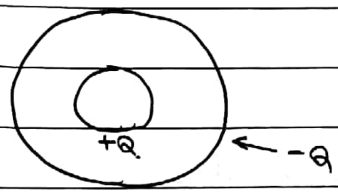
Units of μ_0 :- Inductance / Length

8/7.

Gauss' Law



Inner solid sphere was given +Q. Outer hollow sphere got a -Q total charge.
Even when inner sphere was grounded, (its charge made zero), the -Q on outer sphere remained.
The result does not change on placing dielectric between them.

- Replace inner sphere by point charge.

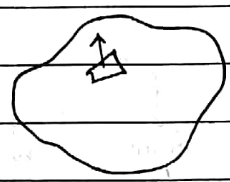
$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{a}_r$$

→ Permittivity of free space $\epsilon_0 = \frac{1}{36\pi} \times 10^{-9}$ F/m

(Capacitance / Length)

$$\vec{D}(r) = \epsilon_0 \vec{E}(r) = \text{Flux density} \quad C/m^2$$

$$= \frac{1}{4\pi}\frac{q}{r^2}\hat{a}_r$$

- 

Surface element $\vec{ds}$ is a vector in normal direction

Flux flowing through the differential surface element $= \vec{D}\cdot\vec{ds}$

Total flux flowing out $= \Psi = \oint \vec{D}\cdot\vec{ds}$

→ Gauss' Law :- $\boxed{\Psi = Q}$ = Charge enclosed.

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* Polarized materials

$$\vec{D} \overset{?}{=} \epsilon \vec{E}$$

They may not even have same direction.

- Mathematically,

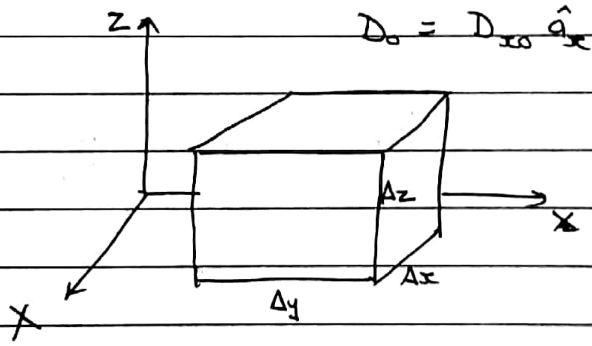
$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \underbrace{\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}}_{\text{Permittivity tensor matrix}} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

If all elements of ϵ matrix are not equal, directions of $\vec{E}$ and $\vec{D}$ are not same.

→ Point form of Gauss Law

$$D_0 = D_{x0}\hat{a}_x + D_{y0}\hat{a}_y + D_{z0}\hat{a}_z$$

$$\triangleq \vec{D} \text{ at centre}$$



$$\oint \vec{D}\cdot\vec{ds} = \int_{top} + \int_{bottom} + \int_{left} + \int_{right} + \int_{front} + \int_{back}$$

$$= D_{front}\,\Delta y \Delta z$$

$$\downarrow$$

$$D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x} \;\ldots\ldots \text{ Taylor series}$$

$$\int_{front} =$$

$$\int_{back} = \left(D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

- $\int_{front} + \int_{back} = \Delta x \Delta y \Delta z \dfrac{\partial D_x}{\partial x} = \Delta v \dfrac{\partial D_x}{\partial x}.$

- $\therefore \int + \int + \int + \int + \int + \int = \Delta v \left(\dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z} \right) = q$

(Gauss law)

- $\therefore \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z} = \dfrac{q}{\Delta v} = \rho_v =$ 'Charge Density'

- Define LHS as ~~gradient~~ divergence of $\vec{D}$.

$$= div(\vec{D}) = \vec{\nabla} \cdot \vec{D} = \left(\frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z \right) \cdot \left(D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z \right)$$

- Gauss Law :- $\boxed{\vec{\nabla} \cdot \vec{D} = \rho_v}$ = '<u>Maxwell's first law</u>'

- Gauss Law also holds for time-varying electric ~~and magnetic~~ fields.

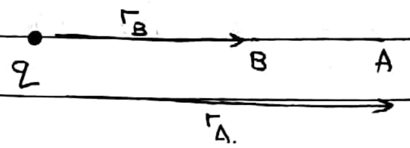
→ $\oint \vec{D} \cdot \vec{ds} = Q = \int \rho_v \, dv$

$$\therefore \boxed{\oint \vec{D} \cdot \vec{ds} = \int \vec{\nabla} \cdot \vec{D} \, dv.}$$

'<u>Divergence theorem</u>' :- Converts areal integral to volume integral.

→ Divergence is a measure of a vector's tendency to radiate out.
Curl turn.

ELECTRIC POTENTIAL



- $\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{a}_r$

- $V_B - V_A = -\int_A^B \vec{E} \cdot \vec{dl}$

$$= -\frac{q}{4\pi\epsilon_0} \int_A^B \frac{dr}{dr^2}$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right).$$

- Define $V(\infty) = 0$

$$V_B = \frac{q}{4\pi\epsilon_0 r_B}$$

- The potential difference between any two points is independent of path traversed.
In a ~~static~~ conservative field, $\int \vec{E} \cdot \vec{dl}$ is independent of path.
- Static fields are conservative

- $\boxed{\oint \vec{E} \cdot \vec{dl} = 0}$

'Maxwell's 2nd law'

Ampere's Circuital Law

- Lorentz Force = Force experienced by moving charge = $q(\vec{v} \times \vec{B})$
  - Used to define $\vec{B}$.

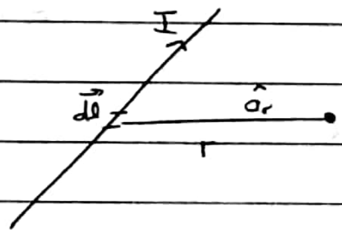
- $\vec{B} = \mu_0 \vec{H}$ in free space

$$\rightarrow \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} & & \\ & \mu & \\ & & \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

Permeability tensor

I Biot Savart Law

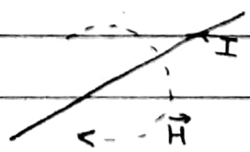
$$\boxed{d\vec{H} = \frac{\cancel{\mu_0} I}{4\pi} \frac{d\vec{l} \times \hat{a}_r}{r^2}}$$



II Ampere's Law

$$\boxed{\oint \vec{H} \cdot d\vec{l} = I}$$

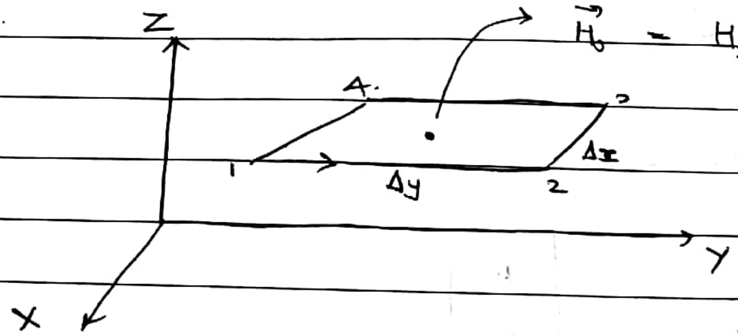
in steady magnetic fields.



'Maxwell's 3rd law'

→ Other form of Ampere's Law.

$\vec{H}_0 = H_{x0}\hat{a}_x + H_{y0}\hat{a}_y + H_{z0}\hat{a}_z$



$$\oint \vec{H}\cdot\vec{dl} = \int_{12} + \int_{23} + \int_{34} + \int_{41}$$

$\int_{12} \rightarrow H_{y12}\,\Delta y$

$H_{y12} \rightarrow H_y\left(\cancel{x_0}\, x_0 + \frac{\Delta x}{2}\right) = H_y(x_0) + \frac{\Delta x}{2}\frac{\partial H_y}{\partial x}$

- On path 12, H_y is assumed to be constant is Δy is very small

- $\int_{34} = H_{y34}(-\Delta y) =$

$H_{y34} \rightarrow H_y\left(x_0 - \frac{\Delta x}{2}\right) = H_y(x_0) - \frac{\Delta x}{2}\frac{\partial H_y}{\partial x}$

- $\oint = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\Delta y = I$

$I \rightarrow J_z\,\Delta x\Delta y$

current density

$$\Rightarrow \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$

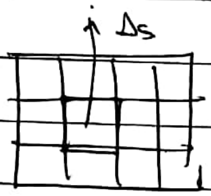
Similarly $\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x$$

- Hence, $\vec{J} = J_x\hat{a}_x + J_y\hat{a}_y + J_z\hat{a}_z = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = \vec{\nabla}\times\vec{H} = \text{"curl}(\vec{H})\text{"}$

$$\boxed{\vec{J} = \vec{\nabla}\times\vec{H}}$$

III. Stokes' Theorem



$$\oint \vec{H} \cdot d\vec{l} = \Delta I = J_N \Delta s$$

$$J_N = (\vec{\nabla} \times \vec{H})_{\text{Normal direction}}$$

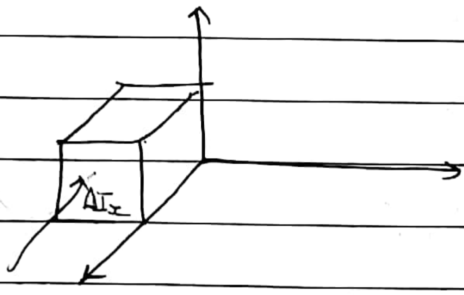
$$= (\vec{\nabla} \times \vec{H}) \cdot \Delta\vec{s}$$

$$= \oint_{\Delta s} \vec{H} \cdot d\vec{l}$$

Stokes' Theorem :- $\boxed{\int (\vec{\nabla} \times \vec{H}) \cdot d\vec{s} = \oint \vec{H} \cdot d\vec{l}}$

Area integral   Line integral

IV. Current



$$\Delta I_x = J_x \Delta s_x = \vec{J} \cdot \vec{\Delta s_x}$$

$$\Delta I_y \qquad \vec{J} \cdot \vec{\Delta s_y}$$

$$\Delta I_y \qquad \vec{J} \cdot \vec{\Delta s_z}$$

$$\Delta I = \vec{J} \cdot (\vec{\Delta s_x} + \vec{\Delta s_y} + \vec{\Delta s_z})$$

$$= \vec{J} \cdot \Delta\vec{s} \qquad \text{where } \Delta\vec{s} = \vec{\Delta s_x} + \vec{\Delta s_y} + \vec{\Delta s_z}$$

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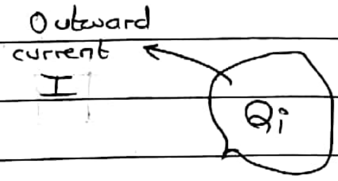
V Continuity Equation



$$I = -\frac{dQ_i}{dt}$$

$$\oint \vec{J}\cdot\vec{ds} = -\frac{d}{dt}\int \rho_v\, dv$$

$$\Rightarrow \int_v \nabla\cdot\vec{J}\, dv = \int \frac{d\rho_v}{dt}\, dv$$

$$\boxed{\nabla\cdot\vec{J} = -\frac{\partial\rho_v}{dt}}$$

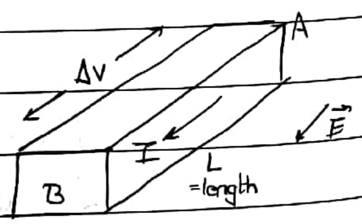
$$\rightarrow \Delta I_x = \frac{\Delta Q}{\Delta t} = \frac{\rho_v\, \Delta S_x\, \Delta x}{\Delta t} = \rho_v\, \Delta S_x\, v_x$$

↳ velocity in x direction

$$\Rightarrow J_x = \rho_v v_x$$

$$\therefore \boxed{\vec{J} = \rho_v \vec{v}}$$

VI Ohm's Law



$$V = IR$$

$$E\cancel{L} = I\,\frac{\rho\cancel{L}}{S}$$

$$= J\rho$$

$$\therefore \boxed{\vec{J} = \sigma\vec{E}} \quad \text{where } \sigma = \frac{1}{\text{Resistivity}}$$

classmate

Date
Page

VI MAXWELL'S FOURTH LAW

From 1^{st} law, $\oint \vec{D} \cdot \vec{ds} = q$

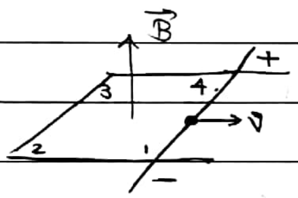
$\vec{\nabla} \cdot \vec{D} = \rho_v$.

Since magnetic monopoles do not exist, $\oint \vec{B} \cdot \vec{ds} = 0$

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

- Maxwell's 1^{st} and 4^{th} law are valid for time-varying fields. 2^{nd} and 3^{rd} laws have to be modified.

VII SECOND LAW IN TIME VARYING $\vec{B}$



$\vec{F} = q(\vec{v} \times \vec{B})$

Define motional emf $= \boxed{\vec{E}_m} = \dfrac{\vec{F}}{q} = \boxed{\vec{v} \times \vec{B}}$

$\oint \vec{E} \cdot \vec{dl} = V_4 - V_1 = -\int_{1}^{4} \vec{E}_m \cdot dl = -Blv$

☆ $\vec{E}$ is zero on other sides

~~But~~ If $\phi = Bxl$

Then $\vec{E}_m = \cancel{-\dfrac{d\phi}{dt}}$ $\boxed{\oint \vec{E} \cdot dl = -\dfrac{d\phi}{dt}}$

$\oint \vec{E} \cdot \vec{dl} = -\dfrac{d}{dt} \int \vec{B} \cdot \vec{ds}$ (writing $\phi = \int \vec{B} \cdot \vec{ds}$)

$\int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{ds} = -\int \dfrac{d\vec{B}}{dt} \cdot \vec{ds}$ .... Stokes Theorem

$$\therefore \boxed{\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}}$$

VIII Maxwell's 3rd Law in Time-varying $\vec{E}$

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}$$

$$= T \quad \text{(say)}$$

$$\therefore \int_{vol} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) \, dv = \int_{vol} T \, dV$$

$$\Rightarrow \oint_S (\vec{\nabla} \times \vec{H}) \, \vec{ds} = \int_{vol} T \, dV$$

area of opening in the volume

As area of opening $\rightarrow 0$, LHS $= 0$, RHS $= 0$

$$\Rightarrow T = \vec{\nabla} \cdot \vec{J} = 0$$

→ Modify :- $\vec{\nabla} \times \vec{H} = \vec{J} + \vec{G}$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{G} = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{G} = -\vec{\nabla} \cdot \vec{J} = \frac{d\rho_v}{dt} = \frac{d}{dt} (\vec{\nabla} \cdot \vec{D})$$

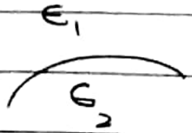
$$\therefore \boxed{\vec{G} = \frac{d\vec{D}}{dt}}$$

Boundary Conditions

→ Vacuum $\xleftarrow[\vec{E}]{}$ (Metal

$\vec{E}$ is always normal to interface

$D_{N1} - D_{N2} = S$ surface charge

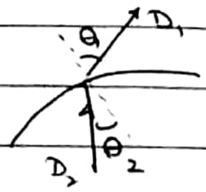


$D_{N1} = D_{N2}$ ... normal $\vec{D}$ or $\vec{B}$

$E_{t1} = E_{t2}$ ... tangential $\vec{E}$ or $\vec{H}$

same for $\vec{B}$ and $\vec{H}$

REFRACTION.



$$\boxed{D_1 = D_2 \sqrt{\cos^2\theta_2 + \frac{\epsilon_1^2}{\epsilon_2^2}\sin^2\theta_2}}$$

$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$

Solution of Maxwell's Equations

Write $\vec{\mathcal{E}}(x,y,z,t) = \vec{E}(x,y,z)\, e^{j\omega t}$

$\vec{\mathcal{H}} \quad \vec{H}$

$\vec{\mathcal{D}} \quad \vec{D}$

$\vec{\mathcal{B}} \quad \vec{B}$

$\vec{\mathcal{J}} \quad \vec{J}$

$\rho \quad \rho$

- Assume time-variation is sinusoidal

- $\frac{d}{dt} \vec{\mathcal{D}} = j\omega \vec{\mathcal{D}}$ ..... similarly for other quantities.

1 $\vec{\nabla} \cdot \vec{\mathcal{D}} = \rho \Rightarrow \boxed{\vec{\nabla} \cdot \vec{D} = \rho}$ (in free space $\rho = 0$)

2 $\vec{\nabla} \cdot \vec{\mathcal{B}} = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$

3 $\vec{\nabla} \times \vec{\mathcal{H}} = \vec{J} + \frac{\partial \vec{\mathcal{D}}}{\partial t} \Rightarrow \boxed{\vec{\nabla} \times \vec{H} = \vec{J} + j\omega \vec{D}}$ ..... in free space $\vec{J} = 0$ (lossless medium)

4 $\vec{\nabla} \times \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t} \Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -j\omega \vec{B}}$

~~In free space~~,

- If $J = 0$, $\vec{\nabla} \times \vec{H} = j\omega \vec{D}$ and $\vec{\nabla} \times \vec{E} = j\omega \vec{B}$

$= j\omega(\epsilon \vec{E}) \qquad = j\omega(\mu \vec{H})$

- $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -j\omega\mu(\vec{\nabla} \times \vec{H})$

$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = j\omega\mu(j\omega\epsilon \vec{E})$

where $\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$ ---- Gradient of f

$\nabla^2 \vec{H} = \frac{\partial^2 H_x}{\partial x^2}\hat{x} + \frac{\partial^2}{\partial y^2} H_y \hat{y} + \frac{\partial^2}{\partial z^2} H_z \hat{z}$ ---- Double Gradient

classmate
Date
Page

25/7

$$\rightarrow \nabla\times(\nabla\times\bar{E}) = \bar{\nabla}(\overbrace{\cancel{\bar{\nabla}}\cdot\bar{E}}^{0 \text{ in free space } (\rho=0)}) - \nabla^2\bar{E}$$

$$\therefore \nabla\times(-j\omega\mu\bar{H}) = -\nabla^2\bar{E}$$

$$\therefore -\nabla^2\bar{E} = j(\nabla\times -\omega\mu\bar{H}) + (\mp\omega\mu\bar{H})(\overbrace{\cancel{\bar{\nabla}}\,\bar{j}}^{0})$$

$$= -j\omega\mu(\nabla\times\bar{H})$$

$$= -j\omega\mu(j\omega\epsilon\bar{E}) \quad \ldots\ 3^{rd} \text{ law}$$

$$\therefore \boxed{\nabla^2\bar{E} + \omega^2\mu\epsilon\bar{E} = 0}$$

Similarly, if you begin with $\nabla\times(\nabla\times\bar{H})$,

$$\boxed{\nabla^2\bar{H} + \omega^2\mu\epsilon\bar{H} = 0}$$

These two equations are called '<u>Helmholtz Equations</u>'.
These are homogeneous equations.

eg Say $\bar{E} = E_x(z)\,\hat{x}$

$\therefore$ Helmholtz equation :- $\dfrac{\partial^2 E_x(z)}{\partial z^2} + k^2 E_x(z) = 0$ where $k = \omega^2\mu\epsilon$

- Solution :- $E_x(z) = E_x^+ e^{-jkz} + E_x^- e^{jkz}$.

- $\vec{\mathbb{E}}(z,t) = E_x(z)e^{j\omega t}$
$= E_x^+ e^{j(\omega t - kz)} + E_x^- e^{j(\omega t + kz)}$

- Real part :- $\vec{\mathbb{E}}(z,t) = \underbrace{E_x^+\cos(\omega t - kz)}_{\text{moves in } +Z} + \underbrace{E_x^-\cos(\omega t + kz)}_{\text{moves in } -Z}$

Wave is travelling in sinusoidal waveform in Z direction.

P.T.O.

- $\boxed{k = \omega\sqrt{\mu\epsilon}}$ = 'Propagation constant'

'Angular frequency' = $2\pi f$

- Wavelength = $\lambda = \frac{2\pi}{k}$

- Velocity (with which one particular phase is moving)

$$\omega t - kz = c$$

$$\omega - k\frac{dz}{dt} = 0$$

$$\therefore \quad \boxed{v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}} = \frac{1}{\sqrt{\mu_0\epsilon_0}} \text{ in free space}$$

• If we use the second Helmholtz equation, we will receive expression for $\vec{H}$.

- For above example,

$$\nabla \times \bar{E} = -j\omega\mu\bar{H}$$

$$\therefore \quad \bar{H} = \frac{-1}{j\omega\mu} \nabla \times \hat{x}\underbrace{\left[E_x^+ e^{-jkz} + E_x^- e^{jkz}\right]}_{\text{Solve}}$$

Solve, put $k/\omega\mu = \frac{1}{\eta}$,

$$\bar{H} = \frac{1}{\eta}\hat{y}\left[E_x^+ e^{-jkz} - E_x^- e^{jkz}\right]$$

where $\eta = \sqrt{\mu/\epsilon}$ = 'Wave Impedance'

in free space & unbounded media

Verify 1. $\bar{E}$ and $\bar{H}$ are always perpendicular, and perpendicular to direction of propagation.

2. If $\bar{E}$ expression has '+', $\bar{H}$ expression will have '−' between terms travelling in opposite direction

3 Amplitude of $\bar{H}$ = $\dfrac{\text{Amplitude of } \bar{E}}{\eta}$

* η has same dimensions as $Z = V/I$

- All these observations are for lossless, uniform, unbounded media.

I Lossy, Uniform, Unbounded Media

$$\bar{J} \neq 0 \quad \text{and} \quad \sigma \neq 0$$

1 $\nabla \times \bar{E} = -j\omega\mu \bar{H}$

2 $\nabla \times \bar{H} = j\omega \epsilon \bar{E} + \sigma \bar{E} = (\sigma + j\omega\epsilon)\bar{E}$

• Modified Helmholtz's equation :-

$$\nabla^2 \bar{E} + \underbrace{\omega^2 \mu \epsilon \left[1 - \frac{j\sigma}{\omega\epsilon}\right]}_{-\gamma^2} \bar{E} = 0$$

$$\nabla^2 \bar{E} - \gamma^2 \bar{E} = 0 \quad \text{, where } \boxed{\gamma = j\omega\sqrt{\mu\epsilon}\sqrt{1 - \frac{j\sigma}{\omega\epsilon}}}$$

.... counterpart for 'k'.

$= \alpha + j\beta$

where $\alpha > 0$

eg Again, consider $\bar{E} = E_x(z)\,\hat{x}$

Solve Helmholtz :- $E_x(z) = E_x^+ e^{-\gamma z} + E_x^- e^{\gamma z}$

$$\boxed{\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right)}}$$

Positive Z wali wave :- $E_x^+ \underbrace{e^{-\alpha z}}_{\text{attenuation in } +Z} e^{-j\beta z}$.

Negative Z wali wave :- $E_x^- \underbrace{e^{\alpha z}}_{\text{attenuation in } -Z} e^{j\beta z}$

$$E = E e^{-\alpha z} e^{j(\omega t - \beta z)}$$

If $\frac{\sigma}{\omega\epsilon} << 1$, then

$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$

$\beta \approx \omega\sqrt{\mu\epsilon}$ = k from before

$$\sqrt{j} = \frac{1+j}{\sqrt{2}}$$

- Again, calculate $\bar{H} = \frac{-1}{j\omega\mu} \nabla \times \bar{E}$

$$\bar{H} = \frac{1}{\eta}\left[E_x^+ e^{-\gamma z} - E_x^- e^{\gamma z}\right]$$

- Previous observations hold.
- $\boxed{\text{New } \eta = \frac{j\omega\mu}{\gamma} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{1 - j\sigma/\omega\epsilon}}}$
  - Value of η depends on frequency :(

Verify ☆ - Because η is complex, $\bar{E}$ and $\bar{H}$ are not in phase wrt time, but still perpendicular in space.

A] EM Waves in Good Conductors

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\sqrt{1 - \frac{j\sigma}{\omega\epsilon}}$$ where σ is very high.

$$= j\omega\sqrt{\mu\epsilon}\sqrt{\frac{-j\sigma}{\omega\epsilon}}$$

$$= \sqrt{\omega\mu\sigma}\sqrt{j}$$

$$\boxed{\gamma = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}}$$

- Skin depth (δ_s) :- Distance after which attenuated to 36.8% ($\frac{1}{e}$ times)

$$e^{-\alpha z} = \frac{1}{e} \Rightarrow \boxed{\delta_s = \frac{1}{\alpha}} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

- Skin effect :-

As frequency increases, $\bar{E}$ cannot penetrate much radially into the wire. Hence, current will only flow more along the circumference of wire.

- $\boxed{\eta = \frac{j\omega\mu}{\gamma} = \frac{1+j}{\sigma \delta_s}}$

For good conductors, $\arg(\eta) = 45°$

poor $0 < \arg(\eta) < 45°$

✳ Why did we not assume $\bar{E} = E_x(x)\,\hat{x}$?

Because $\bar{H} = \frac{-1}{j\omega\mu} \nabla \times \bar{E}$ would become zero

- ~~For so~~ ∴ $\bar{E}$ and $\bar{H}$ cannot vary in direction of propagation, or be constant spatially.

1/8

II. General Solution Of Wave Equation.

$$\nabla^2 \bar{E} + k^2 \bar{E} = 0$$

$$\Rightarrow \frac{\partial^2 \bar{E}}{\partial x^2} + \frac{\partial^2 \bar{E}}{\partial y^2} + \frac{\partial^2 \bar{E}}{\partial z^2} + k^2 \bar{E} = 0$$

Write $\bar{E} = E_x(x,y,z)\,\hat{x} + E_y(x,y,z)\,\hat{y} + E_z(x,y,z)\,\hat{z}$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$$

Similarly for E_y, E_z

* Linear Polarization :-

When directions of $\bar{E}$ and $\bar{H}$ do not change during propagation.

- Write $E_x(x, y, z) = f(x) g(y) h(z)$

$\therefore$ $ghf'' + fhg'' + fgh'' + k^2 fgh = 0$

- $\therefore$ $\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} + k^2 = 0$

Hence, $\frac{f''}{f}$, $\frac{g''}{g}$, $\frac{h''}{h}$ are all constants

By symmetry, all three are negative, but may not be equal.

- $\therefore$ $\frac{f''}{f} = -k_x^2$, $\frac{g''}{g} = -k_y^2$, $\frac{h''}{h} = -k_z^2$

where $k^2 = k_x^2 + k_y^2 + k_z^2$

- $\therefore$

$$f(x) = f^+ e^{-jk_x x} + f^- e^{jk_x x}$$
$$g(y) = g^+ e^{-jk_y y} + g^- e^{jk_y y}$$
$$h(z) = h^+ e^{-jk_z z} + h^- e^{jk_z z}$$

- Consider only the +ve travelling wave.

$\therefore$ $E_x(x, y, z) = A\, e^{-j(k_x x + k_y y + k_z z)}$

where $A = f^+ g^+ h^+$

Wavelength :- Distance between two equiphase planes

Phase velocity :- $V_p = \frac{\omega}{k}$

- Similarly,

$$E_y(x,y,z) = B e^{-j(k_x x + k_y y + k_z z)}$$

and $$E_z(x,y,z) = C e^{-j(k_x x + k_y y + k_z z)}$$

☆ Why are k_x k_y k_z equal in all directions?

- ∴ $$\boxed{\bar{E} = (A\hat{x} + B\hat{y} + C\hat{z})\, e^{-j(\bar{k}\cdot\bar{r})}}$$

$$= \boxed{E_0\, e^{-j(\bar{k}\cdot\bar{r})}}$$

'Plane Wave' – because $\bar{E}$ and $\bar{H}$ form a plane of same phase

where $\bar{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ = 'Wave number vector'

and $\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$ = 'Position vector'

- $\bar{k}$ points along direction of propagation

$$\bar{k} = k\hat{n}$$

$\hookrightarrow k = \sqrt{\mu\epsilon}$

- For planes of equal phase (which are normal to direction of propagation), any point on the plane has the same projection along direction of propagation.

∴ $\bar{k}\cdot\bar{r}$ = constant ---- for equiphase plane

- $$\nabla\cdot\bar{E} = \bar{\nabla}\left(\bar{E}_0\, e^{-j(\bar{k}\cdot\bar{r})}\right)$$

$$= \bar{E}_0\cdot\left(\bar{\nabla} e^{-j\bar{k}\cdot\bar{r}}\right) + e^{-j\bar{k}\cdot\bar{r}}\cancel{(\bar{\nabla}\cdot\bar{E}_0)}^{\,0}$$

0 since E_0 is a constant

$$= E_0\left[\frac{\partial}{\partial x} e^{-j(k_x x + k_y y + k_z z)}\hat{x} + \frac{\partial}{\partial y} e^{-j(} \cdots\cdots \right]$$

$$= E_0\left(-jk_x\hat{x} - jk_y\hat{y} - jk_z\hat{z}\right) e^{-j\bar{k}\cdot\bar{r}}$$

$$= -j\, e^{-j(\bar{k}\cdot\bar{r})}(\bar{E}_0\cdot\bar{k})$$

Since medium is uniform, $\bar{\nabla} \cdot \bar{E} = \bar{\nabla} \cdot \bar{D}$

Since medium does not have charges, $\bar{\nabla} \cdot \bar{E} = 0$

$\therefore \bar{E}_0 \cdot \bar{k} = 0$

$\therefore$ In absence of current and charge in uniform medium,

$$\underline{\underline{\bar{E} \perp \bar{k}}}$$

- $\bar{\nabla} \times \bar{E} = -j\omega\mu\bar{H}$

$$\therefore \bar{H} = \frac{j}{\omega\mu} \bar{\nabla} \times \bar{E}$$

$$= \frac{j}{\omega\mu} \left[\bar{\nabla} \times E_0 e^{-j(\bar{k} \cdot \bar{r})} \right]$$

$$= \frac{j}{\omega\mu} \left[\bar{\nabla} (e^{-j\bar{k} \cdot \bar{r}}) \times E_0 + 0 \right]$$

$$= \frac{j}{\omega\mu} \left[\frac{\partial}{\partial x} e^{-j(\bar{k} \cdot \bar{r})} \hat{x} + \frac{\partial}{\partial y} \text{-----} \right]$$

$$= \frac{j}{\omega\mu} \left[-jk_x e^{-j(\bar{k} \cdot \bar{r})} \hat{x} \text{ -----} \right]$$

$$= \frac{e^{-j(\bar{k} \cdot \bar{r})}}{\omega\mu} \bar{k} \times \bar{E}_0$$

$$= \frac{\bar{k} \times \bar{E}}{\omega\mu} \quad \text{------} \quad \bar{H} \perp \bar{E} \text{ and } \bar{H} \perp \bar{k}$$

$$= \frac{k\hat{n} \times \bar{E}}{\omega\mu}$$

$$\therefore \boxed{\vec{H} = \frac{\hat{n} \times \bar{E}}{\eta}}$$

- $\underline{\bar{E}(x, y, z, t) = \bar{E}_0 e^{j(\omega t - \bar{k} \cdot \bar{r})}}$

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Date ________

Page ________

A] Circular Polarization

$$\bar{E}(x,y,z) = (E_1\hat{x} + E_2\hat{y})\, e^{-j\bar{k}\cdot\bar{r}}$$

Values of E_1 and E_2 can depend on each other.

For simplicity, assume propagation along z. $\Rightarrow \bar{k} = k\hat{z}$

$$\therefore \bar{E} = (E_1\hat{x} + E_2\hat{y})\, e^{-jk_z z}$$

- Impose $E_1 = jE_2 = E_0$ .... E_1 leads E_2 by $90°$

$$\therefore E_2 = -jE_0.$$

$$\therefore \bar{E}(z) = E_0(\hat{x} - j\hat{y})\, e^{-jk_z z}.$$

$$\bar{\mathbb{E}}(z,t) = E_0(\hat{x} - j\hat{y})\, e^{j(\omega t - k_z z)}$$

$$\therefore \bar{\mathbb{E}}(0,t) = E_0(\hat{x} - j\hat{y})\, e^{j\omega t}.$$

$$= E_0\left[\hat{x}\cos\omega t + \hat{y}\cos(\omega t - \pi/2)\right]$$

$$= E_0\left[\hat{x}\cos\omega t + \hat{y}\sin\omega t\right]$$

Angle made by $\bar{\mathbb{E}}(0,t)$ with x axis, $\phi = \tan^{-1}(\tan\omega t)$

$$= \omega t.$$

- When $E_1 = jE_2$, ϕ is increasing with time
  'Right hand circular polarization'
- When $jE_1 = E_2$, $\phi = -\omega t$ decreases with time
  'Left hand circular polarization'

- If E_1 and E_2 magnitudes also vary with time, 'Elliptical polarization'

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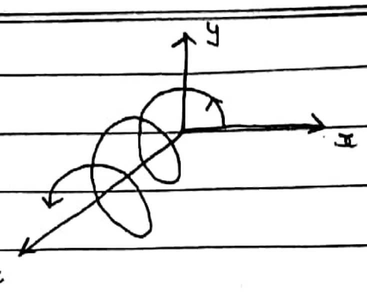
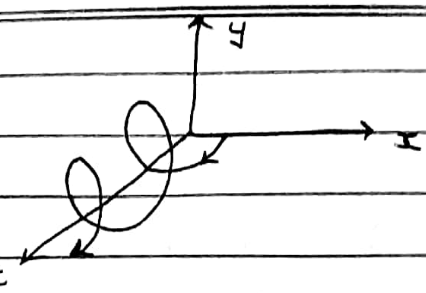
Electric energy = $\frac{1}{2}$ ...

Magnetic energy = $\frac{1}{2}\mu|H|^2$

3/8

- 



RHCP   LHCP

- Locus of electric field is a circle :- $E_x^2 + E_y^2 = E_0^2$

- For $\bar{E}(0,t) = E_1 \cos\omega t\ \hat{x} + E_2 \sin\omega t\ \hat{y}$

  where $|E_1| \overset{?}{=} |E_2|$,

  then the locus of electric field is elliptic - HW

III Poynting's Theorem

$$\bar{\nabla}(\bar{E}\times\bar{H}) = -\bar{E}\cdot(\bar{\nabla}\times\bar{H}) + \bar{H}\cdot(\bar{\nabla}\times\bar{E})$$

$$\therefore \bar{E}\cdot(\bar{\nabla}\times\bar{H}) = \bar{H}\cdot\bar{\nabla}\times\bar{E} - \bar{\nabla}(\bar{E}\times\bar{H})$$

Using $\bar{\nabla}\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t}$

$$\therefore \bar{E}\cdot\bar{J} + \bar{E}\cdot\frac{\partial\bar{D}}{\partial t} = \bar{H}\cdot\left(\frac{-\partial\bar{B}}{\partial t}\right) - \bar{\nabla}\cdot(\bar{E}\times\bar{H})$$

$$\therefore -\oint_s \bar{E}\times\bar{H}\cdot d\bar{s} = \underbrace{\int_{Vol}\bar{E}\cdot\bar{J}\,dv}_{\text{Resistive Power loss}} + \underbrace{\int_{Vol}\epsilon\bar{E}\cdot\frac{\partial\bar{E}}{\partial t}\,dv}_{\text{Rate of change of electric energy}} + \underbrace{\int_{Vol}\mu\bar{H}\cdot\frac{\partial\bar{H}}{\partial t}\,dv}_{\text{Rate of change of magnetic energy}}$$

$\oint_s \bar{E} \times \bar{H} \, d\bar{s}$ = Total power exiting the volume

$\therefore \bar{E} \times \bar{H}$ = Power exiting the volume per unit area
$\triangleq$ 'Poynting Vector' $(\bar{P})$

8/8

* Ohmic Power loss $= P_L = V \cdot I$
$= (E l)(J A)$

$\therefore \boxed{\dfrac{P_L}{Vol} = \bar{E} \cdot \bar{J}}$ ---- as used above.

* Q_1 is present at position 1.
Charges Q_i are sequentially placed at positions i.
Work done :- $W_1 = Q_2 V_{21}$
$W_2 = Q_3 (V_{31} + V_{32})$
?

$$2W_E = Q_1 V_1 + Q_2 V_2 \cdots Q_N V_N = \sum_{k=1}^{N} Q_k V_k$$

$$\therefore W_E = \frac{1}{2} \int_{vol} \rho_v V \, dv$$

Write $\rho_v = \bar{\nabla} \cdot \bar{D}$

$$\therefore W_E = \frac{1}{2} \int_{vol} (\bar{\nabla} \cdot \bar{D}) V \, dv$$

P.T.O.

Time averaged intensity ... avg ... $2n$

$$\bar{\nabla}\cdot(V\bar{D}) = V(\bar{\nabla}\cdot\bar{D}) + \bar{D}\cdot(\bar{\nabla}V)$$

$$\therefore V(\bar{\nabla}\cdot\bar{D}) = \bar{\nabla}\cdot(V\bar{D}) - \bar{D}\cdot(\bar{\nabla}V)$$

$$\therefore W_E = \frac{1}{2}\int_{vol}\left[\bar{\nabla}\cdot(V\bar{D}) - \bar{D}\cdot(\bar{\nabla}V)\right]dv$$

$$W_E = \frac{1}{2}\oint V\bar{D}\,d\bar{s} + \frac{1}{2}\int_{vol}\bar{D}\cdot\bar{E}\,dv$$

$\rightarrow 0$, for some goddamn reason

Electrostatic Work done $\therefore$ $\boxed{W_E = \frac{1}{2}\int_{vol}\bar{D}\cdot\bar{E}\,dv}$ (as used previously)

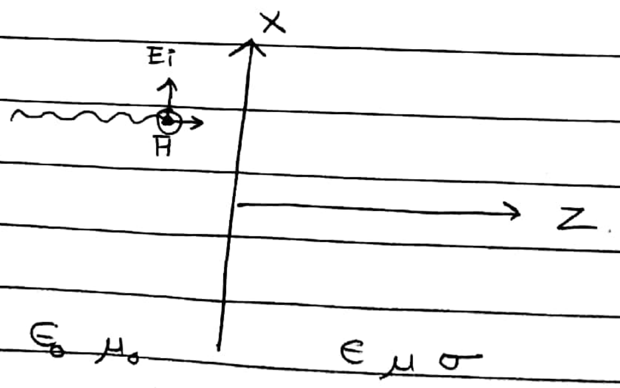
- Similarly, magnetic work done, $\boxed{W_M = \frac{1}{2}\int_{vol}\vec{B}\cdot\bar{H}\,dv}$

* General expression :- $W_E = \frac{1}{2}(\bar{D}\cdot\bar{E}^*) = \frac{1}{2}\epsilon_0|\bar{E}|^2$

$W_M = \frac{1}{2}(\bar{B}\cdot\bar{H}^*) = \frac{1}{2}\mu_0|\bar{H}|^2$

$$\boxed{\bar{S} = \bar{E}\times\bar{H}^*}$$

IV Plane Wave Reflection



Γ and T can be complex

classmate
Date
Page

- Define $k_0 = \omega\sqrt{\mu_0 \epsilon_0}$, $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$

- $\bar{E}_i = \hat{x} \; E_0 e^{-jk_0 z}$, $\bar{H}_i = \hat{y} \; \frac{E_0}{\eta_0} e^{-jk_0 z}$

- TFT :- Γ (reflection coefficient)

$\bar{E}_r = \hat{x} \, \Gamma E_0 e^{jk_0 z}$, $\bar{H}_r = -\hat{y} \; \frac{\Gamma E_0}{\eta_0} e^{jk_0 z}$

- TFT :- T (transmission coefficient)

$\bar{E}_t = \hat{x} \, T E_0 e^{\cancel{-jk_0 z} -\gamma z}$, $\bar{H}_t = \hat{y} \; \frac{T E_0}{\eta} e^{-\gamma z}$

where $\eta = \frac{j\omega\mu}{\gamma}$ and $\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\sqrt{1 - \frac{j\sigma}{\omega\epsilon}}$

- For $z < 0$,

$\bar{E}_i + \bar{E}_r = \hat{x} \; E_0 \left(e^{-jk_0 z} + \Gamma e^{jk_0 z}\right)$

At $z = 0$, $\bar{E}_i + \bar{E}_r = \hat{x} \; E_0 (1 + \Gamma)$ ..... tangential $\bar{E}$ for $z<0$

- For $z > 0$,

Tangential component $= \bar{E}_t$ (at $z = 0$) $= \hat{x} \, T E_0$

Continuity of $\bar{E}$ at $z = 0$

- ~~During Boundary Condition~~ :- ~~Tangential $\bar{E}$ does not change.~~

$\therefore \; \hat{x} \; E_0 (1 + \Gamma) = E_0 T$

$\therefore \; \boxed{1 + \Gamma = T}$

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- For $z<0$, $\bar{H}_i + \bar{H}_r = \frac{E_0}{\eta_0}(1-\Gamma)e^{jk_0 z}\,\hat{y}$

For $z>0$, $\bar{H}_t = \hat{y}\,\frac{TE_0}{\eta}\,e^{-\gamma z}$

Continuity of $\bar{H}$ at $z=0$:- $\quad 1-\Gamma = \frac{\eta_0}{\eta}T$

→ If second medium $(z>0)$ is also lossless,

$$\sigma = 0$$

$$\gamma = j\omega\sqrt{\mu\epsilon} = j\beta \quad (\alpha = 0)$$

$$= jk_0\sqrt{\mu_r\epsilon_r}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{k_0\sqrt{\mu_r\epsilon_r}} = \frac{\lambda_0}{\sqrt{\mu_r\epsilon_r}}$$

$$v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{\mu_r\epsilon_r}}$$

→ Poynting Vector :- (Lossless Medium)

- For $z<0$, $S^- = \bar{E}\times\bar{H}^*$

$$= (\bar{E}_i + \bar{E}_r)\times(\bar{H}_i + \bar{H}_r)^*$$

$$= \hat{x}\,E_0\left(e^{-jk_0 z} + \Gamma e^{jk_0 z}\right)\times\hat{y}\,\frac{E_0^*}{\eta_0}\left(e^{-jk_0 z} - \Gamma^* e^{jk_0 z}\right)^*$$

$$= \hat{z}\,\frac{E_0^2}{\eta_0}\left[1 - |\Gamma|^2 + 2j\,\Gamma\sin(2k_0 z)\right]$$

At $z=0$, $S^- = \hat{z}\,\frac{E_0^2}{\eta_0}\left[1-|\Gamma|^2\right]$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad T = \frac{2\eta_2}{\eta_2 + \eta_1}$$

- For $z > 0$, $S^+ = \bar{E}_t \times \bar{H}_t^*$

$$= \hat{x}\, T E_0 e^{-\gamma z} \times \hat{y}\, \frac{T^* E_0^*}{\eta^*} e^{-\gamma^* z}$$

$$= \hat{z}\, \frac{|T|^2 E_0^2}{\eta} \qquad (\text{at } z = 0)$$

$$= \hat{z}\, \frac{\cancel{4}\, 4\eta^2}{(\eta+\eta_0)^2} \times \frac{E_0^2}{\eta}$$

$$= \hat{z}\, \frac{4\eta\, |E_0|^2}{(\eta+\eta_0)^2}$$

$$= \hat{z}\, \frac{E_0^2}{\eta_0}\left(1 - |\Gamma|^2\right)$$

- $\therefore$ $\boxed{\bar{S}^- = \bar{S}^+}$ at boundary

- $P^- = \frac{1}{2} \mathrm{Re}\left[S^- \cdot \hat{z}\right]$ and $P^+ = \frac{1}{2} \mathrm{Re}\left[S^+ \cdot \hat{z}\right]$

Because of time-averaging of square of sinusoidal wave

- Define $\bar{S}_i \triangleq \bar{E}_i \times \bar{H}_i^* = \hat{z}\, \frac{E_0^2}{\eta_0}$ at $z = 0$

$\bar{S}_r \triangleq \bar{E}_r \times \bar{H}_r^* = -\hat{z}\, \frac{E_0^2 |\Gamma|^2}{\eta_0}$ at $z = 0$

For any z, note that $\bar{S}_i + \bar{S}_r \neq \bar{S}^-$

$\bar{S}_i + \bar{S}_r$ does not have the sinusoidal term

- $\therefore$ $\bar{S}_i + \bar{S}_r = \mathrm{Re}(\bar{S}^-)$ = Real Power transferred
- The imaginary sinusoidal term in $\bar{S}^-$ denotes energy stored in standing waves