Problem 5.31

(a) Starting with the Fourier-transform pair

$$\exp(-\pi t^2) \rightleftharpoons \exp(-\pi f^2) \tag{1}$$

and applying the differentiation property of the Fourier transform to Eq. (1), we write

$$\frac{d}{dt}\exp(-\pi t^2) \rightleftharpoons j2\pi f \exp(-\pi f^2)$$

or, equivalently

$$-2\pi t \exp(-\pi t^2) \rightleftharpoons j2\pi f \exp(-\pi f^2) \tag{2}$$

Multiplying the left-hand side of Eq. (1) by A and invoking the linearity property of the Fourier transform, we go on to write

$$-2\pi t A \exp(-\pi t^2) \rightleftharpoons j2\pi f A \exp(-\pi f^2)$$

Simplifying terms:

$$tA \exp(-\pi t^2) \rightleftharpoons jfA \exp(-\pi f^2) \tag{3}$$

Finally, applying the dilation property of the Fourier transform to Eq. (3). we get

$$A\left(\frac{t}{\tau}\right)\exp\left(-\pi\left(\frac{t}{\tau}\right)^2\right) \rightleftharpoons -j\tau f A \exp(-\pi f^2\tau^2) \tag{4}$$

The left-hand side of this transform pair is recognized as the time function (see Eq. (5.39))

$$v(t) = A\left(\frac{t}{\tau}\right)\exp\left(-\pi\left(\frac{t}{\tau}\right)^2\right) \tag{5}$$

From Fig. 5.22, we see that the maximum value of $v(t)$ is +1. To find this maximum, we differentiate $v(t)$ with respect to time t and set the result equal to zero, obtaining

$$\frac{A}{\tau}\exp\left(-\pi\left(\frac{t}{\tau}\right)^2\right) - A\left(\frac{t}{\tau}\right)(2\pi t/\tau)\exp(-\pi t/\tau^2) = 0$$

Cancelling common terms and solving for t_{max}/τ, we get

$$\frac{t_{max}}{\tau} = \left(\frac{1}{2\pi}\right)^{1/2} \tag{6}$$

Using this value in Eq. (5):

$$v(t_{max}) = A\left(\frac{1}{2\pi}\right)^{1/2}\exp\left(-\frac{1}{2}\right)$$

With $v(t_{max}) = 1$, it follows that

$$A = (2\pi)^{1/2}\exp\left(\frac{1}{2}\right) = 4.1327$$

(b) The formula used to plot the spectrum of Fig. 5.23 is defined by the Fourier transform on the right-hand side of Eq. (4), that is,

$$V(f) = -j2\pi f A \exp(-\pi f^2\tau^2) \tag{7}$$