Problem 2.19

(a) The half-cosine pulse $g(t)$ of Fig. 2.40(a) may be considered as the product of the rectangular function rect(t/T) and the sinusoidal wave $A\cos(\pi t/T)$. Since

$$\mathrm{rect}\left(\frac{t}{T}\right) \rightleftharpoons T\,\mathrm{sinc}(fT)$$

$$A\cos\left(\frac{\pi t}{T}\right) \rightleftharpoons \frac{A}{2}\left[\delta\left(f-\frac{1}{2T}\right)+\delta\left(f+\frac{1}{2T}\right)\right]$$

and multiplication in the time domain is transformed into convolution in the frequency domain, it follows that

$$G(f) = [T\,\mathrm{sinc}(fT)]\bigstar\left\{\frac{A}{2}\left[\delta\left(f-\frac{1}{2T}+f+\frac{1}{2T}\right)\right]\right\}$$

where ★ denotes convolution. Therefore, noting that

$$\mathrm{sinc}(fT)\bigstar\delta\left(f-\frac{1}{2T}\right) = \mathrm{sinc}\left[T\left(f-\frac{1}{2T}\right)\right]$$

$$\mathrm{sinc}(fT)\bigstar\delta\left(f+\frac{1}{2T}\right) = \mathrm{sinc}\left[T\left(f+\frac{1}{2T}\right)\right]$$

we obtain the desired result

$$G(f) = \frac{AT}{2}\left[\mathrm{sinc}\left(fT-\frac{1}{2}\right)+\mathrm{sinc}\left(fT+\frac{1}{2}\right)\right]$$

(b) The half-sine pulse of Fig. 2.40(b) may be obtained by shifting the half-cosine pulse to the right by $T/2$ seconds. Since a time shift of $T/2$ seconds is equivalent to multiplication by $\exp(-j\pi fT)$ in the frequency domain, it follows that the Fourier transform of the half-sine pulse is

$$G(f) = \frac{AT}{2}\left[\mathrm{sinc}\left(fT-\frac{1}{2}\right)+\mathrm{sinc}\left(fT+\frac{1}{2}\right)\right]\exp(-j\pi fT)$$

(c) The Fourier transform of a half-sine pulse of duration aT is equal to

$$\frac{|a|AT}{2}\left[\mathrm{sinc}\left(afT-\frac{1}{2}\right)+\mathrm{sinc}\left(afT+\frac{1}{2}\right)\right]\exp(-j\pi faT)$$

(d) The Fourier transform of the negative half-sine pulse shown in Fig. 2.40(c) is obtained from the result by putting a = -1, and multiplying the result by -1, and so we find that its Fourier transform is equal to

$$-\frac{AT}{2}\left[\mathrm{sinc}\left(fT+\frac{1}{2}\right)+\mathrm{sinc}\left(fT-\frac{1}{2}\right)\right]\exp(j\pi fT)$$

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Problem 2-19 continued

(e) The full-sine pulse of Fig. 2.40(d) may be considered as the superposition of the half-sine pulses shown in parts (b) and (c) of the figure. The Fourier transform of this pulse is therefore

$$
\begin{aligned}
G(f) &= \frac{AT}{2}\left[\operatorname{sinc}\left(fT-\frac{1}{2}\right)+\operatorname{sinc}\left(fT+\frac{1}{2}\right)\right][\exp(-j\pi fT)-\exp(j\pi fT)] \\
&= -jAT\left[\operatorname{sinc}\left(fT-\frac{1}{2}\right)+\operatorname{sinc}\left(fT+\frac{1}{2}\right)\right]\sin(\pi fT) \\
&= -jAT\left[\frac{\sin\left(\pi fT-\frac{\pi}{2}\right)}{\pi fT-\frac{\pi}{2}}+\frac{\sin\left(\pi fT+\frac{\pi}{2}\right)}{\pi fT+\frac{\pi}{2}}\right]\sin(\pi fT) \\
&= -jAT\left[-\frac{\cos(\pi fT)}{\pi fT-\frac{\pi}{2}}+\frac{\cos(\pi fT)}{\pi fT+\frac{\pi}{2}}\right]\sin(\pi fT) \\
&= jAT\left[\frac{\sin(2\pi fT)}{2\pi fT-\pi}-\frac{\sin(2\pi fT)}{2\pi fT+\pi}\right] \\
&= jAT\left[-\frac{\sin(2\pi fT-\pi)}{2\pi fT-\pi}+\frac{\sin(2\pi fT+\pi)}{2\pi fT+\pi}\right] \\
&= jAT[\operatorname{sinc}(2fT+1)-\operatorname{sinc}(2fT-1)]
\end{aligned}
$$