Problem 2.24

(a) The convolution of $g_1(t)$ and $g_2(t)$ is defined by

$$g_1(t)\bigstar g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \tag{1}$$

Differentiating both sides of Eq. (1) with respect to time t:

$$\begin{aligned}\frac{d}{dt}[g_1(t)\bigstar g_2(t)] &= \int_{-\infty}^{\infty} g(\tau)\frac{d}{dt}g_2(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} g_1(\tau)\frac{d}{d(t-\tau)}g_2(t-\tau)d\tau \\ &= g_1(t)\bigstar\left[\frac{d}{dt}g_2(t)\right]\end{aligned}$$

Since convolution is commutative, we may also write

$$\frac{d}{dt}[g_1(t)\bigstar g_2(t)] = \left[\frac{d}{dt}g_1(t)\right]\bigstar g_2(t)$$

In other words, the derivative of a convolution produce of two signals is equivalent to the convolution of one of the signals and the derivative of the other.

(b) Changing variables in Eq. (1), we may write

$$g_1(t)\bigstar g_2(t) = \int_{-\infty}^{\infty} g_1(\lambda)g_2(t-\lambda)d\lambda \tag{2}$$

Integrating both sides of Eq. (2) with respect to t:

$$\int_{-\infty}^{t}[g_1(\tau)\bigstar g_2(\tau)]d\tau = \int_{-\infty}^{t}\int_{-\infty}^{\infty} g_1(\lambda)g_2(\tau-\lambda)\lambda d\tau$$

Interchanging the order of integration and rearranging terms:

$$\int_{-\infty}^{t}[g_1(\tau)\bigstar g_2(\tau)]d\tau = \int_{-\infty}^{\infty} g_1(\lambda)\int_{-\infty}^{\infty} g_2(\tau-\lambda)\tau d\lambda d \tag{3}$$

Recognizing that

$$\int_{-\infty}^{t} g_2(\tau-\lambda)d\lambda = \int_{-\infty}^{t-\lambda} g_2(\tau)d\tau$$

we may rewrite Eq. (3) as

$$\begin{aligned}\int_{-\infty}^{t}[g_1(\tau)\bigstar g_2(\tau)]d\tau &= \int_{-\infty}^{\infty} g_1(\lambda)\int_{-\infty}^{t-\lambda} g_2(\tau)d\tau d\lambda \\ &= g_1(t)\bigstar\left[\int_{-\infty}^{t} g_2(\tau)d\tau\right]\end{aligned}$$

In other words, the integral of a convolution product of two signals is equivalent to the convolution of one of the signals and the integral of the other.