Homework 4: probability inequalities, characteristic functions

EE 325 (DD): Probability and Random Processes, Autumn 2018

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Instructions: Some of these questions will be asked in a quiz in the class on 18/09/18. *If you have queries, then meet the instructor or the TA during office hours.*

1. Let $X \sim \mathcal{N}(0, 4)$ and $Y \sim \mathcal{N}(0, 16)$ be independent Gaussian random variables. Find a non-zero pair α, β such that $X + \alpha Y$ and $X - \beta Y$ are independent. (For this problem, your technique should be restricted to what has been covered in the class so far.)

2. Let $X_1, \ldots, X_n$ be random variables which are not necessarily independent. Let there be a $\sigma > 0$ such that

$$\mathbb{E}(e^{tX_i}) \leq \exp(t^2\sigma^2/2) \text{ for all } t > 0.$$

Then show that

$$\mathbb{E}\left(\max_{1 \leq i \leq n} X_i\right) \leq \sigma\sqrt{2 \log n}.$$

Note that a zero-mean Gaussian random variable with variance σ^2 will satisfy this inequality.

3. Assume that X is a continuous r.v. with $\phi_X(t)$ as the characteristic function. Are $\mathrm{Re}[\phi_X(t)]$, $\mathrm{Im}[\phi_X(t)]$, and $|\phi_X(t)|^2$ valid characteristic functions? For a complex number z, $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$ represent its real and imaginary parts.

4. Let Y be a zero-mean random variable with variance σ^2. Show the one-sided inequality,

$$\mathbb{P}(Y \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

for $a > 0$. (Hint: use the fact that $Y \geq a \Leftrightarrow Y + c \geq a + c$, for any $c \in \mathbb{R}$.)

5. For this problem you may require the Schwarz inequality. Given any two rv X and Y with finite variances, the Schwarz inequality states that

$$[\mathbb{E}(XY)]^2 \leq \left[\mathbb{E}(X^2)\mathbb{E}(Y^2)\right].$$

For a rv Z which is positive, i.e. $Z \geq 0$, show that

$$\mathbb{P}(Z > a) \geq \frac{(\mathbb{E}(Z) - a)^2}{\mathbb{E}(Z^2)},$$

where $a > 0$ is any arbitrary constant. (**Hint:** think of a rv which converts into a probability upon taking expectations.)

6. Construct examples of distributions for X such that,

   (a) The Markov inequality is tight, i.e., there exists a distribution $F_X(x)$ and a point $a \in \mathbb{R}$ such that $\mathbb{P}(X \geq a) = (\mathbb{E}(X)/a)$.

   (b) The Chebyshev inequality is tight, i.e., there exists a distribution $F_X(x)$ and a point $a > 0, a \in \mathbb{R}$ such that $\mathbb{P}(|X - \mathbb{E}(X)| \geq a) = (\sigma_X^2/a^2)$.

7. (Gallager 1.38) If $Y > 0$ and $\mathbb{E}(Y) < \infty$, then show that $\lim_{y\to\infty} y\mathbb{P}(Y \geq y) = 0$.

8. Let $\{X_i\}_{i=1}^{\infty}$ be an IID sequence of random variables, distributed according to the exponential distribution $\text{Exp}(\lambda)$. Show that,

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq n\left(\frac{1}{\lambda}+\epsilon\right)\right) \leq \exp\left(n[\ln(1+\lambda\epsilon)-\lambda\epsilon]\right).$$

Show that the bound is non-trivial or the RHS of the inequality is not equal to 1 for $\epsilon > 0$. (Hint: use Chernoff bound formulation.)

9. (Kullback Leibler divergence between discrete random variables:) Let X and Y be two discrete random variables with non-zero pmf defined on the set of integers $\{1, 2, \ldots, m\}$.

   (a) Show that the function $g(x) = \ln x - x + 1 \leq 0$. (Hint: Show that $g''(x) \leq 0$ and find its unique maxima to establish the inequality.)

   (b) Using $\ln x \leq x - 1$ from part (a), show that,

$$\sum_{i=1}^{m} p_X(i) \ln \frac{p_X(i)}{p_Y(i)} \geq 0, \tag{1}$$

   with equality occuring only if $p_X(i) = p_Y(i)$ for all $i = 1, 2, \ldots, m$. (Hint: Work with the negative of expression on the LHS of the inequality.)