (equivalently)
  - **Are signals in real-world applications bandlimited ?**

  - To answer this question, we need to understand some properties of bandlimited signals

Fourier Analysis

- Sampling
  - **A bandlimited function is infinitely differentiable**
    - Signal f (x) = bandlimited
    - Fourier transform F f (w). Equals 0 outside [−L, L]
    - f(x) can be represented as :

$$f(x) := F^{-1}(Ff)(x) \text{ Fourier inversion theorem}$$

$$= \int_{-\infty}^{\infty} Ff(w)e^{iwx}dw \text{ Definition of inverse Fourier transform}$$

$$= \int_{-L}^{L} Ff(w)e^{iwx}dw \text{ Bandlimited assumption}$$

Fourier Analysis

- Sampling
  - **A bandlimited function is infinitely differentiable**
    - Signal f(x). Fourier transform Ff(w) = 0 outside (-L,L)
    - Following derivatives exist because (a) integration is performed on a bounded domain and (b) integrand is differentiable

Then, the derivative of $f(x)$ is

$$\frac{df(x)}{dx} = \frac{d}{dx}\int_{-L}^{L} Ff(w)e^{iwx}dw = \int_{-L}^{L} Ff(w)\frac{d}{dx}e^{iwx}dw = \int_{-L}^{L}(iw)Ff(w)e^{iwx}dw$$

Then, the k-th derivative of $f(x)$ is

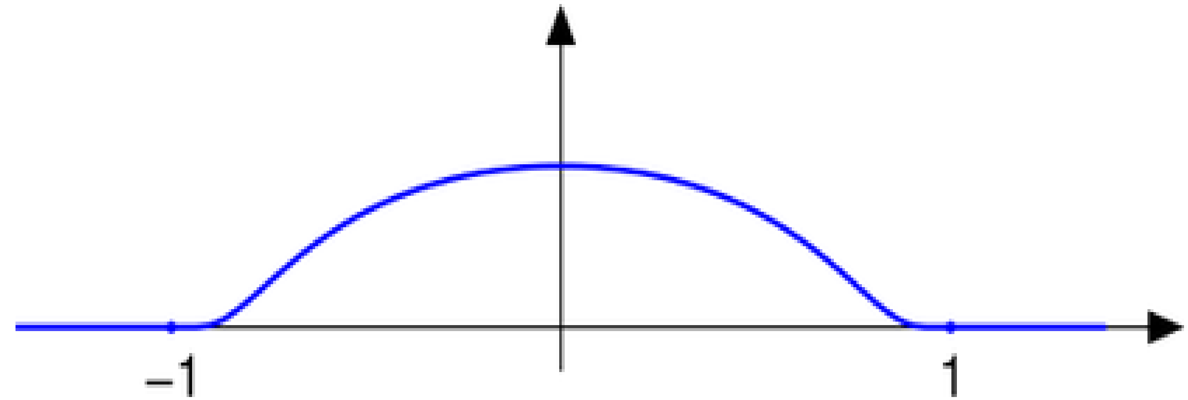
$$\frac{d^k f(x)}{d^k x} = \int_{-L}^{L}(iw)^k Ff(w)e^{iwx}dw$$

Fourier Analysis

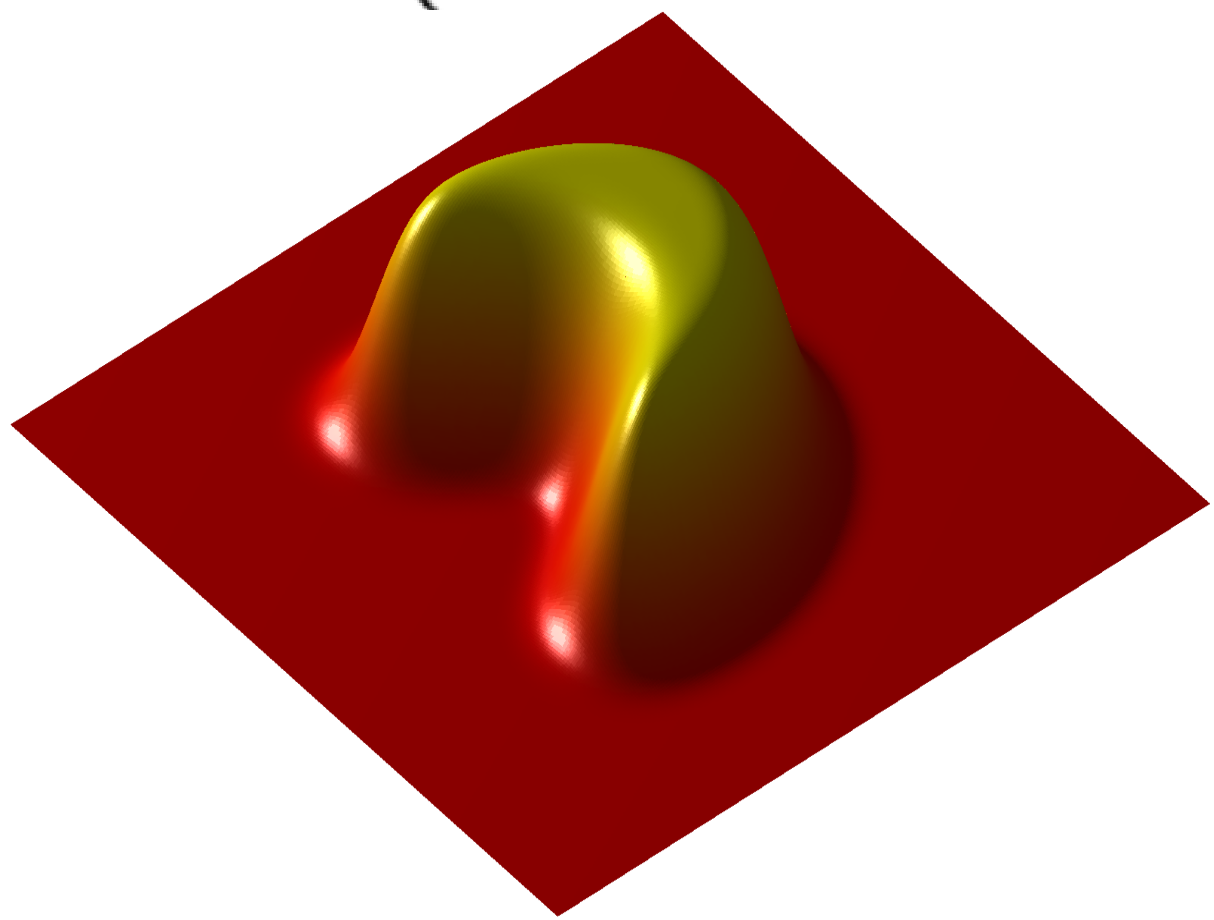
- Sampling
- Give an example of an infinitely differentiable function
  - Sinusoid
  - A sinusoid has *infinite* support
- Give an example of an infinitely differentiable function with *finite* support
  - See next ...

Fourier Analysis

- Sampling
- **Bump function**
  - Infinitely differentiable
  - Finite support
    - Time limited
- Is bump function **bandlimited ?**
  - **No**
  - **Bandlimited-ness is a stronger property than being infinitely differentiable**



$$\Psi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{for } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$



Fourier Analysis

- Bump function is NOT bandlimited
  - Proof by contradiction :
    - Assume bump function (**finite support**) f(x) was bandlimited
      - Ff(w) = 0 when “w” outside (-L,L)
    - Multiplying Ff(w) by box func B(w) over (-L,L) won't change Ff(w)
      - Ff(w) = Ff(w) B(w)
    - But, multiplying Ff(w) by B(w) $\leftrightarrow$ convolving f(x) with sinc(x)
      - Ff(w) = Ff(w) B(w) = F (f * sinc) (w)
    - So, f(x) = (f * sinc)(x)
    - sinc(x) has infinite support
    - **Right hand side = (f * sinc)(x)** $\rightarrow$ **infinite support**
    - **Left hand side** = f(x) $\rightarrow$ **finite support**
    - Contradiction !

Fourier Analysis

- Sampling
  - We just showed that :

    A **space-limited function** (e.g., bump function) **cannot be bandlimited**

  - This implies that :

    A function cannot be bandlimited and space-limited

  - We could, as well, have proved the other way :

    A **bandlimited function cannot be space-limited**

Fourier Analysis

- A bandlimited function is NOT space-limited
  - Proof by contradiction :
    - Assume bandlimited function f(x) was space-limited
      - f(x) = 0 when "x" outside [-M,M]
    - Multiplying f(x) by suitable box function b(x) won't change f(x)
      - f(x) = f(x) b(x)
    - But, multiplying f(x) by b(x) ↔ convolving Ff(w) with sinc(w)
      - f(x) = f(x) b(x) = F^{-1} (Ff * sinc) (x)
    - So, Ff(w) = (Ff * sinc)(w)
    - sinc(w) has infinite support
    - Right hand side = (Ff * sinc)(w) has infinite support
    - Left hand side = Ff(w) → finite support
    - Contradiction !

Fourier Analysis

- Sampling
  - **Space-limiting an infinite-duration bandlimited signal destroys its bandlimited-ness**
    - Assume a (non-zero) function f(x) that is :

      (1) **bandlimited**

      (2) of **infinite support**
    - Then, f(x) has a Fourier transform Ff(w) = 0 outside (-L,L)
    - **Cut-out / space-limit** f(x) by multiplying it with box function b(x)
    - Space-limited function g(x) = f(x) b(x) has Fourier transform Fg(w) = (Ff * Fb)(w)
    - Fb(w) has infinite support in frequency domain
    - Thus, Fg(w) has infinite support in frequency domain
    - Thus, f(x)b(x) is NOT bandlimited

Fourier Analysis

- Sampling
  - Aliasing can be avoided only for bandlimited signals
    - By sampling them at the Nyquist rate

  - **Can aliasing be avoided in the real world ? No !**

    (1) Real-world signals are **NOT of infinite extent**

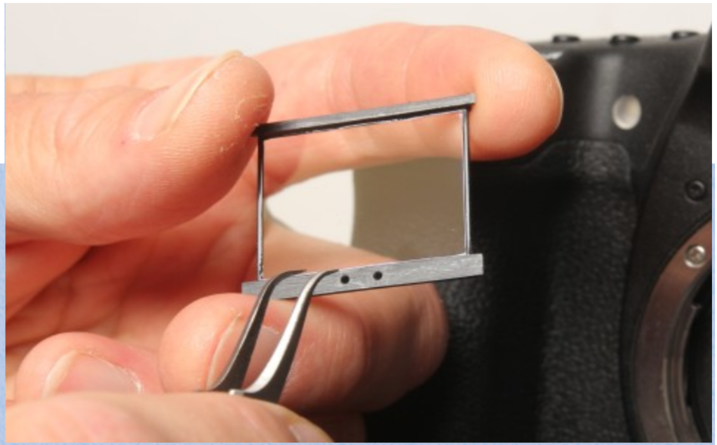
    - Acquiring an infinitely-long signal will take infinite time
    - We can artificially impose infinite extent by **extending samples periodically beyond boundaries (standard trick)**
      - Will this work ? Not for arbitrary signals. Only in some special cases.

    (2) Real-world signals may **NOT** be **infinitely differentiable**

    - Non-differentiability may hold at
      - Points within the part of function observed OR
      - Points on the boundary, due to forced periodicity

Fourier Analysis

- Sampling
  - Some digital cameras have lens systems that blur incoming light signal to reduce aliasing (moire) in photo



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6 STAGES OF TAKING AN EXAM UNPREPARED