Problem 4.20

The filter input is

$$v_1(t) = g(t)s(t)$$

$$= g(t)\cos(2\pi f_c t - \pi k t^2)$$

The complex envelope of $v_1(t)$ is

$$\tilde{v}_1(t) = g(t)\exp(-j\pi k t^2)$$

The impulse response $h(t)$ of the filter is defined in terms of the complex impulse response $\tilde{h}(t)$ as follows

$$h(t) = \mathbf{Re}[\tilde{h}(t)\exp(j2\pi f_c t)]$$

With $h(t)$ defined by

$$h(t) = \cos(2\pi f_c t + \pi k t^2),$$

we have

$$\tilde{h}(t) = \exp(j\pi k t^2)$$

The complex envelope of the filter output is therefore (except for a scaling factor)[1]

$$\tilde{v}_o(t) = \tilde{h}(t) \star \tilde{v}_i(t)$$

$$= \int_{-\infty}^{\infty} g(\tau)\exp(-j\pi k\tau^2)\exp[j\pi k(t-\tau)]^2 d\tau$$

$$= \exp(j\pi k t^2)\int_{-\infty}^{\infty} g(\tau)\exp(-2j\pi k t\tau)d\tau$$

$$= \exp(j\pi k t^2)G(kt) \qquad (1)$$

where in the last line we have used the definition of the Fourier transform to write

$$G(kt) = \int_{-\infty}^{\infty} g(\tau)\exp(-j2\pi k t\tau)d\tau$$

Hence, from Eq. (1), we obtain the

1. It turns out that the scaling factor equals 1/2; to be exact, we should write

$$\tilde{v}_o(t) = \frac{1}{2}\tilde{h}(t) \star \tilde{v}_i(t)$$

For details, see the 4th edition of the book:
S. Haykin, Communication Systems, pp. 725-734, 4th edition, Wiley.

$$\tilde{v}_0(t) = |G(kt)| \qquad (2)$$

Equation (2) shows that the envelope of the filter output is, except for a scaling factor, equal to the magnitude of the Fourier transform of the input signal $g(t)$ with kt playing the role of frequency f.