Problem 4.4

Consider the narrow-band FM wave approximately defined by Eq. (4.17). Building on Problem 4.3, do the following:

(a) Determine the envelope of this modulated wave. What is the ratio of the maximum to the minimum value of this envelope?

(b) Determine the average power of the narrow-band FM wave, expressed as a percentage of the average power of the unmodulated carrier wave.

(c) By expanding the angular argument $\theta(t) = 2\pi f_c t + \phi(t)$ of the narrow-band FM wave $s(t)$ in the form of a power series and restricting the modulation index β to a maximum value of 0.3 radian, show that

$$\theta(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

What is the value of the harmonic distortion for $\beta = 0.3$ radian?

Hint: For small x, the following power series approximation

$$\tan^{-1}(x) \approx x - \frac{1}{3}x^3$$

holds. In this approximation, terms involving x^5 and higher order ones are ignored, which is justified when x is small compared to unity.

Solution

(a) From Eq. (4.17), the narrow-band FM wave is approximately defined by

$$s(t) \approx A_c \cos((2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\sin(2\pi f_m t)) \qquad (1)$$

The envelope of $s(t)$ is therefore

$$\begin{aligned} a(t) &= A_c(1 + \beta^2 \sin^2(2\pi f_m t))^{1/2} \\ &\approx A_c\left(1 + \frac{1}{2}\beta^2 \sin^2(2\pi f_m t)\right)^{1/2} \qquad \text{for small } \beta \end{aligned}$$

The maximum value of $a(t)$ occurs when $\sin^2(2\pi f_m t) = 1$, yielding

$$A_{\text{max}} \approx A_c\left(1 + \frac{1}{2}\beta^2\right)$$

The minimum value of $a(t)$ occurs when $\sin^2(2\pi f_m t) = 0$, yielding

$$A_{\text{min}} = A_c$$

The ratio of the maximum to the minimum value is therefore

$$\frac{A_{\text{max}}}{A_{\text{min}}} \approx \left(1 + \frac{1}{2}\beta^2\right)$$

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Problem 4-4 continued

(b) Expanding Eq. (1) into its individual frequency components, we may write

$$s(t) \approx A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c \cos(2\pi(f_c + f_m)t) - \frac{1}{2}\beta A_c \cos(2\pi(f_c - f_m)t)$$

The average power of $s(t)$ is therefore

$$P_{\text{av}} = \frac{1}{2}A_c^2 + \left(\frac{1}{2}\beta A_c\right)^2 + \left(\frac{1}{2}\beta A_c\right)^2$$

$$= \frac{1}{2}A_c^2(1 + \beta^2)$$

The average power of the unmodulated carrier is

$$P_c = \frac{1}{2}A_c^2$$

Hence,

$$\frac{P_{\text{av}}}{P_c} = 1 + \beta^2$$

(c) The angle $\theta(t)$ is defined by

$$\theta(t) = 2\pi f_c t + \phi(t)$$

$$= 2\pi f_c t + \tan^{-1}(\beta \sin(2\pi f_m t))$$

Setting $\beta = \sin(2\pi f_m t)$

and using the approximation (based on the Hint), we may approximate $\theta(t)$ as

$$\theta(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t) - \frac{1}{3}\beta^3 \sin(2\pi f_m t)$$

Ideally, we should have (see Eq. (4.15))

$$\theta(t) = 2\pi f_c t + \beta \sin(2\pi f_m t)$$

The harmonic distortion produced by using the narrow-band approximation is therefore

$$D(t) = \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

The maximum absolute value of $D(t)$ for $\beta = 0.3$ is therefore

$$D_{\max} = \frac{\beta^3}{3}$$

$$= \frac{0.3^3}{3} = 0.009 \approx 1\%$$

which is small enough for it to be ignored in practice.