End-Semester Examination Solutions

Probability and Random Processes (EE 325), Autumn'17

QUESTION 1

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$. So $f_Y(y) = 0$ for $y \notin (0,2)$ and for $y \in (0,2)$:

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{2} y e^{-xy} dx = \frac{1}{2}.$$

Thus,

$$f_Y(y) = \begin{cases} \frac{1}{2}, & 0 < y < 2, \\ 0, & \text{else.} \end{cases}$$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \begin{cases} y e^{-xy}, & 0 < x < \infty,\ 0 < y < 2, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

$$E\left(e^{X/2} | Y = 1\right) = \int_{x=0}^{\infty} e^{x/2} f_{X|Y}(x|1) dx = \int_{x=0}^{\infty} e^{x/2} e^{-x} dx = 2.$$

QUESTION 2

The Jacobian determinant for the transformation is given by:

$$J(u,z) = \begin{vmatrix} -\sqrt{2z}\sin(u) & \frac{1}{\sqrt{2z}}\cos(u) \\ \sqrt{2z}\cos(u) & \frac{1}{\sqrt{2z}}\sin(u) \end{vmatrix} = -1.$$

So $|J(u,z)| = 1$.

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{f_{U,Z}(u,z)}{|J(u,z)|} \\ &= f_U(u) f_Z(z) \\ &= \left(\frac{1}{2\pi}\right)\left(e^{-z}\right) \\ &= \frac{1}{2\pi}\exp\left(-\frac{(x^2+y^2)}{2}\right). \end{aligned}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right),$$

and similarly $f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-y^2}{2}\right)$. So $f_{X,Y}(x,y) = f_X(x) f_Y(y)$.

QUESTION 3

$$
\begin{aligned}
&\text{Cov}(a_1X_1 + a_2X_2, a_3X_3 + a_4X_4) = \\
&\qquad E\left((a_1X_1 + a_2X_2)(a_3X_3 + a_4X_4)\right) - E(a_1X_1 + a_2X_2)E(a_3X_3 + a_4X_4) \\
&= a_1a_3\left[E(X_1X_3) - E(X_1)E(X_3)\right] + a_1a_4\left[E(X_1X_4) - E(X_1)E(X_4)\right] \\
&\qquad + a_2a_3\left[E(X_2X_3) - E(X_2)E(X_3)\right] + a_2a_4\left[E(X_2X_4) - E(X_2)E(X_4)\right] \\
&= a_1a_3\text{Cov}(X_1, X_3) + a_1a_4\text{Cov}(X_1, X_4) \\
&\qquad + a_2a_3\text{Cov}(X_2, X_3) + a_2a_4\text{Cov}(X_2, X_4)
\end{aligned}
$$

QUESTION 4

The mean vector and covariance matrix of Y are:

$$\mu_{\mathbf{Y}} = A\mu_{\mathbf{X}} = \begin{pmatrix} -4 \\ 2\sqrt{2} \end{pmatrix}$$

and

$$K_{\mathbf{Y}} = AK_{\mathbf{X}}A^T = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

respectively. Also, since $\mathbf{Y} = A\mathbf{X}$ and $\mathbf{X}$ is a Gaussian random vector, $\mathbf{Y}$ is a Gaussian random vector.

(a) Since $K_{\mathbf{Y}}$ is a diagonal matrix, Y_1 and Y_2 are uncorrelated. This and the fact that Y_1 and Y_2 are jointly Gaussian imply that Y_1 and Y_2 are independent.

(b)

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2) = \frac{1}{16\pi}\exp\left(-\frac{(y_1+4)^2}{16}\right)\exp\left(-\frac{(y_2-2\sqrt{2})^2}{16}\right).$$

QUESTION 5

(a) Fix $\epsilon > 0$. We have:

$$
\begin{aligned}
P(|X_n - X| \geq \epsilon) &= P(|X_n - X|^3 \geq \epsilon^3) \\
&\leq \frac{E(|X_n - X|^3)}{\epsilon^3}
\end{aligned}
$$

by Markov's inequality. So:

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) \leq \frac{1}{\epsilon^3} \lim_{n\to\infty} E(|X_n - X|^3) = 0.$$

Thus, $X_1, X_2, X_3, \ldots$ converges to X in probability.

(b) Consider the moving, shrinking rectangles example discussed in class (see slides 3 and 4 of the lecture slides of 16/10/17). It was shown in class that the sequence $X_1, X_2, X_3, \ldots$ in that example does not converge almost surely to any random variable. But:

$$E(|X_n - X|^3) = (1)^3 \alpha_n = \alpha_n,$$

where α_n is the width of the rectangle on which $X_n = 1$. So $\lim_{n\to\infty} E(|X_n - X|^3) = 0$, which shows that the sequence converges to 0 is third-mean sense.

(c) Consider the example on slide 10 of the lecture slides of 16/10/17 with $a_n = n$. It was shown in class that the sequence converges to 0 almost surely for all a_n. But:

$$E(|X_n - X|^3) = (n^3)\left(\frac{1}{n}\right) = n^2.$$

So $\lim_{n\to\infty} E(|X_n - X|^3)$ does not equal 0 and hence the sequence does not converge to 0 in third-mean sense.

QUESTION 6

(a) Since X is uniformly distributed in $[0,1]$:

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{else.} \end{cases}$$

Now:

$$\begin{aligned} E(Y) &= E(e^{-X} + 2X) \\ &= \int_{-\infty}^{\infty} (e^{-x} + 2x) f_X(x) dx \\ &= \int_0^1 (e^{-x} + 2x) dx \\ &= 2 - e^{-1}. \end{aligned}$$

Similarly,

$$\begin{aligned} E(Y^2) &= E\left[(e^{-X}+2X)^2\right] \\ &= E\left[e^{-2X}+4X^2+4Xe^{-X}\right] \\ &= \int_0^1 \left[e^{-2x}+4x^2+4xe^{-x}\right]dx \\ &= \frac{35}{6}-8e^{-1}-\frac{e^{-2}}{2}. \end{aligned}$$

So:

$$\begin{aligned} var(Y) &= E(Y^2)-[E(Y)]^2 \\ &= \frac{35}{6}-8e^{-1}-\frac{e^{-2}}{2}-\left(2-e^{-1}\right)^2 \\ &= \frac{11}{6}-4e^{-1}-\frac{3e^{-2}}{2}. \end{aligned}$$

(b) Let Y be the distance from the shot to the target. Note that Y is uniformly distributed in $[0, 10]$. X is a discrete random variable that takes the values 10, 5, 3 and 0. The p.m.f. of X is as follows. $P(X = 10) = P(Y \leq 1) = \frac{1}{10}$, $P(X = 5) = P(1 < Y \leq 3) = \frac{2}{10}$, $P(X = 3) = P(3 < Y \leq 5) = \frac{2}{10}$ and $P(X = 0) = P(Y > 5) = \frac{5}{10}$. So:

$$\begin{aligned} E(X) &= 0\times P(X=0)+3\times P(X=3)+5\times P(X=5)+10\times P(X=10) \\ &= \frac{13}{5}. \end{aligned}$$

Also:

$$\begin{aligned} E(X^2) &= 0^2\times P(X=0)+3^2\times P(X=3)+5^2\times P(X=5)+10^2\times P(X=10) \\ &= \frac{168}{10}. \end{aligned}$$

Hence:

$$\begin{aligned} var(X) &= E(X^2)-[E(X)]^2 \\ &= \frac{1004}{100}. \end{aligned}$$

Question 7

This is problem 1.14 on pp. 81-82, Haykin, 4th edition, which is part of Homework 10.

QUESTION 8

This is Question 9 from Homework 8.

QUESTION 9

Yes, $X(t)$ is ergodic in the mean. To prove this, note that we showed in class that $X(t)$ is WSS and has autocorrelation function $R_X(\tau) = \begin{cases} A^2\left(1-\frac{|\tau|}{T}\right), & |\tau| < T, \\ 0, & |\tau| \geq T. \end{cases}$ Also, its mean is $\eta_X = 0$. So its autocovariance function is $K_X(\tau) = R_X(\tau) - \eta_X^2 = \begin{cases} A^2\left(1-\frac{|\tau|}{T}\right), & |\tau| < T, \\ 0, & |\tau| \geq T. \end{cases}$

We now use the necessary and sufficient condition proved on slide 12 of the lecture slides of Nov. 6 to prove that $X(t)$ is ergodic in the mean. Consider:

$$\frac{1}{2T'}\int_{-2T'}^{2T'}\left(1-\frac{|\tau|}{2T'}\right)K_X(\tau)d\tau = \frac{A^2}{2T'}\int_{-T}^{T}\left(1-\frac{|\tau|}{2T'}\right)\left(1-\frac{|\tau|}{T}\right)d\tau.$$

Hence,

$$\left|\frac{1}{2T'}\int_{-2T'}^{2T'}\left(1-\frac{|\tau|}{2T'}\right)K_X(\tau)d\tau\right| \leq \frac{A^2}{2T'}\int_{-T}^{T}(1)(1)d\tau = \frac{A^2T}{T'}$$

which converges to 0 as $T' \to \infty$. The above inequality follows from the fact that $0 \leq 1-\frac{|\tau|}{2T'} \leq 1$ and $0 \leq 1-\frac{|\tau|}{T} \leq 1$ for $-T \leq \tau \leq T$. It follows that $X(t)$ is ergodic in the mean.

QUESTION 10

This is problem 7.11.10 on p. 92, Grimmett and Stirzaker.

QUESTION 11

This is problem 7.11.7 on p. 92, Grimmett and Stirzaker. Note that the "first Borel Cantelli lemma" referred to in the solution in the book is the lemma that we discussed in class using slide 3 of the lecture slides of Oct. 24.