Problem 2.21

Express $g(t)$ as

$$g(t) = g_1(t) + g_2(t)$$

where

$$g_1(t) = \frac{1}{\tau}\int_{t-T}^{0} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$g_2(t) = \frac{1}{\tau}\int_{0}^{t+T} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

Therefore,

$$g_1(t+T) = \frac{1}{\tau}\int_{t}^{0} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$= -\frac{1}{\tau}\int_{0}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$= -\frac{1}{\tau}\int_{-\infty}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du + \frac{1}{2}$$

where we have made use of the fact that

$$\frac{1}{\tau}\int_{-\infty}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du = \frac{1}{2}$$

Similarly

$$g_2(t-T) = \int_{0}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$

$$= \frac{1}{\tau}\int_{-\infty}^{t} \exp\left(-\frac{\pi u^2}{\tau^2}\right)du = \frac{1}{2}$$

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Next, noting the following four relationships

$$\mathbf{F}[g_1(t+T)] = G_1(f)\exp(j2\pi fT)$$

$$\mathbf{F}[g_2(t-T)] = G_2(f)\exp(-j2\pi fT)$$

$$\exp\left(-\frac{\pi t^2}{\tau^2}\right) \rightleftharpoons \tau\exp(-\pi\tau^2 f^2)$$

$$\int_{-\infty}^{t} g(u)du = \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f)$$

we find that taking the Fourier transforms of $g_1(t+T)$ and $g_2(t-T)$ respectively yields

$$G_1(f)\exp(j2\pi fT) = -\frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)$$

$$G_2(f)\exp(-j2\pi fT) = \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)$$

Therefore,

$$G_1(f) = \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)\exp(-j2\pi fT)$$

$$G_2(f) = \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)\exp(j2\pi fT)$$

Thus the Fourier transform of $g(t)$ is

$$\begin{aligned} G(f) &= G_1(f) + G_2(f) \\ &= \frac{1}{j2\pi f}\exp(-\pi\tau^2 f^2)[\exp(-j2\pi fT) + \exp(j2\pi fT)] \\ &= \frac{1}{\pi f}\exp(-\pi\tau^2 f^2)\sin(2\pi fT) \\ &= 2T\exp(-\pi\tau^2 f^2)\operatorname{sinc}(2fT) \end{aligned}$$

When τ approaches zero, $G(f)$ approaches the limiting value $2T\operatorname{sinc}(2fT)$, which corresponds to the Fourier transform of a rectangular pulse of unit amplitude and duration $2T$, which is correct.