↳ Let ~~$\det K_{\bar{z}}$~~ $\det(K_{\bar{z}}) = 0$. Then there is $\bar{b} \in \mathbb{R}^n$ s.t. $K_{\bar{z}}\bar{b} = 0$. So, $\bar{b}^T K_{\bar{z}} \bar{b} = 0$ which contradicts +ve definiteness.

↳ Any +ve definite matrix K has a full set of ~~eigenvalues~~ eigenvectors with +ve eigenvalues
ie. $K_{\bar{z}}$ $(n \times n)$ has eigenvectors $\bar{q}_1, \dots \bar{q}_n$ with $\lambda_1, \dots \lambda_n$ eigenvalues $(\lambda_i > 0)$

↳ If eigenvectors $\bar{q}_1, \dots \bar{q}_n$ are orthogonal if $\lambda_1, \dots \lambda_n$ are distinct.

$$\lambda_i \langle \bar{q}_i, \bar{q}_j \rangle = (\bar{q}_i^T \underset{\overset{\shortparallel}{K^T}}{K} \bar{q}_j) = \lambda_j \langle \bar{q}_i, \bar{q}_j \rangle$$

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↳ $\bar{Z} \Rightarrow K_{\bar{z}} = RR^T$, where $\bar{Z} = R\bar{W}$

Theorem (Covariance matrix whitening)

Let $K_{\bar{z}}$ be any cov. matrix. Then it is positive semi-definite. Further, $K_{\bar{z}} = AA^T$ & $\bar{Z}$ can be linearly transformed into a white vector $\bar{W}$.

Corollary (Whitening of Gaussian)

Let $\bar{Z}$ be a Gaussian with cov. matrix $K_{\bar{z}}$. Then $K_{\bar{z}} = Q\Lambda Q^T$, eigenvalue of $K_{\bar{z}}$
Further, $\bar{Z} = A\bar{W}$, where $K_{\bar{z}} = AA^T$ & $\bar{W}$ is white. One choice for A is $Q\Lambda^{+1/2}Q^T$

Scanned by CamScanner

↳ $(Q\Lambda^{-1/2}Q^T)\bar{Z}$ will be white

$$\bar{Z} \longrightarrow \boxed{Q\Lambda^{-1/2}Q^T} \longrightarrow \bar{W}$$

$\bar{Z}$ ↳ coloured gaussian

$\bar{W}$ ↳ white gaussian

Uncorrelated jointly Gaussian r.v. ⇒ independent

Cannot whiten +ve semi-definite. Only +ve definite can be.

$$\bar{Z} = \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\bar{Z} = \begin{pmatrix} z_1 \\ 0 \\ z_2 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

Geometry of Gaussian density

↳ $K_{\bar{Z}}$ is invertible, i.e. $K_{\bar{Z}}$ is +ve definite

↳ $\bar{V} = Q^T\bar{Z}$ is a coordinate transformation

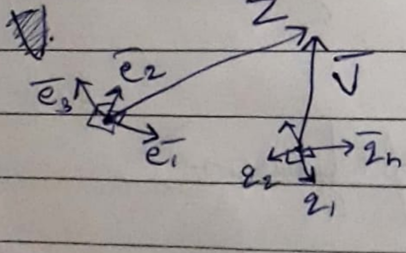
↳ $\bar{Q} = [\bar{q}_1 | \dots | \bar{q}_n]$

orthogonal

$V_1 = \bar{q}_1^T \bar{Z}$

$V_2 = \bar{q}_2^T \bar{Z}$

⋮



$V_n = \bar{q}_n^T \bar{Z}$

coordinate of $\bar{Z}$ in the basis $\{\bar{q}_1, \dots \bar{q}_n\}$

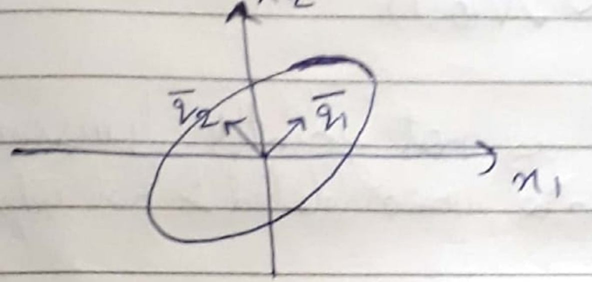
$$f_{\bar{V}}(\bar{v}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(K_{\bar{V}})}} \cdot \exp\left(-\frac{\bar{v}^T K_{\bar{V}}^{-1} \bar{v}}{2}\right)$$

$\Rightarrow K_{\bar{V}} = \bar{Q}^T K_{\bar{Z}} Q = Q^T Q \Lambda Q^T Q = \Lambda$

$\therefore f_{\bar{V}}(\bar{v}) = \dfrac{1}{(2\pi)^{n/2}\sqrt{\lambda_1 \dots \lambda_n}} \cdot \exp\left(-\sum_{i=1}^{n} \dfrac{v_i^2}{2\lambda_i}\right)$

ie. $V_1, \dots V_n$ are independant r.vs.

consider $n=2$. The equiprobable contours in $f_{\bar{Z}}(\bar{x})$ are given by $\bar{x}^T K_{\bar{Z}}^{-1} \bar{x} = \text{const}^n$.



$x^T K_{\bar{Z}}^{-1} \bar{x} = \text{const}^n$

$\bar{v}^T \Lambda^{-1} \bar{v} = \text{const}^n$

$\bar{V} = Q^T \bar{Z} \Rightarrow \bar{Z} = Q\bar{V}$

$\bar{Z} \sim N(\bar{0}, K_{\bar{Z}})$, $K_{\bar{Z}} = Q\Lambda Q^T$

$\det(K_{\bar{Z}}) = \det(\Lambda) \neq 0$

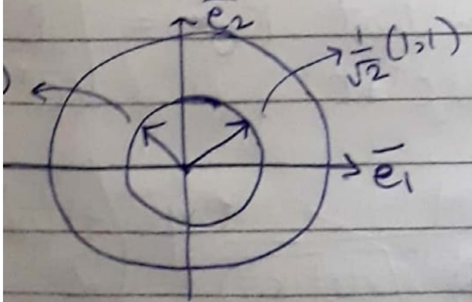
Let $\bar{V} = Q^T \bar{Z}$

$f_{\bar{V}}(\bar{v}) = \dfrac{1}{(2\pi)^{n/2}\sqrt{\det \Lambda}} \exp\left(-\sum_{i=1}^{n} \dfrac{v_i^2}{2\lambda_i}\right) = f_{V_i}(v_i)$

ie. $V_1, \dots V_n$ are independant.

$\rightarrow Q^T \Lambda^{-1/2} \bar{Z}$ will be white.

(compare with $Q \Lambda^{-1/2} Q^T \bar{Z}$)



$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $Q = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$Q\bar{w} \sim N(0, I_n)$

$\hookrightarrow QQ^T = I_n$

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↳ $\bar{Z} \sim \mathcal{N}(\bar{m}, K_{\bar{Z}})$

$$\bar{Z} = \bar{m} + (Q\Lambda^{-1/2}Q^T)\bar{W}$$

↳ Let $\bar{Y}$ is jointly Gaussian. Let $Y_1, \ldots Y_n$ be uncorrelated. Then $Y_1, \ldots Y_n$ are indep.

↳ $X \sim \mathcal{N}(0,1)$, $B \sim$ equiprobable on $\{-1,+1\}$

$\boxed{Y = BX} \sim \mathcal{N}(0,1)$

→ But (X, Y) is not jointly Gaussian $\boxed{(X+Y) \text{ is not gaussian}}$

Though (X, Y) are uncorrelated & not indep.

Conditional distribution

Let $\bar{U}$ be $(m+n)$-dim zero mean jointly Gaussian r.v.
Let $\bar{U} = (\bar{X}^T, \bar{Y}^T)^T$ where $\bar{X}$ is m-dim & $\bar{Y}$ is n-dim.
Let $K_{\bar{U}}$ be non-singular.

Then,

$$K_{\bar{U}} = \left(\begin{array}{c|c} K_{\bar{X}} & K_{\bar{X}\bar{Y}} \\ \hline K_{\bar{Y}\bar{X}} & K_{\bar{Y}} \end{array}\right), \text{ where } K_{\bar{X}\bar{Y}} = E(\bar{X}\bar{Y}^T) = K_{\bar{Y}\bar{X}}^T$$

cross covariance

↳ It can be shown that $K_{\bar{U}}^{-1} = \left(\begin{array}{c|c} B & C \\ \hline C^T & D \end{array}\right)$

$\boxed{B^T = B,\ D^T = D,\ C^T \neq C}$

Then,

$$f_{\bar{X},\bar{Y}}(\bar{x}, \bar{y}) = \underbrace{\frac{1}{(\sqrt{2\pi})^{m+n}\sqrt{\det(K_{\bar{U}})}}} \exp\left(-\frac{1}{2}(\bar{x}^T \bar{y}^T) K_{\bar{U}}^{-1} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}\right)$$

$$= h_1 \exp\left(-\frac{1}{2}(\bar{x}^T B \bar{x} + \bar{x}^T C \bar{y} + \bar{y}^T C^T \bar{x} + \bar{y}^T D \bar{y}\right)$$

and $f_{\bar{Y}}(\bar{y}) = \dfrac{1}{(\sqrt{2\pi})^{n} \sqrt{\det K_{\bar{Y}}}} \exp\left(\dfrac{-\bar{y}^T K_{\bar{Y}}^{-1} \bar{y}}{2}\right)$

Then, pdf of $\bar{X}|(\bar{Y}=\bar{y})$ is,

$$f_{\bar{X}|\bar{Y}}(\bar{x}/\bar{y}) = \frac{f_{\bar{X},\bar{Y}}(\bar{x},\bar{y})}{f_{\bar{Y}}(\bar{y})}$$

$$= h_2(\bar{y}) \exp\left(\frac{-1}{2}(\bar{x}^T B \bar{x} + \bar{x}^T C \bar{y} + \bar{y}^T C^T \bar{x})\right)$$

Since $\int_{\mathbb{R}^m} f_{\bar{X}|\bar{Y}}(\bar{x}/\bar{y}) d\bar{x} = 1$, $h_2(\bar{y})$ can be found at the end.

$$f_{\bar{X}|\bar{Y}}(\bar{x}|\bar{y}) = h_3(\bar{y}).\exp\left(\frac{-1}{2}(\bar{x} + B^{-1}C\bar{y})^T B(\bar{x} + B^{-1}C\bar{y})\right)$$

$$h_3(\bar{y}) = h_2(\bar{y}).\exp\left(+\frac{1}{2}\bar{y}^T C^T B^{-1} C\bar{y}\right)$$

indep^n of $\bar{y}$ (depends on distn of $\bar{Y}$)

$$\left(\begin{array}{c|c} B & C \\ \hline C^T & \Delta \end{array}\right) = K_{\vec{U}}^{-1} = \left(\begin{array}{c|c} K_{\bar{X}} & K_{\bar{X}\bar{Y}} \\ K_{\bar{X}\bar{Y}}^T & K_{\bar{Y}} \end{array}\right)$$

Since $K_{\bar{U}}$ is symm., $B^T = B$ & $\Delta^T = \Delta$

$\therefore$ Observe that $\bar{X}|\bar{Y}=\bar{y} \sim N(-B^{-1}C\bar{y}, B^{-1})$

& therefore $h_3(\bar{y}) = \dfrac{1}{(2\pi)^{m/2}\sqrt{\det B^{-1}}}$

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$\bar{U} = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix}$, $K_{\bar{U}} \rightarrow$ non-singular

$\hookrightarrow$ jointly gauss.

$$K_{\bar{U}} = \left(\begin{array}{c|c} K_{\bar{X}} & K_{\bar{X}\bar{Y}} \\ \hline K_{\bar{X}\bar{Y}}^T & K_{\bar{Y}} \end{array}\right), \quad K_{\bar{U}}^{-1} = \left(\begin{array}{c|c} B & C \\ \hline C^T & \Delta \end{array}\right)$$

B & Δ are +ve definite & symmetric

What is B & C = ?

$$\begin{pmatrix} B & C \\ C^T & D \end{pmatrix}\begin{pmatrix} K_{\bar{X}} & K_{\bar{X}\bar{Y}} \\ K_{\bar{X}\bar{Y}}^T & K_{\bar{Y}} \end{pmatrix} = I$$

$$\therefore BK_{\bar{X}} + CK_{\bar{X}\bar{Y}}^T = I_m$$

$\hookrightarrow \bar{X} = G\bar{Y} + \bar{V}$ (Since $\bar{X}|\bar{Y}=y$ has a mean which is linear in $\bar{y}$ & cov. indep of $\bar{y}$)

$\bar{Y}$ & $\bar{V}$ are indep^n.

co. variance of LHS must be same as that of RHS (like power conservn.)

$$\therefore E(\bar{V}) = E(\bar{X}) - GE(\bar{Y}) = 0$$

$$K_{\bar{X}} = E(XX^T) = GK_{\bar{Y}}G^T + K_{\bar{V}} \quad \{\text{Since } K_{\bar{V}\bar{Y}} = 0\}$$

$$K_{\bar{X}\bar{Y}} = E(\bar{X}\bar{Y}^T) = GK_{\bar{Y}} + 0 = GK_{\bar{Y}}$$

$$\therefore \boxed{G = K_{\bar{X}\bar{Y}} K_{\bar{Y}}^{-1}}$$

$$\therefore K_{\bar{V}} = K_{\bar{X}} - K_{\bar{X}\bar{Y}} . K_{\bar{Y}}^{-1} . K_{\bar{Y}\bar{X}}$$

$\therefore \bar{X} = (K_{\bar{X}\bar{Y}} K_{\bar{Y}}^{-1})\bar{Y} + \bar{V}$; $\bar{V}$ is indep^n of $\bar{Y}$.

Similarly, $\bar{Y} = (K_{\bar{Y}\bar{X}} K_{\bar{X}}^{-1})\bar{X} + \bar{W}$; $\bar{W}$ is indep. of $\bar{X}$

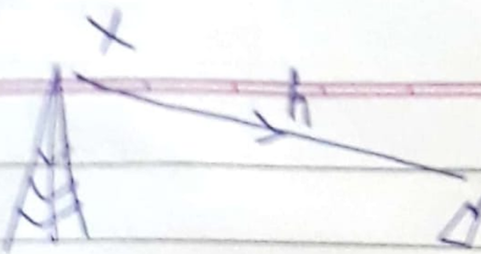
$$W \sim N(\bar{0}, K_{\bar{Y}} - K_{\bar{Y}\bar{X}} K_{\bar{X}}^{-1} K_{\bar{X}\bar{Y}})$$

Observe, $(\bar{X} - G\bar{Y})$ is indep^n. of $\bar{Y}$.

$\hookrightarrow \bar{X}, \bar{Y}$ are jointly Gaussian $\left\{ K_{\bar{U}} = \begin{pmatrix} K_{\bar{X}} & 0 \\ 0 & K_{\bar{Y}} \end{pmatrix} \right.$ → $\bar{0}$ matrix

→ Block diagonal

$\Rightarrow \bar{X} \perp\!\!\!\perp \bar{Y}$ → is indep. of



$$hX + W = Y$$

Gaussian   Gaussian

$$\bar{Y} = H\bar{X} + \bar{W}$$

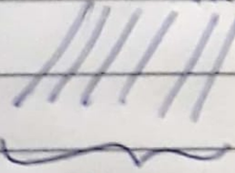
signal   noise   – communication
predictive   innovation   – signal processing

$$\left.\begin{aligned} \bar{X}_{n+1} &= H\bar{X}_n + \bar{W}_n \\ \bar{X}_{n+2} &= H\bar{X}_{n+1} + \bar{W}_{n+1} \end{aligned}\right\} \text{filtering}$$

Mpeg

4kHD } 8 MPixels   10MB

30 images x/s
(8 Mpixels)



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