Problem 4.22

(a) Starting with Eq. (4.15) for sinusoidal FM, we write

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)] \tag{1}$$

where f_m is the modulation frequency and β is the modulation index. Correspondingly, the Fourier transform of $s(t)$ is defined for an arbitrary value of β (see Eq. (4.31))

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m)] \tag{2}$$

where $J_n(\beta)$ is the nth order Bessel function of the first kind. Passing $s(t)$ through a linear channel of transfer function $H(f)$ produces an output signal $y(t)$ whose Fourier transform is defined by

$$Y(f) = H(f)S(f)$$

$$= \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[H(f_c + nf_m)\delta(f - f_c - nf_m) + H(-f_c - nf_m)\delta(f + f_c + nf_m)] \tag{3}$$

Applying the inverse Fourier transform to $Y(f)$ yields the output signal

$$y(t) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[H(f_c + n f_m)\exp(j2\pi(f_c + n f_m)t)] + H(-f_c - n f_m)\exp(-j2\pi(f_c + n f_m)t) \tag{4}$$

For a channel with real-valued impulse response, we have $H(f) = H^*(-f)$ where the asterisk denotes complex conjugation. We may therefore rewrite Eq. (4) as

$$y(t) \approx \frac{1}{2} A_c \sum_{n=-\infty}^{\infty} J_n(\beta)[H(f_c + n f_m)\exp(j2\pi(f_c + n f_m)t)] + H^*(f_c + n f_m)\exp(-j2\pi(f_c + n f_m)t)$$

$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\mathbf{Re}[H(f_c + n f_m)\exp(j2\pi(f_c + n f_m)t)] \tag{5}$$

where **Re** denotes the real-time operator.

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Problem 4-22 continued

(b) Following the development of the universal curve plotted in Fig. 4.9, let $n_{\max}$ denote the largest value of n in Eq. (5) for which the condition

$|J_n(\beta)| > 0.1$

is satisfied so as to preserve the effective frequency content of the FM signal $s(t)$. We may then approximate Eq. (5) as

$$y(t) \approx A_c \sum_{n=-n_{\max}}^{n_{\max}} J_n(\beta)\mathbf{Re}[H(f_c + nf_m)\exp(j2\pi(f_c + nf_m)t)] \qquad (6)$$

Expressing the transfer function $H(f)$ in the polar form

$$H(f) = |H(f)|\exp(j\phi(f)) \qquad (7)$$

we may rewrite Eq. (6) as

$$y(t) \approx A_c \sum_{n=-n_{\max}}^{n_{\max}} J_n(\beta)|H(f_c + nf_m)|\cos(2\pi(f_c + nf_m)t + \phi(f_c + nf_m)) \qquad (8)$$

From the discussion presented in Section 2.7, recall that the transmission of a signal through a linear channel (filter) is distortionless provided that two conditions are satisfied:

(i) The amplitude response $|H(f)|$ is constant over the band $-B \le f \le B$, where B is the channel bandwidth.

(ii) The phase response $\phi(f)$ is a linear function of the frequency f inside the band $-B \le f \le B$.

Accordingly, in the context of our present discussion, the FM transmission through the channel of transfer function $H(f)$ introduces two forms of linear distortion:

(i) *Amplitude distortion*, which arises when the condition

$|H(f_c + nf_m)|$ is constant for $0 \le n \le n_{\max}$

is violated.

(ii) *Phase distortion*, when the condition

$\theta(f_c + nf_m)$ is a linear function of n for $0 \le n \le n_{\max}$

is violated.