End-Semester Examination

Probability and Random Processes (EE 325), Autumn'17

Nov. 11, 2017; Total: 45 marks; Time: 3 hours

Note:

- *You are allowed to use two A4 sheets with handwritten notes on both sides of each sheet.*
- *You are allowed to use any result discussed in class without proof. For all other results, a proof needs to be provided.*

QUESTION 1 (4 MARKS)

Let X and Y be continuous random variables with the following joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} y e^{-xy}, & 0 < x < \infty,\ 0 < y < 2, \\ 0, & \text{else.} \end{cases}$$

Find $f_{X|Y}(x|y)$ and $E\left(e^{X/2}|Y=1\right)$.

QUESTION 2 (4 MARKS)

Let U be uniformly distributed on $(0, 2\pi)$, Z be exponentially distributed with mean 1, and U and Z be independent. Let $X = \sqrt{2Z}\cos(U)$ and $Y = \sqrt{2Z}\sin(U)$. Show that X and Y are independent standard normal random variables.

QUESTION 3 (4 MARKS)

Show that the following holds for arbitrary random variables X_1, X_2, X_3, X_4 and arbitrary constants a_1, a_2, a_3, a_4:

$$\begin{aligned} \text{Cov}(a_1X_1 + a_2X_2, a_3X_3 + a_4X_4) &= a_1a_3\text{Cov}(X_1, X_3) + a_1a_4\text{Cov}(X_1, X_4) \\ &\quad + a_2a_3\text{Cov}(X_2, X_3) + a_2a_4\text{Cov}(X_2, X_4) \end{aligned}$$

QUESTION 4 (2.5 + 1.5 = 4 MARKS)

Let $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be a Gaussian random vector with mean vector $\mu_{\mathbf{X}} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and covariance matrix $K_{\mathbf{X}} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Let $A = \begin{pmatrix} 1 & -1 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = A\mathbf{X}$.

(a) Show that Y_1 and Y_2 are independent random variables.

(b) Find the joint PDF of Y_1 and Y_2.

QUESTION 5 (2 + 1 + 1 = 4 MARKS)

A sequence of random variables $X_1, X_2, X_3, \ldots$ is said to converge to X in *third-mean sense* if $\lim_{n\to\infty} E\left(|X_n - X|^3\right) = 0$.

(a) Show that if a sequence $X_1, X_2, X_3, \ldots$ converges to X in third-mean sense, then it converges to X in probability.

(b) Give an example in which $X_1, X_2, X_3, \ldots$ converges to X in third-mean sense, but not almost surely.

(c) Give an example in which $X_1, X_2, X_3, \ldots$ converges to X almost surely, but not in third-mean sense.

QUESTION 6 (2.5 + 2.5 = 5 MARKS)

(a) Let X be a random variable that is uniformly distributed in $[0, 1]$, and let:

$$Y = e^{-X} + 2X.$$

Find $E(Y)$ and $\mathrm{var}(Y)$.

(b) A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, 3 points if it is between 3 and 5 inches of the target, and 0 points otherwise. Suppose the distance from the shot to the target is uniformly distributed in $[0, 10]$. Let X be the number of points the man receives. Find $E(X)$ and $\mathrm{var}(X)$.

QUESTION 7 (5 MARKS)

A WSS process, $X(t)$, whose autocorrelation function is:

$$R_X(\tau) = \exp(-2v|\tau|),$$

where v is a positive constant, is applied to the low-pass RC filter shown in Fig. 1. Find the power spectral density and autocorrelation function of the random process at the filter output.

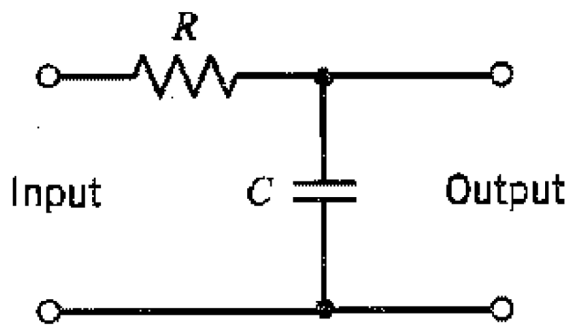

Fig. 1. The figure for Question 7.

QUESTION 8 (1.5 + 2.5 = 4 MARKS)

Let $X_1, \ldots, X_n$ be independent and identically distributed continuous random variables. We say that a record value occurs at time $j \in \{1, \ldots, n\}$ if $X_j \geq X_i$ for all $1 \leq i \leq j$. Let Y be the number of record values. Show that:

(a)

$$E(Y) = \sum_{j=1}^{n} \frac{1}{j}.$$

(b)

$$\mathrm{var}(Y) = \sum_{j=1}^{n} \frac{(j-1)}{j^2}.$$

QUESTION 9 (4 MARKS)

Recall the Amplitude Modulated (AM) process example discussed in class. This process, say $X(t)$, consists of a sequence of pulses of width T each (see Fig. 2). The amplitude of each pulse is A or $-A$ with probability $\frac{1}{2}$ each, independently of the other pulses, where $A > 0$. Also, t_d, which is the start time of the first complete pulse to the right of the origin, is uniformly distributed in $[0, T]$. Is the process $X(t)$ ergodic in the mean? Justify your answer.

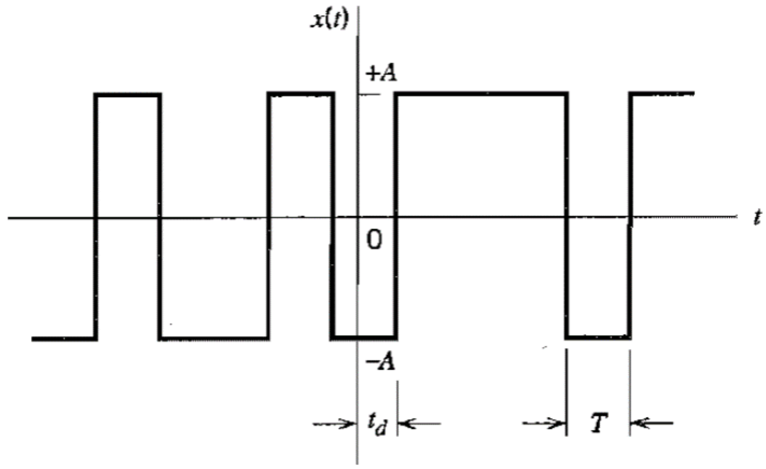

Fig. 2. The figure for Question 9.

QUESTION 10 (4 MARKS)

Show that the sequence X_n converges to 0 in probability if and only if $E\left(\frac{|X_n|}{1+|X_n|}\right) \to 0$ as $n \to \infty$. (*Hint*: You may find the following fact useful: for $u \geq 0$, $g(u) = \frac{u}{1+u}$ is an increasing function.)

QUESTION 11 (3 MARKS)

Show that if $\sum_{n=1}^{\infty} E\left((X_n - X)^2\right) < \infty$, then X_n converges to X almost surely.