EE308 – Communication Systems (Section 1, Autumn 2018)

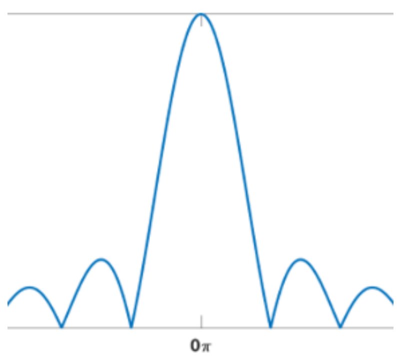
Mid-Semester Exam | September 09, 2018 (11:00–13:00 hrs) | Max. Marks: 25

1. The Fourier transform of the unit step function is given by *[3]*

$$\mathcal{F}(u(t)) = \frac{1}{j2\pi f} + \frac{\delta(f)}{2}.$$

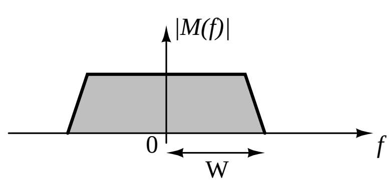
Using this result, and properties of Fourier transforms,

(a) Find the expression for Fourier transform of a rectangular pulse, which has value 1 for $|t| \leq T/2$ and zero elsewhere. Sketch its magnitude response.
Sol: Expression for rectangular pulse is given as: $rect(\frac{t}{T}) = u(t+\frac{T}{2}) - u(t-\frac{T}{2})$.
Fourier transform of this pulse: $\mathcal{F}(rect(\frac{t}{T})) = e^{\frac{j\omega T}{2}}\mathcal{F}(u(t)) - e^{-\frac{j\omega T}{2}}\mathcal{F}(u(t))$
Using the given result: $\mathcal{F}(rect(\frac{t}{T})) = e^{\frac{j\omega T}{2}}(\frac{1}{j2\pi f} + \frac{\delta(f)}{2}) - e^{-\frac{j\omega T}{2}}(\frac{1}{j2\pi f} + \frac{\delta(f)}{2})$
As, $e^{\frac{j\omega T}{2}}\frac{\delta(f)}{2} - e^{-\frac{j\omega T}{2}}\frac{\delta(f)}{2} = 0$,
$\mathcal{F}(rect(\frac{t}{T})) = (e^{\frac{j\omega T}{2}} - e^{-\frac{j\omega T}{2}})\frac{1}{j2\pi f}$
on simplifying, $\mathcal{F}(rect(\frac{t}{T})) = \frac{sin(\pi f T)}{\pi f}$.
Please note that only magnitude (always positive) is required to be plotted.



(b) Derive the expression for the time domain impulse response of the Hilbert transform.
Sol: Hibert transform in frequency domain is given as: $H(f) = -jsgn(f)$
$H(f) = -ju(f) + ju(-f)$
Using duality property of Fourier transform,
$\mathcal{F}^{-1}(u(-f)) = \frac{1}{j2\pi t} + \frac{\delta(t)}{2}$
$\mathcal{F}^{-1}(u(f)) = \frac{1}{-j2\pi t} + \frac{\delta(t)}{2}$
$\mathcal{F}^{-1}(H(f)) = \frac{1}{2\pi t} + j\frac{\delta(t)}{2} + \frac{1}{2\pi t} - j\frac{\delta(t)}{2} = \frac{1}{\pi t}$
Marking Scheme: 2 marks for part (a), 1 mark for part (b). In part (a), 1.5 marks are for correct derivation and 0.5 marks are for correct plot.

2. The magnitude spectrum for real signal $m(t)$ has been shown below. *[3]*

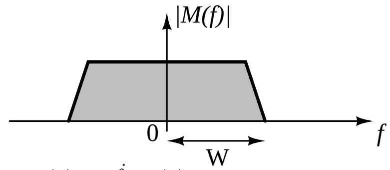


(a) Sketch the magnitude spectrum of $\text{Re}[m(t) + \frac{j}{2}\hat{m}(t)]$, where $\hat{m}(t)$ is the Hilbert transform of $m(t)$. [Caution: don't ignore the factor $\frac{j}{2}$ in the expression above.]

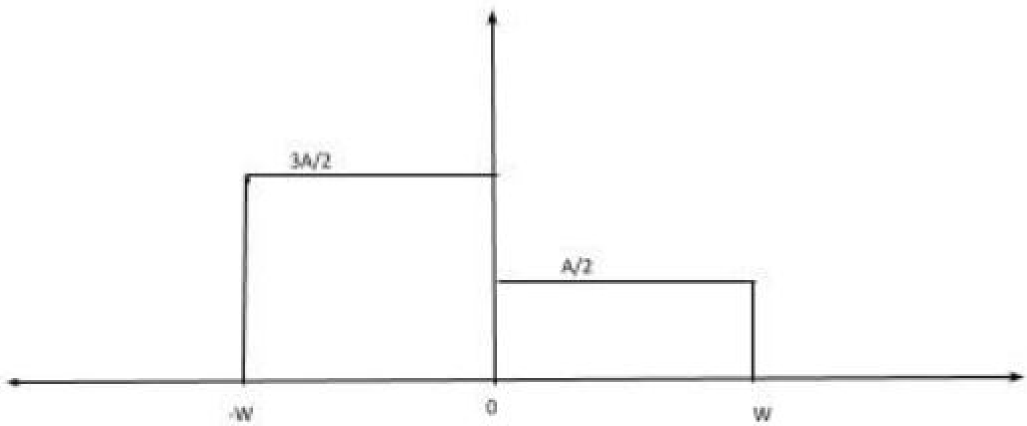
(b) Sketch the magnitude spectrum of $\mathrm{Re}([m(t) - \frac{j}{2}\hat{m}(t)] \times e^{-j2\pi f_c t})$, where $\hat{m}(t)$ is the Hilbert transform of $m(t)$.

Solution:

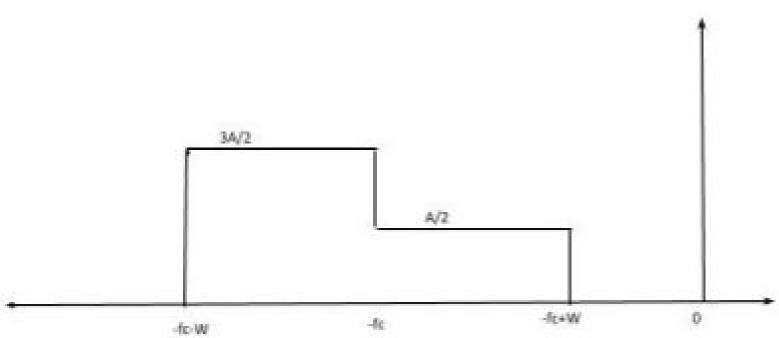
a) Given $m(t)$ is real. This implies that it's hilbert transform $\hat{m}(t)$ will also be real. Thus, $\mathrm{Re}[m(t) + \frac{j}{2}\hat{m}(t)] = m(t)$. Hence, its magnitude spectrum will be same as that of $m(t)$.



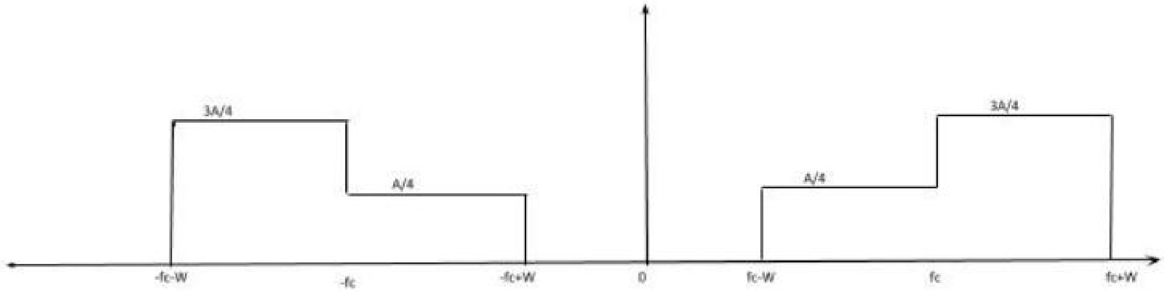
b)Magnitude spectrum of$m(t) - \frac{j}{2}\hat{m}(t)$ is:



Magnitude spectrum of $m(t) - \frac{j}{2}\hat{m}(t))e^{-j2\pi f_c t}$ is:



Magnitude spectrum of $\mathrm{Re}[(m(t) - \frac{j}{2}\hat{m}(t))e^{-j2\pi f_c t}]$ is:



3. The transform $H(f)$ is applied to a real message signal $m(t)$ to obtain $m_1(t)$, and the vestigial sideband signal is obtained as $m_{VSB}(t) = m(t) + jm_1(t)$. Here, [9]

$$H(f) = \begin{cases} j, & \text{for } f < -1\,\text{kHz}; \\ -j, & \text{for } f \geq -1\,\text{kHz}. \end{cases}$$

(a) Given that $m(t)$ real, under what condition will $m_{VSB}(t)$ also be real (explain or prove)?

(b) If $m(t) = \cos(200\pi t) + 0.1 \times \cos(4000\pi t)$, derive the expression for $m_{VSB}(t)$ and sketch its magnitude spectrum (with correct amplitudes).

(c) Find the expression for the transmitted VSB-AM (full carrier) signal $s_{VSB}(t) = \text{Re}[(1 + 0.4 \times m_{VSB}(t)) \times 100e^{-j10^6\pi t}]$, for the $m(t)$ defined in (b).

(d) This signal [i.e. $s_{VSB}(t)$ defined in (c)] is sent through an ideal envelope detector. Write the expression for the signal at the envelope detector output. Using approximations (if applicable), can we say that envelope detection gives back the original message signal (why/why not)?

(e) Will the answer in (d) change if $m(t) = 0.1 \times \cos(200\pi t) + \cos(4000\pi t)$ instead of $\cos(200\pi t) + 0.1 \times \cos(4000\pi t)$ in (b)?

(f) Given that Analog-TV signals are transmitted using similar kind of VSB modulation scheme and can be demodulated successfully using envelope detection, what can we say about the spectrum of Analog-TV baseband signals. [Hint: relate to results from (d) and (e).]

Solution:

(a) For $m_{VSB}(t)$ to be real $m_1(t)$ should be purely imaginery. As a result,

$$M_1(f) = -M_1^*(-f)............................(1)$$

Since $H(f) = -j\text{sgn}(f + 1000)$,

$$\begin{aligned} M_1(f) &= M(f)H(f) \\ &= M(f)[-j\text{sgn}(f + 1000)]................(2) \\ M_1^*(f) &= M^*(f)[j\text{sgn}(f + 1000)] \\ M_1^*(-f) &= M^*(-f)[j\text{sgn}(-f + 1000)] \\ &= M^*(-f)[-j\text{sgn}(f - 1000)] \\ -M_1^*(-f) &= M(f)[j\text{sgn}(f - 1000)]....................(3) \end{aligned}$$

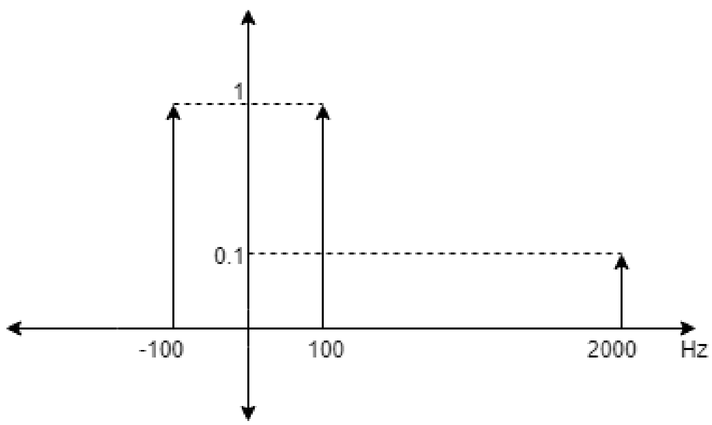
From (1), (2) and (3), $\quad sgn(f-1000) = -sgn(f+1000)$

Above condition holds only for $|f| \leq 1000$. So, $m(t)$ should be bandlimited to 1KHz.

(b)

$$M_{vsb}(f) = M(f)[1 + sgn(f+1000)]$$
$$= \begin{cases} 2M(F) & \text{for } f \geq -1\,\text{kHz;} \\ 0 & \text{for } f < -1\,\text{kHz.} \end{cases}$$

$$M(f) = \frac{1}{2}\delta(f-100) + \frac{1}{2}\delta(f+100) + \frac{0.1}{2}\delta(f-2000) + \frac{0.1}{2}\delta(f+2000)$$
$$\therefore M_{vsb}(f) = \delta(f-100) + \delta(f+100) + 0.1\delta(f-2000)$$



$$m_{vsb}(t) = e^{j200\pi t} + e^{-j200\pi t} + 0.1e^{j4000\pi t}$$
$$= 2\cos(200\pi t) + 0.1e^{j4000\pi t}$$

(c)

$$s_{VSB}(t) = \text{Re}[(1 + 0.4 \times m_{VSB}(t)) \times 100e^{-j10^6\pi t}]$$
$$= \text{Re}[[1 + 0.4 \times (2\cos(200\pi t) + 0.1e^{j4000\pi t})] \times 100e^{-j10^6\pi t}]$$
$$= \text{Re}[(1 + 0.8\cos(200\pi t) + 0.04\cos(4000\pi t) + j0.04\sin(4000\pi t))(\cos(10^6\pi t) - j\sin(10^6\pi t))100]$$
$$= 100\cos(10^6\pi t) + 80\cos(200\pi t)\cos(10^6\pi t) + 4\cos(996 \times 10^3\pi t)$$

(d) From expression of $s_{VSB}(t)$,
Output of envelope detector $= \sqrt{A^2 + B^2}$
where, A $= 100(1 + 0.8\cos(200\pi t) + 0.04\cos(4000\pi t))$ and B $= 4\sin(4000\pi t)$
Since $A^2 >> B^2$, output can be approximated as A.
Hence, obtained original message signal.

(e) In this case, approximation does not hold and so, ans in (d) changes. Original message signal can't be recovered.

(f) Spectrum of Analog-TV baseband signals have significant power in lower frequencies compared to higher frequencies, since they are demodulated successfully using envelope detection.

4. Consider passband signals with carrier frequency f_c and single-tone message frequency f_m. An angle-modulated version of such a signal can be written as

$$s_{p,\,angle-mod}(t) = \cos[2\pi f_c t + \beta cos(2\pi f_m t + \phi_m) + \phi_c],$$

where β is called the modulation index (here all the angles are being expressed in radians). Now consider the following modulated signal with carrier and message frequencies f_c and f_m, respectively:

$$s_p(t) = \sin(2\pi f_c t) + 0.1\cos[2\pi(f_c - f_m)t] + 0.1\alpha\cos[2\pi(f_c + f_m)t].$$

Given that the message signal is $m(t) = \sin(2\pi f_m t)$, answer the following (with the aid of an expression): *[5]*

(a) What is the type of modulation if $\alpha = +1$ (it can be AM DSB-FC, DSB-SC, SSB-SC, SSB-FC, Narrow-Band FM, Wide-Band FM, Phase-Modulation etc.).

(b) What is the type of modulation if $\alpha = 0$.

(c) What is the type of modulation if $\alpha = -1$.

(d) For each of the cases in (a)-(c) that have DSB-AM, FM or PM type of modulation, what is the modulation index?

Solution:

(a)
$$s_p(t) = \sin(2\pi f_c t) + 0.1\cos[2\pi(f_c - f_m)t] + 0.1\cos[2\pi(f_c + f_m)t]$$

This output can be obtained when taking the given message input, along with carrier as $\cos[2\pi f_c t + \phi_c]$.
Clearly it cannot be true for a Wideband FM, therefore, we will observe for Narrowband FM.

$$s(t) = \cos[2\pi f_c t + \phi_c + 2\pi f_m k_f \int \sin(2\pi f_m t)dt]$$
$$s(t) = \cos[2\pi f_c t + \phi_c + k_f\cos(2\pi f_m t)]$$

For Narrowband FM, $k_f \leq 1$

$$s(t) = \cos[2\pi f_c t + \phi_c]\cos[k_f\cos(2\pi f_m t)] - \sin[2\pi f_c t + \phi_c]\sin[k_f\cos(2\pi f_m t)]$$

Since $k_f \leq 1$, $\cos[k_f\cos(2\pi f_m t)] \approx 1$ and $\sin[k_f\cos(2\pi f_m t)] \approx k_f\cos(2\pi f_m t)$.
Therefore,
$$s(t) = \cos[2\pi f_c t + \phi_c] - k_f\cos(2\pi f_m t)\sin[2\pi f_c t + \phi_c]$$

Comparing with our modulated signal $s_p(t)$, we get $\phi_c = -90°$

$$s(t) = \sin[2\pi f_c t] + k_f\cos[2\pi f_m t]\cos[2\pi f_c t]$$

$$s(t) = \sin[2\pi f_c t] + \frac{k_f}{2}[\cos[2\pi(f_c - f_m)t] + \cos[2\pi(f_c + f_m)t]$$

Comparing with $s_p(t)$, we find it to be of the exact form, where $k_f = 0.2$. So this is a **Narrowband FM**.

(b)

$$s_p(t) = \sin(2\pi f_c t) + 0.1\cos[2\pi(f_c - f_m)t]$$

We can observe this expression to have only the Lower sideband from the message signal spectrum. Also, since we have the term for the carrier wave explicitly present, it is also a Full Carrier Modulation. Therefore, this signal is **SSB-FC modulation**.

(c)

$$s_p(t) = \sin(2\pi f_c t) + 0.1\cos[2\pi(f_c - f_m)t] - 0.1\cos[2\pi(f_c + f_m)t]$$

For a DSB-FC AM, with the given message signal, we will get the following modulated signal.

$$s(t) = [1 + k\sin(2\pi f_m t)]\cos[2\pi f_c t + \phi_c]$$

$$s(t) = \cos[2\pi f_c t + \phi_c] + k\sin[2\pi f_m t]\cos[2\pi f_c t + \phi_c]$$

Comparing with $s_p(t)$, we get $\phi_c = -90°$

$$s(t) = \sin[2\pi f_c t] + k\sin[2\pi f_m t]\sin[2\pi f_c t]$$

$$s(t) = \sin[2\pi f_c t] + \frac{k}{2}[\cos[2\pi(f_c - f_m)t] - \cos[2\pi(f_c + f_m)t]]$$

Comparing with $s_p(t)$, we find it to be of the exact form, where $k = 0.2$. So this is a **DSB-FC AM**.

(d) For Part (a), we have Narrowband FM with $k_f = 0.2$.
For Part (c), we have DSB-FC AM with $k = 0.2$.
Note: None of the modulations comply with low Phase-Deviation PM, since the additive terms become sin and not cos.

5. Consider a message signal that has a constant double-sided power spectral density of 100 nW/Hz (within the signal bandwidth) and the baseband signal bandwidth of $W = 100\,\text{kHz}$. This signal is FM modulated and transmitted and the phase of the modulated carrier is

$$\phi_m(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau,$$

where $k_f = 3 \times 10^4$. At the receiver antenna, the received carrier amplitude is $A_c = 1$ the double-sided noise PSD is 1 nW/Hz (you can assume that the received carrier power is $A_c^2/2$, i.e. the receiver resistor is $1\,\Omega$). *[5]*

(a) Find the SNR at the receiver's FM-demodulator output.

(b) Consider a filter H_{de} defined as

$$H_{de}(f) = \begin{cases} 1, & \text{for } |f| < 1\,\text{kHz}; \\ \dfrac{10^3}{|f|}, & \text{for } |f| \geq 1\,\text{kHz}. \end{cases}$$

If the inverse of the filter $H_{de}(f)$ is applied to the signal before modulation and $H_{de}(f)$ applied after the demodulator output to obtain the final demodulated singal, what is the achieved SNR.

(c) What is the disadvantage of using the approach discussed in (b)?

Solution:
Recall that discriminator output is given by $v(t) \approx k_f m(t) + \frac{1}{2\pi A_c}\frac{d}{dt}(n_q(t))$
Hence, average Power of output signal is $P_s = k_f^2 P$, where P is average power of the message. Since double sided PSD of the message, S_m is 100 nW/Hz, the average power of message P is given by
$P_s = k_f^2 S_m(2W) = 18 \times 10^6$ Watts **(0.5 marks)**
The noise in the system is white noise with double sided power spectral denisty $\frac{N_0}{2} = 10^{-9}$. Hence, we get the following situation after BPF

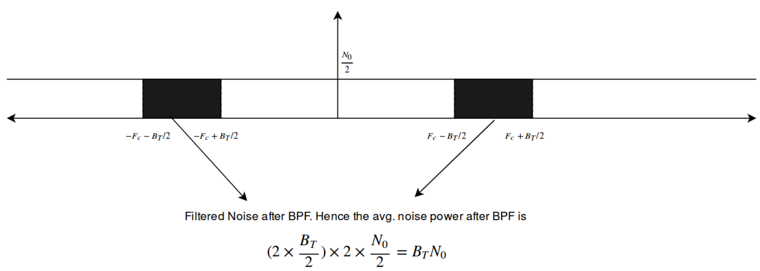


Figure 1: Power spectral density of noise after BPF

Represent noise after BPF as $n(t) = n_i(t)\cos(2\pi f_c t) - n_q(t)\sin(2\pi f_c t)$.
Thus, $\langle n^2(t)\rangle = \frac{\langle n_i^2(t)\rangle}{2} + \frac{\langle n_q^2(t)\rangle}{2}$, since $\langle \cos^2(2\pi f_c t)\rangle = \langle \sin^2(2\pi f_c t)\rangle = 1/2$
Considering $\langle n_i^2(t)\rangle = \langle n_q^2(t)\rangle$, we get $\langle n_q^2(t)\rangle = B_T N_0$. Assuming $n_q(t)$ to be white as well motivates the following double sided power spectral density $S_{n_Q}(f)$:-

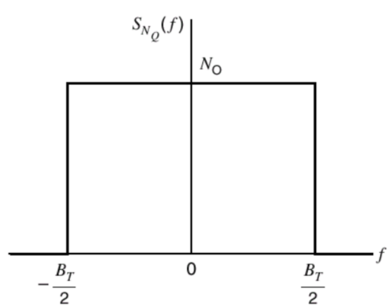


Figure 2: Power spectral density of quadrature phase of noise

Recall that the power spectral density of noise at FM demodulator output is given by:-

$$S_{N_d}(f) = \begin{cases} \frac{S_{n_Q} f^2}{A_c^2} = \frac{N_0 f^2}{A_c^2}, & \text{if } |f| \leq W \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Hence, average power of output noise (P_{n_o}) will be determined by integrating the power spectral density from -W to W to get
$P_{n_o} = \frac{N_0}{A_c^2}\int_{-W}^{W} f^2 df = \frac{2N_0 W^3}{3A_c^2}$ **(1 mark)**
Notice that you need to take $N_0/2 = 10^{-9}$. Substituting the values, we get $P_n = \frac{4}{3} \times 10^6$ Watts
Hence SNR at output will be $\frac{18\times 3}{4} = 13.5$. **(0.5 marks)**
Note: If someone misses a factor of two, only 0.5 marks will be deducted for the final calculation
End of Part A

Since the inverse of $H_{de}(f)$ is applied before modulation, the signal power wont change **(0.5 marks)**
Hence, to get the new SNR, one needs to find the change in noise power.

Average noise power with emphasis will be $\int_{-W}^{W} \frac{N_0}{A_c^2} f^2 |H_{de}(f)|^2 df$ **(0.5 marks)**

$= 2\frac{N_0}{A_c^2}(\int_0^{10^3} f^2 df + \int_{10^3}^{10^5} f^2 \times \frac{10^6}{f^2} df) = 4 \times 10^{-9}(\frac{10^9}{3} + 10^6(10^5 - 10^3)) \approx 400$ Watts **(0.5 marks)**

Hence, the new SNR will be $\frac{18\times10^6}{400} = 4.5 \times 10^4$ **(0.5 marks)**

Note: Even if someone obtains the ratio of improvement in SNR (i.e. $\frac{SNR_{\text{emphasis}}}{SNR_{\text{w/o emphasis}}} = \frac{10^4}{3}$) correctly he/she will be given full marks for the part B of the question, and not taking $\frac{N_0}{2}$ has not been penalized

End of Part B

Since the pre-emphasis filter will be the inverse of the de-emphasis filter, it's gain will be proportional to $|f|$ and hence it will amplify the higher frequencies. Thus, this may lead to overmodulation of higher frequencies, which may distort the signal and also produce clicks due to thresholding effect arising from high gains in message signal.

Note: 0.5 marks have been given if message distortion is mentioned and **0.5 marks** for mentioning clicks arising due to thresholding effects.

End of Question 5
