Problem 5.8

Starting with Eq. (5.9), show that the Fourier transform of the rectangular pulse $h(t)$ is given by

$H(f) = T\,\mathrm{sinc}(fT)\exp(-j\pi fT)$

What happens to $H(f)/T$ as the pulse duration T approaches zero?

Solution

Given

$$h(t) = \begin{cases} 1, & 0 < t < T \\ \frac{1}{2}, & t = 0, t = T \\ 0, & \text{otherwise} \end{cases}$$

the required Fourier transform is

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt \\ &= \int_0^T 1 \cdot \exp(-j2\pi ft)dt \\ &= \left[\frac{\exp(-j2\pi ft)}{-j2\pi f}\right]_{t=0}^{T} \\ &= \frac{1}{j2\pi f} - \frac{\exp(-j2\pi fT)}{j2\pi f} \\ &= \frac{\exp(-j2\pi fT)}{j2\pi f}[\exp(j\pi fT) - \exp(-j\pi fT)] \end{aligned}$$

Since

$$\sin(\pi fT) = \frac{1}{2j}[\exp(j\pi fT) - \exp(-j\pi fT)]$$

it follows that

$$\begin{aligned} H(f) &= \frac{\sin(\pi fT)}{\pi f}\exp(-j\pi fT) \\ &= T \cdot \frac{\sin(\pi fT)}{\pi fT}\exp(-j\pi fT) \\ &= T\,\mathrm{sinc}(fT)\exp(-j\pi fT) \end{aligned}$$