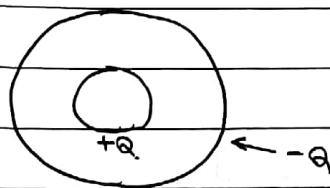


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GAUSS' LAW



Inner solid sphere was given $+Q$. Outer hollow sphere got a $-Q$ total charge.

Even when inner sphere was grounded, (its charge made zero), the $-Q$ on outer sphere remained.

The result does not change on placing dielectric between them.

- Replace inner sphere by point charge.

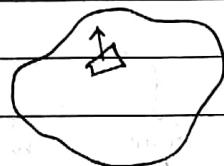
$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{a}_r$$

Permittivity $\epsilon_0 = 1 \times 10^{-9} \text{ F/m}$
of free space

(Capacitance / Length)

$$\vec{D}(r) = \epsilon_0 \vec{E}(r) = \text{Flux density. } \text{C/m}^2 \\ = \frac{1}{4\pi} \frac{q}{r^2} \hat{a}_r$$

- Surface element \vec{ds} is a vector in normal direction.



Flux flowing through the differential surface element
 $= \vec{D} \cdot \vec{ds}$

Total flux flowing out = $\Psi = \oint \vec{D} \cdot \vec{ds}$

→ Gauss' Law :- $\boxed{\Psi = Q}$ = Charge enclosed.

$$\frac{\partial \Phi}{\partial r} \left(\frac{d\phi}{dr} - \frac{\sigma}{\epsilon_0} \right) = \frac{Q}{\epsilon_0 r^2}$$

* Polarized materials

$$\vec{D} \stackrel{?}{=} \epsilon \vec{E}$$

They may not even have same direction.

• Mathematically,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \underbrace{\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}}_{\text{Permittivity tensor matrix}} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

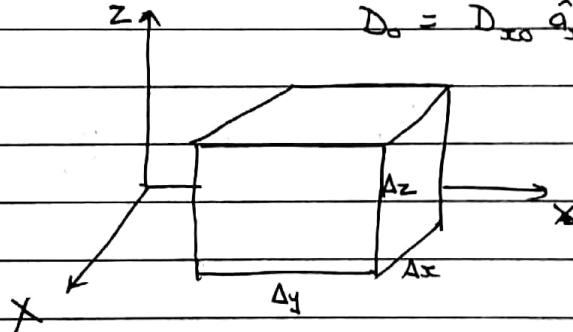
Permittivity tensor matrix.

If all elements of ϵ matrix are not equal, directions of \vec{E} and \vec{D} are not same.

→ Point form of Gauss Law

$$D_o = D_{x0} \hat{a}_x + D_{y0} \hat{a}_y + D_{z0} \hat{a}_z$$

$\stackrel{\Delta}{=} \vec{D}$ at centre



$$\oint \vec{D} \cdot d\vec{s} = \int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{front}} + \int_{\text{back}}$$

$$= D_{x0} \Delta y \Delta z$$

$$D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \dots \text{Taylor series}$$

$$\frac{f}{\text{front}} =$$

$$\int_{\text{back}} = \left(D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$

- $\int_{\text{front}} + \int_{\text{back}} = \Delta x \Delta y \Delta z \frac{\partial D_x}{\partial x} = \Delta v \frac{\partial D_x}{\partial x}$

- $\int + \int + \int + \int + \int + \int = \Delta v \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = q$
(Gauss law)

- $\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \frac{q}{\Delta v} = s_v = \text{'Charge Density'}$

- Define LHS as gradient divergence of \vec{D} :

$$\begin{aligned} &= \text{div}(\vec{D}) = \vec{\nabla} \cdot \vec{D} = \left(\frac{\partial \hat{a}_x}{\partial x} + \frac{\partial \hat{a}_y}{\partial y} + \frac{\partial \hat{a}_z}{\partial z} \right) \\ &\quad \cdot (D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z) \end{aligned}$$

• Gauss Law :- $\boxed{\vec{\nabla} \cdot \vec{D} = s_v} = \text{'Maxwell's first law'}$

• Gauss Law also holds for time-varying electric and magnetic fields

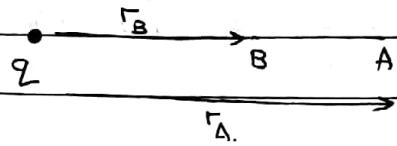
→ $\oint \vec{D} \cdot d\vec{s} = Q = \iint_V dV$

∴ $\boxed{\oint \vec{D} \cdot d\vec{s} = \int \vec{\nabla} \cdot \vec{D} dV}$

'Divergence theorem' - Converts areal integral to volume integral.

→ Divergence is a measure of a vector's tendency to radiate out.
curl.

ELECTRIC POTENTIAL



$$- \vec{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

$$- V_B - V_{BA} = - \int_A^B \vec{E} \cdot d\vec{l}$$

$$= - \frac{q}{4\pi\epsilon_0} \int_A^B \frac{dr}{r^2}$$

$$= - \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right)$$

- Define $V(\infty) = 0$

$$V_B = \frac{q}{4\pi\epsilon_0 r_B}$$

- The potential difference between any two points is independent of path traversed.

In a ~~static~~ field, $\oint \vec{E} \cdot d\vec{l}$ is independent of path.

- Static fields are conservative

$$\oint \vec{E} \cdot d\vec{l} = 0$$

'Maxwell's 2nd law'

AMPERE'S CIRCUITAL LAW

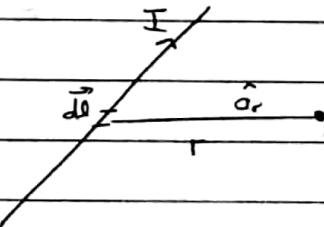
- Lorentz Force = Force experienced by moving charge = $q(\vec{v} \times \vec{B})$
 - Used to define \vec{B} .
- $\vec{B} = \mu_0 \vec{H}$ in free space

$$\rightarrow \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \mu \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

Permeability tensor

I BIOT SAVART LAW

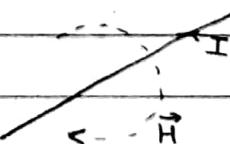
$$d\vec{H} = \frac{\mu I}{4\pi} \frac{dl \times \hat{a}_r}{r^2}$$



II AMPERE'S LAW

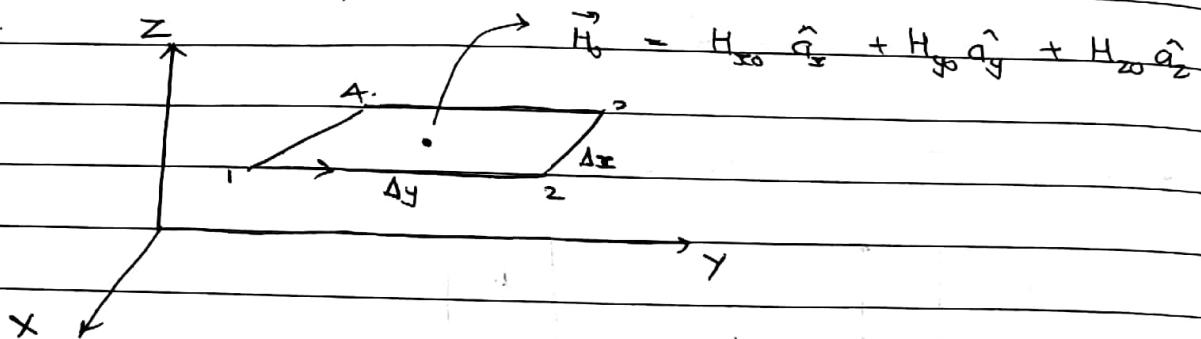
$$\oint \vec{H} \cdot d\vec{l} = I$$

in steady magnetic fields.



'Maxwell's 3rd law'

→ Other form of Ampere's Law.



$$\oint \vec{H} \cdot d\vec{\ell} = \int_{12} + \int_{23} + \int_{34} + \int_{41}$$

$$H_y \Delta y$$

$$H_y \left(x_0 + \frac{\Delta x}{2} \right) = H_y(x_0) + \frac{\Delta x}{2} \frac{\partial H_y}{\partial x}$$

• On path 12, H_y is assumed to be constant if Δy is very small

$$- \int_{34} = H_y \Delta x (-\Delta y) =$$

$$H_y \left(x_0 - \frac{\Delta x}{2} \right) = H_y(x_0) - \frac{\Delta x}{2} \frac{\partial H_y}{\partial x}$$

$$- \oint = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y = I$$

\downarrow
current density
 $J_z \Delta x \Delta y$

$$\Rightarrow \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$

$$\text{Similarly } \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_{xy}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x$$

• Hence, $\vec{J} = J_x \hat{\ell}_x + J_y \hat{\ell}_y + J_z \hat{\ell}_z = \begin{vmatrix} \hat{\ell}_x & \hat{\ell}_y & \hat{\ell}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \vec{\nabla} \times \vec{H} = c \operatorname{curl}(\vec{H})'$

III STOKES' THEOREM



$$\oint \vec{H} \cdot d\vec{l} = \Delta I = J_N \Delta s$$

$$J_N = (\vec{\nabla} \times \vec{H})$$

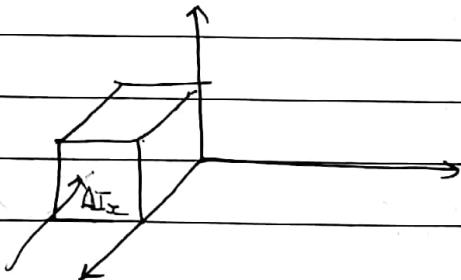
Normal direction.

$$= (\vec{\nabla} \times \vec{H}) \cdot \frac{d\vec{s}}{\Delta s}$$

Stokes' Theorem :-

$\int (\vec{\nabla} \times \vec{H}) \cdot d\vec{s}$	$= \oint \vec{H} \cdot d\vec{l}$
Area integral	Line integral

IV CURRENT



$$\Delta I_x = J_x \Delta s_x = \vec{J} \cdot \vec{\Delta s}_x$$

$$\Delta I_y = \vec{J} \cdot \vec{\Delta s}_y$$

$$\Delta I_z = \vec{J} \cdot \vec{\Delta s}_z$$

$$\Delta I = \vec{J} \cdot (\vec{\Delta s}_x + \vec{\Delta s}_y + \vec{\Delta s}_z)$$

$$= \vec{J} \cdot \vec{\Delta s}$$

$$\text{where } \vec{\Delta s} = \vec{\Delta s}_x + \vec{\Delta s}_y + \vec{\Delta s}_z$$

VI CONTINUITY EQUATION

$$I = - \frac{dQ}{dt}$$

Outward current

I



$$\oint \vec{J} \cdot d\vec{s} = - \frac{d}{dt} \int S_v d\phi$$

$$\Rightarrow \int_S \vec{J} \cdot \vec{dS} dv = \int \frac{dS_v}{dt} dv$$

$\nabla \cdot \vec{J} = - \frac{\partial S_v}{\partial t}$

$$\rightarrow \Delta I_x = \frac{\Delta Q}{\Delta t} = S_v A_{S_x} \Delta v = S_v A_{S_x} v_x$$

velocity in x direction

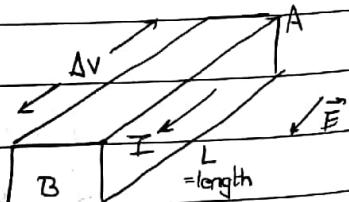
$$\Rightarrow J_x = S_v v_x$$

$\vec{J} = S_v \vec{v}$

VII OHM'S LAW

$$V = IR$$

$$E/V = I \frac{S}{L}$$



$$= JS$$

$\vec{J} = \sigma \vec{E}$

$$\text{where } \sigma = \frac{1}{R}$$

Resistivity

MAXWELL'S FOURTH LAW

From 1st law, $\oint \vec{D} \cdot d\vec{s} = q$
 $\nabla \cdot \vec{D} = \rho$

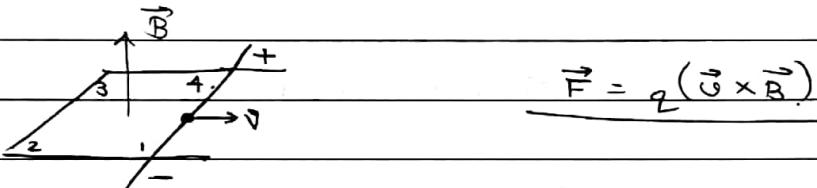
Since magnetic monopoles do not exist, $\oint \vec{B} \cdot d\vec{s} = 0$

$$\oint \vec{B} \cdot d\vec{s} = 0$$

$$\nabla \cdot \vec{B} = 0$$

- Maxwell's 1st and 4th law are valid for time-varying fields.
- 2nd and 3rd laws have to be modified.

SECOND LAW IN TIME VARYING \vec{B}



$$\vec{F} = q(\vec{v} \times \vec{B})$$

Define motional emf $= [\vec{E}_m] = \frac{\vec{F}}{q} = [\vec{v} \times \vec{B}]$

$$\oint \vec{E} \cdot d\vec{l} = V_4 - V_1 = - \int_{\text{left}}^{\text{right}} \vec{E}_m \cdot dl = -BLv$$

$\star \vec{E}$ is zero on other sides

~~If~~ If $\phi = B \cdot L$

Then $\vec{E} = -\frac{d\phi}{dt}$ $\quad \quad \quad \oint \vec{E} \cdot dl = -\frac{d\phi}{dt}$

- $\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{s}$ (writing $\phi = \int \vec{B} \cdot d\vec{s}$)

$\int_S (\nabla \times \vec{E}) \cdot d\vec{s} = - \int \frac{d\vec{B}}{dt} \cdot d\vec{s}$... Stokes Theorem

$\therefore \nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$

VIII

MAXWELL'S 3RD LAW IN TIME-VARYING \vec{E}

$$\vec{\nabla} \times \vec{H} = -\frac{d}{dt} \vec{D}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{D}$$

$$= T \quad (\text{say})$$

\star \star

$$\int_{\text{vol}} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) dV = \int_{\text{vol}} T dV$$

$$\Rightarrow \oint_S (\vec{\nabla} \times \vec{H}) ds = \int_{\text{vol}} T dV$$

area of opening
in the volume

As area of opening $\rightarrow 0$, LHS = 0, RHS = 0

$$\Rightarrow T = \vec{\nabla} \cdot \vec{D} = 0$$

→ Modify :- $\vec{\nabla} \times \vec{H} = \vec{D} + \vec{G}$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{D} + \vec{\nabla} \cdot \vec{G} = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{G} = -\vec{\nabla} \cdot \vec{D} = \frac{dS_V}{dt} = \frac{d}{dt} (\vec{D} \cdot \vec{n})$$

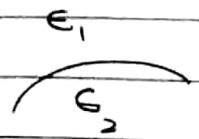
$$\therefore \boxed{\vec{G} = \frac{d\vec{D}}{dt}}$$

BOUNDARY CONDITIONS

→ Vacuum \leftarrow Metal

\vec{E} is always normal to interface

$$D_{N1} - D_{N2} = S \quad \text{surface charge}$$



$$D_{N1} = D_{N2}$$

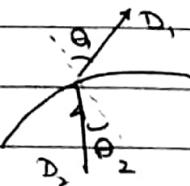
$$E_{t1} = E_{t2}$$

--- normal \vec{D} or \vec{B}

--- tangential \vec{E} or \vec{H}

same for \vec{B} and \vec{H}

REFRACTION.



$$D_1 = D_2 \sqrt{\cos^2 \theta_2 + \frac{\epsilon_1^2}{\epsilon_2^2} \sin^2 \theta_2}$$

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

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SOLUTION OF MAXWELL'S EQUATIONS

Write $\vec{E}(x, y, z, t) = \vec{E}(x, y, z) e^{j\omega t}$

$$\vec{H} = \vec{H}(x, y, z)$$

$$\vec{D} = \vec{D}(x, y, z)$$

$$\vec{B} = \vec{B}(x, y, z)$$

$$\vec{J} = \vec{J}(x, y, z)$$

$$S = S(x, y, z)$$

- Assume time-variation is sinusoidal

- $\frac{d}{dt} \vec{D} = j\omega \vec{D}$... similarly for other quantities.

$$1 \quad \vec{\nabla} \cdot \vec{B} = S \Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = S} \quad (\text{in free space}) \quad S = 0$$

$$2 \quad \vec{E} \cdot \vec{B} = 0 \Rightarrow \boxed{\vec{E} \cdot \vec{B} = 0}$$

$$3 \quad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \boxed{\vec{\nabla} \times \vec{H} = \vec{J} + j\omega \vec{D}} \quad \dots \text{in free space} \quad \vec{J} = 0$$

$$4 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -j\omega \vec{B}} \quad (\text{lossless medium})$$

In free space,

- If $J = 0$, $\vec{\nabla} \times \vec{H} = j\omega \vec{D}$ and $\vec{\nabla} \times \vec{E} = j\omega \vec{B}$
 $= j\omega (\epsilon \vec{E})$
 $= j\omega (\mu \vec{H})$

- $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -j\omega \mu (\vec{\nabla} \times \vec{H})$

$$\Rightarrow \boxed{\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = j\omega \mu (\mu \vec{H})}$$

where $\nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}$... Gradient of f

$$\nabla^2 \vec{F} = \frac{\partial^2 F_x}{\partial x^2} \hat{x} + \frac{\partial^2 F_y}{\partial y^2} \hat{y} + \frac{\partial^2 F_z}{\partial z^2} \hat{z}$$
 ... Double Gradient

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σ in free space ($S=0$)

$$\rightarrow \nabla \times (\nabla \times \bar{E}) = \bar{\nabla}(\bar{\nabla} \cdot \bar{E}) - \nabla^2 \bar{E}$$

$$\nabla \times (-j\omega \mu_0 \bar{H}) = -\nabla^2 \bar{E}$$

$$\therefore -\nabla^2 \bar{F} = j(\nabla \times -\omega_0 \bar{H}) + (\bar{\omega}_0 \bar{H})(\bar{\nabla} \times j)$$

$$= -j\omega \mu (\nabla \times \bar{H})$$

$$= -j\omega \epsilon_1 \quad (j\omega \in \mathbb{E}) \quad \dots \text{3rd law}$$

$$\therefore \quad \nabla^2 \bar{E} + \omega^2 \mu \in \bar{E} = 0$$

Similarly, if you begin with $\nabla \times (\nabla \times \vec{H})$,

$$\nabla^2 \bar{H} + \omega^2 \mu \in \bar{H} = 0$$

These two equations are called 'Helmholtz Equations'.

These are homogeneous equations.

$$c_g \mid \text{Say} \quad \bar{E} = E(z) \hat{\pi}$$

$$\therefore \text{Helmholtz equation} :: \frac{\partial^2 F_r(z)}{\partial z^2} + k^2 F_r(z) = 0 \quad \text{where } k = \omega^2 \mu \epsilon$$

$$\text{Solution : } F(z) = F_{\infty}^+ e^{-jkz} + F_{\infty}^- e^{jkz}.$$

$$\vec{E}(z, t) = E_0(z) e^{j\omega L} \\ = E^+ e^{j(\omega t - kz)} + E^- e^{j(\omega t + kz)}$$

- Real part :- $\vec{E}(z,t) = E_+^+ \cos(\omega t - kz) + E_-^- \cos(\omega t + kz)$

Wave is travelling in sinusoidal waveform in Z direction.

P.T.O.

- $k = \omega \sqrt{\mu \epsilon}$ = "Propagation constant"
 ↗ "Angular frequency" = $2\pi f$

- Wavelength = $\lambda = \frac{2\pi}{k}$

- Velocity (with which one particular phase is moving)

$$\omega t - kz = c$$

$$\omega - k \frac{dz}{dt} = 0$$

$$\therefore \boxed{v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu \epsilon}}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \text{ in free space}$$

- If we use the second Helmholtz equation, we will receive expression for \bar{H} .

- For above example,

$$\nabla \times \bar{E} = -j\omega \mu \bar{H}$$

$$\therefore \bar{H} = -\frac{1}{j\omega \mu} \nabla \times \hat{i} [E_x^+ e^{-jkz} + E_x^- e^{jkz}]$$

Solve

Solve, put $\frac{k}{\omega \mu} = \frac{1}{n}$,

$$\bar{H} = \frac{1}{n} \hat{y} [E_x^+ e^{-jkz} - E_x^- e^{jkz}]$$

where $n = \sqrt{\mu/\epsilon}$, "Wave Impedance"

in free space & unbounded media

- Verify
- \bar{E} and \bar{H} are always perpendicular and perpendicular to direction of propagation.
 - If \bar{E} expression has '+', \bar{H} expression will have '-' between terms travelling in opposite direction

3 Amplitude of \bar{H} = Amplitude of \bar{E}
 π

* π has same dimensions as $Z = V/I$

- All these observations are for lossless, uniform, unbounded media.

I LOSSY, UNIFORM, UNBOUNDED MEDIA

$$\bar{J} \neq 0 \text{ and } \sigma \neq 0$$

$$1 \quad \nabla \times \bar{E} = -j\omega \mu \bar{H}$$

$$2 \quad \nabla \times \bar{H} = j\omega \epsilon \bar{E} + \sigma \bar{E} = (\sigma + j\omega \epsilon) \bar{E}$$

- Modified Helmholtz's equation:-

$$\nabla^2 \bar{E} + \omega^2 \mu \epsilon \underbrace{\left[1 - \frac{j\sigma}{\omega \epsilon} \right]}_{-\gamma^2} \bar{E} = 0$$

$$\nabla^2 \bar{E} + -\gamma^2 \bar{E} = 0, \text{ where } \gamma = j\omega \sqrt{\mu \epsilon} \sqrt{1 - \frac{j\sigma}{\omega \epsilon}}$$

--- counterpart for ' k '

$$= \alpha + j\beta$$

where $\alpha > 0$

eq Again, consider $\bar{E} = E(z) \hat{x}$

$$\text{Solve Helmholtz: } \bar{E}(z) = E^+ e^{-\gamma z} + E^- e^{\gamma z}$$

Positive z wall wave: $E^+ e^{-\alpha z} e^{-j\beta z}$
 attenuation in $+z$

$$\alpha = \omega \sqrt{\mu \epsilon} \sqrt{\left(1 + \frac{\sigma^2}{\omega^2 \epsilon} \right) - 1}$$

Negative z wall wave: $E^- e^{\alpha z} e^{j\beta z}$
 attenuation in $-z$

$$\bar{E} = E e^{-\alpha z} e^{j(\omega t - \beta z)}$$

If $\frac{\sigma}{\omega\epsilon} \ll 1$, then

$$\alpha \approx \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}$$

$\beta \approx \omega \sqrt{\mu\epsilon}$ - k from before

$$\sqrt{j} = \frac{1+j}{\sqrt{2}}$$

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- Again, calculate $\bar{H} = -i \cdot \nabla \times \bar{E}$

$$\bar{H} = i \left[E_x^+ e^{-\gamma z} - E_x^- e^{\gamma z} \right] / n$$

- Previous observations hold.

- New $n = j\omega\mu = \frac{j\omega\mu}{\sqrt{1 - j\sigma/\omega\epsilon}}$

- Value of n depends on frequency \therefore

Verify

- Because n is complex, \bar{E} and \bar{H} are not in phase wrt time, but still perpendicular in space

A]

EM Waves in Good Conductors

$$\begin{aligned}\gamma &= \alpha + j\beta = j\omega\sqrt{\mu\epsilon} \sqrt{1 - \frac{j\sigma}{\omega\epsilon}} \quad \text{where } \sigma \text{ is very high.} \\ &= j\omega\sqrt{\mu\epsilon} \sqrt{-\frac{j\sigma}{\omega\epsilon}} \\ &= \sqrt{\omega\mu\sigma} \sqrt{j}\end{aligned}$$

$$\gamma = (1+j) \sqrt{\frac{\omega\mu\sigma}{2}}$$

- Skin depth (δ_s) :- Distance after which attenuated to 36.8% ($\frac{1}{e}$ time)

$$e^{-\alpha z} = \frac{1}{e} \Rightarrow \delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

- Skin effect :-

As frequency increases, \bar{E} cannot penetrate much radially into the wire.
Hence, current will only flow more along the circumference of wire.

$$\bullet \quad n = \frac{j\omega\mu}{\gamma} = \frac{1+j}{\sigma\delta_s}$$

For good conductors, $\arg(n) = 45^\circ$

poor $0 < \arg(n) < 45^\circ$

* Why did we not assume $\bar{E} = E_x(\hat{x}) \hat{x}$?

Because $\bar{H} = -j\omega\mu \nabla \times \bar{E}$ would become zero

- ~~For so~~ $\therefore \bar{E}$ and \bar{H} cannot vary in direction of propagation,
or be constant spatially.

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II. GENERAL SOLUTION OF WAVE EQUATION.

$$\nabla^2 \bar{E} + k^2 \bar{E} = 0$$

$$\Rightarrow \frac{\partial^2 \bar{E}}{\partial x^2} + \frac{\partial^2 \bar{E}}{\partial y^2} + \frac{\partial^2 \bar{E}}{\partial z^2} + k^2 \bar{E} = 0$$

$$\text{Write } \bar{E} = E_x(x, y, z) \hat{x} + E_y(x, y, z) \hat{y} + E_z(x, y, z) \hat{z}$$

- $\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$

Similarly for E_y, E_z

* Linear Polarization:-

When directions of \bar{E} and \bar{H} do not change during propagation

- Write $E_x(x, y, z) = f(x)g(y)h(z)$

$$\therefore ghf'' + fhg'' + fgh'' + k^2fgh = 0$$

- $\therefore \frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} + k^2 = 0$

Hence, $\frac{f''}{f}, \frac{g''}{g}, \frac{h''}{h}$ are all constants

By symmetry, all three are negative, but may not be equal

- $\therefore \frac{f''}{f} = -k_x^2, \frac{g''}{g} = -k_y^2, \frac{h''}{h} = -k_z^2$

where $k^2 = k_x^2 + k_y^2 + k_z^2$

- $f(x) = f^+ e^{-jk_x x} + f^- e^{jk_x x}$

$$g(y) = g^+ e^{-jk_y y} + g^- e^{jk_y y}$$

$$h(z) = h^+ e^{-jk_z z} + h^- e^{jk_z z}$$

- Consider only the +ve travelling wave.

$$\therefore E_x(x, y, z) = A e^{-j(k_x x + k_y y + k_z z)}$$

where $A = f^+ g^+ h^+$

Wavelength :- Distance between two equiphasic planes

Phase velocity :- $v_p = \frac{\omega}{k}$

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- Similarly,

$$E_y(x, y, z) = B e^{-j(k_x x + k_y y + k_z z)}$$

$$\text{and } E_z(x, y, z) = C e^{-j(k_x x + k_y y + k_z z)}$$

* Why are k_x, k_y, k_z equal in all directions?

- $\bar{E} = (A\hat{i} + B\hat{j} + C\hat{k}) e^{-j(\bar{k} \cdot \bar{r})}$

$$= [E_0 e^{-j(\bar{k} \cdot \bar{r})}]$$

'Plane Wave'
because \bar{E} and \bar{r} form a plane of same phase

where $\bar{k} = k_x \hat{i} + k_y \hat{j} + k_z \hat{k}$ = 'Wave number vector'

and $\bar{r} = x \hat{i} + y \hat{j} + z \hat{k}$ = 'Position vector'

• \bar{k} points along direction of propagation

$$\bar{k} = k \hat{k}$$

$$\hookrightarrow k = \sqrt{\mu \epsilon}$$

• For planes of equal phase (which are normal to direction of propagation), any point on the plane has the same projection along direction of propagation.

$$\therefore \bar{k} \cdot \bar{r} = \text{constant} \quad \text{--- for equiphasic plane}$$

- $\nabla \cdot \bar{E} = \bar{\nabla} (\bar{E}_0 e^{-j(\bar{k} \cdot \bar{r})})$

$$= \bar{E}_0 \cdot (\bar{\nabla} e^{-j(\bar{k} \cdot \bar{r})}) + e^{-j(\bar{k} \cdot \bar{r})} (\cancel{\bar{\nabla} \cdot \bar{E}_0}) \quad \text{since } \bar{E}_0 \text{ is a constant}$$

$$= \bar{E}_0 \left[\frac{\partial}{\partial x} e^{-j(k_x x + k_y y + k_z z)} \hat{i} + \frac{\partial}{\partial y} e^{-j(k_x x + k_y y + k_z z)} \hat{j} + \dots \right]$$

$$= \bar{E}_0 (-j k_x \hat{i} - j k_y \hat{j} - j k_z \hat{k}) e^{-j(\bar{k} \cdot \bar{r})}$$

$$= -i e^{-j(\bar{k} \cdot \bar{r})} (\bar{E}_0 \cdot \bar{k})$$

Since medium is uniform, $\bar{\nabla} \cdot \bar{E} = \bar{\nabla} \cdot \bar{D}$

Since medium does not have charges, $\bar{\nabla} \cdot \bar{E} = 0$.

$$\therefore \bar{E} \cdot \bar{k} = 0$$

\therefore In absence of current and charge in uniform medium,

$$\underline{\bar{E} \perp \bar{k}}$$

- $\bar{\nabla} \times \bar{E} = -j\omega \mu \bar{H}$

$$\therefore \bar{H} = \frac{j}{\omega \mu} \bar{\nabla} \times \bar{E}$$

$$= \frac{j}{\omega \mu} \left[\bar{\nabla} \times E_0 e^{-j(\bar{k} \cdot \bar{r})} \right]$$

$$= \frac{j}{\omega \mu} \left[\bar{\nabla} (e^{-j(\bar{k} \cdot \bar{r})}) \times E_0 + 0 \right]$$

$$= \frac{j}{\omega \mu} \left[\frac{\partial E_0}{\partial x} \hat{x} + \dots \right] e^{-j(\bar{k} \cdot \bar{r})}$$

$$= \frac{j}{\omega \mu} \left[-jk_x e^{-j(\bar{k} \cdot \bar{r})} \hat{x} \dots \right]$$

$$= e^{-j(\bar{k} \cdot \bar{r})} \bar{k} \times \bar{E}_0$$

$\omega \mu$

$$= \bar{k} \times \bar{E} \quad \dots \quad \bar{H} \perp \bar{E} \text{ and } \bar{H} \perp \bar{k}$$

$\omega \mu$

$$= \hat{n} \times \bar{E}$$

$\omega \mu$

\bar{H}	$= \hat{n} \times \bar{E}$
	n

- $\boxed{\bar{E}(x, y, z, t) = E_0 e^{j(\omega t - \bar{k} \cdot \bar{r})}}$

A] Circular Polarization

$$\bar{E}(x, y, z) = (E_1 \hat{x} + E_2 \hat{y}) e^{-jk \cdot r}$$

Values of E_1 and E_2 can depend on each other.

For simplicity, assume propagation along z . $\Rightarrow \vec{k} = k \hat{z}$

$$\therefore \bar{E} = (E_1 \hat{x} + E_2 \hat{y}) e^{-jk_z z}$$

- Impose $E_1 = jE_2 = E_0$ --- E_1 leads E_2 by 90°

$$\therefore E_2 = -jE_0.$$

$$\therefore \bar{E}(z) = E_0 (\hat{x} - j\hat{y}) e^{-jk_z z}.$$

$$\bar{E}(z, t) = E_0 (\hat{x} - j\hat{y}) e^{j(\omega t - k_z z)}$$

$$\therefore \bar{E}(0, t) = E_0 (\hat{x} - j\hat{y}) e^{j\omega t}.$$

$$= E_0 [\hat{x} \cos \omega t + \hat{y} \sin \omega t]$$

$$= E_0 [\hat{x} \cos \omega t + \hat{y} \sin \omega t]$$

Angle made by $\bar{E}(0, t)$ with x axis, $\phi = \tan^{-1}(\tan \omega t)$
 $= \omega t$.

- When $E_1 = jE_2$, ϕ is increasing with time

'Right hand circular polarization'

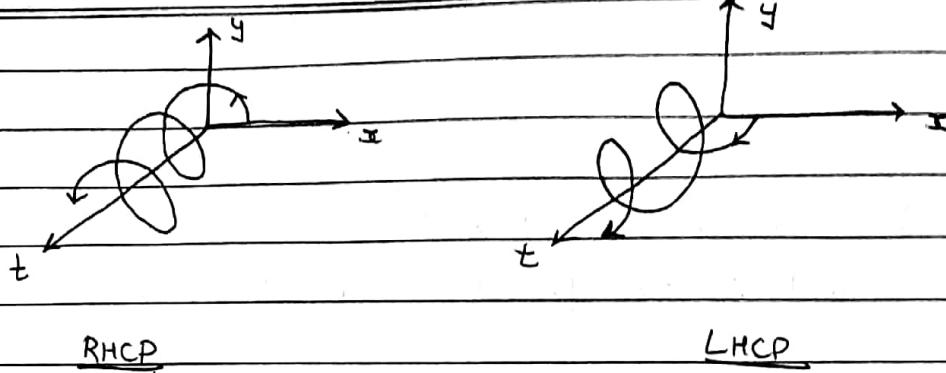
- When $jE_1 = E_2$, $\phi = -\omega t$ decreases with time

'Left hand circular polarization'

- If E_1 and E_2 magnitudes also vary with time, 'Elliptical polarization'

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$$\text{Magnetic energy} = \frac{1}{2} \mu_0 |H|^2$$



- Locus of electric field is a circle :- $E_x^2 + E_y^2 = E_0^2$

- For $\bar{E}(0, t) = E_x \cos \omega t \hat{x} + E_y \sin \omega t \hat{y}$
where $|E_x| \stackrel{?}{=} |E_y|$,

then the locus of electric field is elliptic - HW

III

Poynting's THEOREM

$$\nabla(\bar{E} \times \bar{H}) = -\bar{E} \cdot (\nabla \times \bar{H}) + \bar{H} \cdot (\nabla \times \bar{E})$$

$$\therefore \bar{E} \cdot (\nabla \times \bar{H}) = \bar{H} \cdot \nabla \times \bar{E} - \nabla(\bar{E} \times \bar{H})$$

Using $\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{B}}{\partial t}$

$$\therefore \bar{E} \cdot \bar{J} + \bar{E} \cdot \frac{\partial \bar{B}}{\partial t} = \bar{H} \cdot \left(-\frac{\partial \bar{B}}{\partial t} \right) - \nabla \cdot (\bar{E} \times \bar{H})$$

$$\therefore - \oint \bar{E} \times \bar{H} \cdot d\bar{s} = \underbrace{\int_{\text{Vol}} \bar{E} \cdot \bar{J} dv}_{\substack{\text{Resistive} \\ \text{Power} \text{ (st)}}} + \underbrace{\int_{\text{Vol}} \epsilon_0 \bar{E} \cdot \frac{\partial \bar{B}}{\partial t} dv}_{\substack{\text{Rate of change} \\ \text{of electric} \\ \text{energy}}} + \underbrace{\int_{\text{Surf}} \bar{H} \cdot \frac{\partial \bar{B}}{\partial t} dv}_{\substack{\text{Rate of change} \\ \text{of magnetic} \\ \text{energy}}}$$

$$\oint_s \bar{E} \times \bar{H} ds = \text{Total power exiting the volume}$$

$$\begin{aligned} \bar{E} \times \bar{H} &= \text{Power exiting the volume per unit area} \\ &\triangleq \text{'Pointing Vector' } (\bar{P}) \end{aligned}$$

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* Ohmic Power loss = $P_L = V \cdot I$
 $= (E_0)(J_A)$

$$\therefore P_L = \frac{\bar{E} \cdot \bar{J}}{\text{Vol}} \quad \dots \text{as used above.}$$

* Q_i is present at position i .

Charges Q_i are sequentially placed at positions i .

Work done :- $W_1 = Q_2 V_{21}$

$$W_2 = Q_3 (V_{31} + V_{32})$$

|

|

|

?

|

.

$$2W_E = Q_1 V_1 + Q_2 V_2 \dots Q_N V_N = \sum_{k=1}^N Q_k V_k$$

$$\therefore W_E = \frac{1}{2} \int_{\text{vol}} S_v V d\text{vol}$$

Write $S_v = \bar{V} \cdot \bar{D}$

$$\therefore W_E = \frac{1}{2} \int_{\text{vol}} (\bar{V} \cdot \bar{D}) V d\text{vol}$$

$$\bar{\nabla} \cdot (\bar{v} \bar{D}) = v(\bar{\nabla} \cdot \bar{D}) + \bar{D} \cdot (\bar{\nabla} v)$$

$$\therefore v(\bar{\nabla} \cdot \bar{D}) = \bar{\nabla} \cdot (v \bar{D}) - \bar{D} \cdot (\bar{\nabla} v)$$

$$\therefore W_E = \frac{1}{2} \int_{\text{vol}} [\bar{\nabla} \cdot (v \bar{D}) - \bar{D} \cdot (\bar{\nabla} v)] dv$$

$$W_E = \frac{1}{2} \cancel{\int v \bar{D} ds} + \frac{1}{2} \int_{\text{vol}} \bar{D} \cdot \bar{E} ds$$

o, for some goddamn reason

Electrostatic
Work done

$$W_E = \frac{1}{2} \int_{\text{vol}} \bar{D} \cdot \bar{E} ds \quad (\text{as used previously})$$

- Similarly, magnetic work done, $W_M = \frac{1}{2} \int_{\text{vol}} \bar{B} \cdot \bar{H} ds$

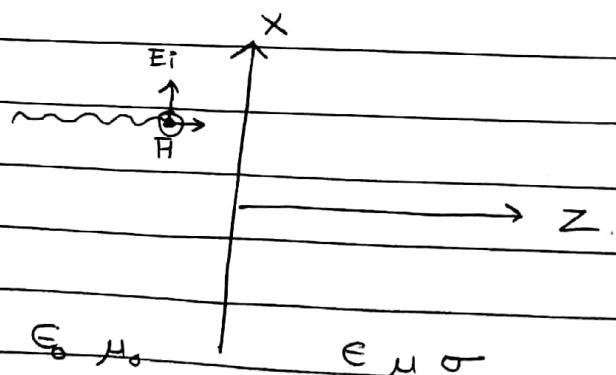
* General expression :- $W_E = \frac{1}{2} (\bar{D} \cdot \bar{E}^*) = \frac{1}{2} \epsilon_0 |\bar{E}|^2$

$$W_M = \frac{1}{2} (\bar{B} \cdot \bar{H}^*) = \frac{1}{2} \mu_0 |\bar{H}|^2$$

$$\bar{S} = \bar{E} \times \bar{H}^*$$

IV

PLANE WAVE REFLECTION



μ and T can be complex

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- Define $k_0 = \omega \sqrt{\mu_0 \epsilon_0}$, $R_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$

- $\bar{E}_p = \hat{x} E_0 e^{-jk_0 z}$, $\bar{H}_p = \hat{y} H_0 e^{-jk_0 z}$

- TFT :- r (reflection coefficient)

$$\bar{E}_r = \hat{x} r E_0 e^{jk_0 z}, \quad \bar{H}_r = -\hat{y} r H_0 e^{jk_0 z}$$

- TFT :- T (transmission coefficient)

$$\bar{E}_t = \hat{x} T E_0 e^{-\gamma z}, \quad \bar{H}_t = \hat{y} T H_0 e^{-\gamma z}$$

where $\gamma = j\omega \mu$ and $\gamma = \alpha + j\beta = j\omega \sqrt{\mu \epsilon} \sqrt{1 - \frac{j\sigma}{\omega}}$

- For $z < 0$,

$$\bar{E}_p + \bar{E}_r = \hat{x} E_0 (e^{-jk_0 z} + r e^{jk_0 z})$$

At $z = 0$, $\bar{E}_p + \bar{E}_r = \hat{x} E_0 (1 + r)$ tangential \bar{E} for $z < 0$

- For $z > 0$,

Tangential component = \bar{E}_t (at $z = 0$) = $\hat{x} T E_0$

Continuity of \bar{E} at $z = 0$

- During Boundary Condition :- Tangential \bar{E} does not change.

$$\therefore \hat{x} E_0 (1 + r) = E_0 T$$

$$\therefore 1 + r = T$$

For $z < 0$, $\bar{H}_l + \bar{H}_r = \frac{E_0}{h_0} (1 - r) e^{jk_0 z} \hat{y}$

For $z > 0$, $\bar{H}_l = \frac{\hat{y}}{h} TE_0 e^{-jk_0 z}$

Continuity of \bar{H} at $z=0$:- $|1 - r| = \frac{n_0 T}{n}$

→ If second medium ($z > 0$) is also lossless,

$$\sigma = 0$$

$$\gamma = j\omega \sqrt{\mu_r \epsilon_r} = j\beta \quad (\alpha = 0)$$

$$= jk_0 \sqrt{\mu_r \epsilon_r}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{k_0 \sqrt{\mu_r \epsilon_r}} = \frac{\lambda_0}{\sqrt{\mu_r \epsilon_r}}$$

$$v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

→ Poynting Vector :- (Lossless Medium)

For $z < 0$, $S^- = \bar{E} \times \bar{H}^*$

$$= (\bar{E}_l + \bar{E}_r) \times (\bar{H}_l + \bar{H}_r)^*$$

$$= \hat{x} E_0 (-e^{-jk_0 z} + r e^{jk_0 z}) \times \hat{y} E_0^* \frac{(e^{-jk_0 z} - r e^{jk_0 z})}{h_0}^*$$

$$= \hat{z} \frac{E_0^2}{h_0} [1 - |r|^2 + 2j r \sin(2k_0 z)]$$

At $z > 0$, $S^- = \hat{z} \frac{E_0^2}{h_0} [1 - |r|^2]$

$$T = \frac{n_2 - n_1}{n_2 + n_1}$$

$$T = \frac{2n_2}{n_2 + n_1}$$

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- For $z > 0$, $S^+ = \bar{E}_t \times \bar{H}_E^*$

$$= \hat{\epsilon} T E_0 e^{-\gamma z} \times \hat{j} \frac{T^* E^*}{h^*} e^{-\gamma^* z}$$

$$= \hat{\epsilon} \frac{|T|^2 E_0^2}{h} \quad (\text{at } z=0)$$

$$= \hat{\epsilon} \frac{4n^2}{(n+n_0)^2} \times \frac{E_0^2}{h}$$

$$= \hat{\epsilon} \frac{4n |E_0|^2}{(n+n_0)^2}$$

$$= \hat{\epsilon} \frac{E_0^2 (1 - |r|^2)}{h_0}$$

- ∴ $\bar{S}^- = \bar{S}^+$ at boundary

★ • $P^- = \frac{1}{2} \operatorname{Re} [\bar{S}^- \cdot \hat{z}]$ and $P^+ = \frac{1}{2} \operatorname{Re} [S^+ \cdot \hat{z}]$

Because of time-averaging of square of sinusoidal wave

- Define $\bar{S}_r \triangleq \bar{E}_r \times \bar{H}_r^* = \hat{\epsilon} \frac{E_0^2}{h_0}$ at $z=0$

$$\bar{S}_r \triangleq \bar{E}_r \times \bar{H}_r^* = -\hat{\epsilon} \frac{E_0^2 |r|^2}{h_0} \text{ at } z=0$$

For any z , note that $\bar{S}_r + \bar{S}_i \neq \bar{S}^-$

$\bar{S}_r + \bar{S}_i$ does not have the sinusoidal term

- ∴ $\bar{S}_i + \bar{S}_r = \operatorname{Re}(\bar{S}^-) = \text{Real Power transferred}$

- The imaginary sinusoidal term in \bar{S}^- denotes energy stored in standing waves