Problem 4.25

(a) The loop filter in the second-order phase-locked loop (PLL) is defined by

$$H(f) = 1 + \frac{a}{jf} \tag{1}$$

where a is a filter parameter. The Fourier transform of phase error $\phi_e(t)$ (i.e., the phase difference between the phase of the FM signal applied to the PLL and the phase of the FM signal produced by the VCO) in the PLL is defined by (see part (a) of Problem 4.7)

$$\Phi_e(f) = \frac{1}{1 + L(f)}\Phi_1(f) \tag{2}$$

where the loop transfer function is itself defined by

$$L(f) = K_0 \frac{H(f)}{jf} \tag{3}$$

The $\Phi_1(f)$ in Eq. (2) is the Fourier transform of the angle $\phi_1(t)$ in the FM signal applied to the PLL. Substituting Eq. (1) into (3) and expanding terms, we get

$$\Phi_e(f) = \left(\frac{(jf)^2/aK_0}{1 + [(jf)/a] + [(jf)^2/aK_0]}\right)\Phi_1(f) \tag{4}$$

Define the *natural frequency* of the loop

$$f_n = \sqrt{aK_0} \tag{5}$$

and the *dampling factor*

$$\zeta = \sqrt{\frac{K_0}{4a}} \tag{6}$$

We may then recast Eq. (4) in terms of the loop parameters f_n and ζ as follows:

$$\Phi_e(f) = \left(\frac{(jf/f_n)}{1 + 2\zeta(jf/f_n) + (jf/f_n)^2}\right) \tag{7}$$

(b) Suppose the FM signal applied to the PLL is a single-tone modulating signal, for which the phase input is defined by

$$\phi_1(t) = \beta\sin(2\pi f_m t) \tag{8}$$

Then, invoking the use of Eq. (7), we find that the corresponding phase error $\phi_e(t)$ is defined by

$$\phi_e(t) = \phi_{eo}\cos(2\pi f_m t + \psi) \tag{9}$$

where

$$\phi_{eo} = \frac{(\Delta f/f_m)(f_m/f_n)}{[(1 - (f_m/f_n)^2)^2 + 4\zeta^2(f_m/f_n)^2]^{1/2}} \tag{10}$$

and

$$\psi = \frac{\pi}{2} - \tan^{-1}\left[\frac{2\zeta(f_m/f_n)}{1 - (f_m/f_n)^2}\right] \tag{11}$$

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Problem 4-25 continued

One other thing we need to do is to evaluate the Fourier transform of the PLL output $v(t)$. For this purpose, we first note that the Fourier transform of $v(t)$ is related to $\Phi_1(f)$ as follows (see part (b) of Problem 4.7)

$$V(f) = \frac{jf}{k_v}\frac{L(f)}{1+L(f)}\Phi_1(f) \tag{12}$$

where k_v is the frequency sensitivity of the VCO. Using Eqs. (1), (3), and (12), we may write

$$V(f) = \left(\frac{(jf/f_n)[1+2\zeta(jf/f_n)]}{1+2\zeta(jf/f_n)+(jf/f_n)^2}\right)\Phi_1(f) \tag{13}$$

In light of the PLL theory presented herein, we may make two important observations for an incoming FM signal of fixed frequency deviation produced by a sinusoidal modulating signal $m(t)$:

(i) The frequency response that defines the phase error $\phi_e(t)$ is representative of a band-pass filter, as shown by Eq. (10).

(ii) The frequency response of the PLL output $v(t)$ is representative of a low-pass filter, as shown by Eq. (13).

Therefore, by appropriately choosing the damping factor ζ and natural frequency f_n, which determine the frequency response of the PLL, it is possible to restrain the phase error $\phi_e(t)$ to always remain small and yet, at the same time, the modulating (message) signal is reproduced at the PLL output with minimum distortion.