Problem 3.10

Using Eqs. (3.22) and (3.23), show that for positive frequencies the spectra of the two kinds of SSB modulated waves are defined as follows:

(a) For the upper SSB,

$$S(f) = \begin{cases} \frac{A_c}{2}M(f-f_c) & \text{for } f \geq f_c \\ 0 & \text{for } 0 < f \leq f_c \end{cases}$$

(b) For the lower SSB,

$$S(f) = \begin{cases} 0 & \text{for } \quad f > f_c \\ \frac{A_c}{2}M(f-f_c) & \text{for } 0 < f \leq f_c \end{cases}$$

(c) Write down the formulas for these two kinds of SSB modulation that pertain to negative frequencies.

Solution

According to Eq. (3.24):

$$s(t) = \frac{A_c}{2}m(t)\cos(2\pi f_c t) \mp \frac{A_c}{2}\hat{m}(t)\sin(2\pi f_c t)$$

where $\hat{m}(t)$ is the Hilbert transform of $m(t)$. Taking the Fourier transform of $s(t)$:

$$S(f) = \frac{A_c}{4}(M(f-f_c)+M(f+f_c)) \pm \frac{A_c}{4j}(\hat{M}(f-f_c)-\hat{M}(f+f_c))$$

From Eq. (3.22):

$$\hat{M}(f) = -jM(f)\operatorname{sgn}(f)$$

Hence,

$$\begin{aligned} S(f) &= \frac{A_c}{4}(M(f\text{-}f_c)+M(f\text{+}f_c)) \pm (\frac{A_c}{4j}(-jM(f\text{-}f_c)\operatorname{sgn}(f\text{-}f_c)+jM(f\text{+}f_c)\operatorname{sgn}(f\text{+}f_c)) \\ &= \frac{A_c}{4}(M(f-f_c) \mp M(f-f_c)\operatorname{sgn}(f-f_c)) + \frac{A_c}{4}(M(f+f_c) \pm M(f+f_c)\operatorname{sgn}(f+f_c)) \\ &= \frac{A_c}{4}(1 \mp \operatorname{sgn}(f-f_c))M(f-c_c) + \frac{A_c}{4}(1 \pm \operatorname{sgn}(f+f_c))M(f+f_c) \qquad (1) \end{aligned}$$

By definition:

$$\operatorname{sgn}(f-f_c) = \begin{cases} 1 & \text{for } f > f_c \\ -1 & \text{for } f < f_c \end{cases}$$

and

$$\operatorname{sgn}(f+f_c) = \begin{cases} 1 & \text{for } f > -f_c \\ -1 & \text{for } f < -f_c \end{cases}$$

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Problem 3.10 continued

(a) From Eq. (3.24), recall that the minus sign in this formula corresponds to upper SSB. Hence, for the upper SSB, we have

$$S(f) = \frac{A_c}{4}(1 + \mathrm{sgn}(f - f_c))M(f - f_c) + \frac{A_c}{4}(1 - \mathrm{sgn}(f + f_c))M(f + f_c) \qquad (2)$$

where the term containing $M(f - f_c)$ pertains to positive frequencies and the term containing $M(f + f_c)$ pertains to negative frequencies. Therefore for positive frequencies and $f \geq f_c$, Eq. (2) simplifies to

$$S(f) = \frac{A_c}{4}(1 + 1)M(f - f_c)$$

$$= \frac{A_c}{2}M(f - f_c) \qquad (3)$$

For $0 \leq f \leq f_c$, $S(f) = 0$.

(b) From Eq. (3.24), also recall that the plus sign in this formula corresponds to the lower SSB, for which we find that for $f \leq f_c$:

$$S(f) = \frac{A_c}{4}(1 - \mathrm{sgn}(f - f_c))M(f - f_c) + \frac{A_c}{4}(1 + \mathrm{sgn}(f + f_c))M(f + f_c)$$

Therefore for positive frequencies and $f \leq f_c$, we have

$$S(f) = \frac{A_c}{4}(1 + 1)M(f - f_c)$$

$$= \frac{A_c}{2}M(f - f_c) \qquad (4)$$

On the other hand, for $f > f_c$ we have $S(f) = 0$.

(c) For negative frequencies, we focus on terms containing $M(f + f_c)$, in light of which we get the following results:

(i) For upper SSB:

$$S(f) = \begin{cases} \frac{A_c}{4}M(f + f_c) & \text{for } f \leq -f_c \\ 0 & \text{for } -f_c < f < 0 \end{cases} \qquad (5)$$

(ii) For lower SSB:

$$S(f) = \begin{cases} 0 & \text{for } f < -f_c \\ \frac{A_c}{4}M(f + f_c) & \text{for } -f_c < f < 0 \end{cases} \qquad (6)$$