Problem 5.28

(a) Consider a periodic waveform $x(t)$ whose Fourier transform is defined by

$$X(f) = \sum_{k=-m}^{m} c_k \delta(f - k f_0) \tag{1}$$

where f_0 is the fundamental frequency of $x(t)$. In effect, we are assuming that $x(t)$ is the result of prefiltering a periodic signal with period $1/f_0$ and all harmonic components in excess of the mth component have been suppressed. The highest frequency of $x(t)$ is therefore mf_0.

Suppose now $x(t)$ is purposely sampled at the rate

$$f_s = (1-a) f_0 \tag{2}$$

where $0 < a < 1$. The sampling rate f_s is clearly less than the Nyquist rate $2mf_0$, hence the possibility of aliasing. From Eq. (5.2) in the text, recall that the Fourier transform of the sampled version of $x(t)$ is defined by

$$\begin{aligned} \frac{1}{f_s} X_\delta(f) &= \sum_{i=-\infty}^{\infty} X(f - i f_s) \\ &= \sum_{i=-\infty}^{\infty} X(f - i f_0 + a i f_0) \end{aligned} \tag{3}$$

Substituting Eq. (1) into (3) yields

$$\frac{1}{f_s} X_\delta(f) = \sum_{i=-\infty}^{\infty} \sum_{k=-m}^{\infty} c_k \delta(f - (i+k) f_0 + a i f_0) \tag{4}$$

To proceed further with this equation, we will use *induction* to solve Problem 5.28.

(i) Let $m = 1$, for which Eq. (1) reads as

$$X(f) = c_0 \delta(f) + c_1 [\delta(f - f_0) + \delta(f + f_0)] \tag{5}$$

This spectrum represents a sinusoidal wave of amplitude $2c_1$, superimposed on a dc bias of c_0; see Fig. 1(a). For this case, Eq. (4) simplifies to

$$\begin{aligned} \frac{1}{f_s} X_\delta(f) &= \sum_{i=-\infty}^{\infty} \sum_{k=-1}^{\infty} c_k \delta(f - (i+k) f_0 + a f_0) \\ &= \sum_{i=-\infty}^{\infty} [c_0 \delta(f - i f_0 + a i f_0) \\ &\quad + c_i \delta(f - (i+1) f_0 + a i f_0) \\ &\quad + c_i \delta(f - (i-1) f_0 + a i f_0)] \end{aligned} \tag{6}$$

Evaluating Eq. (5) yields the sampled spectrum depicted in Fig. 1(b).

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Problem 5-28 continued

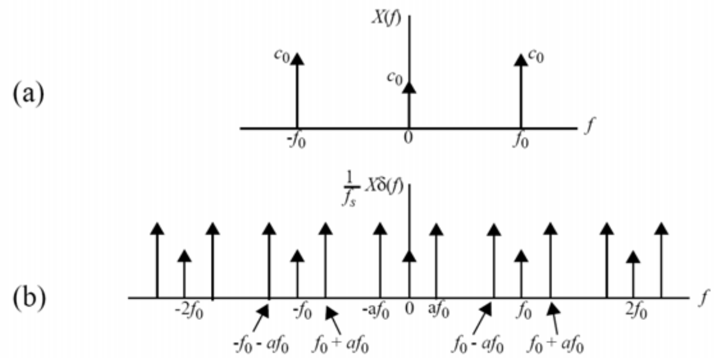

Figure 1

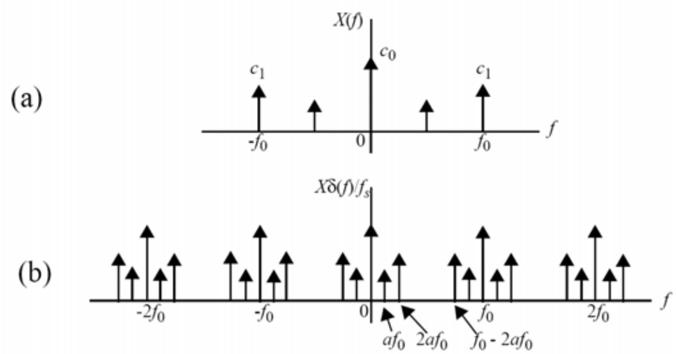

Figure 2

(ii) Next, let $m = 2$, for which we deduce that the relationship between the original spectrum $X(f)$ and the sampled spectrum $X_\delta(f)/f_s$ is pictured as shown in Fig. 2. The results displayed here follow from the evaluation of Eq. (4) for $m = 2$.

Based on the results depicted in Figs. 1 and 2, we may draw the following conclusions:

- The part of the spectrum $X_\delta(f)/f_s$ centered on the origin $f = 0$ is a compressed version of the original spectrum $X(f)$.
- The original spectrum $X(f)$ can be recovered from $X_\delta(f)/f_s$ by using a low-pass filter, provided there is no spectral overlap. In both figures, there is no spectral overlap. For this to be so, in Fig. 1(b) with $m = 1$ we must choose

$$(f_0 - af_0) > af_0$$

or

$$a < \frac{1}{2} \tag{7}$$

In the case of Fig. 2(b) with $m = 2$, we must choose

$$(f_0 - 2af_0) > 2af_0$$

or

$$a < \frac{1}{4} \tag{8}$$

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Generalizing these two results, we may say that spectral overlap in the sampled spectrum $X_\delta(f)/f_s$ is avoided provided that we choose

$$a < \frac{1}{2m}$$

However, the choice of $1/2m$ does not leave any room for the design of a realizable low-pass reconstruction filter. This last provision is made by choosing

$$a < \frac{1}{2M+1} \qquad (9)$$

- From Fourier transform theory, we recall that spectral compression in the frequency domain corresponds to signal expansion in the frequency domain. We therefore conclude that provided the choice of parameter a satisfies Eq. (9), then we may use the scheme described in Fig. 5.28 to expand the time display of a periodic waveform with highest frequency component mf_0 and do so with a realizable reconstruction filter, provided that parameter a satisfies the condition of Eq. (9).