Digital Image Processing

Fourier Analysis – 1

Series, Transform, Properties

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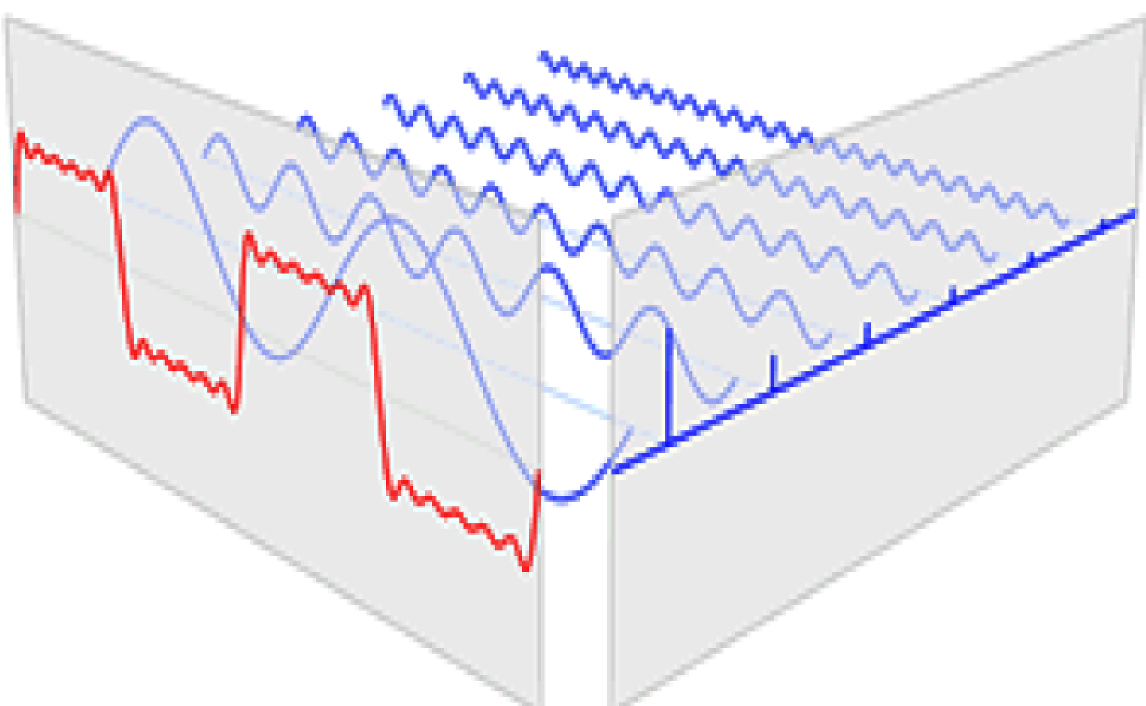
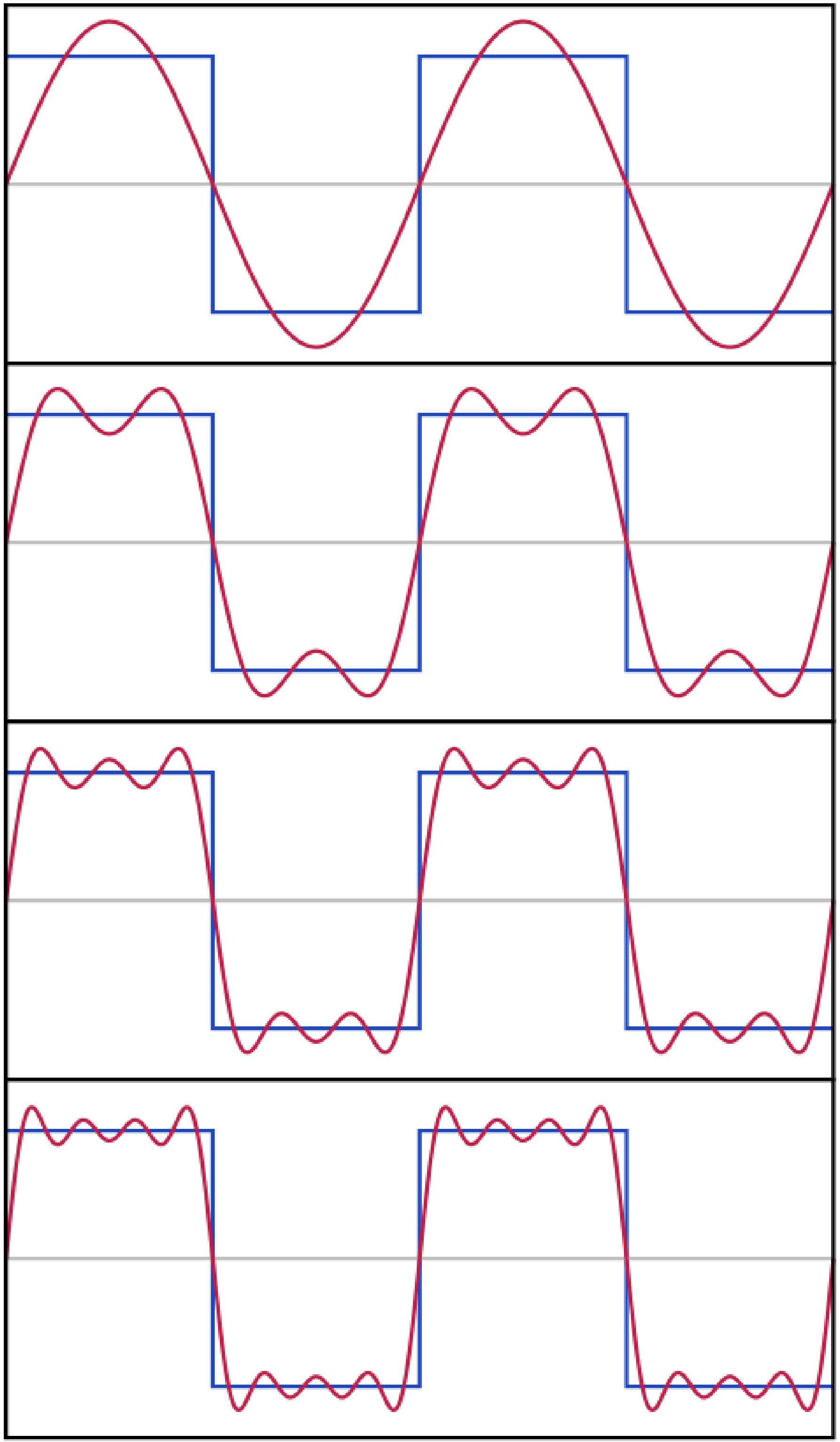
Fourier Analysis

- Joseph Fourier
  - Mathematician and physicist
  - 1768 – 1830
  - Invented Fourier series for studying heat transfer
  - PhD Guide : Lagrange
  - Took part in French revolution
- First application of Fourier analysis to digital images in 1960s

Fourier Analysis

- Fourier Series
  - Represent a function as a linear combination (scaling + superposition) of sinusoidal functions





Fourier Analysis

- Sinusoidal waves
  - $f(t) = \cos(wt)$
  - $f(t) = \sin(wt)$
  - Period = $2\pi/w$
  - Frequency = 1 / period = $w/(2\pi)$
  - Larger ω → shorter period and higher frequency

Fourier Analysis

- Complex-valued sinusoidal waves
  - Complex-valued waves e^{int}
  - n = **integer**
  - **Harmonic frequencies**
    - **Fundamental** frequency : (n = 1)
      - 1st harmonic
      - This isn't the lowest frequency
      - Lowest frequency : (n = 0) $\rightarrow$ exp (i 0 t) = 1 = constant "wave"
    - Other frequencies = **integer** multiples of fundamental frequency
      - m = …, -2, -1, 0, 1, 2, ...

Fourier Analysis

- Assume a function **f(t) defined on domain = [0, 2π]**
- Assume that f(t) can be represented as a
  linear combination of
  complex sinusoidal waves of harmonic frequencies
  $f(t) := \sum_{n \in I} c_n e^{int}$, where c_n is complex and $0 \leq t \leq 2\pi$
  - Frequencies = n / 2π
- c_n = coefficients (complex)
- Problem
  - Given f(t)
  - Find coefficients c_n

Fourier Analysis

- Problem
  - Given f(t)
  - Find coefficients c_n
- Important Observation 1
  - Take integral of complex wave e^{int} with e^{-imt}, s.t. $m \neq n$

$$\int_0^{2\pi} e^{int} e^{-imt} dt = \int_0^{2\pi} e^{i(n-m)t} dt$$

$$= \frac{e^{i(n-m)t}}{i(n-m)} \Bigg|_0^{2\pi} = \frac{1}{i(n-m)} (e^{i(n-m)2\pi} - e^0)$$

$$= \frac{1}{i(n-m)} (1-1) = 0$$

Fourier Analysis

- Problem
  - Given f(t)
  - Find coefficients c_n
- Important Observation 2
  - Take integral of complex wave e^{int} with e^{-imt}, s.t. m = n

$$\int_0^{2\pi} e^{int} e^{-int} dt = \int_0^{2\pi} e^0 dt = 2\pi$$

Fourier Analysis

- Problem
  - Given f(t)
  - Find coefficients c_n
- Solution ?
  - We can get $\mathbf{c_n}$ by integrating product of f(t) with $\mathbf{e^{-int}}$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt$$

Fourier Analysis

- Simpler example to gain insights
  - Consider 3D Euclidean space
  - Consider 3 unit vectors orthogonal to each other
    - $\langle \bar{i}, \bar{j}, \bar{k} \rangle$ is an orthogonal basis
    - Basis = Set of linearly-independent vectors that can be used to represent any other vector as a linear combination
      - Coordinate system
  - Consider an arbitrary vector in 3D that can be represented as a linear combination of basis vectors

    $$\bar{x} = x_1 \bar{i} + x_2 \bar{j} + x_3 \bar{k}$$
  - Problem
    - Given x
    - How do we find x1, x2, x3 ?

Fourier Analysis

- Simpler example to gain insights
  - Consider an arbitrary vector in 3D that can be represented as a linear combination of basis vectors

    $$\overline{x} = x_1\overline{i} + x_2\overline{j} + x_3\overline{k}$$

  - Problem
    - Given x
    - How do we find x1, x2, x3 ?
  - Solution
    - Take dot product of x with each basis vector

Fourier Analysis

- Dot product = inner product
  - Summation of component-wise products of vector values
- How does inner product generalize to the space of functions ?
  - Integration of products of function values
  - If functions are **real** valued: $\langle u, v \rangle = \int_a^b u(x)v(x)dx$
  - If functions are **complex** valued: $\langle \psi, \chi \rangle = \int_a^b \psi(x)\overline{\chi(x)}dx$
    - Bar denotes conjugate

Fourier Analysis

- Inner product of e^{int} with e^{imt}
  - Integral of complex wave e^{int} with e^{-imt}
    - Constant when $m = n$
    - 0 with $m <> n$
- **Set of complex waves $\{ e^{int} \}$ = an orthogonal set**
  - Actually: An orthogonal basis in Hilbert space of square-integrable complex functions defined on $[0,2\pi]$
- **Hilbert space**
  = Vector space of functions + defined inner product

Fourier Analysis

- **Vector space** defines operations of addition, scaling on elements of this space (called vectors)

Axiom	Meaning
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of addition	For every $\mathbf{v} \in \mathrm{V}$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [nb 2]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Fourier Analysis

- **Inner-product** definition must satisfy 3 conditions :
  - (1) Conjugate Symmetry : $\langle f, g \rangle = \langle g, f \rangle^*$
  - (2) Linearity in first argument :

$$\langle a_1 f_1 + a_2 f_2, g \rangle = a_1 \langle f_1, g \rangle + a_2 \langle f_2, g \rangle$$

  - (3) Positive definite : $\langle f, f \rangle \geq 0$ with equality iff $f = 0$
    - A zero function is a function that is almost-everywhere zero
  - Check that the inner product on complex-valued functions satisfies these 3 conditions

Fourier Analysis

- Assume that f(t) can be represented as a linear combination of complex sinusoidal waves of harmonic frequencies

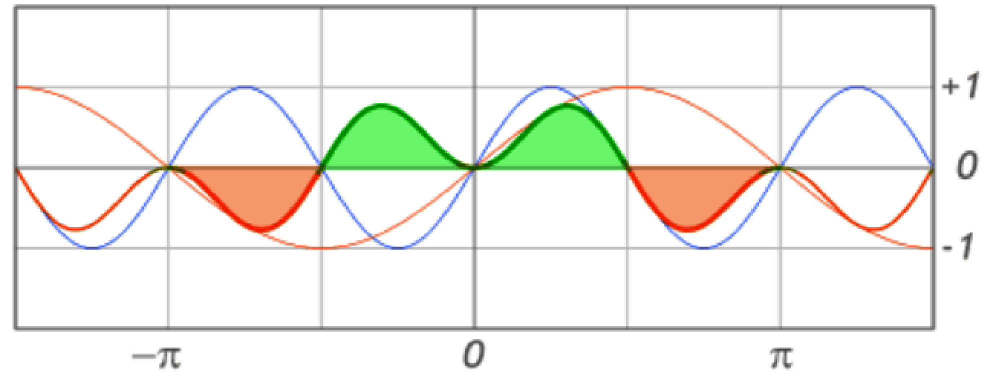
  $f(t) := \sum_{n \in I} c_n e^{int}$, where c_n is complex and $0 \leq t \leq 2\pi$
- Problem
  - Given f(t)
  - How do we find coefficients c_n ?
- Solution
  - Inner product of f(t) with e^{int}

    = integral of product of f(t) with **conjugate of e^{int}**

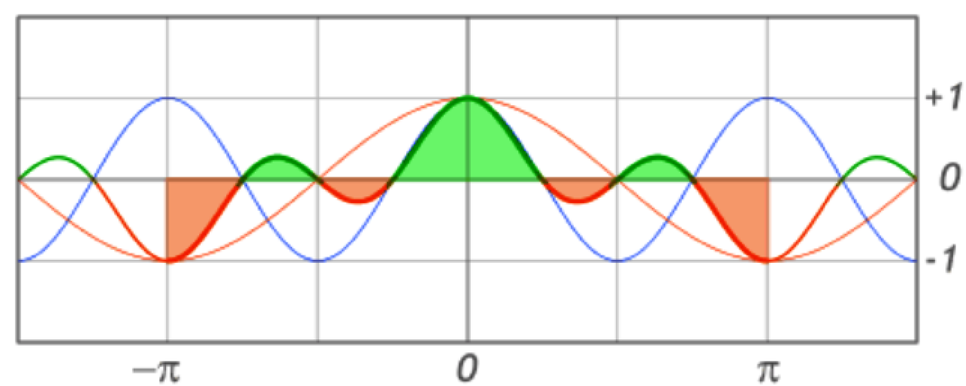
    = integral of product of f(t) with e^{-int}

Fourier Analysis

- Fourier **Series**
  - Orthogonal basis
    - Inner product of real-valued functions
      - Integral f(x) g(x)
    - Set of all **sine waves**
      - Integral sin(mx) sin(nx)
        - Pi; if integer m = n
        - 0; otherwise
    - Set of all **cosine waves**
      - Integral cos(mx) cos(nx)
        - Pi; if integer m = n
        - 0; otherwise



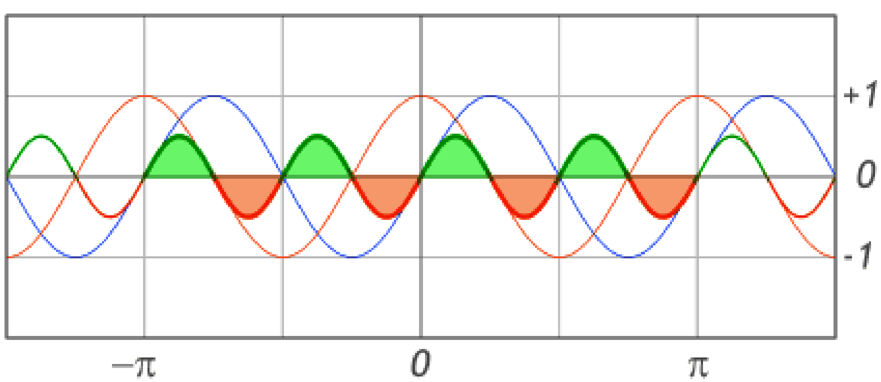
$$\int_{-\pi}^{+\pi} \sin(2x)\sin(1x)\ dx = 0$$



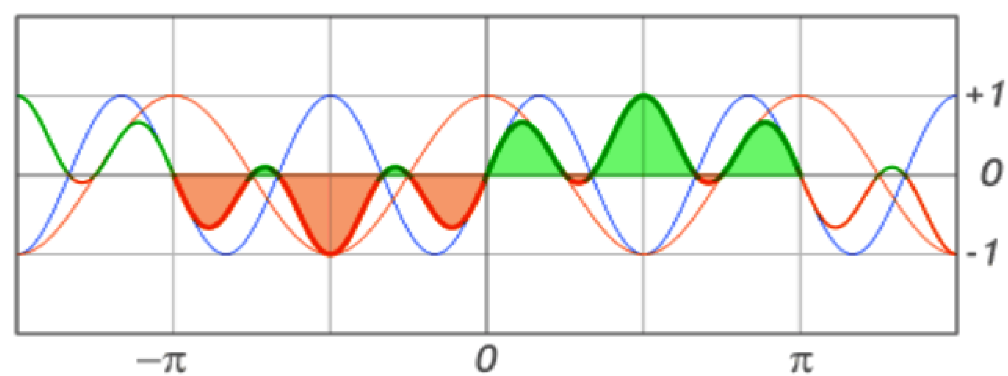
$$\int_{-\pi}^{+\pi} \cos(2x)\cos(1x)\ dx = 0$$

Fourier Analysis

- Fourier **Series**
  - Orthogonal basis
    - Set of all **sine waves and cosine waves**
      - Integral sin (mx) cos (nx)
        - 0; for any integer m, n



$$\int_{-\pi}^{+\pi} \sin(2x)\cos(2x)\ dx = 0$$



$$\int_{-\pi}^{+\pi} \sin(3x)\cos(2x)\ dx = 0$$

Fourier Analysis

- Do we need **both** sine waves **and** cosine waves ?
  - Yes
  - Because we need to represent functions with non-zero phase

$$\sin(wt + \phi) \;:=\; \sin(wt)\cos(\phi) + \cos(wt)\sin(\phi)$$

  - Observe: The set of all sine waves cannot represent a cosine wave !
    - Why not ?
    - What is the intuition ?
    - What is an algebraic argument / proof ?

Fourier Analysis

- Fourier **Series**
  - Orthogonal basis
    - In case of complex functions
      - Inner product uses conjugate
      - Integral f(x) g*(x)
    - Set of all **complex waves**
      - Integral of exp (i n x) exp (- i m x)

        = Integral of (cos(nx) + i sin(nx)) (cos(mx) - i sin(mx))
        - 2 Pi; if m = n
        - 0; otherwise

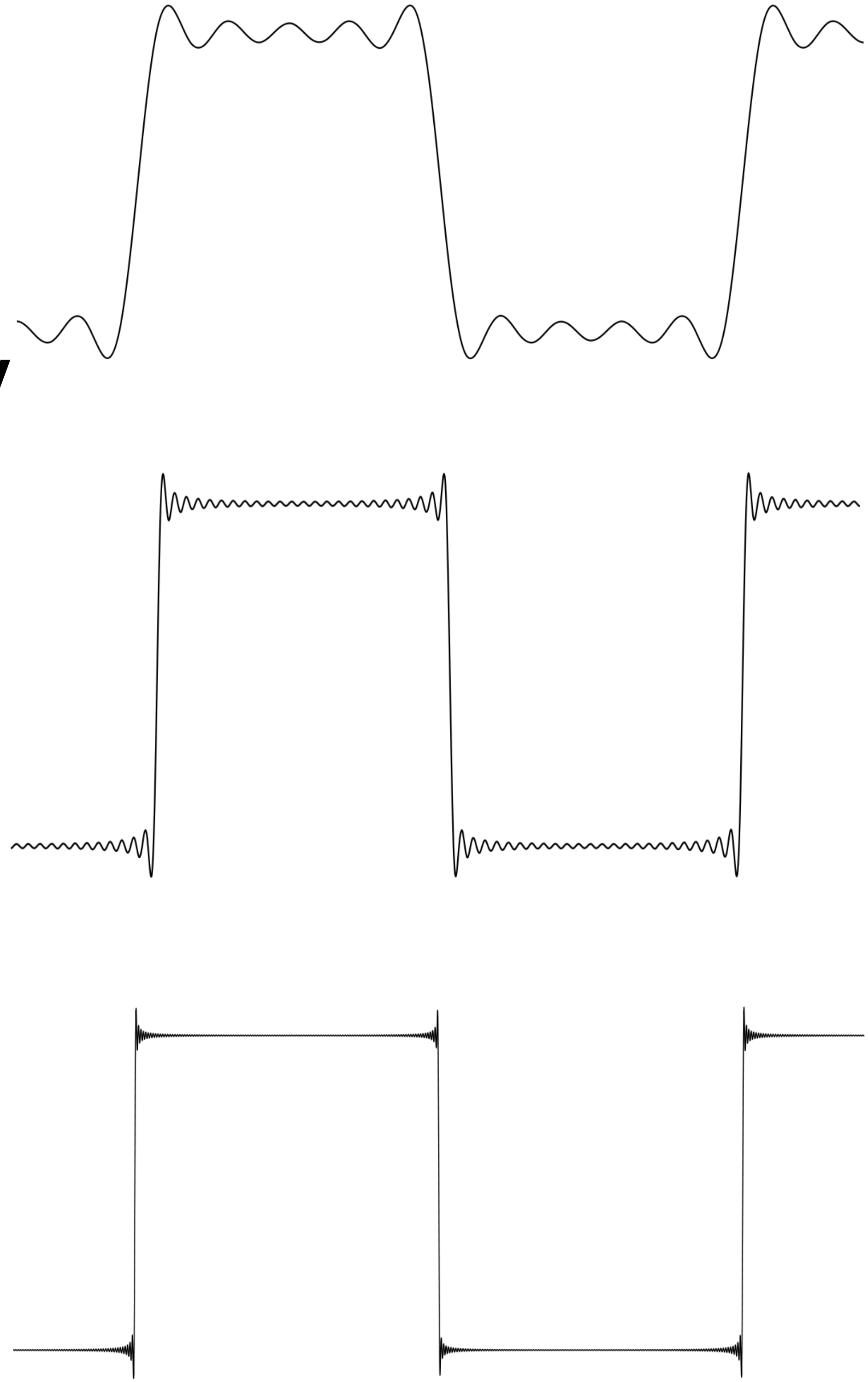
Fourier Analysis

- Fourier **Series**
  - Functions with **jump (or step) discontinuity**
    - Left limit exists, finite
    - Right limit exists, finite
    - These are unequal
  - **Gibbs Phenomenon**

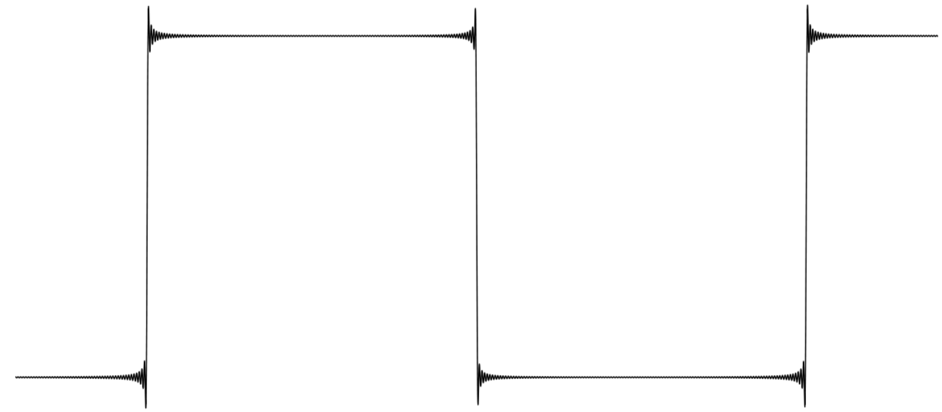
    (1) **Overshoot** and **undershoot** around jump discontinuity

    - Approx. 9% of jump magnitude

    (2) **Oscillations** around discontinuity → "**ringing**"



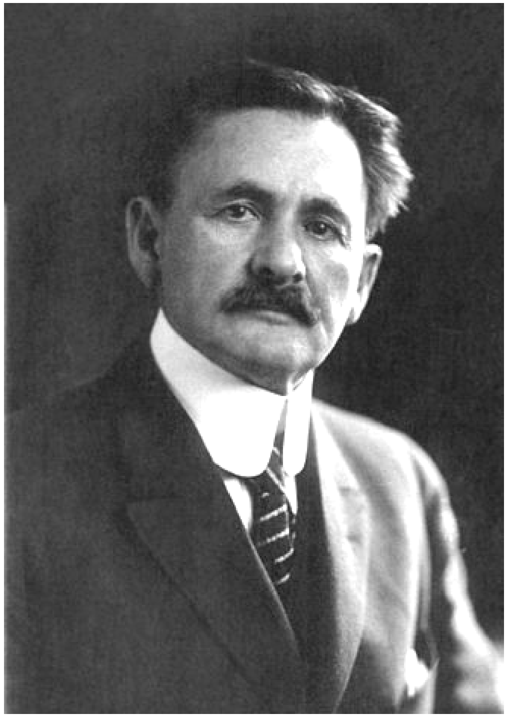
Fourier Analysis



- Fourier **Series**
  - **Gibbs Phenomenon**
    - Concerns
      Convergence of the sequence of approximated functions
      (as a Fourier series using waves of increasing frequencies),
      to the original function f(x)
    - Series for f(x) converges to f(x), at all x,
      except at jump discontinuities
      - Original function : $f(x-) \rightarrow a$. $f(x+) \rightarrow b > a$.
      - Series : $f(x-) \rightarrow a - 0.09(b-a)$. $f(x+) \rightarrow b + 0.09(b-a)$. $f(x) = (a+b) / 2$
    - What does this mean in practice ?
      - As $n \rightarrow \infty$, **integral of squared error** $\rightarrow 0$
      - Mismatches at isolated points → don't change practical system behavior
      - But, convergence is “infinitely slow” → large 'n' needed

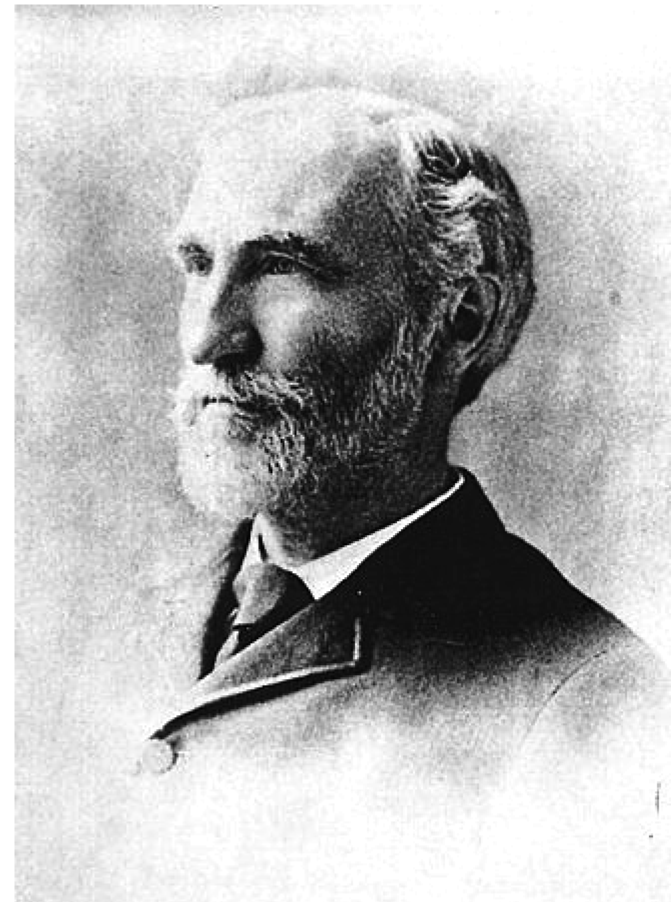
Fourier Analysis

- Albert Michelson (1852 – 1931)
  - In 1898, built machine to show Fourier series representation using large (finite) 'n'
    - Perhaps didn't see ringing and over/under shooting because of low quality of graphs output by machine
  - Experimental physicist
    - Speed of light
    - Relativity
  - Nobel Prize in Physics in 1907
    - First American to receive Nobel Prize in sciences



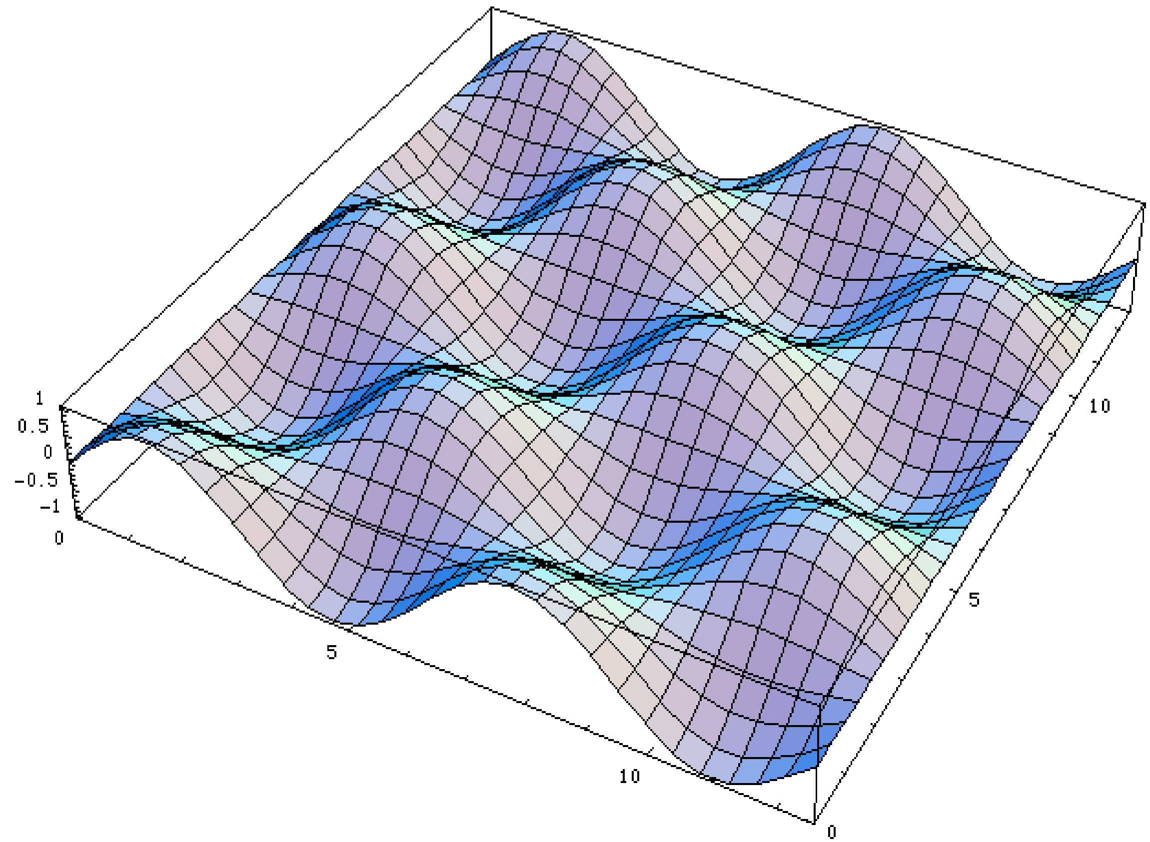
Fourier Analysis

- Josiah Gibbs (1839 – 1903)
  - Explains Fourier-series convergence phenomenon mathematically (1899)
  - Einstein called him
    “Greatest mind in American history”
    - First American PhD in engineering; at Yale University
  - Scientist
    - Physics, chemistry, mathematics
    - Thermodynamics
      - Gibbs free energy
      - Statistical mechanics
    - Invented vector calculus

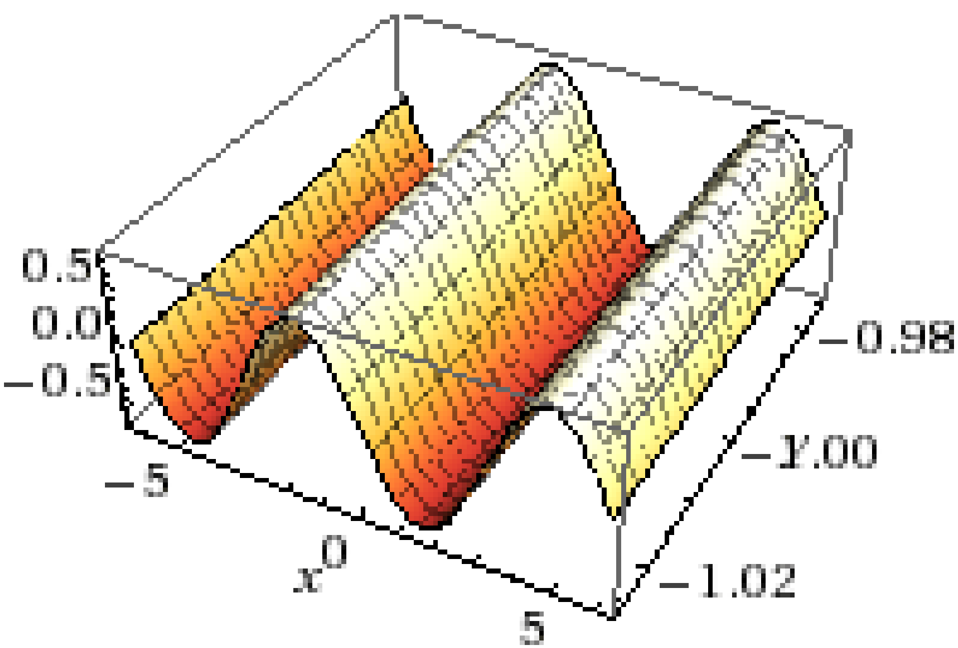


Fourier Analysis

- Fourier series (2D)
  - A 2D function →



  - A basis element



Fourier Analysis

- **Periodic** functions
  - If a function defined on $0 \leq t \leq 2\pi$ has the form

    $$f(t) := \sum_{n \in I} c_n e^{int}$$

    where cn is complex

    and $|c_1| > 0$ or $|c_{-1}| > 0$

    Then, f(t) is periodic with **period 2π**
  - Inner-product integral limits will be
    - $0 \to 2\pi$
  - **Fundamental frequency: 1 / 2π**
  - Frequencies = ..., -2, -1, 0, 1, 2, ...
  - **Separation** between harmonic frequencies: **1 / 2π**

Fourier Analysis

- Periodic functions
  - **Period can be modified by (re)scaling time axis**
    e.g., f(t) = $\sum_{n \in I} c_n e^{-int2\pi/T}$
    with $|c_1| > 0$ or $|c_{-1}| > 0$
    has **period** 2π / (2π/T) = **T**
  - Inner-product integral limits can be
    - $0 \rightarrow T$
  - **Fundamental frequency = 1/T**
  - Frequencies = ..., -2/T, -1/T, 0, 1/T, 2/T, ...
  - **Separation** between harmonic frequencies = **1/T**

Fourier Analysis

- Periodic functions
  - **Period interval can be modified by shifting time axis**

    e.g., $t' \leftarrow t - t0$
  - Then, interval that contains one period changes
    from [0, T]
    to [t0, t0 + T]
- **Fourier series decomposes signals that are :**
  defined on interval [t0, t0 + T],
  defined periodically outside interval
  - Inner-product integral limits become t0 and t0 + T
  - Inner product of wave with itself = T
  - Coefficients obtained by scaling down integral by T

Fourier Analysis

- **Going from Fourier Series → Fourier Transform**
  - Think of a **Fourier series** where **period T** → ∞
  - Separation between harmonic frequencies → 0
  - **Fundamental frequency (1st-harmonic frequency) → 0**

- **Fourier transform** extends Fourier series :
  - Allows **all real frequencies** ("n" needn't be integer)
  - **Doesn't assume** signal to be **periodic**

Fourier Analysis

- **Fourier Transform**
  - Definition:
    Fourier transform of an absolutely-integrable function f (x) is defined for each **real number w** as :

    $$F f(w) := \int_{x=-\infty}^{\infty} f(x) e^{-iwx} dx$$

    where
  - w is frequency
  - F f (w) = amplitude of complex wave having frequency w
    - Absolute integrability is a sufficient (not necessary) condition for existence of Ff(w)

Fourier Analysis

- **Linearity** of the Fourier transform
  - $F\,(f + g)\,(w) = F\,f\,(w) + F\,g\,(w)$
    - Proof follows from definition
  - $F\,(a\,f)\,(w) = a\,F\,f\,(w)$
    - Proof follows from definition

Fourier Analysis

- **Inverse** Fourier Transform
  - For a function h(w),
    the inverse Fourier transform is defined
    for reach real number x as :

$$F^{-1}h(x) := \frac{1}{2\pi}\int_{w=-\infty}^{\infty} h(w)e^{iwx}dw$$

Fourier Analysis

- **Fourier Inversion Theorem**
  - If f (x) is continuous : $\forall x, F^{-1}(Ff)(x) := f(x)$
    - Start with f(x) → Define Ff(w) via FT → Define g(y) via IFT
    - Then, g(y) = f(y), for all y

If $Ff(w) := \int_{x=-\infty}^{\infty} f(x)e^{-iwx}dx$ Fourier Transform

And If $g(y) := \int_{w=-\infty}^{\infty} Ff(w)e^{iwy}dw$ Inverse Fourier Transform

$$= \int_{w=-\infty}^{\infty} \left(\int_{x=-\infty}^{\infty} f(x)e^{-iwx}dx \right) e^{iwy}dw$$

$$= \int_{w=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f(x)e^{-iw(x-y)}dxdw$$

Then, $g(y) = f(y), \forall y$

Fourier Analysis

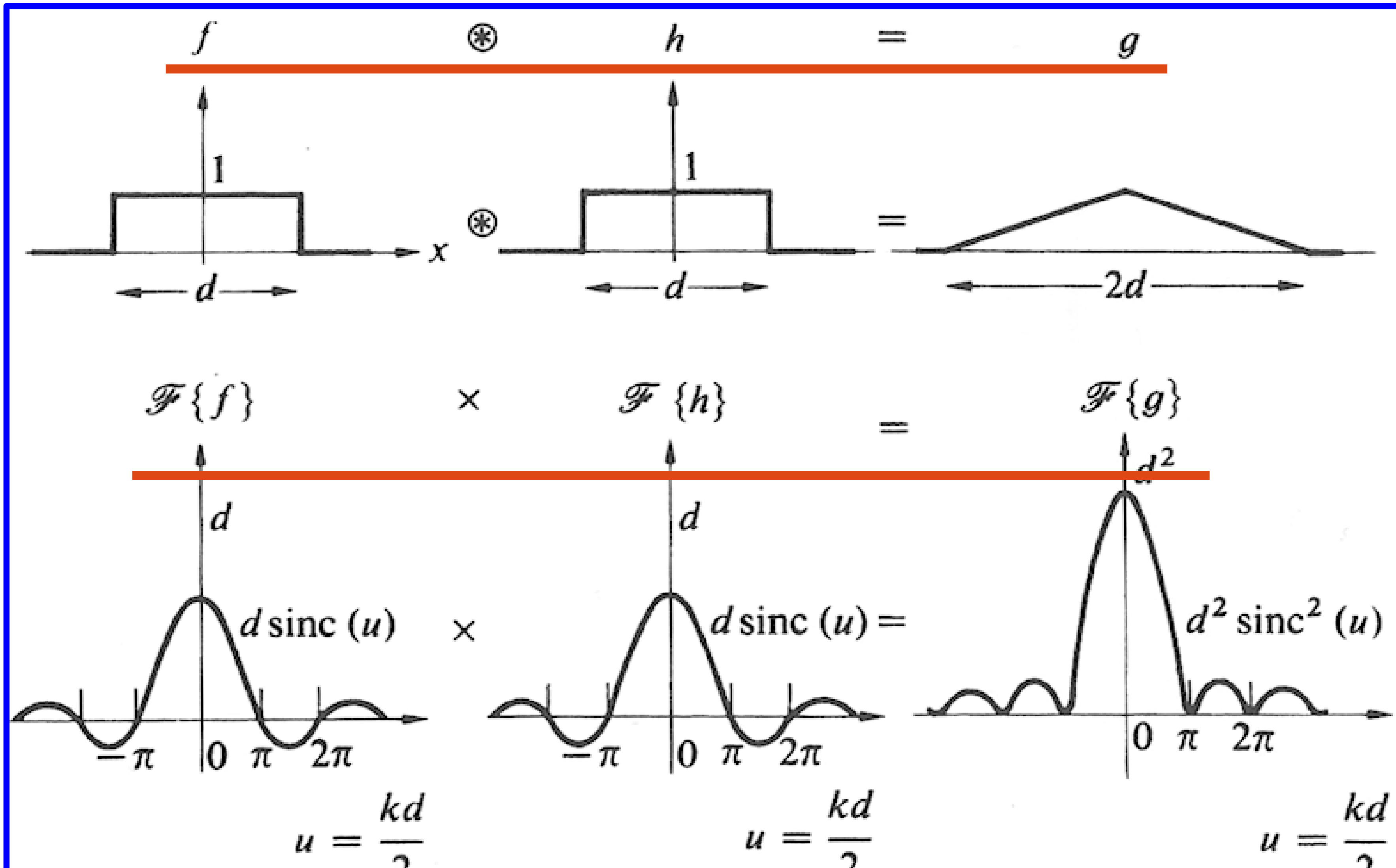
- **Fourier Transform and Convolution**
  - Let f(x) and g(x) be 2 functions with Fourier transforms Ff(w) and Fg(w)
  - **Theorem:**
    Product of Fourier transforms of f(.) and g(.)
    = Fourier transform of convolution of f(.) and g(.)

    $$Ff(w) \cdot Fg(w) = F(f * g)(w)$$
  - Illustration and Proof … next

Fourier Analysis

- Fourier Transform and Convolution

$$Ff(w) \cdot Fg(w) = F(f * g)(w)$$



Fourier Analysis

- Fourier Transform and Convolution
  - Product of Fourier transforms of f(.) and g(.)
    = Fourier transform of convolution of f(.) and g(.)

$$Ff(w) \cdot Fg(w) := \int_{x=-\infty}^{\infty} f(x) e^{-iwx} dx \int_{y=-\infty}^{\infty} g(y) e^{-iwy} dy$$

$$= \int_{x=-\infty}^{\infty} f(x) e^{-iwx} dx \int_{s=-\infty}^{\infty} g(s-x) e^{-iw(s-x)} ds$$ Substitute s := y + x

$$= \int_{s=-\infty}^{\infty} \left(\int_{x=-\infty}^{\infty} f(x) g(s-x) dx \right) e^{-iws} ds$$ Rearranging terms

$$= \int_{s=-\infty}^{\infty} (f * g)(s) e^{-iws} ds$$ Definition of convolution

$$= F(f * g)(w)$$ Definition of Fourier transform

Fourier Analysis

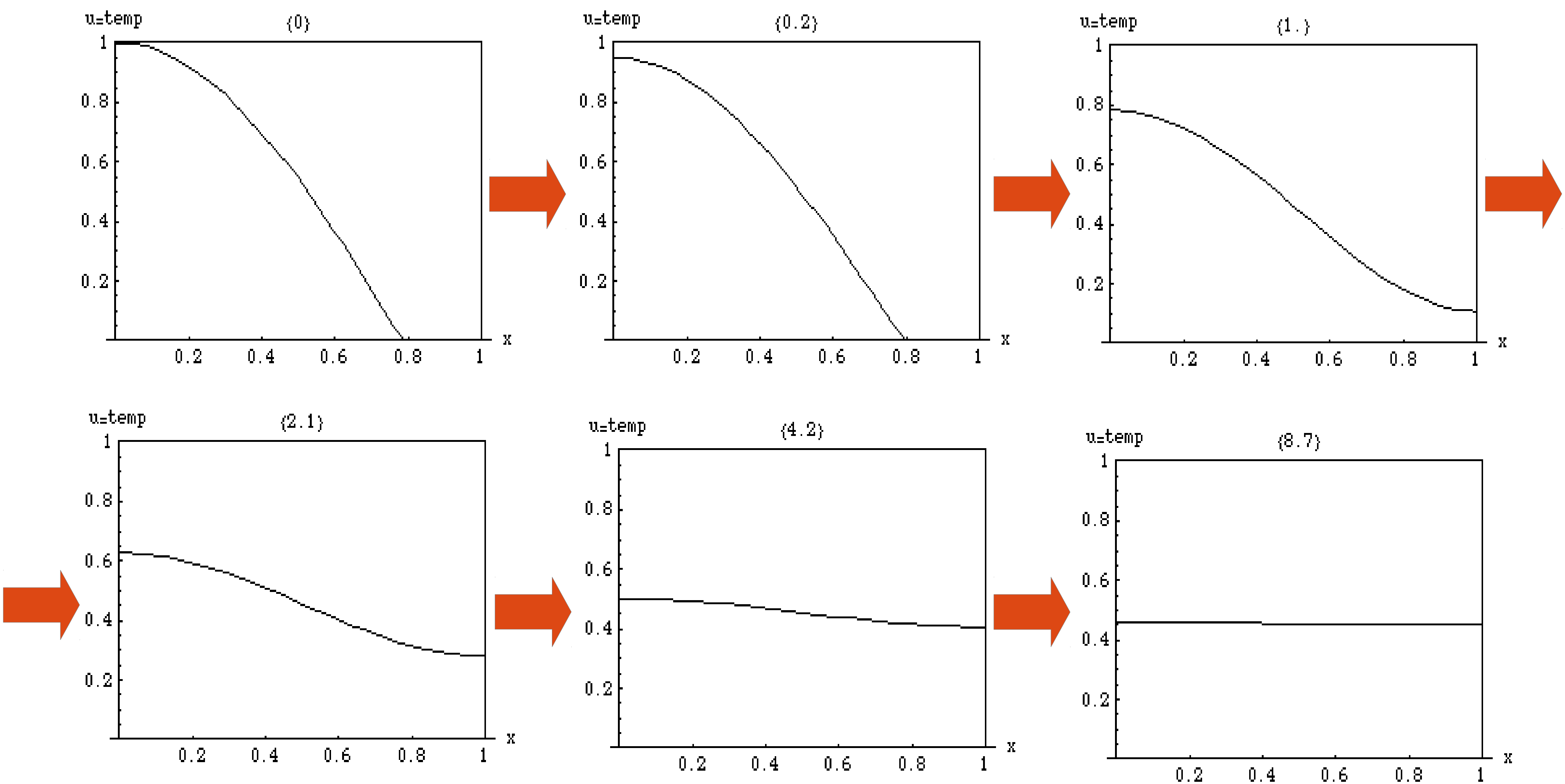
- Fourier Transform and Convolution

(1) $Ff(w) \cdot Fg(w) = F(f * g)(w)$

(2) Similarly, F (f(x) . g(x)) = (Ff * Fg) (w)

Fourier Analysis

- Why was Fourier analysis invented ?
  - To study heat flow (diffusion of thermal energy)
  - The Analytic Theory of Heat. J Fourier. 1822.



Fourier Analysis

- Heat equation
  - Consider a function over space and time : f (x, t)
    - where x = location, t = time
    - e.g., temperature distribution within an object over time
  - Rate of change (in time) of f(x) is proportional to second spatial derivative (Laplacian) of f(x) :

$$f_t(x,t) = \alpha f_{xx}(x,t)$$

- **Problem**
  - **Given**: f (x, t = 0), for all x
  - **Find**: f (x, t = T), for all x
  - **Strategy**: Analyze in frequency domain !

Fourier Analysis

- Heat equation

$$f_t(x,t) = \alpha f_{xx}(x,t)$$

  - **Fourier transform** (over x) of the **left hand side** is:

$$\begin{aligned} F f_t(w,t) &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} f(x,t) e^{-iwx} dx \\ &= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} f(x,t) e^{-iwx} dx \\ &= \frac{\partial}{\partial t} F f(w,t) \end{aligned}$$

    - Switching order of integral (w.r.t. ‘x’) and derivative (w.r.t. ‘t’)

Fourier Analysis

- Heat equation $$f_t(x,t) = \alpha f_{xx}(x,t)$$
  - **Fourier transform** of the **right hand side** = ?
    - Simple way to find Fourier transform of derivatives of f(x)

$$f(x) = \int_{w=-\infty}^{\infty} F(w) e^{iwx} dw$$ Fourier inversion theorem

$$f_x(x) = \int_{w=-\infty}^{\infty} (iw) F(w) e^{iwx} dw$$ Differentiate both sides w.r.t. x

$$f_{xx}(x) = \int_{w=-\infty}^{\infty} (iw)^2 F(w) e^{iwx} dw$$ Differentiate both sides w.r.t. x

Thus, the Fourier transform of f_{xx} is:

$$\int_{x=-\infty}^{\infty} f_{xx}(x) e^{-iwx} dx = (iw)^2 F(w)$$

Fourier Analysis

- Heat equation $f_t(x,t) = \alpha f_{xx}(x,t)$
  - **In the Fourier domain**, the heat equation is:

$$\frac{\partial Ff(w,t)}{\partial t} = \alpha(-w^2)Ff(w,t)$$

$$\frac{\partial Ff(w,t)}{Ff(w,t)} = \alpha(-w^2)\partial t$$

  - Now integrate RHS from t = 0 to t = T, and integrate LHS from Ff(w,0) to Ff(w,T)

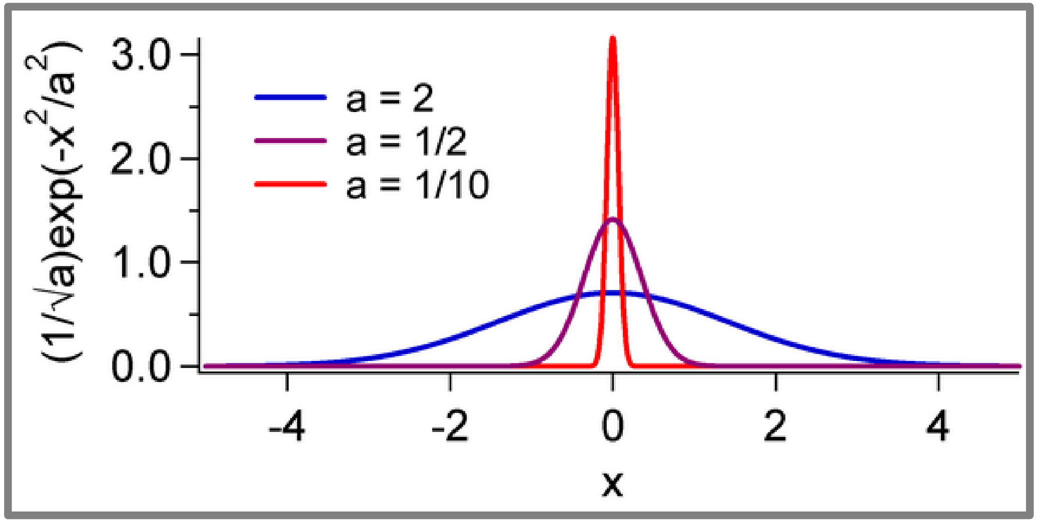
$$[\log Ff(w,t)]_{t=0}^{T} = -\alpha w^2 t|_{t=0}^{T} = -\alpha w^2 T$$

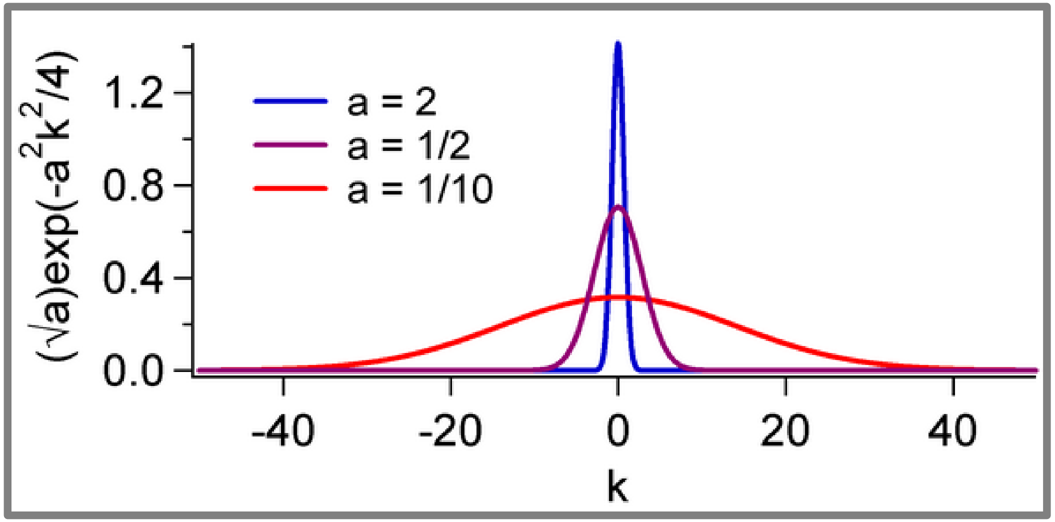
$$Ff(w,t=T) = e^{-\alpha w^2 T} Ff(w,t=0)$$

  - What does this tell us ?

Fourier Analysis

- Heat equation $f_t(x,t) = \alpha f_{xx}(x,t)$
  - In Fourier domain $Ff(w,t=T) = e^{-\alpha w^2 T} Ff(w,t=0)$
  - **Fourier** transform of function at time t=T equals Fourier transform of initial function, at t=0, **multiplied** by Gaussian (scaled)
    - Variance of the Gaussian proportional to (1 / T)
  - In **spatial** domain, function at time t=T equals **convolution** of initial function, at t=0, with … what ?
    - Inverse Fourier transform of Gaussian in frequency domain = ?





Fourier Analysis

- Heat equation $f_t(x,t) = \alpha f_{xx}(x,t)$
  - In Fourier domain
  $$F f(w, t=T) = e^{-\alpha w^2 T} F f(w, t=0)$$
  - **Fourier** transform of function at time t=T equals Fourier transform of initial function, at t=0, **multiplied** by Gaussian (scaled)
    - Variance of the Gaussian proportional to (1 / T)
  - In **spatial** domain, function at time t=T equals **convolution** of initial function, at t=0, with Gaussian
    - Variance of Gaussian proportional to T
  - Thus, a function evolving based on the heat equation → function undergoing increasing Gaussian smoothing

Fourier Analysis

- Heat equation
  - The heat equation is a partial differential equation (PDE) that defines “isotropic” diffusion on the function
    - Isotropic → same in all directions
    - NOT edge preserving
    - **Linear** filter (convolution in spatial domain)
  - “Anisotropic” diffusion
    - Diffuse / average intensities along the edge, NOT across the edge
    - **Nonlinear** filtering in spatial domain