Problem 3.8

As just mentioned, the phase discriminator in the Costas receiver of Fig. 3.16 consists of a multiplier followed by a time-averaging unit. Referring to this figure, do the following:

(a) Assuming that the phase error ϕ is small compared to one radian, show that the output $g(t)$ of the multiplier component is approximately $\frac{1}{4}\phi m^2(t)$.

(b) Furthermore, passing $g(t)$ through the time-averaging unit defined by

$$\frac{1}{2T}\int_{-T}^{T} g(t)dt$$

where the averaging interval $2T$ is long enough compared to the reciprocal of the bandwidth of $g(t)$, show that the output of the phase discriminator is proportional to the phase-error ϕ multiplied by the dc (direct current) component of $m^2(t)$. The amplitude of this signal (acting as the control signal applied to the voltage-controlled oscillator in Fig. 3.16) will therefore always have the same algebraic sign as that of the phase error ϕ, which is how it should be.

Solution

(i) Referring to the Costas receiver in Fig. 3.16 in the text, we see that the output of the in-phase channel is $\frac{1}{2}A_c\cos\phi m(t)$ and the output of the quadrature channel is $\frac{1}{2}A_c\sin\phi m(t)$. The output of the multiplier in the phase discriminator is therefore

$$\begin{aligned} g(t) &= \left(\frac{1}{2}A_c\cos\phi m(t)\right)\left(\frac{1}{2}A_c\sin\phi m(t)\right) \\ &= \frac{1}{4}\sin\phi\cos\phi m^2(t) \qquad (1) \end{aligned}$$

If the phase error ϕ is small compared to one radian, we may use the approximations:

$\sin\phi \approx \phi$

$\cos\phi \approx 1$

in which case the multiplier output $g(t)$ simplifies approximately to $\frac{1}{4}\phi m^2(t)$.

(ii) Passing $g(t)$ through the time-averaging unit yields the phase discriminator output

$$\begin{aligned} v(t) &= \frac{1}{2T}\int_{-T}^{T} g(t)dt \\ &\approx \frac{1}{2T}\int_{-T}^{T} \frac{1}{4}\phi m^2(t)dt \\ &= \frac{\phi}{8T}\int_{-T}^{T} m^2(t)dt \\ &= \frac{1}{4}\phi P_0 \end{aligned}$$

where

$$P_0 = \frac{1}{2T}\int_{-T}^{T} m^2(t)dt$$

is the dc component of $m^2(t)$ or, equivalently, the average power of $m(t)$.