Problem 6.3

Given that $P(f)$ is the Fourier transform of a pulse-like function $p(t)$, we may state the following theorem:[1]

The pulse $p(t)$ decreases asymptotically with time as $1/t^{k+1}$ provided that the following two conditions hold:

1. The first k-1 derivatives of the Fourier transform $P(f)$ with respect to frequency f are all continuous.
2. The kth derivative of $P(f)$ is discontinuous.

Demonstrate the validity of this theorem for the three different values of α plotted in Fig. 6.3(a).

Solution

Consider first the idealized Nyquist channel for which $\alpha = 0$. With the brick-wall characteristic of this limiting case, it is immediately apparent that the Fourier transform $P(f)$ has *no* continuous derivatives with respect to f. Hence, according to the theorem, the inverse Fourier transform $p(t)$ decreases asymptotically as $1/|t|$; this is confirmed by the formula of Eq. (6.14), where the numerator ranges between -1 and +1, whereas the denominator is proportional to t.

Consider next the case of a raised-cosine pulse $p(t)$ defined in Eq. (6.19), rewritten here as

$$p(t) = \sqrt{E}\ \frac{\sin(2\pi B_0 t)}{2\pi B_0 t}\left(\frac{\cos(2\pi\alpha B_0 t)}{1-16\alpha^2 B_0^2 t^2}\right)$$

In this case, we readily see that $p(t)$ decreases asymptotically as $1/|t|^3$, for $0 < \alpha \leq 1$. Examining the two plots shown in Fig. 6.3(a), we see that the first derivative of $P(f)$ for this range of values of α is continuous, but the second derivative is discontinuous. Here again validity of the theorem is established.

1. For a detailed discussion of this theorem, see Gitlin, Hayes and Weinstein (1992), p.258.